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Singularities of the moduli spaces of certain Abelian surfaces

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Introduction

The space of principally polarized abelian surfaces has been studied extensively. Gottschling [2, 3], see also [1], has determined the singularities of this space. They occur in codimension 2 and 3. The moduli spaces of abelian surfaces with non-principal polarization have been studied much less. It turns out that there again the situation is very complicated. However, in this situation we have a natural notion of level structure. Our main objects of interest in this paper are the moduli spaces $\mathscr{A}_{1,p}$ of $(1, p)$-polarized abelian surfaces with level structure, i.e., of triples (A, H, α) where A is an abelian surface defined over the complex numbers, H is a polarization of type $(1, p)$ on A, and α is an according level structure on A. (For definitions see, e.g., [4].)

Since singularities in moduli spaces of abelian surfaces arise from surfaces with non-trivial automorphism groups, and since the presence of a level structure breaks many of these symmetries, one can hope to be able to classify the singularities of the spaces $\mathscr{A}_{1,p}$. Our aim is to show that this is indeed the case. In fact, we determine the singularities of a suitable toroidal compactification of $\mathscr{A}_{1,p}$.

For simplicity we shall always assume that p is a prime and that $p \neq 2$. In order to describe our results we introduce some notation which we will use throughout the paper. Let

$$\mathscr{S}_2 = \{Z \in M(2 \times 2, \mathbf{C}) | Z = {}^tZ, \operatorname{Im} Z > 0\}$$

be the *Siegel upper half space of degree* 2. Then, the *symplectic group* defined with respect to the standard symplectic form J,

$$\operatorname{Sp}(4, \mathbf{Q}) = \{g \in \operatorname{GL}(4, \mathbf{Q}) \,|\, gJ{}^tg = J\}, \qquad J = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix},$$

acts properly discontinuously on $\mathcal{S}_2$. Namely, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbf{Q})$, where A, B, C, D denote 2×2 matrices, maps $Z \in \mathcal{S}_2$ to $(AZ + B)(CZ + D)^{-1}$. We also note that if g is symplectic and of this form, then its inverse can be expressed as $g^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$.

In $\mathrm{Sp}(4, \mathbf{Q})$ we consider the arithmetic subgroup

$$\Gamma_{1,p} = \left\{ g \in \mathrm{Sp}(4, \mathbf{Q}) \mid g - \mathbf{1}_4 \in \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ p\mathbf{Z} & p\mathbf{Z} & p\mathbf{Z} & p^2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \end{bmatrix} \right\}$$

of symplectic transformations which preserve $(1, p)$-polarizations and level structures. By taking the quotient of $\mathcal{S}_2$ with respect to the left action of $\Gamma_{1,p}$ we obtain

$$\mathcal{A}_{1,p} = \Gamma_{1,p} \backslash \mathcal{S}_2,$$

the *moduli space of* $(1, p)$*-polarized abelian surfaces with level structure.* $\mathcal{A}_{1,p}$ has at most quotient singularities. We denote by $\mathcal{A}^*_{1,p}$ a suitable toroidal compactification of $\mathcal{A}_{1,p}$ which we shall briefly describe at the beginning of section 2. See chapter I of our forthcoming book [5] for details about this compactification which is intended to be an analogue for the Igusa compactification of the moduli space of principally polarized abelian surfaces.

Denote by $\Gamma_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}) \middle| a \equiv d \equiv 1(p),\ b \equiv c \equiv 0(p) \right\}$ the *principal congruence subgroup of level* p in $\mathrm{SL}(2, \mathbf{Z})$ which acts on $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\}$. The quotient $X^\circ(p) = \Gamma_1(p) \backslash \mathcal{S}_1$ is called (*open*) *modular curve of level p.* Its natural compactification to a smooth curve by adding points at the $(p^2 - 1)/2$ cusps is denoted by $X(p)$ and also called *modular curve of level p.*

A non-isolated 3-dimensional singularity is said to be of *transversal type* 'X' *along* C, where C is a smooth curve, if it locally is isomorphic to a product of C and an isolated surface singularity of type 'X'. The surface singularities of interest to us are the rational double point A_1 which is isomorphic to $\mathbf{C}^2/\{\pm \mathbf{1}_2\}$, and the rational triple point $C_{3,1}$ which is isomorphic to $\mathbf{C}^2/\{\rho^k \mathbf{1}_2 | k = 0, 1, 2\}$, $\rho = e^{2\pi i/3}$. The singularity types A_1 resp. $C_{3,1}$ occur at the vertices of the affine cones over the rational normal curves of degree 2 in $\mathbf{P}_2$ resp. of degree 3 in $\mathbf{P}_3$.

An isolated 3-dimensional cyclic quotient singularity is of type '$\frac{1}{n}(q_1, q_2, q_3)$' if it is determined by the action of the diagonal matrix $\mathrm{diag}(\zeta^{q_1}, \zeta^{q_2}, \zeta^{q_3})$, $\zeta = e^{2\pi i/n}$, as a generator of $\mathbf{Z}_n$. Note that a cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1)$

occurs at the vertex of the cone over the Veronese surface in $\mathbf{P}_5$. See [8, §1] for details.

We summarize our results in

THEOREM (2.15, 3.4). *$\mathscr{A}^*_{1,p}$ contains two disjoint curves C_1^* and C_2^* isomorphic to $X(p)$ such that $\mathscr{A}^*_{1,p}$ is singular with transversal A_1-type along C_1^*, and with transversal $C_{3,1}$-type along C_2^*. The complement $\mathscr{A}^*_{1,p} - (C_1^* \cup C_2^*)$ contains only isolated cyclic quotient singularities which all lie on the boundary $\mathscr{A}^*_{1,p} - \mathscr{A}_{1,p}$, namely $(p^2-1)/2$ singularities of each of the types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$.*

The first section is devoted to characterizing the singularities in the moduli space $\mathscr{A}_{1,p}$ in case $p \geqslant 5$. In doing so we rely on work of Gottschling ([2, 3]) and Ueno ([9]). In section 2 we study the toroidal compactification $\mathscr{A}^*_{1,p}$ of $\mathscr{A}_{1,p}$ in order to determine its singularities on the boundary $\mathscr{A}^*_{1,p} - \mathscr{A}_{1,p}$. In the final section we treat the case $p = 3$ which is slightly different from $p \geqslant 5$.

1. Singularities in the moduli space $\mathscr{A}_{1,p}$

We first determine the isotropy subgroups contained in $\Gamma_{1,p}$ corresponding to fixed points in $\mathscr{S}_2$ up to conjugacy, then study their respective fixed varieties, and finally characterize the corresponding quotient singularities in $\mathscr{A}_{1,p}$.

DEFINITION 1.1. The following four matrices are elements of finite order in $\Gamma_{1,p}$:

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

PROPOSITION 1.2. *Every non-trivial element of finite order in $\Gamma_{1,p}$, $p \geqslant 5$, is conjugate with respect to* Sp(4, **Z**) *to exactly one of the following eight matrices:*

(a) *Involutions: S and T.*
(b) *Elements of order* 4: *U and U^{-1}.*
(c) *Elements of order* 6: *V and V^{-1}.*
(d) *Elements of order* 3: *V^2 and V^{-2}.*

Furthermore, $U^2 = V^3 = S$.

Proof. Gottschling has determined the conjugacy classes of all elements of finite order in Sp(4, **Z**) ([2]; see [9, I, §2; II, Appendix] for representatives of all 56 conjugacy classes). Since $\Gamma_{1,p}$ is a subgroup of Sp(4, **Z**) we only need to characterize which of these classes contain elements of $\Gamma_{1,p}$. Two necessary conditions are given by

LEMMA 1.3. *Suppose that $M \in \mathrm{Sp}(4, \mathbf{Z})$ is conjugate to an element of $\Gamma_{1,p}$ and let $\chi_M(\lambda) = \det(M - \lambda \mathbf{1}_4)$ be the characteristic polynomial of M. Then,*

(1) *$\chi_M(\lambda)$ is divisible by $(\lambda - 1)^2$ modulo p;*
(2) *$\chi_M(\lambda)$ is divisible by $\lambda - 1$ modulo p^2.*

Proof. We may assume $M \in \Gamma_{1,p}$. With the diagonal matrix $F = \mathrm{diag}(1, 1, 1, p)$, we obtain

$$F \cdot (M - \lambda \mathbf{1}_4) \cdot F^{-1} \in \begin{bmatrix} \mathbf{Z}-\lambda & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ p\mathbf{Z} & p\mathbf{Z}-(\lambda-1) & p\mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z}-\lambda & \mathbf{Z} \\ p\mathbf{Z} & p\mathbf{Z} & p\mathbf{Z} & p\mathbf{Z}-(\lambda-1) \end{bmatrix}$$

from which both statements of the lemma are easy consequences. □

We continue with the proof of Proposition 1.2. For the convenience of the reader we reproduce part of the classification of matrices of finite order in Sp(4, **Z**) according to their characteristic polynomials from Ueno [9, II, p. 198]:

Name	characteristic polynomial	# conjugacy classes	order in Sp(4, **Z**)
I(1)	$(\lambda-1)^4$	1	1
I(2)	$(\lambda+1)^4$	1	2
II(1)	$(\lambda+1)^2(\lambda-1)^2$	2	2
II(2)	$(\lambda^2+1)^2$	4	4
II(3)	$(\lambda^2+\lambda+1)^2$	3	3
II(4)	$(\lambda^2-\lambda+1)^2$	3	6
III(1)	$(\lambda^2-\lambda+1)(\lambda+1)^2$	2	6
III(2)	$(\lambda^2+\lambda+1)(\lambda-1)^2$	2	3
III(3)	$(\lambda^2+\lambda+1)(\lambda+1)^2$	2	6
III(4)	$(\lambda^2-\lambda+1)(\lambda-1)^2$	2	6
III(5)	$(\lambda^2+1)(\lambda-1)^2$	2	4
III(6)	$(\lambda^2+1)(\lambda+1)^2$	2	4
IV(1)	λ^4+1	4	8
IV(2)	$\lambda^4+\lambda^2+1$	8	6
IV(3)	$\lambda^4-\lambda^2+1$	2	12
IV(4)	$\lambda^4+\lambda^3+\lambda^2+\lambda+1$	4	5
IV(5)	$\lambda^4-\lambda^3+\lambda^2-\lambda+1$	4	10
IV(6)	$(\lambda^2+\lambda+1)(\lambda^2+1)$	4	12
IV(7)	$(\lambda^2-\lambda+1)(\lambda^2+1)$	4	12

Applying 1.3(1) with $p \geqslant 5$ yields that the classes I(2), II(2), II(3), II(4), III(1), III(3), III(6) and all of the classes IV(k) with the possible exception only of IV(4) in case $p = 5$ do not contain elements of $\Gamma_{1,p}$. Moreover, class IV(4) can be excluded even if $p = 5$ by 1.3(2).

Of the remaining classes, I(1) is the identity. II(1) contains two involutions which are conjugate to S and T of (a). For the element conjugate to S this is obvious, for the other representative given by Ueno we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} = T$$

Finally, III(2), III(4), and III(5) correspond to (d), (c), and (b), respectively. In these cases Ueno's representatives are those of the proposition. □

PROPOSITION 1.4. *Let* $Z \in \mathscr{S}_2$. *Then, the isotropy subgroup* $\{g \in \Gamma_{1,p} \mid g(Z) = Z\}$ *is either trivial or conjugate with respect to* Sp(4, **Z**) *to one of the four cyclic groups generated by* S, T, U, *and* V.

Proof. Suppose that $G \subset \Gamma_{1,p}$ is a non-trivial isotropy group. It is shown in [2, Proof of Lemma 2] that with respect to GL(4, **C**) the group G is conjugate to a group of matrices $\begin{pmatrix} \bar{X} & 0 \\ 0 & X \end{pmatrix}$ with unitary $X \in \mathrm{U}(2)$. We may thus identify G with a subgroup G' of U(2). Moreover, since by 1.2 every element $\neq 1$ of G has $+1$ as an eigenvalue of multiplicity 2, all elements of G' are *pseudo-reflections*, i.e., one of their eigenvalues is $+1$.

We claim that a finite subgroup of U(2) which contains only pseudo-reflections is cyclic. Let A, $B \in \mathrm{U}(2)$ be two non-trivial pseudo-reflections such that AB is also a pseudo-reflection. We may assume $A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $B = (b_{ij})$ with eigenvalues 1 and β. Since AB is a pseudo-reflection and $\det(AB) = \alpha\beta$, we find $\mathrm{tr}(AB) = b_{11} + \alpha\beta_{22} = 1 + \alpha\beta$. Wtih $\mathrm{tr}(B) = b_{11} + b_{22} = 1 + \beta$ this implies $(\alpha - 1)(\beta - b_{22}) = 0$, hence $b_{22} = \beta$ and $b_{11} = 1$, and hence $B = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ because B is unitary. It follows that A and B generate a cyclic group which is enough to prove the claim.

Consequently, G is Sp(4, **Z**)-conjugate to one of the cyclic groups generated by the elements listed in 1.2. Finally, observe that if G contains gV^2g^{-1}, $g \in \mathrm{Sp}(4, \mathbf{Z})$, then also gVg^{-1} is in G, as V and V^2 have the same fixed-point set on $\mathscr{S}_2$. We have to show that $gVg^{-1} \in \Gamma_{1,p}$ which follows from $V = \frac{2}{3}1_4 + \frac{2}{3}V^2 - \frac{1}{3}V^4$ since it is easily seen that $\frac{2}{3}A + \frac{2}{3}B - \frac{1}{3}C \in \mathrm{Sp}(4, \mathbf{Z})$ lies in $\Gamma_{1,p}$ if A, B, C are in $\Gamma_{1,p}$. (Here we have used $3 \nmid p$. The statement, however, remains true for $p = 3$, cf. the proof of 3.3.) □

DEFINITION 1.5. (1) Denote the fixed varieties corresponding to the four non-trival isotropy groups of Proposition 1.4 by

$$\mathscr{H}_1 = \mathrm{Fix}(S) = \left\{ Z \in \mathscr{S}_2 \,\middle|\, Z = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix},\ \tau_1, \tau_3 \in \mathscr{S}_1 \right\},$$

$$\mathscr{H}_2 = \mathrm{Fix}(T) = \left\{ Z \in \mathscr{S}_2 \,\middle|\, Z = \begin{pmatrix} \tau_1 & -\frac{1}{2}\tau_3 \\ -\frac{1}{2}\tau_3 & \tau_3 \end{pmatrix},\ \tau_1, \tau_3 \in \mathscr{S}_1 \right\},$$

$$\mathscr{C}_1 = \mathrm{Fix}(U) = \left\{ Z \in \mathscr{S}_2 \,\middle|\, Z = \begin{pmatrix} i & 0 \\ 0 & \tau \end{pmatrix},\ \tau \in \mathscr{S}_1 \right\},$$

$$\mathscr{C}_2 = \mathrm{Fix}(V) = \left\{ Z \in \mathscr{S}_2 \,\middle|\, Z = \begin{pmatrix} \rho & 0 \\ 0 & \tau \end{pmatrix},\ \tau \in \mathscr{S}_1 \right\},\ \rho = e^{2\pi i/3}.$$

(2) The respective images of the fixed loci $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{C}_1$, and $\mathscr{C}_2$ under the natural projection from $\mathscr{S}_2$ onto $\mathscr{A}_{1,p}$ will be called H_1, H_2, C_1, and C_2.

REMARKS 1.6. (1) The points of $H_1 \subset \mathscr{A}_{1,p}$ correspond to $(1,p)$-polarized abelian surfaces A which are determined by period matrices of the form $\begin{pmatrix} 1 & 0 & \tau_1 & 0 \\ 0 & p & 0 & \tau_3 \end{pmatrix}$ and which are hence products of elliptic curves $A \cong E_{\tau_1} \times E'_{\tau_3}$ where $E_\tau = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ and $E'_\tau = \mathbf{C}/(p\mathbf{Z}+\mathbf{Z}\tau)$. These surfaces carry product polarizations which are trivial on the first factor and of degree p on the second factor.

(2) The points of $H_2 \subset \mathscr{A}_{1,p}$ correspond to $(1,p)$-polarized *bielliptic* abelian surfaces, i.e., suitable covers of Jacobians of genus-2 curves which admit an elliptic involution. For a precise definition see [6].

(3) The surfaces $\mathscr{H}_i$, resp. H_i, $i = 1, 2$, are examples of *Humbert surfaces* in the sense of [1] or [5].

(4) Any abelian surface which admits a non-trivial involution corresponds to a point in $H_1 \cup H_2$ ([6, Proposition 2.3]). Since by 1.2 every non-trivial isotropy group in $\Gamma_{1,p}$ contains an involution this shows that $\mathscr{A}_{1,p}$ is non-singular outside $H_1 \cup H_2$. Proposition 1.4 implies $H_1 \cap H_2 = \emptyset$.

(5) The curves C_1 and C_2 in H_1 parametrize product surfaces in the sense of (1) of the form $E_i \times E'_\tau$ resp. $E_\rho \times E'_\tau$. Since the second factor is endowed with a level-p structure, C_1 and C_2 are isomorphic to the (non-compact) modular curve $X^\circ(p) = \Gamma_1(p)\backslash\mathscr{S}_1$. Proposition 1.4 implies $C_1 \cap C_2 = \emptyset$.

All elements in $\Gamma_{1,p}$ conjugate to either U or V lead in $\mathscr{A}_{1,p}$ to images of their respective fixed loci which parametrize abelian surfaces admitting non-involutory automorphisms. The following proposition shows, however, that no other loci arise in this way than the curves C_1 and C_2. Namely, every $\mathrm{Sp}(4,\mathbf{Z})$-translate of $\mathscr{C}_i \subset \mathscr{S}_2$ which is the fixed locus of an element in $\Gamma_{1,p}$ is mapped onto C_i in $\mathscr{A}_{1,p}$, $i = 1, 2$.

PROPOSITION 1.7. *If* $X \in \Gamma_{1,p}$ *is conjugate to* U *(resp.* V*) with respect to* $\mathrm{Sp}(4, \mathbf{Z})$, *then it is also conjugate to* U *(resp.* V*) with respect to* $\Gamma_{1,p}$.

Proof. Suppose that $X = gUg^{-1} \in \Gamma_{1,p}$ with $g = (g_{ij}) \in \mathrm{Sp}(4, \mathbf{Z})$. We construct $\tilde{g} \in \Gamma_{1,p}$ such that $X = \tilde{g}U\tilde{g}^{-1}$ by letting $\tilde{g} = gh$ and choosing $h \in \mathrm{Sp}(4, \mathbf{Z})$ appropriately. Since $U^2 = S$ it follows from $gUg^{-1} \in \Gamma_{1,p}$ that $gSg^{-1} \in \Gamma_{1,p}$ and this implies

$$\begin{vmatrix} g_{21} & g_{23} \\ g_{11} & g_{13} \end{vmatrix} \equiv \begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33} \end{vmatrix} \equiv \begin{vmatrix} g_{21} & g_{23} \\ g_{41} & g_{43} \end{vmatrix} \equiv 0(p).$$

Because of $\det(g) = 1$, a property of symplectic matrices, at least one of the remaining three 2×2 minors lying on the first and third columns of g must be $\not\equiv 0(p)$. But then necessarily $g_{21} \equiv g_{23} \equiv 0(p)$. From $\det(g) = 1$ it then follows that at least one of g_{22} and g_{24} is invertible modulo p^2. We can find relatively prime integers a and c such that $ag_{22} + cg_{24} \equiv 1(p^2)$ and hence there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$ such that $(g_{22}, g_{24}) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (1, 0)(p^2)$. Consequently,

$$gh - \mathbf{1}_4 \in \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ p\mathbf{Z} & p^2\mathbf{Z} & p\mathbf{Z} & p^2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{bmatrix},$$

where we let

$$h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

It follows that $\tilde{g} = gh$ lies in $\Gamma_{1,p}$. (The remaining congruence conditions can be deduced from $\tilde{g}^{-1}\tilde{g} = \mathbf{1}_4$ using the fact that $\tilde{g}$ is symplectic.) Since h commutes with U we also have $\tilde{g}U\tilde{g}^{-1} = gUg^{-1} = X$. We have thus proven the assertion about U. For V one argues in exactly the same way. □

We state the main result of this section using the preceding definitions:

THEOREM 1.8. *Assume that* $p \geqslant 5$. *Then, the moduli space* $\mathcal{A}_{1,p}$ *is singular with transversal* A_1*- resp.* $C_{3,1}$*-type along the smooth curves* C_1 *and* C_2. *The complement* $\mathcal{A}_{1,p} - (C_1 \cup C_2)$ *is non-singular.*

Proof. All that remains to be investigated is how the isotropy groups of Proposition 1.4 act on $\mathcal{S}_2$ with respect to local coordinates around their fixed points. In general, if $Z_0 = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ is a point in $\mathcal{S}_2$ having a non-trivial isotropy group we will introduce local coordinates (x, y, z) in a neighbourhood of $0 \in \mathbf{C}^3$ by $Z = \begin{pmatrix} \tau_1+x & \tau_2+y \\ \tau_2+y & \tau_3+z \end{pmatrix}$. For arbitrary fixed points of S, T, U, and V taken from the fixed loci $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{C}_1$, and $\mathcal{C}_2$ (cf. Definition 1.5) we find by straightforward computation

$$S(Z) = Z_0 + \begin{pmatrix} x & -y \\ -y & z \end{pmatrix},$$

$$T(Z) = Z_0 + \begin{pmatrix} x+2y+z & -y-z \\ -y-z & z \end{pmatrix},$$

$$U(Z) = Z_0 + \begin{pmatrix} -x & -iy \\ -iy & z \end{pmatrix} + \text{(higher order terms)},$$

$$V(Z) = Z_0 + \begin{pmatrix} \rho^2 x & -\rho y \\ -\rho y & z \end{pmatrix} + \text{(higher order terms)}.$$

Hence, with respect to local coordinates and neglecting higher order terms, i.e., looking at the tangent space instead of a neighbourhood of the fixed point and thus linearizing the action, we obtain

$$S\colon (X, Y, Z) \mapsto (X, -Y, Z),$$

$$T\colon (X, Y, Z) \mapsto (X + 2Y + Z, -Y - Z, Z),$$

$$U\colon (X, Y, Z) \mapsto (-X, -iY, Z),$$

$$V\colon (X, Y, Z) \mapsto (\rho^2 X, -\rho Y, Z).$$

The two involutions locally act by reflections—for T this is easily seen by checking the characteristic polynomial of the matrix describing the linearized action—, hence by a well known theorem of Chevalley the quotients by their actions are smooth. The two cyclic groups generated by U and V both contain the cyclic group of order 2 generated by S. Dividing out this reflection group we get cyclic groups of order 2 resp. 3 which in the (X, Y)-plane give rise to quotient singularities of type A_1 resp. $C_{3,1}$ while in the Z-direction along the curve $\mathcal{C}_1$ resp. $\mathcal{C}_2$ the action is trivial. Hence the claim. □

2. Singularities on the boundary of $\mathscr{A}^*_{1,p}$

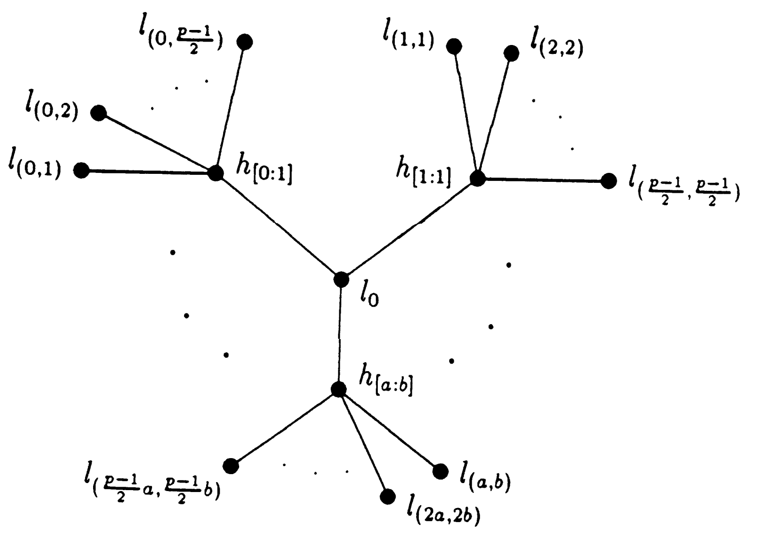
In this section we are concerned with the singularities on the boundary of a particular toroidal compactification $\mathscr{A}^*_{1,p}$ of the moduli space $\mathscr{A}_{1,p}$. In our construction of $\mathscr{A}^*_{1,p}$ we mimic the description of the Igusa compactification in the principally polarized case which Namikawa has given in [7]. A brief outline of it follows. However, since our question is local in nature, not much will be said about the global structure of $\mathscr{A}^*_{1,p}$, nor will there be proofs since a detailed exposition will be given in [5]. We refer to [7] for generalities on toroidal compactification.

The starting point for the compactification of $\mathscr{A}_{1,p}$ is the *Tits building* of $\Gamma_{1,p}$, a graph made up of the $\Gamma_{1,p}$-conjugacy classes of parabolic subgroups of $\Gamma_{1,p}$, or equivalently the $\Gamma_{1,p}$-cosets of non-trivial isotropic subspaces of $\mathbf{Q}^4$ with respect to the symplectic form J, where edges are drawn for inclusion relations.

PROPOSITION 2.1 ([5]). (1) *The $\Gamma_{1,p}$-equivalence classes of isotropic lines in $\mathbf{Q}^4$ are represented by the line l_0 generated by* $(0,0,1,0)$ *and the* $(p^2-1)/2$ *lines $l_{(a,b)}$ generated by* $(0, a/p, 0, b)$ *where a and b are relatively prime integers representing* $(a,b) \in (\mathbf{Z}_p \times \mathbf{Z}_p - \{(0,0)\})/\pm 1$.

(2) *The $\Gamma_{1,p}$-equivalence classes of isotropic planes in $\mathbf{Q}^4$ are represented by the $p+1$ planes $h_{[a:b]}$ spanned by l_0 and $l_{(a,b)}$, where $[a:b] \in \mathbf{P}_1(\mathbf{Z}_p)$.*

(3) *The Tits building of $\Gamma_{1,p}$ is the following graph*



Then, $\mathscr{A}^*_{1,p}$ is obtained from $\mathscr{A}_{1,p}$ by adding boundary components at infinity which are indexed by the vertices of the Tits building. There is a one-to-one correspondence between boundary components of codimension k and $\Gamma_{1,p}$-

classes of isotropic subspaces of rank k in $\mathbf{Q}^4$ with adjacency of boundary components corresponding to edges in the Tits building. In particular, (with (a, b) and $[a:b]$ as in 2.1):

$$\mathscr{A}^*_{1,p} - \mathscr{A}_{1,p} = D^\circ_{l_0} \cup \bigcup_{(a,b)} D^\circ_{l_{(a,b)}} \cup \bigcup_{[a:b]} E_{h_{[a:b]}} = D_{l_0} \cup \bigcup_{(a,b)} D_{l_{(a,b)}}$$

where the D°_l's are open, irreducible surfaces, and the E_h's are connected, compact curves which are reducible. By D_l we denote the Zariski closure of D°_l in $\mathscr{A}^*_{1,p}$. The component D_{l_0} is called *central* boundary component while the other components $D_{l_{(a,b)}}$ are referred to as *peripheral* boundary components.

Let r be an isotropic subspace in $\mathbf{Q}^4$. Then, the boundary component associated with r can—in the present context—be described as follows: Let $P_r \subset \Gamma_{1,p}$ be the parabolic subgroup stabilizing r. In P_r there is an intrinsically defined normal subgroup $P'_r \lhd P_r$ such that $N = P'_r \backslash \mathscr{S}_2$ is a toroidal "neighbourhood of infinity." The partial compactification of $\mathscr{A}_{1,p}$ in the "direction r" is defined by the choice of a suitable toroidal embedding $\bar{N}$ of N such that the action of $P''_r = P_r/P'_r$ on N extends to $\bar{N}$. Then, $P''_r \backslash (\bar{N} - N)$ is the boundary component belonging to r, and $P''_r \backslash \bar{N}$ will become an open neighbourhood of it in $\mathscr{A}^*_{1,p}$. (Of course, compatibility conditions need to be satisfied for the various partial compactifications to glue together.)

Concretely, in our situation P'_r is always conjugate to a lattice consisting of matrices $\begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \in \Gamma_{1,p}$, $B = {}^tB$, which act on $\mathscr{S}_2$ by translations $Z \mapsto Z + B$, so that the quotient map $e_r: \mathscr{S}_2 \to P'_r \backslash \mathscr{S}_2$ is conveniently expressed by exponential functions. $N = P'_r \backslash \mathscr{S}_2$ is locally around infinity isomorphic to either $\mathscr{S}_1 \times \mathbf{C} \times \mathbf{C}^*$ if r is an isotropic line, or to $(\mathbf{C}^*)^3$ if r is a plane. In the first case we simply choose the trivial partial compactification $\mathscr{S}_1 \times \mathbf{C} \times \mathbf{C}$ while in the second case we need to be more careful—see 2.11 below. In either case we obtain a smooth space $\bar{N}$ on which P''_r acts properly discontinuously, so in order to find the singularities of $\mathscr{A}^*_{1,p}$ on the boundary component associated to r it is sufficient to consider the fixed points of the P''_r-action on $\bar{N}$ lying on $\bar{N} - N$.

We also note that under the larger group $\Gamma^\circ_{1,p}$ of symplectic transformations preserving only $(1, p)$-polarizations all lines $l_{(a,b)}$ are equivalent, and all planes $h_{[a:b]}$ are equivalent. Since we naturally choose the partial compactifications belonging to these subspaces to be identified accordingly, we may restrict ourselves to looking at $\mathscr{A}^*_{1,p}$ locally around $D^\circ_{l_0}$, $D^\circ_{l_{(0,1)}}$, and $E_{h_{[0,1]}}$.

In Propositions 2.2, 2.5, and 2.11 below we summarize the technical details of the construction of these components. The computations involved are always straightforward but sometimes lengthy.

PROPOSITION 2.2 ([5]). *The stabilizing subgroup of l_0 in $\Gamma_{1,p}$ is*

$$P_{l_0} = \left\{ \begin{bmatrix} \varepsilon & m & q & pn \\ 0 & a & * & pb \\ 0 & 0 & \varepsilon & 0 \\ 0 & p^{-1}c & * & d \end{bmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p),\ m, n, q \in \mathbf{Z},\ \varepsilon = \pm 1 \right\}$$

where entries '' are determined by the conditions of symplecticity. It contains the lattice* $P'_{l_0} = \left\{ \begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \middle| B = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix},\ q \in \mathbf{Z} \right\}$. *Hence, the corresponding toroidal neighbourhood of infinity is the image of $\mathcal{S}_2$ under the map*

$$e_{l_0}\colon \mathcal{S}_2 \to \mathbf{C}^* \times \mathbf{C} \times \mathcal{S}_1, \qquad \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (e^{2\pi i \tau_1}, \tau_2, \tau_3)$$

An open neighbourhood of $D^\circ_{l_0}$ in $\mathcal{A}^*_{1,p}$ is then obtained as the quotient of a neighbourhood of $\{0\} \times \mathbf{C} \times \mathcal{S}_1$ by the induced action of $P''_{l_0} = P_{l_0}/P'_{l_0}$ on $\mathbf{C} \times \mathbf{C} \times \mathcal{S}_1$. In particular, we may identify P''_{l_0} and its action on $\mathbf{C} \times \mathbf{C} \times \mathcal{S}_1$ as follows:

$$P''_{l_0} \cong \left\{ \begin{bmatrix} 1 & \varepsilon m & \varepsilon n \\ 0 & \varepsilon a & \varepsilon b \\ 0 & \varepsilon c & \varepsilon d \end{bmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p),\ m, n \in \mathbf{Z},\ \varepsilon = \pm 1 \right\},$$

$$\begin{bmatrix} 1 & \varepsilon m & \varepsilon n \\ 0 & \varepsilon a & \varepsilon b \\ 0 & \varepsilon c & \varepsilon d \end{bmatrix} : \begin{bmatrix} t_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \mapsto \begin{bmatrix} t'_1 \\ \tau'_2 \\ \tau'_3 \end{bmatrix} = \begin{bmatrix} t_1 e^{2\pi i \varepsilon [m\tau_2 - \tau'_2(p^{-1}c\tau_2 + c\varepsilon n - d\varepsilon m)]} \\ (\varepsilon\tau_2 + m\tau_3 + pn)(cp^{-1}\tau_3 + d)^{-1} \\ p(ap^{-1}\tau_3 + b)(cp^{-1}\tau_3 + d)^{-1} \end{bmatrix}$$

REMARK 2.3. It follows that $D^\circ_{l_0}$ is isomorphic to the open *Kummer modular surface* $K^\circ(p)$ which is defined to be the quotient of the Shioda modular surface $S^\circ(p)$ by the involution acting simultaneously on all fibers of $S^\circ(p)$ by the natural involution $x \mapsto -x$ of elliptic curves. $K^\circ(p)$ is a smooth surface if $p \geqslant 2$.

PROPOSITION 2.4. *$\mathcal{A}^*_{1,p}$ is non-singular locally around $D^\circ_{l_0}$.*

Proof. Suppose that we have $(t_1, \tau_2, \tau_3) \in \mathbf{C} \times \mathbf{C} \times \mathcal{S}_1$ and

$$g = \begin{bmatrix} 1 & \varepsilon m & \varepsilon n \\ 0 & \varepsilon a & \varepsilon b \\ 0 & \varepsilon c & \varepsilon d \end{bmatrix} \in P''_{l_0} \text{ such that } g(t_1, \tau_2, \tau_3) = (t_1, \tau_2, \tau_3).$$

The invariance of τ_3 implies that $(1/p)\tau_3$ is a fixed point of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$ acting on $\mathcal{S}_1$. It follows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{1}_2$ since $\Gamma_1(p)$ has only trivial fixed points on $\mathcal{S}_1$.

If $\varepsilon = +1$, then invariance of τ_2 implies $m = n = 0$, and hence $g = 1$. Now assume that $\varepsilon = -1$. Then, from the second coordinate we obtain $\tau_2 = \frac{1}{2}(m\tau_3 + pn)$. The fixed points arising in this way are just those of the "Kummer involution" (cf. 2.3) and its P''_{l_0}-conjugates, i.e., the 2-division points in the τ_2-plane with respect to the lattice $p\mathbf{Z} + \mathbf{Z}\tau_3$. The element g acts by

$$g(t_1, \tau_2, \tau_3) = (t_1 e^{-2\pi i m(2\tau_2 - m\tau_3 - pn)}, \ -\tau_2 + m\tau_3 + pn, \tau_3)$$

which locally around $\tau_2 = \frac{1}{2}(m\tau_3 + pn)$ is equivalent to a reflection. Hence, the quotient space is smooth. □

PROPOSITION 2.5 ([5]). *The stabilizing subgroup of* $l_{(0,1)}$ *in* $\Gamma_{1,p}$ *is*

$$P_{l_{(0,1)}} = \left\{ \begin{bmatrix} a & 0 & b & * \\ pm & 1 & pn & p^2q \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}),\ m, n, q \in \mathbf{Z} \right\}$$

where entries '' are determined by the conditions of symplecticity. It contains the lattice* $P'_{l_{(0,1)}} = \left\{ \begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \middle| B = \begin{pmatrix} 0 & 0 \\ 0 & p^2 q \end{pmatrix}, q \in \mathbf{Z} \right\}$. *Hence, the corresponding toroidal neighbourhood of infinity is the image of* $\mathcal{S}_2$ *under the map*

$$e_{l_{(0,1)}}: \mathcal{S}_2 \to \mathcal{S}_1 \times \mathbf{C} \times \mathbf{C}^*, \quad \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (\tau_1, \tau_2, e^{2\pi i \tau_3/p^2})$$

An open neighbourhood of $D^\circ_{l_{(0,1)}}$ *in* $\mathcal{A}^*_{1,p}$ *is then obtained as the quotient of a neighbourhood of* $\mathcal{S}_1 \times \mathbf{C} \times \{0\}$ *by the induced action of* $P''_{l_{(0,1)}} = P_{l_{(0,1)}}/P'_{l_{(0,1)}}$ *on* $\mathcal{S}_1 \times \mathbf{C} \times \mathbf{C}$. *In particular, we may identify* $P''_{l_{(0,1)}}$ *and its action on* $\mathcal{S}_1 \times \mathbf{C} \times \mathbf{C}$ *as follows:*

$$P''_{l_{(0,1)}} \cong \left\{ \begin{bmatrix} 1 & pm & pn \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}),\ m, n \in \mathbf{Z} \right\},$$

$$\begin{bmatrix} 1 & pm & pn \\ 0 & a & b \\ 0 & c & d \end{bmatrix} : \begin{bmatrix} \tau_1 \\ \tau_2 \\ t_3 \end{bmatrix} \mapsto \begin{bmatrix} \tau'_1 \\ \tau'_2 \\ t'_3 \end{bmatrix} = \begin{bmatrix} (a\tau_1 + b)(c\tau_1 + d)^{-1} \\ (\tau_2 + pm\tau_1 + pn)(c\tau_1 + d)^{-1} \\ t_3 e^{2\pi i [pm\tau_2 - \tau'_2(c\tau_2 + p(cn - dm))]p^{-2}} \end{bmatrix}$$

REMARK 2.6. It follows that $D^{\circ}_{l_{(0,1)}}$ is isomorphic to the open Kummer modular surface $K^{\circ}(1)$ which contains four singular points.

DEFINITION 2.7. For $i = 1, 2$, denote by C_i^* the Zariski closure of the curve C_i in $\mathscr{A}^*_{1,p}$. (Cf. 1.5(2) for the definition of C_i.)

PROPOSITION 2.8. *There are precisely four singular points of* $\mathscr{A}^*_{1,p}$ *on* $D^{\circ}_{l_{(0,1)}}$, *namely* (1) *Two non-isolated singular points* Q_1 *and* Q_2, *represented by the points* $(i, 0, 0)$ *and* $(\rho, 0, 0)$, $\rho = e^{2\pi i/3}$, *of* $\mathscr{S}_1 \times \mathbf{C} \times \mathbf{C}$ *in the setting of* 2.5, *respectively. For* $i = 1, 2$, *the point* Q_i *is the point of intersection of* C_i^* *with* $D^{\circ}_{l_{(0,1)}}$ *where* C_i^* *is smooth at* Q_i. *The singularities of* $\mathscr{A}^*_{1,p}$ *around these points are of transversal* A_1-*type along* C_1^* *and of transversal* $C_{3,1}$-*type along* C_2^*.

(2) *Two isolated cyclic quotient singularities* Q_1' *and* Q_2' *which are represented by* $(i, \frac{p}{2}(1-i), 0)$ *resp.* $(\rho, \frac{p}{3}(1-\rho), 0)$ *in* $\mathscr{S}_1 \times \mathbf{C} \times \mathbf{C}$, *and which are of type* $\frac{1}{2}(1, 1, 1)$ *resp.* $\frac{1}{3}(1, 2, 1)$. (*Cf. the introduction.*)

Proof. Suppose that we have $(\tau_1, \tau_2, t_3) \in \mathscr{S}_1 \times \mathbf{C} \times \mathbf{C}$ and

$$g = \begin{bmatrix} 1 & pm & pn \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \in P''_{l_{(0,1)}} \text{ such that } g(\tau_1, \tau_2, t_3) = (\tau_1, \tau_2, t_3).$$

From the first coordinate we read off that $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$ acting on $\mathscr{S}_1$ leaves τ_1 fixed. We then either have $g_0 = \pm \mathbf{1}_2$ which act trivially on $\mathscr{S}_1$, or $\tau_1 = i$ and $g_0 = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $\tau_1 = \rho$ and $g_0 \in \left\{\pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right\}$.

The case of $g_0 = \pm \mathbf{1}_2$ can be dealt with in exactly the same way as in the proof of 2.4. Since it is easy to see that g_0 and g_0^{-1} give rise to the same singularities, for the remaining cases we just have to look into the following three possibilities:

(a) $\tau_1 = i$, $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, $\tau_2 = \frac{p}{2}(m+n)(1-i) - pn$ and modulo the $P''_{l_{(0,1)}}$-action these points belong to the two orbits represented by $\tau_2 = 0$ and $\tau_2 = \frac{p}{2}(1-i)$.

Let $\tau_2 = 0$, i.e., $m = n = 0$. This element g leaves all the points fixed in $\{i\} \times \{0\} \times \mathbf{C}$ which contains the image of $\mathscr{C}_1$. With respect to local coordinates introduced by $(i + x, y, z)$ around $(i, 0, 0)$ the action of g is $g: (x, y, z) \mapsto (-x, -iy, z) + \cdots$ up to first order. Hence the claim about transversal A_1-type along C_1^* locally around Q_1.

Now let $\tau_2 = \frac{p}{2}(1-i)$, i.e., $m = 1$ and $n = 0$. In local coordinates defined around $(i, \frac{p}{2}(1-i), 0)$ by $(i + x, \frac{p}{2}(1-i) + y, z)$ the action of g is $(x, y, z) \mapsto (-x, \frac{p}{2}(1-i)x - iy, -z) + \cdots$ which after a change of base is equivalent to $(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto (-\tilde{x}, i\tilde{y}, -\tilde{z})$. After dividing out the reflection induced by g^2 we see that we get a cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$ at Q_1'.

(b) $\tau_1 = \rho$, $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Then, $\tau_2 = p(m-n) - pn\rho$, hence modulo $P''_{l_{(0,1)}}$ all these points are equivalent to $\tau_2 = 0$, corresponding to $m = n = 0$. Again we find a point-wise fixed curve $\{\rho\} \times \{0\} \times \mathbf{C}$ containing the image of $\mathscr{C}_2$. With coordinates defined by $(\rho + x, y, z)$ around $(\rho, 0, 0)$ we get $(x, y, z) \mapsto (\rho^2 x, -\rho y, z) + \cdots$ for the action of g. Hence, the resulting singularities are of transversal $C_{3,1}$-type along C_2^*.

(c) $\tau_1 = \rho$, $g_0 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then, $\tau_2 = \frac{p}{3}(m+n)(1-\rho) - pn$, and modulo $P''_{l_{(0,1)}}$ these points belong to two orbits, namely those represented by $\tau_2 = 0$ and $\tau_2 = \frac{p}{3}(1-\rho)$. (Note that the orbits of $\frac{p}{3}(1-\rho)$ and $2p/3(1-\rho)$ coincide. This can be seen by applying the Kummer involution $\tau_2 \mapsto -\tau_2$.) In the case $\tau_2 = 0$ the corresponding element g is the square of the one considered in (b) above. Hence, let $\tau_2 = \frac{p}{3}(1-\rho)$, and define local coordinates around $(\rho, \frac{p}{3}(1-\rho), 0)$ by $(\rho + x, \frac{p}{3}(1-\rho) + y, z)$. Then, $g\colon (x, y, z) \mapsto (\rho x, -\frac{p}{3}\rho(1-\rho)x + \rho^2 y, \rho z) + \cdots$ and this defines a cyclic quotient singularity of type $\frac{1}{3}(1,2,1)$ at Q_2' as claimed. □

Now that we know the structure of $\mathscr{A}_{1,p}^*$ around the open codimension-1 boundary components it remains to study a neighbourhood of the codimension-2 boundary component $E_{h_{(0,1)}}$. Unlike in the two preceding cases we cannot just use a trivial embedding of the toroidal neighbourhood. Before giving the details we state a few facts about torus embeddings and define the fan determining the particular torus embedding which we are going to use.

Let $T \cong (\mathbf{C}^*)^r$ be an algebraic torus of rank r, and denote by $M = \mathrm{Hom}(T, \mathbf{C}^*)$ and $N = \mathrm{Hom}(\mathbf{C}^*, T)$ its respective groups of characters and 1-parameter subgroups. M and N are lattices of rank r, naturally dual to each other, with associated real vector spaces $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. If $\sigma \subset N_{\mathbf{R}}$ is a cone defined by rational hyperplanes, then define $X_\sigma = \mathrm{Spec}\,\mathbf{C}[\sigma^\vee \cap M]$ where $\mathbf{C}[\sigma^\vee \cap M]$ is the semi-group ring defined by $\sigma^\vee = \{x \in M_{\mathbf{R}} | x \cdot y \geqslant 0$ for all $y \in \sigma\}$, the cone dual to σ. More generally, let Σ be a *fan* (also called a *rational partial polyhedral decomposition*) in $N_{\mathbf{R}}$, i.e., a collection of cones which do not contain linear subspaces such that with $\sigma \in \Sigma$ all faces of σ are in Σ, and such that $\sigma_1 \cap \sigma_2$ is a common face for any two $\sigma_1, \sigma_2 \in \Sigma$. If σ' is a face of $\sigma \in \Sigma$, a natural open inclusion $X_{\sigma'} \hookrightarrow X_\sigma$ is induced by $\sigma'^\vee \supset \sigma^\vee$. ($T = X_{\{0\}} \hookrightarrow X_\sigma$ is a special case.) The *torus embedding* X_Σ *associated to* Σ is defined to be the scheme obtained from patching together all X_σ, $\sigma \in \Sigma$, using the identifications $X_{\sigma'} \hookrightarrow X_\sigma$ coming from faces $\sigma' \subset \sigma$.

DEFINITION 2.9 (Cf. [7, §7]). Denote by Symm(2) the 3-dimensional real vector space of symmetric 2×2 matrices, and by $\mathrm{Symm}_+(2)$ the cone of positive definite matrices therein. Then, the *2nd Voronoi decomposition* Σ of $\mathrm{Symm}_+(2)$ is defined to be the fan consisting of all $\mathrm{GL}(2, \mathbf{Z})$-translates of the cone

$$\sigma_0 = \mathbf{R}_+ \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{R}_+ \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{R}_+ \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \subset \overline{\mathrm{Symm}_+(2)}$$

together with all respective faces, where $\mathrm{GL}(2, \mathbf{Z})$ acts on Symm(2) by $g(M) = {}^t g^{-1} M g^{-1}$ for $g \in \mathrm{GL}(2, \mathbf{Z})$ and $M \in \mathrm{Symm}(2)$.

REMARK 2.10 ([7, (6.14)]). Let N be the lattice of integral matrices in Symm(2). Then, the 3-dimensional torus embedding X_Σ associated with the 2nd Voronoi decomposition Σ is smooth. Furthermore, the action of $\mathrm{GL}(2, \mathbf{Z})$ on the embedded torus $T = N \otimes \mathbf{C}^*$ induced from the $\mathrm{GL}(2, \mathbf{Z})$-action on N naturally extends to an action on X_Σ. (This is so because $g(\sigma) \in \Sigma$ holds for every $g \in \mathrm{GL}(2, \mathbf{Z})$ and $\sigma \in \Sigma$.)

PROPOSITION 2.11 ([5]). *The stabilizing subgroup of $h_{[0:1]}$ in $\Gamma_{1,p}$ is*

$$P_{h_{[0:1]}} = \left\{ \begin{pmatrix} {}^t A^{-1} & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}(2, \mathbf{Z}),\ A \in \begin{pmatrix} \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & 1 + p\mathbf{Z} \end{pmatrix} \right\} \cdot P'_{h_{[0:1]}},$$

where

$$P'_{h_{[0:1]}} = \left\{ \begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \middle| B = {}^t B,\ B \in \begin{pmatrix} \mathbf{Z} & p\mathbf{Z} \\ p\mathbf{Z} & p^2\mathbf{Z} \end{pmatrix} \right\}.$$

The corresponding toroidal neighbourhood of infinity is the image of $\mathscr{S}_2$ under

$$e_{h_{[0:1]}}: \mathscr{S}_2 \to T = (\mathbf{C}^*)^3,\ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (e^{2\pi i \tau_1},\ e^{2\pi i \tau_2 / p},\ e^{2\pi i \tau_3 / p^2}).$$

The group $P''_{h_{[0:1]}} = P_{h_{[0.1]}} / P'_{h_{[0:1]}}$ can be identified with

$$G = \left\{ g \in \mathrm{GL}(2, \mathbf{Z}) \middle| g \in \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ p\mathbf{Z} & 1 + p\mathbf{Z} \end{pmatrix} \right\} \subset \mathrm{GL}(2, \mathbf{Z})$$

by sending the 2×2 matrix A in the description of $P_{h_{[0.1]}}$ above to FAF^{-1} where $F = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. With respect to coordinates (t_1, t_2, t_3) on T, G then acts on T by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \mapsto \begin{bmatrix} t_1^{d^2} t_2^{-2cd} t_3^{c^2} \\ t_1^{-bd} t_2^{ad + bc} t_3^{-ac} \\ t_1^{b^2} t_2^{-2ab} t_3^{a^2} \end{bmatrix}.$$

This action corresponds to the action of $G \subset \mathrm{GL}(2, \mathbf{Z})$ given in 2.9 if the lattice of integral matrices in Symm(2) *is identified with* $\mathrm{Hom}(\mathbf{C}^*, T)$ *by sending* $\begin{pmatrix} e_1 & e_2 \\ e_2 & e_3 \end{pmatrix}$ *to the 1-parameter subgroup* $\lambda \mapsto (\lambda^{e_1}, \lambda^{e_2}, \lambda^{e_3})$.

An open neighbourhood of $E_{h_{[0:1]}}$ *in* $\mathscr{A}^*_{1,p}$ *is then obtained as the quotient of a neighbourhood of* $X_\Sigma - T$ *in* X_Σ *by the induced action of* G, *where* $T \hookrightarrow X_\Sigma$ *denotes the torus embedding determined by the 2nd Voronoi decomposition* Σ.

PROPOSITION 2.12. $\mathscr{A}^*_{1,p}$ *is non-singular locally around* $E_{h_{[0:1]}}$.

Proof. (I) The description for the action of G on T given in 2.11 allows to define an action of $\mathrm{GL}(2, \mathbf{Z})$ on T, and by definition of Σ this action extends to X_Σ. In order to prove smoothness of $\mathscr{A}^*_{1,p}$ around $E_{h_{[0:1]}}$ it suffices to show that every non-trivial isotropy group in G of a point in $X_\Sigma - T$ is generated by pseudo-reflections. We will prove the stronger assertion that every $g \neq \pm \mathbf{1}_2$ in $\mathrm{GL}(2, \mathbf{Z})$ which is conjugate to an element of G and has a fixed point on $X_\Sigma - T$ acts locally like a reflection.

Let $\sigma_0 \in \Sigma$ be defined as in 2.9. We first observe that it is sufficient to consider only fixed points in the affine piece $X_{\sigma_0} \subset X_\Sigma$ because every point $x \in X_\Sigma$ lies in some $X_{g(\sigma_0)}$, $g \in \mathrm{GL}(2, \mathbf{Z})$, and there is a natural isomorphism $g: X_{\sigma_0} \xrightarrow{\cong} X_{g(\sigma_0)}$ by which fixed points in $X_{g(\sigma_0)}$ correspond to fixed points (with respect to conjugate group elements) in X_{σ_0}. Since X_{σ_0} is isomorphic to $\mathbf{C}^3$ with $(\mathbf{C}^*)^3$ being the image of T, we have to look for fixed points lying on the three axes $\mathbf{C} \times \{0\} \times \{0\}$, $\{0\} \times \mathbf{C} \times \{0\}$, and $\{0\} \times \{0\} \times \mathbf{C}$. However, since the stabilizing subgroup of the cone σ_0 in $\mathrm{GL}(2, \mathbf{Z})$ permutes the generators of σ_0 [7, (8.7)], and hence also permutes the three axes in X_{σ_0}, by a similar argument it suffices to consider only one of them, e.g., $\{0\} \times \{0\} \times \mathbf{C}$. Our problem now splits up into two parts: Fixed points in $\{0\} \times \{0\} \times \mathbf{C}^*$ representing generic points of $E_{h_{[0:1]}}$, and $(0,0,0)$ as a fixed point, representing so-called *deepest points* of $E_{h_{[0:1]}}$.

Before treating the two cases separately, we make two observations which will become useful. Firstly, note that if $x \in X_{\sigma_0}$ is a fixed point of $g \in \mathrm{GL}(2, \mathbf{Z})$, then there exists an open neighbourhood W of x in X_{σ_0} such that $g(W) \subset X_{\sigma_0}$. (A direct consequence of continuity.) In order to make use of this fact we need to understand where points of X_{σ_0} are mapped to by elements $g \in \mathrm{GL}(2, \mathbf{Z})$. Conveniently using the coordinate functions t_1, t_2, t_3 on T for generators of the lattice $M = \mathrm{Hom}(T, \mathbf{C}^*)$ we find that the dual cone $\sigma_0^\vee$ is generated by the characters $t_1 t_2$, $t_2 t_3$, and t_2^{-1} in M. If we use them to represent coordinate functions T_1, T_2, T_3 on $X_{\sigma_0} \cong \mathbf{C}^3$, then the actual embedding of the torus T in X_{σ_0} is expressed by

$$\iota: T = (\mathbf{C}^*)^3 \hookrightarrow X_{\sigma_0} = \mathbf{C}^3, \quad (t_1, t_2, t_3) \mapsto (t_1 t_2, t_2 t_3, t_2^{-1}).$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ we formally define $\varphi_g = \iota \circ g \circ \iota^{-1}$ as a rational morphism from X_{σ_0} to itself. Wherever φ_g is a map it describes the part of the action of g that takes place inside X_{σ_0}. Concretely,

$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \mapsto \begin{bmatrix} T_1^{d(d-b)} T_2^{c(c-a)} T_3^{(c+d)[(c+d)-(a+b)]} \\ T_1^{b(b-d)} T_2^{a(a-c)} T_3^{(a+b)[(a+b)-(c+d)]} \\ T_1^{bd} T_2^{ac} T_3^{(a+b)(c+d)} \end{bmatrix}$$

The second observation we want to make is the following lemma which is a trivial consequence of the definition of G in 2.11:

LEMMA 2.14. *If $g \in \mathrm{GL}(2, \mathbf{Z})$ is conjugate to an element of G, then its characteristic pair* $(\mathrm{tr}(g), \det(g))$ *is either congruent to* $(0, -1)$ *or to* $(2, 1)$ *modulo p.*

(II) Fixed points on $\{0\} \times \{0\} \times \mathbf{C}^*$. Assume that the point (T_1, T_2, T_3) with $T_1 = T_2 = 0$ and $T_3 \neq 0$ is fixed by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using the fact that φ_g is defined in an open neighbourhood of the fixed point, we obtain from $T_1 = 0$ and (2.13) that $bd = 0$, hence $b = 0$ or $d = 0$. Similarly, $T_2 = 0$ implies $ac = 0$, so $a = 0$ or $c = 0$. Of these four cases, obviously those where $a = b = 0$ or $c = d = 0$ cannot lead to invertible matrices. The cases $b = c = 0$ resp. $a = d = 0$ together with $\det(g) = \pm 1$ lead to the following eight possibilities:

$$g \in \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

Finally, 2.14 shows that if $g \in \mathrm{GL}(2, \mathbf{Z})$ is conjugate to an element of G and has a fixed point on $\{0\} \times \{0\} \times \mathbf{C}^*$, then g must be one of the following:

$$g \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

We shall describe the action of these matrices on X_{σ_0} in part (IV) below.

(III) The "deepest point" $(0, 0, 0) \in X_{\sigma_0}$. If (T_1, T_2, T_2) with $T_1 = T_2 = T_3 = 0$ is fixed by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then arguing as in (II) we obtain from $T_1 = 0$ with (2.13) that $d(d - b) \geqslant 0$, $b(b - d) \geqslant 0$, and $bd \geqslant 0$ hold simultaneously. This implies that $b = 0$, or $d = 0$, or $b = d$. Similarly, $T_2 = 0$ yields $a = 0$, or $c = 0$, or $a = c$, and finally from $T_3 = 0$ we get $a + b = 0$, or $c + d = 0$, or $a + b = c + d$. This basically gives us 27 different cases to consider. However, one easily sees that only the following six cases can occur if the matrix g is invertible:

1. $b = 0,\ c = 0,\ a + b = c + d \Rightarrow g = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. $b = 0,\ a = c,\ c + d = 0 \Rightarrow g = \pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

3. $d = 0,\ a = 0,\ a + b = c + d \Rightarrow g = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4. $d = 0,\ a = c,\ a + b = 0 \Rightarrow g = \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

5. $b = d,\ a = 0,\ c + d = 0 \Rightarrow g = \pm\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

6. $b = d,\ c = 0,\ a + b = 0 \Rightarrow g = \pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$

Again, 2.14 helps to cut down the number of possibilities in our case:

$$g \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

(Note that we have used $p \neq 3$ when we excluded the matrices of cases 4 and 5.)

(IV) We describe how the matrices found in (II) and (III) act on X_{σ_0}. In the four non-trivial cases we compute with (2.13):

$$\pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}: (T_1, T_2, T_3) \mapsto (T_1, T_3, T_2)$$

$$\pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}: (T_1, T_2, T_3) \mapsto (T_3, T_2, T_1)$$

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: (T_1, T_2, T_3) \mapsto (T_2, T_1, T_3)$$

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: (T_1, T_2, T_3) \mapsto (T_1 T_3^2, T_2 T_3^2, T_3^{-1})$$

The first three matrices obviously act by reflections. They are also easily seen to be conjugate to each other. The fourth matrix leaves the two hyperplanes defined by $T_3 = 1$ and $T_3 = -1$ fixed and is also acting like a reflection locally around them. This concludes the proof of Proposition 2.12. □

Putting together the results of this section and of Theorem 1.8 we conclude

THEOREM 2.15. *For $p \geqslant 5$, the singular locus of $\mathscr{A}_{1,p}^*$ contains two smooth, compact curves C_1^* and C_2^* isomorphic to $X(p)$ which are the Zariski closures of the two modular curves of degree p in $\mathscr{A}_{1,p}$ parametrizing polarized products of type $E_i \times E'$ resp. $E_\rho \times E'$, where E' is an elliptic curve with level-p structure. $\mathscr{A}_{1,p}^*$ is singular with transversal A_1-type along C_1^*, and is singular with transversal $C_{3,1}$-type along C_2^*. Both curves C_1^* and C_2^* intersect each of the $(p^2-1)/2$ peripheral* *boundary components* $D_{l_{(a,b)}}$ *in precisely one point, and do not meet the central boundary component* D_{l_0}. *Outside* $C_1^* \cup C_2^*$ *the singularities of* $\mathcal{A}_{1,p}^*$ *consist of exactly one isolated cyclic quotient of type* $\frac{1}{2}(1,1,1)$ *and one of type* $\frac{1}{3}(1,2,1)$ *on each of the* $(p^2-1)/2$ *peripheral boundary components.*

3. The case $p=3$

In this section we study the space $\mathcal{A}_{1,3}^*$. It turns out that Theorem 2.15 is still valid for $\mathcal{A}_{1,3}^*$ although some points of $\mathcal{S}_2$ actually have more complicated isotropy groups in $\Gamma_{1,3}$ than they have in $\Gamma_{1,p}$ for $p \geqslant 5$. We first rephrase some of the results of section 1 in terms of automorphism groups of abelian surfaces:

THEOREM 3.1. *Assume that* $p \geqslant 5$ *and let* (A, H, α) *be a* $(1,p)$*-polarized abelian surface with level structure. Then:*

(1) *If* (A, H, α) *is neither a polarized product nor bielliptic, then* $\mathrm{Aut}(A,H,\alpha)$ *is trivial.*
(2) *If* (A, H, α) *is bielliptic, then* $\mathrm{Aut}(A, H, \alpha) \cong \mathbf{Z}_2$.
(3) *If* (A, H, α) *is a polarized product* $A = E \times E'$ *of elliptic curves with product polarization coming from a trivial polarization on* E *and one of degree* p *on* E', *then* $\mathrm{Aut}(A,H,\alpha) \cong \mathrm{Aut}(E)$ *which is either* $\mathbf{Z}_2$ *for* E *generic, or* $\mathbf{Z}_4$ *if* $E \cong E_i$, *or* $\mathbf{Z}_6$ *if* $E \cong E_\rho$, *where* $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$.

DEFINITION 3.2. Let

$$W = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

an element of $\Gamma_{1,3}$ of order 3. Let

$$\mathscr{C}_3 = \mathrm{Fix}(W) = \left\{ Z \in \mathcal{S}_2 \,\middle|\, Z = \begin{pmatrix} \frac{1}{3}\tau_3 & -\frac{1}{2}\tau_3 \\ -\frac{1}{2}\tau_3 & \tau_3 \end{pmatrix}, \ \tau_3 \in \mathcal{S}_1 \right\} \subset \mathcal{H}_2,$$

and denote by C_3 the image of $\mathscr{C}_3$ under the natural projection from $\mathcal{S}_2$ onto $\mathcal{A}_{1,3}$.

THEOREM 3.3. *Let* (A, H, α) *be a* $(1,3)$*-polarized abelian surface with level structure. Then:*

(1) *If* (A, H, α) *is neither a polarized product nor bielliptic, then* $\mathrm{Aut}(A,H,\alpha) = \{1\}$.

(2) *If* (A, H, α) *is bielliptic, then either* $\mathrm{Aut}(A, H, \alpha) \cong \mathbf{Z}_2$ *if* (A, H, α) *does not lie on* C_3, *or else* $\mathrm{Aut}(A, H, \alpha) \cong S_3$, *the symmetric group acting on three symbols.*

(3) *If* (A, H, α) *is a polarized product* $A = E \times E'$ *as in* 3.1(3), *then* $\mathrm{Aut}(A, H, \alpha) \cong \mathrm{Aut}(E)$.

Proof. Firstly, note that an analogue of 1.2 is valid for $p = 3$ which differs from 1.2 only by the existence of one additional conjugacy class of matrices of order 3 in $\Gamma_{1,3}$, represented by W. The arguments for the proof essentially remain unchanged, the only exception being that Ueno's class II(3) can no longer be excluded. Of the three $\mathrm{Sp}(4, \mathbf{Z})$-conjugacy classes II(3)a, II(3)b, and II(3)c belonging to it, the first one is represented by

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

The (2, 4)-entry of gMg^{-1} for $g = (g_{ij}) \in \mathrm{Sp}(4, \mathbf{Z})$ is $(g_{21} + g_{23})^2 + (g_{22} + g_{24})^2 - 3(g_{21}g_{23} + g_{22}g_{24})$ which must be congruent to 0 modulo 9 for gMg^{-1} to be in $\Gamma_{1,3}$. This, however, implies $g_{21} \equiv g_{22} \equiv g_{23} \equiv g_{24} \equiv 0\ (3)$ which clearly contradicts the invertibility of g. Hence, the class II(3)a does not contain elements of $\Gamma_{1,3}$, and the same is true for the class II(3)b since its elements are the inverses of those of II(3)a. As for the last class II(3)c this one in particular contains W as can be seen from the following explicit conjugation of Ueno's representative

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1} = W$$

Secondly, 1.4 carries over to $p = 3$ now stating that every non-trivial isotropy subgroup of $\Gamma_{1,3}$ not containing a conjugate of W is conjugate to one of the cyclic groups generated by S, T, U, and V. We need a new argument to prove that the cyclic groups generated by $\mathrm{Sp}(4, \mathbf{Z})$-conjugates of V^2 do not occur. From the (2, 4)-entry of $gV^2g^{-1} \in \Gamma_{1,3}$ we obtain $g_{21}^2 + g_{23}^2 + g_{21}g_{23} \equiv 0\ (9)$ for $g = (g_{ij})$ which implies $g_{21} \equiv g_{23} \equiv 0\ (3)$. A lengthy computation then shows that this is indeed sufficient for $gVg^{-1} \in \Gamma_{1,3}$. Hence, if an isotropy group contains gV^2g^{-1}, then it also contains gVg^{-1}.

Our next step is to show that in analogy to 1.7 every $\mathrm{Sp}(4, \mathbf{Z})$-conjugate of W lying in $\Gamma_{1,3}$ is a $\Gamma_{1,3}$-conjugate of W as well. Suppose that $gWg^{-1} \in \Gamma_{1,3}$ and

$g \in \mathrm{Sp}(4, \mathbf{Z})$. Our aim is—like in the proof of 1.7—to find $h \in \mathrm{Sp}(4, \mathbf{Z})$ such that $gh \in \Gamma_{1,3}$ and $hWh^{-1} = W$. Of the matrices h commuting with W we will make use of

$$h_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad h_2 = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The (2, 4)-entry of gWg^{-1} is $3g_{21}g_{23} + 2g_{21}g_{24} - 6g_{22}g_{23} - 3g_{22}g_{24} \equiv 0\ (9)$ which readily implies $g_{21} \equiv 0\,(3)$ or $g_{24} \equiv 0\,(3)$. Using this together with the conditions obtained from the first three entries of the second row of $gWg^{-1} - \mathbf{1}_4$ all being $\equiv 0\,(3)$ we deduce that $g_{21} \equiv g_{24} \equiv 0\,(3)$. Then, again exploiting the (2, 4)-entry of gWg^{-1} we see that either $g_{22} \equiv 0\,(3)$ or $g_{23} \equiv 0\,(3)$. Substituting g by one of $\pm g$ or $\pm gh_1$, we may assume that $g_{22} \equiv 1\,(3)$ and $g_{23} \equiv 0\,(3)$. Finally, substituting g by gh_2^k for some integer k we can achieve $g_{24} \equiv 0\,(9)$ in addition to the other congruences satisfied by g. As was noted in the proof of 1.7 this suffices to conclude that $g \in \Gamma_{1,3}$.

We thus see that the only differences between the cases $p \geqslant 5$ and $p = 3$ occur with points lying on $C_3 \subset H_2$ in $\mathscr{A}_{1,3}$. Denote the isotropy group of a point in $\mathscr{S}_2$ over C_3 by G. Like in the proof of 1.4 G is equivalent to a subgroup of U(2). From Gottschling's proof in [2, p. 123] we see that matrices in U(2) corresponding to conjugates of W have ρ and ρ^2 as their eigenvalues. In particular, the image of G in U(2) does not contain scalar matrices and hence is isomorphic to its induced group of automorphisms of the sphere $\mathbf{P}_1$. Thus G either is cyclic, or dihedral, or the group of symmetries of a regular polyhedron. Since G contains only elements of order 2 and 3, this leaves the dihedral group $D_3 \cong S_3$ and the tetrahedral group as the only possibilities. But since G contains the dihedral group of order 6 generated by T and W, it cannot be tetrahedral. So, $G \cong \langle W, T \rangle \cong S_3$. □

COROLLARY 3.4. *Theorem* 2.15 *remains valid if* $p = 3$.

Proof. (I) We first prove the analogue of Theorem 1.8 for $p = 3$. In view of 3.3 it suffices to show that the isotropy groups G of points $Z \in \mathscr{C}_3$ are generated by elements which locally around Z act like reflections. This, however, is clear since G is a dihedral group of order 6 and as such is generated by the two involutions T and WTW^{-1} which act like reflections—cf. the proof of 1.8.

(II) The compactification procedure for obtaining $\mathscr{A}_{1,p}^*$ from $\mathscr{A}_{1,p}$ described in section 2 goes through unchanged for $p = 3$. Also, the statements and proofs of 2.4 and 2.8 remain valid as they are. The only difference to the case $p \geqslant 5$ arises with deepest points on codimension-2 boundary components where now additional elements of order 3 can appear (cf. the proof of 2.12).

Consider a deepest point and assume that we are in the situation of part (III) of the proof of 2.12. The isotropy group G of $(0,0,0)$ then is contained in the set

$$H = \left\{\mathbf{1}_2, \pm\begin{pmatrix}1 & -1\\0 & -1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 1\\-1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\1 & -1\end{pmatrix}\right\}$$

and we assume that it contains the last two elements which are of order 3 and which leave the curve determined by $T_1 = T_2 = T_3$ point-wise fixed. Since C_3 lies inside H_2 there must also be an involution present in G. It is easy to see that any one of the six possible involutions together with the elements of order 3 generates a dihedral group of order 6 which then acts as a group generated by reflections just as in (I) above. Finally, since this dihedral group contains one element from each of the three pairs of involutions in H, it must already be the whole of G because otherwise G would contain $-\mathbf{1}_2$, a contradiction to $G \subset H$. □

REMARK 3.5. Consider the polarized product $E \times E'$ where $E = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\frac{1}{3}\tau_3)$ and $E' = \mathbf{C}/(3\mathbf{Z} + \mathbf{Z}\tau_3)$ which without its polarization of type $(1,3)$ is the product of two isomorphic elliptic curves. If $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ is as a subgroup of $E \times E'$ identified with the points $(\omega_i, -\omega_i')$, $i = 1, \ldots, 4$, where $\omega_1, \ldots, \omega_4$ denote the 2-torsion points on E and $\omega_1', \ldots, \omega_4'$ are the corresponding points on $E' \cong E$, then it is easy to see that the abelian variety $A = E \times E'/G$ is bielliptic and corresponds to $\begin{pmatrix}\frac{1}{3}\tau_3 & -\frac{1}{2}\tau_3\\ -\frac{1}{2}\tau_3 & \tau_3\end{pmatrix} \in \mathscr{C}_3$. (This is a special case of a construction in [6, §4].) We note without proof that using this identification one can show that $C_3 \cong X^\circ(3)$.

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