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Introduction

For a class $\mathscr{K}$ of partial orders and a cardinal κ, let $\mathrm{FA}_\kappa(\mathscr{K})$ be the following statement.

If $\mathscr{P}$ is a poset in $\mathscr{K}$ and $\mathscr{D}$ a family of κ dense subsets of $\mathscr{P}$ then there exists a filter G in $\mathscr{P}$ such that $G \cap D \neq \emptyset$, for every $G \in \mathscr{D}$.

Let $\mathrm{FA}(\mathscr{K})$ be $\mathrm{FA}_\kappa(\mathscr{K})$, for all $\kappa < 2^{\aleph_0}$. Thus Martin's Axiom (MA) is just FA(ccc) while the Proper Forcing Axiom is $\mathrm{FA}_{\aleph_1}$ (proper) (for information on PFA see [Ba] or [Sh]). In the proof of the consistency of MA one performs a finite support iteration of ccc posets and it is known that such iterations of arbitrary length preserve all cardinals. Thus, one is able to show that MA is consistent with $2^{\aleph_0}$ arbitrary large. However if one wants to iterate non-ccc posets and preserve $\aleph_1$ countable support iteration is needed and such iterations make CH true at limit stages of cofinality ω_1. This puts severe restrictions on the value of the continuum in such models. In fact, it is known that some extensions of $\mathrm{MA}_{\aleph_1}$ imply that $2^{\aleph_0} = \aleph_2$. The weakest such axiom is $\mathrm{FA}_{\aleph_1}$ for the class of partial orders of size $\leqslant 2^{\aleph_0}$ which can be written as a composition of a σ-closed and a ccc poset (see [Ve]).

Thus, the problem arises to find a class $\mathscr{K}$ of posets richer than the class of all ccc posets for which $\mathrm{FA}(\mathscr{K})$ is consistent with the continuum being large. A weaker version of this question is that of the consistency of $\mathrm{FA}_{\aleph_1}(\mathscr{K})$ together with $2^{\aleph_0} > \aleph_2$. This latter problem was addressed by Groszek and Jech ([GJ]) who isolated a class $\mathscr{K}$ of posets possessing a certain type of fusion property and proved the consistency of $\mathrm{FA}_{\aleph_1}(\mathscr{K})$ with $2^{\aleph_0} > \aleph_2$. Their approach was to consider an iteration along a suitably chosen ω_2-like directed set instead of the usual linear iterations.

In this paper their result is extended for the case of the partial order $\mathscr{S}$ of all perfect trees ([Sa]). Thus, starting with a regular cardinal $\kappa > \aleph_1$ such that $\kappa^{<\kappa} = \kappa$, a ccc generic extension of the universe is found satisfying $\mathrm{MA} + 2^{\aleph_0} = \kappa$ together with the following statement $\mathrm{CCC}(\mathscr{S})$.

For every family $\mathscr{D}$ of $2^{\aleph_0}$ dense subsets of $\mathscr{S}$ there is a ccc perfect subposet $\mathscr{P}$ of $\mathscr{S}$ such that $D \cap \mathscr{P}$ is dense in $\mathscr{P}$, for all $D \in \mathscr{D}$.

Clearly, CCC($\mathscr{S}$) in the presence of MA_κ implies $\mathrm{FA}_\kappa(\mathscr{S})$. This generic extension is constructed as follows. One performs a standard finite support iteration forcing MA + $2^{\aleph_0} = \kappa$. Simultaneously, one generates many powerfully ccc suborders of $\mathscr{S}$. This is done following a construction of Jensen ([Jen]), but instead of $\diamondsuit$ forcing is used at each stage to adjoin to each of the current perfect posets a sufficiently generic tree. At limit steps, one 'seals' the ccc of all the posets produced so far by forcing with the ω-power of their product. The delicate part of the argument is showing that at a limit stage of cofinality ω_1 this 'sealing' poset is ccc. The argument is in fact reminiscent of some applications of $\diamondsuit$. It then remains to verify that for every $\mathscr{D}$, a family of $2^{\aleph_0}$ dense subsets of $\mathscr{S}$, one of the ccc posets constructed captures the density of each $D \in \mathscr{D}$. In fact, in the above construction one can replace $\mathscr{S}$ by any of the standard posets for adding a real, indeed any of the fusion posets from [GJ].

One consequence of CCC($\mathscr{S}$) concerns a question in the theory of degrees of constructibility. While it is well-known that Sacks, Laver, and Silver forcing notions introduce minimal reals over the ground model no such example of a ccc poset is known in ZFC (Jensen ([Jen]) needed $\diamondsuit$ for his construction; Groszek ([Gr]) has constructed such an example under CH). Another one involves an old problem of von Neumann (see [Ma, problem 261]) who asked whether there is a weakly-distributive countably generated ccc complete Boolean algebra which is not a measure algebra. CCC($\mathscr{S}$) resolves both of these questions. Namely, it implies that there is a weakly distributive ccc poset which adjoins a real of minimal degree over V. Such a poset clearly cannot add random reals. To obtain it one simply applies CCC($\mathscr{S}$) to a suitable family of $2^{\aleph_0}$ dense subsets of $\mathscr{S}$ to extract a ccc suborder of $\mathscr{S}$ having the required properties. It is also shown that CCC($\mathscr{S}$) implies that $\mathscr{S}$ preserves all cardinals, and that in the presence of MA it makes the ideal of s_0-sets $< 2^{\aleph_0}$-complete.

Finally, on the negative side, PFA is shown to imply the failure of CCC($\mathscr{S}$). A similar argument is used to answer a question of Miller who asked if $\mathrm{MA}_{\aleph_1}$ implies that the ideal of s_0-sets is closed under $\aleph_1$-unions. Namely, starting with a model of PFA a generic extension is constructed which has the same subsets of ω_1 as the ground model and which contains a dense set D of perfect trees such that every real belongs to the closure of at most countably many members of D. This easily implies that the reals can be covered by $\aleph_1$ s_0-sets. In addition, it shows that $\mathrm{MA}_{\aleph_1}$ does not imply $\mathrm{FA}_{\aleph_1}(\mathscr{S})$.

The paper is organized as follows. Section 1 contains the basic properties of perfect posets and some technical lemmas. The consistency of MA + $2^{\aleph_0} = \kappa$ + CCC($\mathscr{S}$) is proved in Section 2. The above-mentioned applications are deduced in Section 3. Finally, Section 4 contains the proof that PFA refutes CCC($\mathscr{S}$) and the consistency proof of $\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} = \aleph_2 + \neg \mathrm{FA}_{\aleph_1}(\mathscr{S})$.

Our forcing terminology is mostly standard and can be found in either [Jec] or [Ku]. We denote the ground model by V. If $\mathbb{P}$ is a poset then $V^{\mathbb{P}}$ is the Boolean-valued universe using RO($\mathbb{P}$). A subset D of $\mathbb{P}$ is called *predense* iff for every $p \in \mathbb{P}$ there exists $q \in D$ such that p and q are compatible. If $\mathbb{P}$ and $\mathbb{Q}$ are partial orders and $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ we identify $V^{\mathbb{P}}$ with a subuniverse of $V^{\mathbb{Q}}$.

1. Basic properties of perfect posets

In this section we present the basic definitions and properties of perfect posets and prove some lemmas that will be needed subsequently. Much of the material is well-known and is reproduced for the convenience of the reader. For more details see [Ab], [BL], and [Mi].

Perfect trees

Let $2^{<\omega}$ denote the set of all finite $\{0,1\}$-sequences ordered by extension. $T \subseteq 2^{<\omega}$ is called a *perfect tree* if it is an initial segment of $2^{<\omega}$ and every element of T has two incomparable extensions in T. Let $\mathscr{S}$ denote the poset of all perfect trees partially ordered by inclusion. Thus, $\mathscr{S}$ is the well-known Sacks forcing ([Sa]). For $T \in \mathscr{S}$ let $[T]$ denote the set of all infinite branches through T. Then $[T]$ is a perfect subset of 2^{ω}. For $s \in T$ let $T_s = \{t \in T: s \subseteq t \text{ or } t \subseteq s\}$. If T_i is a perfect tree, for $i < n$, we shall identify $\langle T_0, \ldots, T_{n-1}\rangle$ with the perfect tree

$$\{\langle t_0, \ldots, t_{n_1}\rangle: t_i \in T_i \text{ and } \mathrm{lh}(t_i) = \mathrm{lh}(t_j), \text{ for } i, j < n\}$$

If $\sigma = \langle s_0, \ldots, s_{n-1}\rangle \in T^n$ let $T(\sigma)$ denote $\langle T_{s_0}, \ldots, T_{s_{n-1}}\rangle$.

Meet and join

Given two perfect trees T and S, the *meet* $T \wedge S$ is defined as the largest perfect tree contained in both T and S, if it exists, or else $\varnothing$. Thus $[T \wedge S]$ is simply the perfect kernel of $[T] \cap [S]$. The *join* $T \vee S$ is simply $T \cup S$. The following facts are easily verified:

(a) $(T \wedge S)_s = T_s \wedge S_s$,
(b) $(T_1 \vee T_2) \wedge S = (T_1 \wedge S) \vee (T_2 \wedge S)$.

Perfect posets

A collection $\mathscr{P}$ of perfect trees is called a *perfect poset* provided:

(a) $(2^{<\omega})_s \in \mathscr{P}$, for all $s \in 2^{<\omega}$,
(b) if $T, S \in \mathscr{P}$ then $T \vee S \in \mathscr{P}$, and if in addition, $T \wedge S \neq \varnothing$ then $T \wedge S \in \mathscr{P}$.

The order on $\mathscr{P}$ is inclusion. Note that (b) implies that any two members of $\mathscr{P}$ are compatible in $\mathscr{P}$ if and only if they are compatible in $\mathscr{S}$. If $\mathscr{P}$ is a perfect poset and T a perfect tree let $\mathscr{P}[T]$ be the collection of all joins of finite subsets of $\mathscr{P} \cup \{T_s: s \in T\}$. Under certain conditions $\mathscr{P}[T]$ will be a perfect poset. If $\mathscr{P}$ and $\mathscr{Q}$ are perfect posets let $\mathscr{P} \times \mathscr{Q}$ be the collection of all $\langle T, S \rangle$ where $T \in \mathscr{P}$ and $S \in \mathscr{Q}$. Then $\mathscr{P} \times \mathscr{Q}$ is not a perfect poset since it is not closed under joins. However, the collection of all joins of finite subsets of $\mathscr{P} \times \mathscr{Q}$ forms a perfect poset.

The amoeba poset $\mathscr{A}(\mathscr{P})$

To each perfect poset $\mathscr{P}$ we associate the *amoeba poset* $\mathscr{A}(\mathscr{P})$. Elements of $\mathscr{A}(\mathscr{P})$ are pairs (T, n), where $T \in \mathscr{P}$ and $n \in \omega$. Say that $(T, n) \leqslant (S, m)$ iff $T \leqslant S$, $n \geqslant m$, and $T \cap 2^m = S \cap 2^m$. Note that $\mathscr{A}(\mathscr{P})$ is ccc iff any finite power of $\mathscr{P}$ is.

$\mathscr{A}(\mathscr{P})$-generic trees

Suppose $\mathscr{P}$ is a perfect poset and V is a universe of set theory containing $\mathscr{P}$. If G an $\mathscr{A}(\mathscr{P})$-generic filter over V let

$$T(G) = \bigcup \{T \cap 2^n: (T, n) \in G\}.$$

Then, by genericity, $T(G)$ is a perfect tree and is called the *$\mathscr{A}(\mathscr{P})$-generic tree* derived from G. If $\mathscr{P}_i$ is a perfect poset, for $i < n$, and G is $\times_{i<n}\mathscr{A}(\mathscr{P}_i)$ generic one canonically defines a sequence $\langle T_i(G): i < n \rangle$ of generic perfect trees.

Complete extensions

Let $\mathscr{P}$ and $\mathscr{Q}$ be posets such that $\mathscr{P}$ is a suborder of $\mathscr{Q}$, and let V be a universe of set theory. We say that $\mathscr{Q}$ is a *complete extension* of $\mathscr{P}$ over V iff for every $n < \omega$, every $D \in V$ which is predense in $\mathscr{P}^n$ is also predense in $\mathscr{Q}^n$.

$\mathscr{A}(\mathscr{P})$-weakly generic trees

Suppose $\mathscr{P}$ is a perfect poset and V a universe of set theory. We say that a perfect tree T is *$\mathscr{A}(\mathscr{P})$-weakly generic* over V if for every $D \in V$ which is predense in $\mathscr{P}^n$, for some n, every $m < \omega$, and 1-1 sequence $\sigma \in (T \cap 2^m)^n$ there exists a finite subset X of D such that $T(\sigma) \leqslant \bigvee X$.

Note that if T is $\mathscr{A}(\mathscr{P})$-generic over a universe V containing $\mathscr{P}$ then it is also weakly generic over V. Moreover a perfect subtree of a weakly generic tree is itself weakly generic. This is not necessarily true for generic trees and is the main reason for introducing the notion of weak genericity.

Let $\mathscr{P}$ be a perfect poset, V a universe of set theory, and $\mathscr{F}$ a collection of perfect posets. Say that a perfect tree T is *$\mathscr{A}(\mathscr{P})$-weakly generic* over $(V, \mathscr{F})$ if for every poset $\mathscr{Q}$ which is a finite product of members of $\mathscr{F}$, every $D \in V$ which is predense in $\mathscr{P}^n \times \mathscr{Q}$, every $S \in \mathscr{Q}$, $m < \omega$, and a 1-1 sequence $\sigma \in (T \cap 2^m)^n$, there exists $R \leqslant S$, and a finite subset X of D such that $\langle T(\sigma), R\rangle \leqslant \bigvee X$.

One can generalize the above definitions for product of perfect posets. Thus, suppose $\mathscr{P}_i$ is a perfect poset, for $i < n$. A sequence $\langle T_i : i < n\rangle$ of perfect trees is called $\times_{i<n}\mathscr{A}(\mathscr{P}_i)$-*weakly generic* over V if for every $\langle n_i : i < n\rangle \in \omega^n$, every $D \in V$ a predense subset of $\times_{i<n}\mathscr{P}_i^{n_i}$, all $m < \omega$, and 1-1 sequences $\sigma_i \in (T_i \cap 2^m)^{n_i}$, for $i < n$, there exists a finite subset X of D such that $\langle T_i(\sigma_i) : i < n\rangle \leqslant \bigvee X$. Similarly one defines what it means that a sequence $\langle T_i : i < n\rangle$ is weakly generic over $(V, \mathscr{F})$.

We now prove some lemmas that will be useful in the proof of the main theorem. They may appear somewhat technical but the reader should bear in mind that we are trying to extend perfect posets while preserving the predensity of certain sets. Instead of adjoining a generic tree T to a perfect poset one wants to adjoin a perfect subtree S of T. Unfortunately, S need not be generic itself. However, weak genericity suffices. An additional complication is that we in fact have a collection of perfect posets which we would like to extend while preserving predense subsets of finite products of these posets. Special care needs to be taken since we are not extending all of these posets at the same time. This is the reason for introducing the notion of weak genericity over $(V, \mathscr{F})$. Note that Lemma 3 below is analogous to the product lemma in forcing.

LEMMA 1. *Let $\mathscr{P}$ be a perfect poset and V be a universe of set theory containing $\mathscr{P}$. Suppose T is $\mathscr{A}(\mathscr{P})$-weakly generic over V. Then $\mathscr{P}[T]$ is a perfect poset.*

Proof. Clearly, (a) in the definition of perfect posets is satisfied. To verify (b) it suffices to show that for every $R \in \mathscr{P}$ there is $m \in \omega$ and $A \subseteq T \cap 2^m$ such that: $R \wedge T$ equals $\bigvee\{T_s : s \in A\}$. To that end define D_0 to be the set of all $S \in \mathscr{P}$ such that $S \leqslant R$, and D_1 to be the set of all $S \in \mathscr{P}$ such that $S \cap R$ is finite. Clearly, $D_0 \cup D_1$ is dense in $\mathscr{P}$. By the weak genericity of T pick a finite subset X of $D_0 \cup D_1$ such that $T \leqslant \bigvee X$. Let $X_i = X \cap D_i$, for $i = 0, 1$. Find m sufficiently large such that if $S_0 \in X_0$ and $S_1 \in X_1$ then $S_0 \cap S_1 \subseteq 2^{<m}$. Finally, let A be the set of $s \in T \cap 2^m$ such that $T_s \leqslant \bigvee\{S : S \in X_0\}$. □

LEMMA 2. *Suppose $\mathscr{P}$ is a perfect poset, $\mathscr{F}$ is a collection of perfect posets containing $\mathscr{P}$, and $\mathscr{Q}$ is a finite product of members of $\mathscr{F}$. Assume that a perfect tree T is $\mathscr{A}(\mathscr{P})$-weakly generic over $(V, \mathscr{F})$, where V is a universe of set theory. Then $\mathscr{P}[T] \times \mathscr{Q}$ is a complete extension of $\mathscr{P} \times \mathscr{Q}$ over V.*

Proof. Suppose that $D \in V$ is a predense subset of $\mathscr{P}^n \times \mathscr{Q}^m$. Without loss of generality $m = 1$. Let $\langle S_0, \ldots, S_{n-1}, Q\rangle$ be an element of $\mathscr{P}[T]^n \times \mathscr{Q}$. We may assume that for some $l \leqslant n$ and some $p < \omega$, there exists a 1-1 sequence $\sigma \in (T \cap 2^p)^l$ such that $S_i = T_{\sigma(i)}$ for $i < l$, and that $S \in \mathscr{P}$, for $i \geqslant l$. Let $R = \langle S_l, \ldots, S_{n-1}, Q\rangle$. Then applying the weak genericity of T to D and R one obtains an element of D which is compatible with $\langle S_0, \ldots, S_{n-1}, Q\rangle$. □

Suppose $\mathscr{P}_i$ is a perfect poset, for $i < n$. Let $\langle T_i: i < n\rangle$ be $\times_{i<n}\mathscr{A}(\mathscr{P}_i)$-weakly generic over $(V, \mathscr{F})$, and let $\mathscr{Q}$ be as in the lemma. Then, one shows in exactly the same way that $(\times_{i<n}\mathscr{P}_i[T_i]) \times \mathscr{Q}$ is a complete extension of $(\times_{i<n}\mathscr{P}_i) \times \mathscr{Q}$ over V.

LEMMA 3. *Suppose T is $\mathscr{A}(\mathscr{P})$-weakly generic over $(V, \mathscr{F})$. Let W be a universe of set theory containing $V \cup \mathscr{F} \cup \{T\}$, and let $\mathscr{Q}$ be any perfect poset in $\mathscr{F}$. Assume that S is $\mathscr{A}(\mathscr{Q})$-generic over W. Then $\langle T, S\rangle$ is $\mathscr{A}(\mathscr{P}) \times \mathscr{A}(\mathscr{Q})$-weakly generic over $(V, \mathscr{F})$.*

Proof. Let $\mathscr{R}$ be any finite product of posets in $\mathscr{F}$, let $D \in V$ be predense in $\mathscr{P}^n \times \mathscr{Q}^m \times \mathscr{R}$. By a density argument, using the fact that all the relevant information is in W, it suffices to show that for every $(Q, h) \in \mathscr{A}(\mathscr{Q})$ there exist $(Q^*, h) \in \mathscr{A}(\mathscr{Q})$, $R^* \in \mathscr{R}$, and a finite subset X of D such that $(Q^*, h) \leqslant (Q, h)$, $R^* \leqslant R$, and for every 1-1 sequences $\tau \in (T \cap 2^h)^n$ and $\sigma \in (Q^* \cap 2^h)^m$

$$\langle T(\tau), Q^*(\sigma), R^*\rangle \leqslant \bigvee X.$$

To that end fix an enumeration $\langle \sigma_i: i < N\rangle$ of 1-1 sequences in $(Q \cap 2^h)^m$. Inductively build: a sequence $\langle Q_i: i \leqslant N\rangle$ of perfect trees, a decreasing sequence $\langle R_i: i \leqslant N\rangle$ of conditions in $\mathscr{R}$, and a sequence $\langle X_i: i \leqslant N\rangle$ of finite subsets of D as follows.

Set $Q_0 = Q$, $R_0 = R$, and $X_0 = \varnothing$. Suppose Q_i, R_i, and X_i have been constructed. Considering $\langle Q_i(\sigma_i), R_i\rangle$ as an element of $\mathscr{Q}^m \times \mathscr{R}$ and using the weak genericity of T, find $\langle Q^*_{i+1}, R_{i+1}\rangle \leqslant \langle Q_i(\sigma_i), R_i\rangle$ and $X_{i+1} \in [D]^{<\omega}$ such that for every 1-1 sequence $\tau \in (T \cap 2^h)^{(n)}$

$$\langle T(\tau), Q^*_{i+1}, R_{i+1}\rangle \leqslant \bigvee X_{i+1}.$$

Then set

$$Q_{i+1} = Q^*_{i+1} \cup \{(Q_i)_s: s \in (Q_i \backslash Q^*_{i+1}) \cap 2^h\}.$$

In the end set $Q^* = Q_N$, $R^* = R_N$, and $X = \bigcup \{X_i: i \leqslant N\}$. □

The higher dimensional analog of this lemma is proved in exactly the same way. Thus, if $\langle T_0, \ldots, T_{n-1}\rangle$ is $\times_{i<n}\mathscr{A}(\mathscr{P}_i)$-weakly generic sequence of trees over $(V, \mathscr{F})$ and if $\langle S_0, \ldots, S_{m-1}\rangle$ is $\times_{j<m}\mathscr{A}(\mathscr{Q}_j)$-generic over W which contains $V \cup \mathscr{F}$ then $\langle T_0, \ldots, T_{n-1}, S_0, \ldots, S_{m-1}\rangle$ is weakly generic over $(V, \mathscr{F})$.

LEMMA 4. *Let V be a universe of set theory, $\mathscr{P}$ and $\mathscr{Q}$ perfect posets, and $\mathscr{F}$ a collection of perfect posets such that $\mathscr{Q} \in \mathscr{F}$. Suppose that $\langle T, S\rangle$ is $\mathscr{A}(\mathscr{P}) \times \mathscr{A}(\mathscr{Q})$-weakly generic over $(V, \mathscr{F})$, and that S_r does not belong to V, for every $r \in S$. Then T is $\mathscr{A}(\mathscr{P})$-weakly generic over $(V, \mathscr{F} \cup \{\mathscr{Q}[S]\})$.*

Proof. Let $\mathscr{R}$ be a finite product of posets in $\mathscr{F}$, and let $D \in V$ be a predense subset of $\mathscr{P}^n \times \mathscr{Q}[S]^m \times \mathscr{R}$, for some $n, m < \omega$. Since $\mathscr{Q}[S] \cap V = \mathscr{Q} \cap V$, it follows that D is actually predense in $\mathscr{P}^n \times \mathscr{Q}^m \times \mathscr{R}$.

Suppose $\langle Q_0, \ldots, Q_{m-1}, R\rangle \in \mathscr{Q}[S]^m \times \mathscr{R}$ and $l < \omega$ are given. We have to find

$$\langle Q_0^*, \ldots, Q_{m-1}^*, R^*\rangle \leqslant \langle Q_0, \ldots, Q_{m-1}, R\rangle$$

and finite $X \subseteq D$ such that for every 1-1 sequence $\tau \in (T \cap 2^l)^n$

$$\langle T(\tau), Q_0^*, \ldots, Q_{m-1}^*, R^*\rangle \leqslant \bigvee X.$$

We may assume that for some $r \leqslant m$, there is for each $i < r$ an $s_i \in S$ such that $Q_i = S_{s_i}$, and that $Q_i \in \mathscr{Q}$, for $i \geqslant r$. Then find $p \geqslant l$ and a 1-1 sequence $\langle t_0, \ldots, t_{r-1}\rangle$ in $(S \cap 2^p)^r$ such that for all $i < r$, $s_i \subseteq t_i$. Using the fact that $\langle T, S\rangle$ is weakly generic and that $\mathscr{Q} \in \mathscr{F}$ we find

$$\langle Q_r^*, \ldots, Q_{m-1}^*, R^*) \leqslant \langle Q_r, \ldots, Q_{m-1}, R\rangle$$

and finite $X \subseteq D$ such that for all 1-1 sequences $\tau \in (T \cap 2^p)^n$ and $\sigma \in (S \cap 2^p)^r$

$$\langle T(\tau), S(\sigma), Q_r^*, \ldots, Q_{m-1}^*, R^*\rangle \leqslant \bigvee X.$$

Set then $Q_i^* = S_{t_i}$ for $i < r$. Then $\langle Q_0^*, \ldots, Q_{m-1}^*, R^*\rangle$ and X are as required. □

Like the previous two, this lemma can be generalized to higher dimensions. Thus, if $\mathscr{P}_i$, for $i < n$, and $\mathscr{Q}_j$, for $j < m$, are perfect posets and $\langle T_0, \ldots, T_{n-1}, S_0, \ldots, S_{m-1}\rangle$ is a sequence of $(\times_{i<n}\mathscr{A}(\mathscr{P}_i)) \times (\times_{j<m}\mathscr{A}(\mathscr{Q}_j))$-weakly generic trees over $(V, \mathscr{F})$, and if $(S_j)_r \notin V$ for every $r \in S_j$, then $\langle T_0, \ldots, T_{n-1}\rangle$ is $\times_{i<n}\mathscr{A}(\mathscr{P}_i)$-weakly generic over $(V, \mathscr{F} \cup \{\mathscr{Q}_j[S_j]: j < m\})$.

2. Consistency of CCC($\mathscr{S}$) + MA + $2^{\aleph_0} = \kappa$

In this section we prove the relative consistency of CCC($\mathscr{S}$) + MA together with the continuum arbitrary large. Starting with a cardinal $\kappa > \aleph_1$ such that $\kappa^{<\kappa} = \kappa$, we perform a standard finite support iteration of ccc posets of length κ forcing MA + $2^{\aleph_0} = \kappa$. Simultaneously, in order to obtain CCC($\mathscr{S}$), we generate many ccc perfect posets. Let H_κ be the collection of all sets hereditarily of size $< \kappa$. For each $\alpha \leqslant \kappa$ and $f: \alpha \to H_\kappa$ which is in the ground model, a perfect poset $\mathscr{P}_f$ will be defined in the 2α-th stage of the iteration. Special care will have to be taken to ensure that all these posets satisfy the ccc and that for every family $\mathscr{D}$ of κ dense open subsets of $\mathscr{S}$ in the extension there exists $f \in (H_\kappa^\kappa)^V$ such that $\mathscr{P}_f$ captures the density of all sets in $\mathscr{D}$.

THEOREM 1. *Assume* ZFC. *Let κ be a cardinal $> \aleph_1$ such that $\kappa^{<\kappa} = \kappa$. Then there exists a ccc poset $\mathbb{P}$ such that $V^{\mathbb{P}}$ satisfies* CCC($\mathscr{S}$) + MA + $2^{\aleph_0} = \kappa$.

Proof. As described in the introduction we shall build inductively a finite support iteration $\langle \mathbb{P}_\alpha; \mathbb{Q}_\alpha: \alpha < \kappa\rangle$ of ccc posets of size $\leqslant \kappa$. We shall assume that the whole iteration is embedded in some reasonable way in H_κ, and for every $f \in (H_\kappa^{<\kappa})^V$ define a name $\mathscr{P}_f$ for a perfect poset and a name T_f for a perfect tree. If $\mathrm{dom}(f) = \alpha$, then $\mathscr{P}_f$ will be defined in $V^{\mathbb{P}_{2\alpha}}$ and T_f will be defined in $V^{\mathbb{P}_{2\alpha+1}}$.

To begin, let $\mathscr{P}_{<>}$ be the collection of all joins of finite subsets of $\{(2^{<\omega})_s : s \in 2^{<\omega}\}$, and set $\mathbb{Q}_0 = \mathscr{A}(\mathscr{P}_{<>})$. In $V^{\mathbb{P}_1}$, let $T_{<>}$ be the generic tree introduced by $\mathbb{Q}_0$. For odd ordinals α let $\mathbb{Q}_\alpha$ be the ccc poset generated by some fixed bookkeeping device which guarantees that all ccc posets of size $< \kappa$ appear at some stage. This will make sure that MA + $2^{\aleph_0} = \kappa$ holds at the end.

Now suppose we are at an even ordinal $\delta = 2\alpha$ and $\langle \mathbb{P}_\xi; \mathbb{Q}_\xi : \xi < \delta \rangle$ has already been defined as well as $\mathscr{P}_f$ and T_f, for all $f \in (H_\kappa^{<\alpha})^V$.

CASE 1. α is a successor, say $\alpha = \beta + 1$. Suppose $f \in (H_\kappa^\alpha)^V$. If $f(\beta)$ is a canonical $\mathbb{P}_{2\alpha}$-term for a perfect subtree of $T_{f \upharpoonright \beta}$ define

$$\mathscr{P}_f = \mathscr{P}_{f \upharpoonright \beta}[f(\beta)].$$

Otherwise let $\mathscr{P}_f = \mathscr{P}_{f \upharpoonright \beta}$. By Lemma 1, $\mathscr{P}_f$ is a perfect poset. Let $\mathbb{Q}_{2\alpha}$ be the finite support product of the $\mathscr{A}(\mathscr{P}_f)$, for $f \in (H_\kappa^\alpha)^V$. In $V^{\mathbb{P}_{2\alpha+1}}$ for each f let T_f be the $\mathscr{A}(\mathscr{P}_f)$-generic tree introduced by $\mathbb{Q}_{2\alpha}$ provided that $\mathscr{P}_f = \mathscr{P}_{f \upharpoonright \beta}[f(\beta)]$. If $\mathscr{P}_f = \mathscr{P}_{f \upharpoonright \beta}$ let $T_f = T_{f \upharpoonright \beta}$.

CASE 2. α is a limit ordinal. For each $f \in (H_\kappa^\alpha)^V$ let

$$\mathscr{P}_f = \bigcup_{\xi < \alpha} \mathscr{P}_{f \upharpoonright \xi}.$$

Let $\mathbb{Q}_{2\alpha}^0$ be the finite support product of ω copies of the $\mathscr{P}_f$, and let $\mathbb{Q}_{2\alpha}^1$ be the finite support product of the $\mathscr{A}(\mathscr{P}_f)$, for $f \in (H_\kappa^\alpha)^V$. Finally, let $\mathbb{Q}_{2\alpha} = \mathbb{Q}_{2\alpha}^0 \times \mathbb{Q}_{2\alpha}^1$. In $V^{\mathbb{P}_{2\alpha+1}}$, similarly to Case 1, for $f \in (H_\kappa^\alpha)^V$ define T_f to be the $\mathscr{A}(\mathscr{P}_f)$-generic tree introduced by $\mathbb{Q}_{2\alpha}^1$, unless for some $\xi < \alpha$ $\mathscr{P}_f = \mathscr{P}_{f \upharpoonright \xi}$. In that case let T_f be $T_{f \upharpoonright \xi}$ for such ξ.

This completes the inductive construction. Let the final poset $\mathbb{P}$ be the direct limit of $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$.

LEMMA 5. *$\mathbb{Q}_\alpha$ is ccc in $V^{\mathbb{P}_\alpha}$, for all $\alpha < \kappa$.*

Proof. For odd ordinals α this is part of the definition of $\mathbb{Q}_\alpha$. We prove by induction the statement for even ordinals $\alpha < \kappa$. To simplify notation let V_α denote the boolean-valued universe $V^{\mathbb{P}_\alpha}$. Notice that if α is a limit ordinal and $f \in (H_\kappa^\alpha)^V$, then $\mathbb{Q}_{2\alpha}$ includes as a factor the product of ω copies of $\mathscr{P}_f$ and hence $\mathscr{P}_f$ is σ-centered in $V_{2\alpha+1}$. Also, if $\alpha = \beta + 1$, $f \in (H_\kappa^\alpha)^V$, and $\mathscr{P}_{f \upharpoonright \beta}$ is σ-centered in $V_{2\beta+1}$, then $\mathscr{P}_f$ is σ-centered in $V_{2\alpha}$. Finally, if $\mathscr{P}_f$ is σ-centered, then so is $\mathscr{A}(\mathscr{P}_f)$. All of this implies that the least ordinal α for which the lemma fails is of cofinality ω_1. Then $\alpha = 2\alpha$, and by a standard argument, we may assume for some $f_0, \ldots, f_{k-1} \in (H_\kappa^\alpha)^V$ and $n < \omega$, the poset $\times_{i<k} \mathscr{P}_{f_i}^n$ is not ccc in $V_{2\alpha}$. Fix such $f_0, \ldots, f_{k-1}$ and n, with k being minimal. It follows that if $i < k$ then $\mathscr{P}_{f_i}$ is a proper extension of $\mathscr{P}_{f_i \upharpoonright \xi}$, for all $\xi < \alpha$. Suppose $A \in V_{2\alpha}$ is an uncountable maximal antichain in $\times_{i<k} \mathscr{P}_{f_i}^n$. By a genericity argument it follows that if $i < k$, and if $\xi < \alpha$ is a sufficiently closed ordinal of countable cofinality then $\mathscr{P}_{f_i \upharpoonright \xi}$ has a countable dense subset. Thus, by a Skolem-closure argument, we can find a limit ordinal $\delta < \alpha$ such that, setting $A_\delta = A \cap \times_{i<k}\mathscr{P}^n_{f_i \upharpoonright \delta}$, the following hold:

(i) A_δ belongs to the model $V_{2\delta}$,
(ii) A_δ is a maximal antichain in $\times_{i<k}\mathscr{P}^n_{f_i \upharpoonright \delta}$,
(iii) $\mathscr{P}_{f_i \upharpoonright \delta}$ is a proper extension of $\mathscr{P}_{f_i \upharpoonright \xi}$, for $\xi < \delta$ and $i < k$.

We shall show that A_δ remains a maximal antichain in $\times_{i<k}\mathscr{P}^n_{f_i}$. Hence, $A = A_\delta$, contradicting the assumption that A is uncountable. To that end we shall prove a more general fact.

LEMMA 6. *Suppose η is such that $\delta \leqslant \eta < \alpha$, and let $\mathscr{F}_\eta = \{\mathscr{P}_{f_i \upharpoonright \eta} : i < k\}$. Then the following hold*:

(a) $\times_{i<k}\mathscr{P}_{f_i \upharpoonright \eta}$ *is a complete extension of* $\times_{i<k}\mathscr{P}_{f_i \upharpoonright \delta}$ *over* $V_{2\delta}$,
(b) $\langle T_{f_0 \upharpoonright \eta}, \dots, T_{f_{k-1} \upharpoonright \eta}\rangle$ *is* $\times_{i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \eta})$*-weakly generic over* $(V_{2\delta}, \mathscr{F}_\eta)$.

Proof. By a simultaneous induction on η. For $\eta = \delta$, (a) is immediate, while (b) follows from property (iii) of δ above, since then $\langle T_{f_0 \upharpoonright \delta}, \dots, T_{f_{k-1} \upharpoonright \delta}\rangle$ is $\times_{i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \delta})$-generic over $V_{2\delta}$ and $\mathscr{F}_\delta \in V_{2\delta}$.

Suppose $\eta = \xi + 1$ and the lemma is known for ξ. Let us assume that for some $l \leqslant k$ $f_i(\xi)$ is a perfect subtree of $T_{f_i \upharpoonright \xi}$ iff $l \leqslant i < k$. Thus $\mathscr{P}_{f_i \upharpoonright \eta} = \mathscr{P}_{f_i \upharpoonright \xi}$, for $i < l$, and $\mathscr{P}_{f_i \upharpoonright \eta} = \mathscr{P}_{f_i \upharpoonright \xi}[f_i(\xi)]$, for $l \leqslant i < k$. So, $\langle f_l(\xi), \dots, f_{k-1}(\xi)\rangle$ is $\times_{l \leqslant i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \xi})$-weakly generic over $(V_\delta, \mathscr{F}_\xi)$. Then, by the generalization of Lemma 2, it follows that $\times_{i<k}\mathscr{P}_{f_i \upharpoonright \eta}$ is a complete extension of $\times_{i<k}\mathscr{P}_{f_i \upharpoonright \xi}$ over $V_{2\delta}$. To prove (b), notice that

$\langle T_{f_0 \upharpoonright \xi}, \dots, T_{f_{l-1} \upharpoonright \xi}, f_l(\xi), \dots, f_{k-1}(\xi)\rangle$ is $\times_{i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \xi})$-weakly generic over $(V_{2\delta}, \mathscr{F}_\xi)$.

Hence, by Lemma 4, $\langle T_{f_0 \upharpoonright \xi}, \dots, T_{f_{l-1} \upharpoonright \xi}\rangle$ is $\times_{i<l}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \xi})$-weakly generic over $(V_{2\delta}, \mathscr{F}_\xi \cup \{\mathscr{P}_{f_i \upharpoonright \eta} : l_i \leqslant i < k\})$. Now, if $i < l$ then $T_{f_i \upharpoonright \eta} = T_{f_i \upharpoonright \xi}$. On the other hand $\langle T_{f_l \upharpoonright \eta}, \dots, T_{f_{k-1} \upharpoonright \eta}\rangle$ is the $\times_{l \leqslant i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \eta})$-generic sequence of trees over $V_{2\delta}$ adjoined by $\mathbb{Q}_{2\eta}$. Hence, by the multidimensional version of Lemma 3, $\langle T_{f_0 \upharpoonright \eta}, \dots, T_{f_{k-1} \upharpoonright \eta}\rangle$ is $\times_{i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \eta})$-weakly generic over $(V_{2\delta}, \mathscr{F}_\eta)$. This completes the proof for successor ordinals η.

Suppose now η is a limit ordinal. Then for $i < k$, $\mathscr{P}_{f_i \upharpoonright \eta} = \bigcup_{\xi<\eta}\mathscr{P}_{f_i \upharpoonright \xi}$. Hence (a) is immediate. To prove (b), let us assume that for some $l \leqslant k$, for every $i < k$ there is $\xi < \eta$ such that $\mathscr{P}_{f_i \upharpoonright \eta} = \mathscr{P}_{f_i \upharpoonright \xi}$ if and only if $i < l$. Pick $\xi < \eta$ sufficiently large such that if $i < l$ then $\mathscr{P}_{f_i \upharpoonright \eta} = \mathscr{P}_{f_i \upharpoonright \xi}$. It follows that if $\xi \leqslant \zeta \leqslant \eta$ and $i < l$ then $T_{f_i \upharpoonright \zeta} = T_{f_i \upharpoonright \xi}$. Together with the inductive assumption this implies that $\langle T_{f_0 \upharpoonright \eta}, \dots, T_{f_{l-1} \upharpoonright \eta}\rangle$ is $\times_{i<l}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \eta})$-weakly generic over $(V_{2\delta}, \mathscr{F}_\eta)$. Finally, $\langle T_{f_l \upharpoonright \eta}, \dots, T_{f_{k-1} \upharpoonright \eta}\rangle$ is the $\times_{l \leqslant i<k}\mathscr{A}(\mathscr{P}_{f_i \upharpoonright \eta})$-generic sequence of trees introduced by $\mathbb{Q}_{2\eta}$. Hence, by the generalization of Lemma 3 again, $\langle T_{f_0 \upharpoonright \eta}, \dots, T_{f_{k-1} \upharpoonright \eta}\rangle$ is

$\times_{i<k}\mathscr{A}(\mathscr{P}_{f_i\restriction\eta})$-weakly generic over $(V_{2\delta}, \mathscr{F}_\eta)$. This completes the proof of Lemmas 5 and Lemma 6. □

LEMMA 7. *$V^{\mathbb{P}}$ satisfies* CCC($\mathscr{S}$).

Proof. For $f \in (H_\kappa^\kappa)^V$ let us define:

$$\mathscr{P}_f = \bigcup_{\alpha<\kappa} \mathscr{P}_{f\restriction\alpha}.$$

Since for each $\alpha < \kappa$, $\mathscr{P}_{f\restriction\alpha}$ is σ-centered and $\mathrm{cof}(\kappa) > \aleph_1$, it follows that $\mathscr{P}_f$ is ccc in $V^{\mathbb{P}}$ being a κ-increasing union of σ-centered posets. Suppose that $\langle D_\alpha : \alpha < \kappa\rangle \in V^{\mathbb{P}}$ is a sequence of dense open subsets of $\mathscr{S}$. We shall build $f \in (H_\kappa^\kappa)^V$ such that $D_\alpha \cap \mathscr{P}_f$ is dense in $\mathscr{P}_f$, for every $\alpha < \kappa$. Although $\langle D_\alpha : \alpha < \kappa\rangle$ is a member of $V^{\mathbb{P}}$ it is important to arrange that the whole construction of f takes place entirely in V. Since the whole iteration is embedded in H_κ^V in a reasonable way, every $\mathbb{P}$-term for a real is essentially a countable object, modulo elements of κ, and hence belongs to H_κ^V. Thus, let us fix some enumeration $\langle \tau_\alpha : \alpha < \kappa\rangle$ of all $\mathbb{P}$-terms for perfect trees and an enumeration $\langle \sigma_\alpha : \alpha < \kappa\rangle$ of all $\mathbb{P}$-terms for ordinals $< \kappa$. We build f recursively as follows. Suppose $f\restriction\delta$ has been defined for some ordinal $\delta < \kappa$. Pick the least pair of ordinals $\langle \alpha, \beta\rangle$ such that τ_α and σ_β are $\mathbb{P}_{2\delta}$-terms, and

$$\Vdash \tau_\alpha \in \mathscr{P}_{f\restriction\delta} \text{ and there is no } T \in D_{\sigma_\beta} \cap \mathscr{P}_{f\restriction\delta} \text{ with } T \leqslant \tau_\alpha.$$

For simplicity, let us assume that every condition in $\mathbb{P}_{2\delta+1}$ forces that $T_{f\restriction\delta} \leqslant \tau_\alpha$. Now, since D_{σ_β} is forced to be a dense open subset of $\mathscr{S}$, there exists $\mu > \delta$ and a $\mathbb{P}_{2\mu}$-term τ such that

$$\Vdash \tau \in D_{\sigma_\beta} \quad \text{and} \quad \tau \leqslant T_{f\restriction\delta}.$$

Let us extend $f\restriction\delta$ to $\mu + 1$ by setting $f(\xi)$ equal to some canonical term for $\varnothing$, for $\delta \leqslant \xi < \mu$, and setting $f(\mu) = \tau$. One can show that thus defined f works. This completes the proof of Lemma 7 and Theorem 1. □

3. Applications

The idea of thinning out Sacks forcing to a ccc poset is due to Jensen ([Jen]), who used it to produce a non-constructible Π_2^1 singleton of minimal degree over L. The principle CCC($\mathscr{S}$) is an attempt to capture the combinatorial content of Jensen's construction compatible with MA$+\neg$CH. We now show that it can be used to extract ccc suborders of Sacks forcing having some of the properties of $\mathscr{S}$ itself. Recall that a poset $\mathscr{P}$ is called *weakly distributive* if and only if any function $f \in \omega^\omega$ which belongs to $V^{\mathscr{P}}$ is dominated by some $g \in \omega^\omega$ from the ground model V. If V is a universe of set theory and x is a real which does not belong to V, x is said to be minimal over V provided for every $y \in V[x]$, either $y \in V$ or $V[y] = V[x]$.

PROPOSITION 1. *Assume* CCC($\mathcal{S}$). *Then there is a weakly distributive ccc poset $\mathcal{P}$ which adds a minimal real over the ground model.*

Proof. Let $\mathcal{A}$ denote the collection of all countable antichains in $\mathcal{S}$. For each sequence $A = \langle A_n: n < \omega \rangle$ of elements of $\mathcal{A}$ let $D(A)$ denote the set of all $T \in \mathcal{S}$ such that either for every $n < \omega$ there is $F \in [A_n]^{<\omega}$ such that $T \leqslant \bigvee F$ or else there exists n such that $T \wedge R = \emptyset$, for every $R \in A_n$. By a fusion argument one shows that $D(A)$ is a dense subset of $\mathcal{S}$. Let $\mathcal{P}$ be a ccc perfect poset such that $D(A) \cap \mathcal{P}$ is dense in $\mathcal{P}$, for all $A \in \mathcal{A}^\omega$.

CLAIM 1. *$\mathcal{P}$ is weakly distributive.*

Proof. Let τ be a $\mathcal{P}$-name for an element of ω^ω. For each $n < \omega$, let A_n be a maximal antichain of elements of $\mathcal{P}$ deciding $\tau(n)$. Since $\mathcal{P}$ is ccc each A_n is countable. Let $A = \langle A_n: n < \omega \rangle$ and fix $R \in D(A) \cap \mathcal{P}$. Since A_n is a maximal antichain in $\mathcal{P}$, for each n, there exists $F_n \in [A_n]^{<\omega}$ such that $R \leqslant \bigvee F_n$. Define the function $g_R \in \omega^\omega$ by letting

$$g_R(n) = \sup\{m < \omega: \text{for some } T \in F_n\ T \Vdash \tau(n) = m\}$$

Then it follows that $R \Vdash \tau \leqslant g_R$. □

Let us say that τ is a *canonical name* for a real if it is a countable subset of $\mathcal{S} \times 2^{<\omega}$ such that: if $(S, s), (T, t) \in \tau$ and s and t are incompatible, then $S \wedge T = \emptyset$; and if $n < m$ then for every $(S, s) \in \tau$ with $s \in 2^n$ there exists $(T, t) \in \tau$ with $t \in 2^m$ such that $T \leqslant S$ and $s \subset t$. For a canonical name τ let $D(\tau)$ be the collection of all perfect trees T such that: either there exists $f \in 2^\omega$ such that if $(S, s) \in \tau$ and $S \wedge T \neq \emptyset$, then $s \subset f$, or for every n there exists $m \geqslant n$ and a 1-1 function $\varphi: T \cap 2^n \to 2^m$ such that if $(R, r) \in \tau$ with $r \in 2^m$ and $R \wedge T \neq \emptyset$, then $r = \varphi(s)$ for some $s \in T \cap 2^n$. Again, by a usual fusion argument, one shows that $D(\tau)$ is a dense subset of $\mathcal{S}$. Suppose now that $\mathcal{P}$ is a perfect ccc poset such that $D(\tau) \cap \mathcal{P}$ is dense in $\mathcal{P}$, for all canonical names τ.

CLAIM 2. *$\mathcal{P}$ adds a minimal real over the ground model.*

Proof. Suppose a is a $\mathcal{P}$-term for a new real. For each $s \in 2^{<\omega}$ let A_s be a maximal antichain contained in $\{T \in \mathcal{P}: T \Vdash s \subset a\}$, and let $\tau = \{(T, s): T \in A_s\}$. Since a is forced not to belong to the ground model, every element of $D(\tau)$ satisfies the second clause in the definition of $D(\tau)$. Now, given $T \in G \cap D(\tau)$, where G is a generic filter, one recovers the generic real as $\bigcup \{\varphi^{-1}(t): t \subset a\}$. Thus, $V[a] = V[G]$. □

Our next application answers a question of Baumgartner who asked if it is consistent with $2^{\aleph_0} > \aleph_2$ that $\mathcal{S}$ preserves the continuum.

PROPOSITION 2. *Assume* CCC($\mathcal{S}$). *Then $\mathcal{S}$ preserves all cardinals.*

Proof. Since $\mathcal{S}$ has the $(2^{\aleph_0})^+$-cc, the only cardinals it could collapse are $\leqslant 2^{\aleph_0}$. Suppose $\kappa < \lambda \leqslant 2^{\aleph_0}$ are cardinals and f is an $\mathcal{S}$-name for a function mapping κ onto λ. For each $\alpha < \lambda$, let

$$D_\alpha = \{T \in \mathscr{S}: \text{for some } \xi < \kappa\ T \Vdash f(\xi) = \alpha\}.$$

Then D_α is a dense open subset of $\mathscr{S}$. Now by applying CCC($\mathscr{S}$) we obtain a ccc poset $\mathscr{P}$ such that $D_\alpha \cap \mathscr{P}$ is dense in $\mathscr{P}$, for all $\alpha < \lambda$. Then $\mathscr{P}$ collapses λ to κ. Contradiction. □

Recall that $A \subseteq 2^\omega$ is called an s_0-set iff for every perfect subset P of 2^ω there exists a perfect $Q \subseteq P$ such that $A \cap Q = \varnothing$. Thus, A is an s_0-set iff the set of all perfect trees T such that $[T] \cap A = \varnothing$ is dense in $\mathscr{S}$. A standard fusion argument shows that the collection of s_0-sets is a σ-complete ideal. We now show that MA + CCC($\mathscr{S}$) implies that this ideal is $< 2^{\aleph_0}$-complete. In the next section we shall show that MA does not suffice for this result.

PROPOSITION 3. *Assume* MA + CCC($\mathscr{S}$). *Then the union of less than continuum s_0-sets is an s_0-set.*

Proof. It suffices to show that for every $\mathscr{X}$, a collection of s_0-sets of size $\kappa < 2^{\aleph_0}$, there exists a perfect set P disjoint from $\bigcup \mathscr{X}$. For each $A \in \mathscr{X}$ let $D(A)$ be the collection of all $T \in \mathscr{S}$ such that $[T] \cap A = \varnothing$. By CCC($\mathscr{S}$) find a ccc perfect poset $\mathscr{P}$ such that $D(A) \cap \mathscr{P}$ is dense in $\mathscr{P}$, for all $A \in \mathscr{X}$. Now consider the amoeba poset $\mathscr{A}(\mathscr{P})$. It satisfies the ccc and for every $A \in \mathscr{X}$ the set $E(A)$ of all $(T, n) \in \mathscr{A}(\mathscr{P})$ such that $T \in D(A)$ is dense. For each $s \in 2^{<\omega}$ let F_s be the set of all $(T, n) \in \mathscr{A}(\mathscr{P})$ such that either $s \notin T$ or there exist distinct $t_1, t_2 \in T \cap 2^n$ such that $s \subseteq t_1$, t_2. Now, applying MA_κ to $\mathscr{P}$ and the union of $\{E(A): A \in \mathscr{X}\}$ and $\{F_s: s \in 2^{<\omega}\}$, one obtains a perfect tree T such that $[T] \cap A = \varnothing$, for all $A \in \mathscr{X}$. □

4. PFA and the negation of CCC($\mathscr{S}$)

In the previous sections we have seen that CCC($\mathscr{S}$) is relatively consistent with MA_κ, and consequently, so is $\mathrm{FA}_\kappa(\mathscr{S})$. Although $\mathrm{FA}_{\aleph_1}(\mathscr{S})$ follows from PFA we show that CCC($\mathscr{S}$) does not. We then show that $\mathrm{MA}_{\aleph_1}$ does not imply $\mathrm{FA}_{\aleph_1}(\mathscr{S})$. This is accomplished by starting with a model of PFA and generically adjoining a dense subset D of $\mathscr{S}$, such that every $r \in 2^\omega$ is a branch through at most countably many members of D. This implies that $\mathrm{FA}_{\aleph_1}(\mathscr{S})$ fails. The forcing does not add new subsets of ω_1 and hence $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$ remains to hold in the extension. For this argument we need only assume a fragment of PFA whose consistency does not require any large cardinal assumptions. We shall indicate how this can be established at the end of this section.

Let us first make some definitions which will facilitate the proofs. Given a function $f: \omega \to \omega$ such that $f(n) > n$, for all $n \in \omega$, say that a perfect tree T is *f-thin* provided for all $n \in \omega$: $\mathrm{card}(T \restriction f(n)) \leqslant \mathrm{card}(T \restriction n) + 1$. A sequence $\langle f_\alpha: \alpha < \kappa \rangle$ of functions in ω^ω is called a *weak scale* provided it is increasing and unbounded in ω^ω, $\leqslant_*$. The following two lemmas will be needed subsequently.

LEMMA 8. *Suppose $X \subseteq 2^\omega$ has cardinality $< 2^{\aleph_0}$, and $f \in \omega^\omega$ is such that for all $n < \omega$, $f(n) > n$. Then every perfect tree T has an f-thin perfect subtree S such that $[S] \cap X = \varnothing$.*

Proof. First find an f-thin perfect subtree R of T. Then build a collection $\langle R_x : x \in 2^\omega \rangle$ of perfect subtrees of R such that if $x \neq y$ then $[R_x] \cap [R_y] = \varnothing$. Then for some $x \in 2^\omega$, $[R_x] \cap X = \varnothing$. Set $S = R_x$. □

Suppose κ is an ordinal. Let us say that a subset C of $[\kappa]^n$ is *cofinal* iff for every $\alpha < \kappa$ there exists $x \in C$ with $\min(x) > \alpha$.

LEMMA 9. *Let $\langle f_\alpha : \alpha < \kappa \rangle$ be a weak scale and $\langle r_\alpha : \alpha < \kappa \rangle$ an enumeration of 2^ω. Suppose for every $\alpha < \kappa$, T_α is an f_α-thin tree such that $r_\xi \notin [T_\alpha]$, for $\xi < \alpha$. Let C be a cofinal subset of $[\kappa]^n$. Then there are $x, y \in C$ such that if $\alpha \in x$ and $\beta \in y$ then $[T_\alpha] \cap [T_\beta] = \varnothing$.*

Proof. For $x \in [\kappa]^n$ let T_x be the union of the T_α, for $\alpha \in x$. Let $f_x \in \omega^\omega$ be the pointwise infimum of the f_α, for $\alpha \in x$. It follows that for every m,

$$\operatorname{card}(T_x \upharpoonright f_x(m)) \leqslant (2^m + 1)n.$$

Since $\{f_x : x \in C\}$ is unbounded in $\omega^\omega, \leqslant_*$, there exists m such that the set $I = \{f_x(m) : x \in C\}$ is infinite. This implies that for every $l \in \omega$ there exists $x \in C$ such that $\operatorname{card}(T_x \upharpoonright l) \leqslant (2^m + 1)n$. By an application of König's lemma, find $F \subset \kappa$ of size $(2^m + 1)n$ such that for every $l \in \omega$ there is $x \in C$ such that

$$T_x \upharpoonright l \subseteq \{r_\alpha \upharpoonright l : \alpha \in F\}.$$

Now pick any $y \in C$ with $\min(y) > \sup(F)$. Then, $r_\alpha \notin [T_y]$, for all $\alpha \in F$. Fix l sufficiently large so that $r_\alpha \upharpoonright l \notin T_y$, for all $\alpha \in F$, and $x \in C$ such that $T_x \upharpoonright l$ is contained in $\{r_\alpha \upharpoonright l : \alpha \in F\}$. It follows that $[T_x] \cap [T_y] = \varnothing$, as desired. □

THEOREM 2. *Assume* PFA. *Then there is a family $\mathscr{D}$ of $2^{\aleph_0}$ dense open sets of $\mathscr{S}$ such that for every ccc suborder $\mathscr{P}$ of $\mathscr{S}$, there exists $D \in \mathscr{D}$ such that $D \cap \mathscr{P} = \varnothing$.*

Proof. Let us fix a weak scale $\langle f_\alpha : \alpha < \omega_2 \rangle$ such that $f_\alpha(n) > n$, for every α, and n. Let also $\langle r_\alpha : \alpha < \omega_2 \rangle$ be an enumeration of 2^ω. For $\alpha < \omega_2$ let:

$$D_\alpha = \{T \in \mathscr{S} : T \text{ is } f_\alpha\text{-thin and } r_\xi \notin [T], \text{ for all } \xi < \alpha\}.$$

Then, by Lemma 8, D_α is a dense open subset of $\mathscr{S}$. Let $\mathscr{D} = \{D_\alpha : \alpha < \omega_2\}$. Suppose $\mathscr{P}$ is a subposet of $\mathscr{S}$ such that $D_\alpha \cap \mathscr{P} \neq \varnothing$, for all α. We shall find a proper poset $\mathscr{R}$ which introduces an uncountable antichain to $\mathscr{P}$. An application of PFA then implies that $\mathscr{P}$ is not ccc.

To begin, select $T_\alpha \in D_\alpha \cap \mathscr{P}$, for each $\alpha < \omega_2$. Let $\mathscr{C}$ be the usual σ-closed collapse of ω_2 to have cardinality $\aleph_1$. In $V^{\mathscr{C}}$ fix a cofinal subset C of ω_2^V of order type ω_1, and let $\mathscr{E}^*$ be the poset of all $F \in [C]^{<\omega}$ such that for every distinct $\alpha, \beta \in F$ $[T_\alpha] \cap [T_\beta] = \varnothing$, ordered by reverse inclusion. A standard Δ-system argument together with Lemma 9 implies that $\mathscr{E}^*$ is ccc. Pick a condition $F \in \mathscr{E}^*$ which forces that the generic filter is uncountable and let $\mathscr{E}$ be $\mathscr{E}^*$ below F. It follows that $\mathscr{R} = \mathscr{C} * \mathscr{E}$ introduces an uncountable antichain to $\mathscr{P}$. $\square$

THEOREM 3. *Assume* PFA. *Then there is a poset* $\mathscr{P}$ *such that* $V^{\mathscr{P}}$ *satisfies* $\mathrm{MA} + 2^{\aleph_0} = \aleph_2 + \neg \mathrm{FA}_{\aleph_1}(\mathscr{S})$.

Proof. As in the proof of Theorem 2, let us fix an enumeration $\langle r_\alpha : \alpha < \omega_2 \rangle$ of 2^ω and a weak scale $\langle f_\alpha : \alpha < \omega_2 \rangle$ in ω^ω such that $f_\alpha(n) > n$, for all α, n. Define the poset $\mathscr{P}$ as follows: $p \in \mathscr{P}$ iff p is a function which maps some $\alpha < \omega_2$ into $\mathscr{S}$ such that:

(a) $p(\xi)$ is f_ξ-thin, for $\xi < \alpha$,
(b) $r_\eta \notin [p(\xi)]$, for $\eta < \xi < \alpha$,
(c) there is a partition $\alpha = \bigcup_{i \in \omega} X_i$ such that for every i, and $\xi, \eta \in X_i$, if $\xi \neq \eta$ then $[p(\xi)] \cap [p(\eta)] = \varnothing$.

We shall show that $\mathscr{P}$, ordered under reverse inclusion is the required poset. Clearly, $\mathscr{P}$ is σ-closed. Assuming PFA we show that more is true.

LEMMA 10 (PFA). $\mathscr{P}$ *does not add new* ω_1*-sequences of ordinals.*

Proof. It suffices to show that if $\mathscr{D}$ is a family of $\aleph_1$ dense open subsets of $\mathscr{P}$ then $\bigcap \mathscr{D} \neq \varnothing$. Let G be the canonical $\mathscr{P}$-generic filter and let $g = \bigcup G$. Then g maps ω_2^V to $\mathscr{S}$, and every proper initial segment of g is in $\mathscr{P}$. Moreover, for every $D \in \mathscr{D}$ there exists $\delta < \omega_2^V$ such that $g \restriction \delta \in D$. Let $\mathscr{C}$ be the usual σ-closed collapse of ω_2^V to have cardinality $\aleph_1$. In $V^{\mathscr{P} * \mathscr{C}}$ define the poset $\mathscr{Q}^*$ as follows. Let $F \in \mathscr{Q}^*$ iff $F \in [\omega_2^V]^{<\omega}$ and for every distinct $\xi, \eta \in F$, $[g(\xi)] \cap [g(\eta)] = \varnothing$. The order is reverse inclusion. Finally, let $\mathscr{Q}$ be the finite support product of ω copies of $\mathscr{Q}^*$.

LEMMA 11. $\mathscr{Q}$ *has the ccc in* $V^{\mathscr{P} * \mathscr{C}}$.

Proof. Live in $V^{\mathscr{P} * \mathscr{C}}$. Suppose $\{q_\alpha : \alpha < \omega_1\}$ is an uncountable antichain in $\mathscr{Q}$. We may assume that for some n, $q_\alpha \in (\mathscr{Q}^*)^n$ for all α. Let

$$q_\alpha = \langle F_\alpha^0, \ldots, F_\alpha^{n-1} \rangle.$$

By a counting and a Δ-system argument we may assume that $\mathrm{card}(F_\alpha^i) = k_i$ does not depend on α, and by throwing away the root of the Δ-system, that $F_\alpha^i \cap F_\beta^i = \varnothing$, for distinct α and β. For $\alpha < \omega_1$, let m_α be such that for $i < n$ and any distinct $\xi, \eta \in F_\alpha^i$ $g(\xi) \cap g(\eta) \subseteq 2^{<m_\alpha}$. Let

$$F_\alpha^i = \{a_{\alpha,0}^i, \ldots, a_{\alpha,k_i-1}^i\}_<$$

be the increasing enumeration. By going to suitable subsequence again, we may assume that $m_\alpha = m$, for all α, and that there exist $t_i^j \subseteq 2^m$ such that $g(a_{\alpha,j}^i) \cap 2^m = t_j^i$, for all α. Note that $t_{j_1}^i$ and $t_{j_2}^i$ are disjoint for $j_1 \neq j_2$. Moreover, we may assume that if $\alpha < \beta$ then $a_{\alpha,j}^i < a_{\beta,j}^i$, for all $i < n$, and $j < k_i$. For each i let $l_i \leqslant k_i$ be the least j, if it exists, such that the set

$$A_j^i = \{a_{\alpha,j}^i : \alpha < \omega_1\}$$

is unbounded in ω_2^V, and let δ be the supremum of $\bigcup\{A_j^i : j < l_i\}$. Since $\delta < \omega_2^V$, and $g \restriction \delta \in \mathscr{P}$, there is a partition $\delta = \bigcup_{i<\omega} X_i$ such that for all i and distinct ξ, $\eta \in X_i$, $[g(\xi)] \cap [g(\eta)] = \varnothing$. By shrinking one more time we may assume that for some $f : \omega \times \omega \to \omega$, all α and $j < l_i$, $a_{\alpha,j}^i \in X_{f(i,j)}$. This implies that for all $i < n$, $j < l_i$, and distinct α and β, $[g(a_{\alpha,j}^i] \cap [g(a_{\beta,j}^i] = \varnothing$. Finally, for each α let

$$E_\alpha = \{a_{\alpha,j}^i : i < n \text{ and } j \geqslant l_i\}.$$

Then $\{E_\alpha : \alpha < \omega_1\}$ is cofinal in ω_2^V. Hence, by Lemma 9, there are α and $\beta < \omega_1$ such that for all $i < n$ and $j \geqslant l_i$, $[g(a_{\alpha,j}^i] \cap [g(a_{\beta,j}^i] = \varnothing$. This implies that q_α and q_β are compatible. This contradiction finishes the proof of Lemma 11. □

Now, to prove Lemma 10, notice that in $V^{\mathscr{P}*\mathscr{C}*\mathscr{Q}}$ g satisfies all the properties in the definition of $\mathscr{P}$ except that its domain is ω_2^V. By applying PFA to a suitable family of $\aleph_1$ dense subsets of $\mathscr{P} * \mathscr{C} * \mathscr{Q}$ we can find a 'copy' of g in V, i.e. for some $\delta < \omega_2^V$ a condition $p \in \mathscr{P}$ with domain δ such that for every $D \in \mathscr{D}$ there exists $\alpha < \delta$ such that $p \restriction \alpha \in D$. This p clearly belongs to $\bigcap \mathscr{D}$, as required. □

To finish the proof of Theorem 3, note that since $V^{\mathscr{P}}$ has the same subsets of ω_1 as V, it satisfies $\text{MA} + 2^{\aleph_0} = \aleph_2$. If G is the generic filter let D be $\bigcup\{\text{ran}(p) : p \in G\}$. Then, by genericity and Lemma 8, it follows that D is a dense subset of $\mathscr{S}$. Moreover, by (c) in the definition of $\mathscr{P}$, and the fact that no subsets of ω_1 are added, every $r \in 2^\omega$ is a branch through at most countably many members of D. Using the fact that every perfect tree has $2^{\aleph_0}$ incomparable subtrees one can split D into disjoint dense sets D_α, for $\alpha < \omega_1$. Then there cannot exist a filter H in $\mathscr{S}$ which intersects all the D_α since this would yield a real belonging to the closure of uncountably many members of D. Thus, $\text{FA}_{\aleph_1}(\mathscr{S})$ fails. □

REMARKS. Note that in the above model the reals can be covered by $\aleph_1$ s_0-sets. For if one sets $A_\alpha = 2^\omega \setminus \bigcup D_\alpha$, for $\alpha < \omega_1$, then the A_α are s_0-sets whose union covers 2^ω. Moreover, forcing with $\mathscr{S}$ over this model collapses $2^{\aleph_0}$. To see this let $\langle T_\alpha : \alpha < \omega_2 \rangle$ be the sequence of generic trees added by $\mathscr{P}$. If now r is $\mathscr{S}$-generic over $V^{\mathscr{P}}$ then the set $\{\alpha : r \in [T_\alpha]\}$ is a cofinal subset of ω_2^V of order type ω_1.

In the proof of the consistency of PFA large cardinals are used. However, for the proof of Theorem 3 this is not necessary. Namely, starting with a model of GCH one performs a countable support iteration $\langle \mathbb{P}_\alpha ; \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$ of proper posets of size $\leqslant \aleph_1$, forcing with all such posets along the way. Simultaneously, one generates a weak scale $\langle f_\alpha : \alpha < \omega_2 \rangle$, an enumeration $\langle r_\alpha : \alpha < \omega_2 \rangle$ of 2^ω, and an enumeration $\langle p_\alpha : \alpha < \omega_2 \rangle$ of the poset $\mathscr{P}$ from Theorem 4 as defined in the final model $V^{\mathbb{P}_{\omega_2}}$. One takes care that at each stage δ the sequences $\langle f_\alpha : \alpha < \delta \rangle$, $\langle r_\alpha : \alpha < \delta \rangle$, and $\langle p_\alpha : \alpha < \delta \rangle$ belong to the current model $V^{\mathbb{P}_\delta}$. At stage δ of cofinality ω_1 one does the following. If $\langle f_\alpha : \alpha < \delta \rangle$ is a weak scale in $V^{\mathbb{P}_\delta}$, $\langle r_\alpha : \alpha < \delta \rangle$ is an enumeration of the reals in $V^{\mathbb{P}_\delta}$, and if $\text{dom}(p_\alpha)$ is less than δ, for all $\alpha < \delta$, consider the poset $\mathscr{P}_\delta = \{p_\alpha : \alpha < \delta\}$, ordered under reverse extension. Suppose $\mathscr{P}_\delta$ is σ-closed and for each $\alpha < \delta$, the set of $p \in \mathscr{P}_\delta$ with $\mathrm{dom}(p) \geqslant \alpha$ is dense in $\mathscr{P}_\delta$. Let $\mathscr{Q}_\delta$ be defined in $V^{\mathbb{P}_\delta * \mathscr{P}_\delta}$ in the same way $\mathscr{Q}$ was defined in Lemma 10 replacing ω_2 by δ and ignoring the collapsing poset $\mathscr{C}$. Then let $\mathbb{Q}_\delta = \mathscr{P}_\delta * \mathscr{Q}_\delta$. If any of these conditions are not satisfied let $\mathbb{Q}_\delta$ be the trivial poset. Note that $\mathbb{Q}_\delta$ has size $\leqslant \aleph_1$ and the proof of Lemma 11 shows that it is proper. A standard Skolem closure argument shows that the poset $\mathscr{P}$ defined in $V^{\mathbb{P}_{\omega_2}}$ is $\aleph_1$-Baire. Thus, forcing with $\mathscr{P}$ over $V^{\mathbb{P}_{\omega_2}}$ one obtains the desired result.

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