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The restriction to SL_2 of a supercuspidal representation of GL_2

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0. Introduction

Let F be a p-adic field, let $W = W(F)$ be the absolute Weil group of F and let $\mathscr{A}_N^0(F)$ be the set of (equivalence classes of) complex irreducible N-dimensional representations of W. Then it is a conjecture of Langlands that, among other things, there should be a natural map $\sigma \mapsto \pi(\sigma)$ from $\mathscr{A}_N^0(F)$ to the set $\mathscr{A}^0(G)$ of irreducible supercuspidal representations of $G = GL_N(F)$. Let M be a subgroup of finite index in $F^\times$ and let G_M be the subgroup of G consisting of elements g with $\det g$ in M. Then one expects the following to hold:

(0.1) The restriction of $\pi(\sigma)$ to G_M is reducible if and only if there exists an extension E/F and a representation τ of $W(E)$ such that

(i) $N_{E/F}E^\times = M$ and
(ii) $\sigma = \mathrm{Ind}_{E/F}\tau$

(see [K_3] for a more detailed discussion).

As a consequence one should therefore expect

(0.2) The restriction of $\pi(\sigma)$ to $\bar{G} = SL_N(F)$ is reducible if and only if $\sigma = \mathrm{Ind}_{E/F}\tau$ where E/F is prime cyclic.

It is interesting to note that (0.2) has apparently been verified only in case $N = 2$ and that in this case, the result follows from the criterion of Labesse–Langlands [L–L]:

(0.3) An irreducible supercuspidal representation π of $G = GL_2(F)$ restricts irreducibly to $\bar{G}$ if and only if π is exceptional; that is, π cannot be constructed by the method of Weil representations.

It is the purpose of this paper to give a local proof for (0.3) in the hope of eventually proving (0.2) in general, Kazhdan's $\pi(\theta)$ [Ka] replacing the Weil construction. The paper is organized as follows. In Section 1, we review that construction of supercuspidal representations of $G = GL_2(F)$ given in [K_1],

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recasting things in the language of principal orders [B-F]. In Section 2, we use Mackey's theorem to give a preliminary decomposition of $\pi|_{\bar{G}}$, where π is a supercuspidal representation of G. In Section 3, we give our main result, Theorem 3.6, which gives a necessary and sufficient condition that $\pi|_{\bar{G}}$ remain irreducible. Since this condition is the same as the condition given in [K$_2$] that π be exceptional, we have verified (0.3).

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In what follows we let F be a local field of residual characteristic p and let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers in F and $P = P_F$ be the maximal ideal of $\mathcal{O}$ with generator $\varpi = \varpi_F$. We denote by $k = k_F$ the residue class field $\mathcal{O}/P$ with $q = q_F$ elements. We let $U^t = U^t_F$ (t a positive integer) be the subgroup of $U = U_F = \mathcal{O}^\times$ of elements of the form $1 + a$ where $a \in P^t$. For an element x in F we denote the valuation of x by $v(x) = v_F(x)$.

We denote by G the group $\mathrm{GL}_2(F)$ of two by two invertible matrices with coefficients in F and by $\bar{G}$ the subgroup $\mathrm{SL}_2(G)$ of G of matrices of determinant 1. Furthermore, given H a subgroup of G, $\bar{H}$ will stand for the group $\bar{G} \cap H$.

1. Construction of the supercuspidal representations of G

We adopt here the point of view of Bushnell and Fröhlich. (See [B-F] for general definitions and proofs.)

Let A be the ring $M_2(F)$ of two by two matrices over F, so that G is the multiplicative group $A^\times$.

DEFINITION 1.1. A subring $\mathcal{A}$ of A is a *principal* order in A if

(i) $\mathcal{A}$ is a free $\mathcal{O}$-submodule of A of rank 4.
(ii) The Jacobson radical $P_{\mathcal{A}}$ of $\mathcal{A}$ is principal as a left (so as a right) ideal of $\mathcal{A}$. We denote by $\Pi = \Pi_{\mathcal{A}}$ a generator of $P_{\mathcal{A}}$.

We observe that the group of invertible fractional ideals of $\mathcal{A}$ is cyclic and generated by $P_{\mathcal{A}}$.

We denote by $U_{\mathcal{A}}$ the subgroup $\mathcal{A}^\times$ of G and we set $U^t_{\mathcal{A}} = 1 + P^t_{\mathcal{A}}$ for $t \geqslant 1$.

PROPOSITION 1.2. *$K_{\mathcal{A}} = N_G(U_{\mathcal{A}}) = \langle \Pi \rangle \ltimes U_{\mathcal{A}}$ is a maximal, open, compact modulo-center subgroup of G and all such subgroups are of this form.*

EXAMPLES. Write $[A_{ij}]$ for the set of matrices $[a_{ij}]$ with a_{ij} in A_{ij}; as usual if r is a real number, let $[r]$ denote the integer part of r

(1) $\mathcal{A} = M_2(\mathcal{O})$

$P^t_{\mathcal{A}} = P^t M_2(\mathcal{O})$

$U_{\mathcal{A}} = \mathrm{GL}_2(\mathcal{O})$

$K_{\mathcal{A}} = F^\times \mathrm{GL}_2(\mathcal{O})$

(2) $\mathscr{A} = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ P & \mathcal{O} \end{bmatrix}$

$$P^t_{\mathscr{A}} = \begin{bmatrix} P^{[(t+1)/2]} & P^{[t/2]} \\ P^{[t/2]+1} & P^{[(t+1)/2]} \end{bmatrix}$$

$$U_{\mathscr{A}} = \begin{bmatrix} U & \mathcal{O} \\ P & U \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$$

The standard method for constructing principal orders is via lattice chains.

Let $V = F \oplus F$. Then by an $\mathcal{O}$-lattice in V we mean a free rank two $\mathcal{O}$-submodule of V.

A *uniform lattice chain* is a set of $\mathcal{O}$-lattices $L = \{L_i\}_{i \in \mathbb{Z}}$ in V such that $L_i \supsetneq L_{i+1}$ for all integers i and for which there exists an integer $e = e_F(L)$ such that for all integers i

(i) $PL_i = L_{i+e}$

(ii) $\dim_k(L_i/L_{i+1}) = 2/e$. The integer e is called the period of L and obviously is equal to 1 or 2.

Two uniform lattice chains L and L' are *equivalent* if there exists an integer t such that $L_{i+t} = L'_i$ for all i. There is a natural action of G on the set of uniform lattice chains. This action is transitive for lattices of fixed period.

Let $\mathscr{A}(L)$ be the set of g in A satisfying $gL_i \subset L_i$ for all i, and, for m an integer, let $P^m(L)$ be the set of g in A satisfying $gL_i \subset L_{i+m}$ for all i. In particular let $P(L) = P^1(L)$. We have

PROPOSITION 1.3.

(i) *$\mathscr{A}(L)$ is a principal order in A, and all principal orders in A can be obtained in this way.*

(ii) *The radical of $\mathscr{A}(L)$ is $P(L)$ and $P(L)^m = P^m(L)$.*

(iii) *Π in $P(L)$ generates $P(L)$ if and only if $\Pi L_i = L_{i+1}$ for all i.*

(iv) *Let L and L' be uniform lattice chains in V. Then $\mathscr{A}(L) = \mathscr{A}(L')$ if and only if L and L' are equivalent. Furthermore, $e(L) = e(L')$ if and only if there exists a g in G such that the uniform lattice chain $gL = \{gL_i\}$ is equivalent to L', and this is the case if and only if $g\mathscr{A}(L)g^{-1} = \mathscr{A}(L')$.*

Let now $\mathscr{A}$ be a principal order in A. Then, by the above, $\mathscr{A} = \mathscr{A}(L)$ for some uniform lattice chain L in V. Set $e(\mathscr{A}) = e(L)$. We note then that $e(\mathscr{A})$ does not depend on the choice of L and determines $\mathscr{A}$ up to conjugation.

Note. Let L be the uniform lattice chain defined by $L_0 = \mathcal{O} \oplus \mathcal{O}$, $e(L) = 1$ and let L' be the uniform lattice chain defined by $L'_0 = \mathcal{O} \oplus \mathcal{O}$, $L'_1 = \mathcal{O} \oplus P$, $e(L') = 2$.

Then we get, respectively, the orders (1) and (2) of the above example. These orders are the only ones, up to conjugation by G.

A character ψ of F has conductor P^m if P^m is contained in ker ψ and P^{m-1} is not contained in ker ψ. Let us consider a character ψ of F of conductor P. Fix an integer $n \geqslant 1$. If b is an element of $P_{\mathscr{A}}^{-n}$, we define a character of $U_{\mathscr{A}}^{t}$, where $n+1/2 \leqslant t \leqslant n+1$, by the formula $\psi_b(x) = \psi(\operatorname{tr} b(x-1))$, x in $U_{\mathscr{A}}^{t}$. We have then

LEMMA 1.4. *The map* $b \mapsto \psi_b$ *induces an isomorphism of* $P_{\mathscr{A}}^{-n}/P_{\mathscr{A}}^{1-t}$ *with the topological dual* $(U_{\mathscr{A}}^{t}/U_{\mathscr{A}}^{n+1})^{\wedge}$ *of* $U_{\mathscr{A}}^{t}/U_{\mathscr{A}}^{n+1}$.

We need to describe now certain elements, the generic elements, which play an important role in describing the supercuspidal representations. We present here a special case of a more general definition (see for example [K-M]).

DEFINITION 1.5. An element b in $P_{\mathscr{A}}^{-n}$ is $\mathscr{A}$-generic of level $-n$ if

(i) $E = F[b]$ is a subfield of A of degree 2 over F.
(ii) $E^{\times} \subset K_{\mathscr{A}}$.
(iii) If E/F is ramified, then $v_E(b) = -n$ is odd. If E/F is unramified, then $b\varpi^n$ generates the ring of integers $\mathcal{O}_E$ in E.

We note here that if b is generic of level $-n$, then the set $b + P_E^{1-t}$ consists entirely of generic elements.

We have the following from [K_1].

PROPOSITION 1.6. *Let* π *be an irreducible supercuspidal representation of* G. *Then there is a principal order* $\mathscr{A}$ *of* A *and a character* χ *of* $F^{\times}$ *such that either*

(i) $e(\mathscr{A}) = 1$ *and* $\pi \otimes (\chi \cdot \det)$ *restricted to* $U_{\mathscr{A}}$ *contains a representation* σ *with* $U_{\mathscr{A}}^{1} < \ker \sigma$ *such that* σ, *viewed as a representation of* $\mathrm{GL}_2(\mathcal{O}/P)$ *is cuspidal, or*
(ii) *There is an integer* $n \geqslant 1$ *and a generic element* b *of level* $-n$ *such that* $\pi \otimes (\chi \cdot \det)$ *restricted to* $U_{\mathscr{A}}^{n}$ *contains* ψ_b.

We have now

PROPOSITION 1.7. *Let* π *be an irreducible supercuspidal representation of* G. *Then* π *is equivalent to a representation of the form* Ind σ *where* σ *is an irreducible representation of* $K_{\mathscr{A}}$, $\mathscr{A}$ *being a principal order of* A.

Let us consider more closely the case where π is an irreducible supercuspidal representation of G such that for some $n \geqslant 1$, π restricted to U_A^n contains ψ_b with b generic of level $-n$. Then in fact π, when restricted to $U_{\mathscr{A}}^{[n/2]+1}$, contains ψ_b, b as above. Now, the normalizer in $K_{\mathscr{A}}$ of ψ_b is $H = E^{\times} U_{\mathscr{A}}^{[n+1/2]}$. If $n+1$ is even, then π is induced from $\theta\psi_b$ on H where θ is a character of $E^{\times}$ which extends the restriction of ψ_b to $U_{\mathscr{A}}^{[n+1/2]} \cap E^{\times}$ while if $n+1$ is odd, then π is induced from a finite dimensional representation of H.

2. Restriction to $\bar{G}$ of a supercuspidal representation of G

Let π be an irreducible supercuspidal representation of G. If π satisfies (i) of Proposition 1.6, then $\pi = \mathrm{Ind}_K^G \sigma$ where $K = F^\times \mathrm{GL}_2(\mathcal{O})$. We may apply Mackey's theorem to $\pi|_{\bar{G}} = \mathrm{Res}_{\bar{G}}^G(\mathrm{Ind}_K^G \sigma)$ to get

$$\pi|_{\bar{G}} = \bigoplus_{z \in \bar{G}\backslash G/K} \mathrm{Ind}_{zKz^{-1}\cap \bar{G}}^{\bar{G}} \sigma^z = \bigoplus_{z \in G/\bar{G}K} (\mathrm{Ind}_{\bar{K}}^{\bar{G}} \sigma)^z$$

(since $\bar{G} \lhd G$). In this case we have that $G/\bar{G}K \simeq F^\times/\det K = F^\times/F^{\times 2}U_F$ so that $G/\bar{G}K$ has order 2 and $\pi|_{\bar{G}}$ is reducible.

On the other hand, if π satisfies (ii) of Proposition 1.6, then, as above, $\pi = \mathrm{Ind}_H^G \eta$ with $H = E^\times U_{\mathscr{A}}^{[(n+1)/2]}$ and Mackey's theorem gives us again in this case

$$\pi|_{\bar{G}} = \bigoplus_{z \in G/\bar{G}H} (\mathrm{Ind}_{\bar{H}}^{\bar{G}} \eta)^z$$

LEMMA 2.1. $G/\bar{G}H \simeq F^\times/N_{E/F}(E^\times)U_F^{[[(n+1)/2]-1/e]+1}$.

Proof. We have that $G/\bar{G}H \simeq F^\times/\det H$, but $\det H = \det E^\times \det U_{\mathscr{A}}^{[(n+1)/2]}$. Now the result follows from the fact that $\det U_{\mathscr{A}}^m = U_F^{[(m-1)/e]+1}$ (see [B-F]) noting that $\det E^\times = N_{E/F}(E^\times)$.

From this and using properties of the norm (see [S]) we get

LEMMA 2.2. *If $d = d(E/F)$ is the differential exponent of the quadratic extension E/F, then*

$$G/\bar{G}H \simeq \begin{cases} F^\times/N_{E/F}(E^\times) & \text{if } \left[\dfrac{[(n+1)/2]-1}{e}\right] + 1 \geqslant d \\ \{1\} & \text{if } \left[\dfrac{[(n+1)/2]-1}{e}\right] + 1 < d \end{cases}$$

Note that for E unramified over F, we have $d = 0$ hence $G/\bar{G}H$ is of order 2 and $\pi|_{\bar{G}}$ is reducible. If E/F is ramified, then $n = 2r - 1$ with $r \geqslant 1$ and we have

$$G/\bar{G}H \simeq \begin{cases} F^\times/N_{E/F}(E^\times) & \text{if } \left[\dfrac{r+1}{2}\right] \geqslant d \\ \{1\} & \text{if } \left[\dfrac{r+1}{2}\right] < d. \end{cases}$$

It follows then that $\pi|_{\bar{G}}$ is reducible either when π is induced from $F^\times \mathrm{GL}_2(\mathcal{O})$ or there exists, as above, a ramified extension E/F with $d \leqslant [(r+1)/2]$. In what follows we will therefore limit ourselves to the case that π is constructed as above with E/F ramified and $d > [(r+1)/2]$. In this case we know that $\pi|_{\bar{G}} = \mathrm{Ind}_{\bar{H}}^{\bar{G}} \bar{\eta}$. Furthermore we have

LEMMA 2.3. ([K-Sa]). *Let τ be an irreducible constituent of* $\mathrm{Ind}_{\bar{H}}^{\bar{K}}\bar{\eta}$ *where* $\bar{\eta}=\eta|_{\bar{H}}$. *Then* $\mathrm{Ind}_{\bar{K}}^{\bar{G}}\tau$ *is irreducible.*

LEMMA 2.4. $\bar{K}_{\mathscr{A}} = \bar{U}_{\mathscr{A}}$.

Proof. Let L be a lattice chain associated with the principal order $\mathscr{A}$ and let x be in $\bar{K}_{\mathscr{A}}$. Then for some integer $i \geqslant 0$, one gets i, $xL_t = L_{t+i}$ for all t. In particular $1=v(\det x)=[L_0\colon xL_0]=[L_0\colon L_i]$ and so $i=0$. Thus x is in $U_{\mathscr{A}}$, hence in $\bar{U}_{\mathscr{A}}$.

It follows that we may restrict our attention to $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\eta}$.

We recall now that if $t \geqslant (n+1)/2$, then $b \mapsto \psi_b$ gives us an isomorphism between $P_{\mathscr{A}}^{-n}/P_{\mathscr{A}}^{1-t}$ and $(U_{\mathscr{A}}^t/U_{\mathscr{A}}^{n+1})^\wedge$.

Let S be any $\mathscr{A}$-module in A and let S^* be the set of all x in A such that $\mathrm{tr}(xS) \subset P_F$. Then $(S/T)^* \simeq T^*/S^*$ and $(S+T)^* \simeq S^* \cap T^*$.

Denote by S^0 the set of all x in S such that $\mathrm{tr}\,x=0$.

LEMMA 2.5. $(A^0)^* = F$.

Proof. Since A^0 is the kernel of a functional, we have that $\dim_F A^0=\dim_F(A)-1$, from which $\dim_F(A^0)^*=1$. But given that $F\subset(A^0)^*$ we get $(A^0)^* = F$.

PROPOSITION 2.6. $(\overline{U_{\mathscr{A}}^r}/\overline{U_{\mathscr{A}}^{n+1}})^\wedge \simeq P_{\mathscr{A}}^{-n}/[F\cap P_{\mathscr{A}}^{-n}+P_{\mathscr{A}}^{1-r}]$.

Proof. There is a natural embedding of $\overline{U_{\mathscr{A}}^r}/\overline{U_{\mathscr{A}}^{n+1}}$ in $U_{\mathscr{A}}^r/U_{\mathscr{A}}^{n+1}$. Furthermore the restriction map from $(U_{\mathscr{A}}^r/U_{\mathscr{A}}^{n+1})^\wedge$ to $(\overline{U_{\mathscr{A}}^r}/\overline{U_{\mathscr{A}}^{n+1}})^\wedge$ is surjective so that the map which sends b in $P_{\mathscr{A}}^{-n}$ to $\bar{\psi}_b=\psi_b|_{\overline{U_{\mathscr{A}}^r}}$ in $(\overline{U_{\mathscr{A}}^r}/\overline{U_{\mathscr{A}}^{n+1}})^\wedge$ is also surjective. Our proposition now follows from the fact that the kernel of this map is

$$P_{\mathscr{A}}^{-n}\cap(P_{\mathscr{A}}^r\cap A^0)^*=P_{\mathscr{A}}^{-n}\cap(P_{\mathscr{A}}^{1-r}+(A^0)^*)=P_{\mathscr{A}}^{1-r}+F\cap P_{\mathscr{A}}^{-n}.$$

Let $\tilde{N}$ be the set of x in $K_{\mathscr{A}}$ such that $xbx^{-1} \equiv b \bmod(F+P_{\mathscr{A}}^{1-r})$. We have

PROPOSITION 2.7. $\bar{\tilde{N}}$ *is the normalizer of* $\bar{\psi}_b$ *in* $\bar{U}_{\mathscr{A}}$.

Proof. Let x be in $\bar{\tilde{N}}$. Then $xbx^{-1}-b$ lies in $F\cap P_{\mathscr{A}}^{-n}+P_{\mathscr{A}}^{1-r}$. It follows from Proposition 2.6 that $\psi_{xbx^{-1}-b}=1$ on $\overline{U_{\mathscr{A}}^r}$, i.e., that $\psi_{xbx^{-1}} \equiv \psi_b$ on $\overline{U_{\mathscr{A}}^r}$. Thus $\bar{\psi}_b^x=\bar{\psi}_b$.

PROPOSITION 2.8. $H \lhd \tilde{N}$.

Proof. Let x be in $\tilde{N}$ and h be in H. Then there exists c in F such that $xbx^{-1}=b+c(\bmod P_{\mathscr{A}}^{1-r})$. But given that $P_{\mathscr{A}}^{1-r} \lhd K_{\mathscr{A}}$ we have $x^{-1}bx \equiv b-c(\bmod P_{\mathscr{A}}^{1-r})$. It follows that

$$\begin{aligned} xhx^{-1}bxh^{-1}x^{-1} &\equiv xh(b-c)h^{-1}x^{-1} \\ &\equiv xhbh^{-1}x^{-1}-c \\ &\equiv xbx^{-1}-c \\ &\equiv b(\bmod P_{\mathscr{A}}^{1-r}). \end{aligned}$$

COROLLARY 2.9. $\bar{H} \lhd \bar{\bar{N}}$.

We also have

PROPOSITION 2.10. $\bar{\bar{N}}/\bar{H}$ *has exponent* 2.

Proof. We must prove that x in $\bar{\bar{N}}$ implies that x^2 is in $\bar{H}$. If we take traces on $xbx^{-1} \equiv b + c(\mathrm{mod}\, P_{\mathscr{A}}^{1-r})$ we get that $2c$ lies in $\mathrm{tr}\, P_{\mathscr{A}}^{1-r} = P_F^{[2-r/2]}$, so that $2c$ lies in P_E^{1-r}. On the other hand

$$x^2bx^{-2} \equiv x(xbx^{-1})x^{-1} \equiv x(b + c)x^{-1} \equiv b + 2c(\mathrm{mod}\, P_{\mathscr{A}}^{1-r}).$$

Thus $x^2bx^{-2} \equiv b(\mathrm{mod}\, P_{\mathscr{A}}^{1-r})$, i.e., x^2 lies in $\bar{H}$.

COROLLARY 2.11. $\bar{\bar{N}}/\bar{H}$ *is abelian.*

We want to prove now that if $[(r+1)/2] < d$, then $\bar{\bar{N}}/\overline{U^r}$ is abelian. To this end let Q be the set of matrices $\begin{bmatrix} F^\times & F \\ 0 & 1 \end{bmatrix}$. Then $G = E^\times Q$ (see for example [*P*]). Since $E^\times \subset H$, it follows that coset representatives of $\bar{\bar{N}}/\bar{H}$ can be picked in Q. Take x_1, x_2 in $\bar{\bar{N}}$. Then we may write $x_1 = x_1'h_1$, $x_2 = x_2'h_2$, x_1' and x_2' elements of Q and h_1 and h_2 elements of H. Let us denote by $[x, y]$ the commutator of x and y and by $[x, y]^z$ the conjugate by z of the above commutator. We want to show that $[x_1, x_2]$ lies in $\overline{U^r_{\mathscr{A}}}$. Since we have in general that $[uv, wz] = [v, w]^u[v, z]^{uw}[u, w][u, z]^w$ we get

$$[x_1, x_2] = [h_1, x_2']^{x_1'}[h_1, h_2]^{x_1'x_2'}[x_1', x_2'][x_1', h_2]^{x_2'}.$$

Also, if s_1 and s_2 are integers greater than or equal to 1 and x lies in U^{S_1} and y lies in U^{S_2}, then $[x, y]$ lies in $U^{S_1+S_2}$. Moreover since $\bar{\bar{N}}/\bar{H}$ is abelian $[x_1', x_2']$ lies in $H \cap Q = U^r_{\mathscr{A}} \cap Q \subset U^r_{\mathscr{A}}$ from which we see that all we need to prove is that $[x', h]$ lies in $U^r_{\mathscr{A}}$ for x' in $Q \cap \bar{\bar{N}}$ and h in H. We need first the following lemma.

LEMMA 2.12. *Suppose* $[(r+1)/2] < d$. *Let* x *be an element of* $\bar{\bar{N}}$ *and write* $xbx^{-1} \equiv b + c(\mathrm{mod}\, P_{\mathscr{A}}^{1-r})$ *for some* c *in* F. *Then* c *lies in* $P_E^{[(r+1)/2]-n}$.

Proof. Since b is an element of $P_{\mathscr{A}}^{-n}$ we have that $xbx^{-1}b^{-1} \equiv 1 + cb^{-1}(\mathrm{mod}\, P_{\mathscr{A}}^{1-r+n})$. But $1-r+n=r$ so that $xbx^{-1}b^{-1} = 1 + cb^{-1} + \beta$, where β lies in $P^r_{\mathscr{A}}$. Since c lies in $P_{\mathscr{A}}^{-n} \cap F$ and n is odd while c must have even valuation we have that c lies in $P_{\mathscr{A}}^{-n+1}$, i.e., cb^{-1} lies in $P_{\mathscr{A}}$, from which $1 + cb^{-1}$ is a unit and $xbx^{-1}b^{-1} = (1 + cb^{-1})(1 + (1 + cb^{-1})^{-1}\beta)$, where the last factor belongs to $U^r_{\mathscr{A}}$. If we take determinants we get that $N_{E/F}(1 + cb^{-1})$ is in $\det U^r_{\mathscr{A}} = U_F^{[(r+1)/2]}$. Now, by [S], the norm induces an isomorphism $N: U_E/U_E^{[(r+1)/2]} \simeq U_F//U_F^{[(r+1)/2]}$ from which it follows that $1 + cb^{-1}$ lies in $U_E^{[(r+1)/2]}$, so that c lies in $P_E^{[(r+1)/2]-n}$.

COROLLARY 2.13. *If* x *lies in* $\bar{\bar{N}}$, *then* x *lies in* $E^\times U_{\mathscr{A}}^{[(r+1)/2]}$.

Proof. Since $xbx^{-1} \equiv b(\mathrm{mod}\, P_{\mathscr{A}}^{[(r+1)/2]-n})$ the result follows from [C].

PROPOSITION 2.14. *If* $[(r+1)/2] < d$, *then* $\bar{\bar{N}}/\overline{U^r}$ *is abelian.*

Proof. By the above we need to prove that if x lies in $\bar{\bar{N}}$ and h lies in $\bar{H}$, then $[x,h]$ lies in $U^r_{\mathscr{A}}$. But $h=su$ with s in $E^\times$ and u in $U^r_{\mathscr{A}}$. Taking determinants we get $N_{E/F}(s)$ lies in det $U^r_{\mathscr{A}} = U_F^{[(r+1)/2]}$ and as in the above lemma we get that s is in $U_E^{[(r+1)/2]}$. Now by Corollary 2.13, $x=tv$ with t in $E^\times$ and v in $U_{\mathscr{A}}^{[(r+1)/2]}$, so that $[x,h]=[v,s]^t[v,u]^{ts}[t,s][t,u]^s$ from which the result follows.

We state now a result that is going to be needed later.

LEMMA 2.15. *Let* E/F *be a quadratic ramified extension and let* α *be an element of* E *such that* $v_E(\alpha)=1$. *Then either* $d=2v_F(\mathrm{tr}\,\alpha)$ *and* $2v_F(\mathrm{tr}\,\alpha)<2v_F(2)+1$ *or* $d=2v_F(2)+1$ *and* $v_F(\mathrm{tr}\,\alpha)\geqslant v_F(2)+1$.

Proof. We know that $d=v_E(\alpha-\alpha^\sigma)$ where σ is the nontrivial element of the Galois group of the extension. But then $d=v_E(\alpha+\alpha^\sigma-2\alpha^\sigma)$ and since $\alpha+\alpha^\sigma$ has even valuation (being an element of F) and $2\alpha^\sigma$ has valuation $2v_F(2)+1$, an odd number, we get that $d=\min\{2v_F(\mathrm{tr}\,\alpha),\ 2v_F(2)+1\}$.

COROLLARY 2.16. *Let* b *in* E *have valuation* $-n$, n *odd and let* $s=\mathrm{tr}\,b$. *Then* $d=\min\{1+n+2v_F(s),\ 2v_F(2)+1\}$.

Proof. Apply Lemma 2.15 to $\alpha=\varpi_F^{(1+n)/2}b$.

3. The number of components of $\pi|_{\bar{G}}$

Let M be the normalizer of $\bar{\eta}$ in $\bar{\bar{N}}$. Since $\bar{H} \lhd \bar{\bar{N}}$ we have by Clifford theory that if τ is an irreducible subrepresentation of $\mathrm{Ind}_{\bar{H}}^{M}\bar{\eta}$, then $\mathrm{Ind}_M^{\bar{U}_{\mathscr{A}}}\tau$ is irreducible. In this section we will consider the decomposition of $\mathrm{Ind}_{\bar{H}}^{M}\bar{\eta}$. To begin with we note that $M=N_{\bar{\bar{N}}}(\bar{\eta})$ depends only on ψ_b. In fact, x is in M if and only if $\bar{\eta}^x=\bar{\eta}$, i.e., if and only if $[x,\bar{H}]<\ker\bar{\eta}$. But given that $\bar{\bar{N}}/\overline{U^r}$ is abelian, this is the case if and only if $[x,\bar{H}]<\ker\bar{\eta}\cap\overline{U^r_{\mathscr{A}}}=\ker\bar{\psi}_b$.

In what follows it will be convenient to choose a specific lattice chain, namely the one which is obtained from the lattice chain of the note before Lemma 1.4 by conjugation by $\begin{bmatrix}\varpi^{[(n+1)/2]} & 0\\ 0 & 1\end{bmatrix}$.

For this new lattice we have

$$P^l_{\mathscr{A}}=\begin{bmatrix}P^{[(l+1)/2]} & P^{[l/2]+(n+1)/2}\\ P^{[l/2]+(l-n)/2} & P^{[(l+1)/2]}\end{bmatrix}$$

and b can be picked to be

$$b=\begin{bmatrix}0 & 1\\ -\Delta & s\end{bmatrix}$$

where

$$v_F(\Delta) = -n,\ v_F(s) \geqslant \frac{1-n}{2} = 1 - r.$$

Having chosen this chain, our first goal is to describe the group $\bar{\tilde{N}}/\bar{H}$. To this end let x be an element of $\bar{\tilde{N}}$; then $xbx^{-1} \equiv b + c_x(\mathrm{mod}\ P_{\mathscr{A}}^{1-r})$ where $c_x = c$ is in F. As in Proposition 2.10 we get that $2c$ lies in $P_F^{[(2-r)/2]}$. Also since $xbx^{-1}b^{-1} \equiv 1 + cb^{-1}(\mathrm{mod}\ P_{\mathscr{A}}^{r})$ we get as in Lemma 2.12 that $N(1+cb^{-1}) \equiv 1(\mathrm{mod}\ P_F^{[(r+1)/2]})$ from which $ctrb^{-1} + c^2Nb^{-1}$ lies in $P_F^{[(r+1)/2]}$, so that $cs/\Delta + c^2/\Delta$ lies in $P_F^{[(r+1)/2]}$, i.e., $cs + c^2$ lies in $P_F^{[(r+1)/2]-n} = P_F^{1-r+[(1-r)/2]}$.

It follows that

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\Delta & s \end{bmatrix}\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} - \begin{bmatrix} 1 & 0 \\ -\Delta & s \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} -2c & 0 \\ -c^2 - cs & 0 \end{bmatrix}$$

belongs to $P_{\mathscr{A}}^{1-r}$. Thus $x_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ is in $\bar{\tilde{N}}$. Furthermore $xx_c^{-1}bx_cx^{-1} \equiv x(b-c)x^{-1} \equiv b(\mathrm{mod}\ P_{\mathscr{A}}^{1-r})$ so that xx_c^{-1} lies in $\bar{H}$.

On the other hand in order that x_c lies in $\bar{H}$ we must have that c lies in $P_F^{[r/2]+1-r}$ because $\bar{H} \cap \mathbf{Q} = U_{\mathscr{A}}^r \cap \mathbf{Q}$ where $\mathbf{Q} = \begin{bmatrix} F^\times & 0 \\ F & 1 \end{bmatrix}$. Write $C = \begin{bmatrix} 1 & 0 \\ F & 1 \end{bmatrix}$ and let $\tilde{C}_b = \bar{\tilde{N}} \cap C$. Let $\tilde{\mathfrak{C}}_b$ be the inverse image of $\tilde{C}_b$ under the map $c \mapsto x_c$ so that $\tilde{\mathfrak{C}}_b$ is the additive group which consists of elements c in F such that $v_F(2c) \geqslant [(2-r)/2]$ and $v_F(cs + c^2) \geqslant 1 - r + [(1-r)/2]$.

We denote by $\mathfrak{C}_b$ the inverse image of $C_b = M \cap C$ under the $c \mapsto x_c$ map. By the above $\mathfrak{C}_b/P_F^{[r/2]+1-r}$ is isomorphic to the group $M/\bar{H}$, the isomorphism being induced by $c \mapsto x_c$.

We note now that x_c lies in M if and only if $[x_c, \bar{H}] \cap U_{\mathscr{A}}^r \subset \ker \psi_b$.

LEMMA 3.1. *The number of irreducible components of* $\mathrm{Ind}_{\bar{H}}^M \bar{\eta}$, *thus of* $\pi|_{\bar{G}}$, *is equal to* $[\mathfrak{C}_b : P_F^{[r/2]+1-r}]$, *the index of* $P_F^{[r/2]+1-r}$ *in* $\mathfrak{C}_b$.

Proof. We will show that $\bar{\eta}$ extends to M. Our result will then follow from the fact that $M/\bar{H}$ is abelian so that the number of irreducible components of $\mathrm{Ind}_{\bar{H}}^M \bar{\eta}$ is given by $[M : \bar{H}] = [\mathfrak{C}_b : P_F^{[r/2]+1-r}]$.

To see that $\bar{\eta}$ extends to M it is enough to show that if x and y are in M, then $[x, y]$ lies in the kernel of $\bar{\eta}$. But from the above $x = t_1h_1$, $y = t_2h_2$, where

$$t_1 = \begin{bmatrix} 1 & 0 \\ c_1 & 1 \end{bmatrix} \quad \text{and} \quad t_2 = \begin{bmatrix} 1 & 0 \\ c_2 & 1 \end{bmatrix}.$$

Thus $[x,y]=[h_1,t_2]^{t_1}[h_1,h_2]^{t_1t_2}[t_1,t_2][t_1,h_2]^{t_2}$. But $[t_1,t_2]=1$ and the other factors are in the kernel of $\bar{\eta}$, whence our result.

We want to describe the group C_b. To this end we have

LEMMA 3.2. *Let $\beta=t(1+ub)$ be an element of E and let k be in $U^r_{\mathscr{A}}$. Suppose that βk lies in $\bar{H}$. Then $[x_c,\beta k]$ lies in the kernel of ψ_b if and only if $-c^2ut^2/N(\beta)+c(N(\beta)-1)/N(\beta)$ lies in the kernel of ψ, where $N(\beta)$ denotes the image of β in F under the norm map of the extension E/F.*

Proof. Since $[x_c,\beta k]=[x_c,\beta][x_c,k]^{\beta}$ we have that if $h=\beta k$ is in $\bar{H}$, then

$$\psi_b([x_c,\beta k]) = \psi_b([x_c,\beta])\psi_b([x_c,k]).$$

First, we observe that

$$\psi_b([x_c,k]) = \psi(-c\,tr(k-1)).$$

Furthermore, since k is in $U^r_{\mathscr{A}}$ and $\det k=N(\beta^{-1})$ we have that

$$N(\beta^{-1}) = 1 + \mathrm{tr}(k-1) + \det(k-1).$$

Also $v_F(c\det(k-1))\geqslant 1$ from which $\psi([x_c,k])=\psi(-c(N(\beta^{-1})-1)$ and so $\psi_b([x_c,k])=\psi(c(N(\beta)-1)/N(\beta))$.

Next,

$$[x_c,\, t(1+bu)] = \begin{bmatrix}1 & 0\\ c & 1\end{bmatrix}(1+bu)\begin{bmatrix}1 & 0\\ -c & 1\end{bmatrix}\frac{1+b^{\sigma}u}{N(1+bu)}$$

(σ the nontrivial element of $\mathrm{Gal}(E/F)$) so that

$$\psi_b([x_c,\, t(1+bu)]) = \psi\left(\frac{-ut^2c^2}{N(t(1+bu))}\right).$$

The result now follows.

Let c be in $\mathfrak{C}_b$ and set $s_1=s-c$. We consider the polynomial $x^2-s_1x+\Delta$. We observe that if we multiply this polynomial by ϖ^{n+1} we get

$$(\varpi^{(n+1)/2}x)^2 - s_1\varpi^{(n+1)/2}(\varpi^{(n+1)/2}x) + \varpi^{n+1}\Delta,$$

a polynomial in the new variable $\varpi^{(n+1)/2}x$ such that the constant term has valuation one, while the coefficient of the first degree term has valuation greater than or equal to one, thus, this new polynomial is an Eisenstein polynomial from which it follows that if b_1 is a root of $x^2-s_1x+\Delta$, then $E_1=F[b_1]$ is a quadratic ramified extension of F. If $\beta=a+tb$ is in E, we set $\beta_1=\beta_1(\beta)=a+tb_1$ in E_1. Also we will denote by N_1 the norm map of the extension E_1/F and set $d_1=d(E_1/F)$.

We can now restate Lemma 3.2.

LEMMA 3.3. *With notation as in Lemma* 3.2, $[x_c, \beta k]$ *lies in the kernel of* ψ_b *if and only if* $c(N_1(\beta_1)-1)/N(\beta)$ *lies in the kernel of* ψ.

Also, we have

LEMMA 3.4. $[x_c, h]$ *lies in* $U^r_{\mathscr{A}}$ *if* h *lies in* $\bar{H}$.

Proof. Let $h=\beta k$ where β is in $E^\times$ and k is in $U^r_{\mathscr{A}}$. Then $[x_c, k]$ is in $U^r_{\mathscr{A}}$, so that $[x_c, h]$ is in $U^r_{\mathscr{A}}$ if and only if $[x_c, \beta]$ is in $U^r_{\mathscr{A}}$. Now Corollary 2.13 implies that x_c lies in $E^\times U^{[(r+1)/2]} \cap C = U^{[(r+1)/2]}_{\mathscr{A}} \cap C$. Since $[(r+1)/2] < d$ and $N(\beta)$ is in $U_F^{[(r+1)/2]}$ we have that β is in $U_E^{[(r+1)/2]}$ so that $[x_c, \beta]$ lies in $U^{[(r+1)/2]+[(r+1)/2]}$ from which the result follows.

We recall here that the character ψ of F has conductor P_F. If $p=2$ we will assume furthermore that $\psi(x^2+x)=1$ for x in $\mathcal{O}_F$. If E/F is quadratic, $\omega_{E/F}$ will denote the nontrivial character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$ and c_E will denote an element of F for which $\omega_{E/F}(x) = \psi(c_E(x-1))$ for x in $U_F^{[(d+1)/2]}$.

We improve now Lemma 3.3.

LEMMA 3.5. *Let* β *be such that* $N(\beta)$ *is in* $U_F^{[(r+1)/2]}$. *Then* $\psi(c(N_1(\beta_1)-1)/N(\beta))=1$
if and only if $\psi(c(N_1(\beta_1)-1))=1$.

Proof. Let us recall first that $v_E(c) \geqslant [(r+1)/2]-n$. On the other hand we have that

$$\frac{c(N_1(\beta_1)-1)}{N(\beta)} = c(N_1(\beta_1)-1)(1+z) = c(N_1(\beta_1)-1)+c(N_1(\beta_1)-1)z$$

for some z in $P_F^{[(r+1)/2]}$. We consider first the case when $[(r+1)/2] < d_1$. Then $N_1(\beta_1)-1$ lies in $P_F^{[(r+1)/2]}$ (see [S]), so that $c(N_1(\beta_1)-1)z$ has F-valuation greater than or equal to

$$\frac{1}{2}\left[\frac{r+1}{2}\right] - r + \frac{1}{2} + \left[\frac{r+1}{2}\right] + \left[\frac{r+1}{2}\right]$$

which is greater than or equal to 1, and so $c(N_1(\beta_1)-1)z$ lies in the kernel of ψ. Next, assume $d_1=[(r+1)/2]$. Then $U_F^{[(r+1)/2]} = N_1(U_{E_1}^{[(r+1)/2]})$ from which it follows as before that $c(N_1(\beta_1)-1)z$ has valuation greater than or equal to 1. Finally, if $d_1 < [(r+1)/2]$, then $c(N_1(\beta_1)-1)z$ has F-valuation greater than or equal to

$$\frac{1}{2}\left[\frac{r+1}{2}\right] - r + \frac{1}{2} + \frac{1}{2}\left[\frac{r+1}{2}\right] + \frac{d_1}{2} - \frac{1}{2} + \left[\frac{r+1}{2}\right]$$

which is greater than or equal to 1 whence our result.

We may now state and prove our main result. We recall from Lemma 3.1 that the number of components of $\pi|_{\bar{G}}$ is equal to $[\mathfrak{C}_b : P_F^{[r/2]+1-r}]$.

THEOREM 3.6. *Let π be an irreducible supercuspidal representation of G which satisfies (ii) of Proposition 1.6 and such that $[(r+1)/2]<d$. If $[(r+1)/2]<d<\frac{2}{3}(r+1)$, then $[\mathfrak{C}_b : P_F^{[r/2]+1-r}]\geqslant 2$ with equality if the congruence $x^3-sx^2+\Delta\equiv 0(\mathrm{mod}\, P_F^{1-n})$ has no solution of valuation greater than $d/2-r$.*

If $\frac{2}{3}(r+1)\leqslant d$, then $[\mathfrak{C}_b : P_F^{[r/2]+1-r}]=1$ if and only if the equation $x^3-sx^2+\Delta=0$ has no solution.

Proof. (1) We first suppose that $[(r+1)/2]<d<\frac{2}{3}(r+1)$. We consider several cases.

Let c be in $\mathfrak{C}_b$.

(i) $d_1\leqslant[(r+1)/2]$. We have in this case that c must lie in $P_F^{[r/2]+1-r}$, because $U_F^{[(r+1)/2]}=N_1(U_{E_1}^{2[(r+1)/2]+1-d_1})\subset N_1(U_{E_1}^{[(r+1)/2]})$. Thus $P_F^{[(r+1)/2]}\subset N_1(U_{E_1}^{[(r+1)/2]})-1$, from which $cP_F^{[(r+1)/2]}\subset\ker\psi$ (see Lemma 3.5), so that $v_F(c)\geqslant 1-[(r+1)/2]$, i.e., c lies in $P_F^{[r/2]+1-r}$.

(ii) $[(r+1)/2]<d_1<d$. This cannot occur. To see this, we note first that if $r-d_1/2\geqslant d_1$ then $U_F^{r-d_1/2}=N_1(U_{E_1}^{2r-d_1+1-d_1})$. Further, given that $2r-2d_1+1\geqslant[(r+1)/2]$ we get $U_F^{r-d_1/2}\subset N_1(U_{E_1}^{[(r+1)/2]})$, i.e., $P_F^{r-d_1/2}\subset N_1(E_1^{[(r+1)/2]})-1$. On the other hand it follows from Corollary 2.16 that $v_F(c)=d_1/2-r$, so that $\mathcal{O}=cP_F^{r-d_1/2}\subset\ker\psi$, a contradiction.

(iii) We will show now that $[(r+1)/2]<d<d_1<\frac{2}{3}(r+1)$ cannot occur. Here $U_F^{r-d/2}\subset U_F^{r-d_1/2}=N_1(U_{E_1}^{2(r-d_1)+1})\subset N_1(U_{E_1}^{[(r+1)/2]})$. But Corollary 2.16 implies now that $v_F(c)=d/2-r$ so that $\mathcal{O}=cP_F^{r-d/2}\subset\ker\psi$, a contradiction.

(iv) We assume now that $[(r+1)/2]<d=d_1<\frac{2}{3}(r+1)$. Set $c_1=c_{E_1}$. Then $v_F(c_1)=1-d$. Also given that $U_F^d\subset\{N_1(\beta_1)\}$ we have $v_F(c)\geqslant 1-d$. Now $s=s_1+c_1+(c-c_1)$ implies that $s\equiv s_1+c_1(\mathrm{mod}\, P_F^{1-d})$. But $d/2+1-r\leqslant 1-d$, so that $s\equiv s_1+c_1(\mathrm{mod}\, P_F^{d/2+1-r})$. We can apply then Proposition 4.1 of $[K_2]$ to get an extension E_2/F, $s_2=\mathrm{Tr}_{E_2/F}(b_2)$ and $c_2=c_{E_2}$ such that $s\equiv s_2+c_2(\mathrm{mod}\, P_F^{-[(r-1)/2]})$. Conversely, suppose there exists an extension E_2/F with different $d_2=d$ such that $s\equiv s_2+c_2(\mathrm{mod}\, P_F^{-[(r-1)/2]})$ and $\Delta=\Delta_2$ where $s_2=\mathrm{Tr}_{E_2/F}(b_2)$, $\Delta_2=N_{E_2/F}(b_2)$ for some b_2 in E_2. Since we are in the case $d/2<[(r+1)/2]$ we have $N_{E_2/F}(\beta_2)=N_2(\beta_2)$ is in $U_F^{[(r+1)/2]}\subset U_F^{d/2}$ for β_2 in $U_{E_2}^{[(r+1)/2]}$ where β_2 corresponds to β in $U_E^{[(r+1)/2]}$ under the map described after Lemma 3.2. We have in this way that $N_2(\beta_2)-1$ is in $P_F^{d/2}$, so that $\psi(c_2(N_2(\beta_2)-1)=1$. We will see now that c_2 lies in $\mathfrak{C}_b$ but not in $P_F^{[r/2]+1-r}$. In fact, $v_F(2c_2)\geqslant[(2-r)/2]$ since $v_F(2c_2)\geqslant d-1/2+1-d\geqslant[(2-r)/2]$ (see Corollary 2.16). Also $v_F(sc_2+c_2^2)\geqslant[(1-r)/2]+1-r$ since $v_F(c_2(c_2+s))=1-d+v_F(c_2+s)$, $v_F(s)\geqslant d/2-r$ and $d/2-r<1-d$ we have that

$$v_F(c_2(c_2+s))\geqslant 1-d+\frac{d}{2}-r\geqslant\left[\frac{1-r}{2}\right]+1-r.$$

c_2 is not in $P_F^{[r/2]+1-r}$ since $v_F(c_2)=1-d<[r/2]+1-r$.

(v) Finally we consider the case $[(r+1)/2]<d<\frac{2}{3}(r+1)\leqslant d_1$.

In this case $d=1+n+2v_F(s)<1+n+2v_F(s_1)$, so that $v_F(s)<v_F(s_1)$ and $v_F(s)=v_F(c)=d/2-r$. Also $U_F^{d_1}=N_1(U_{E_1}^{d_1+1})\subset N_1(U_{E_1}^{[(r+1)/2]})$ which implies that $cP_F^{d_1}\subset\ker\psi$ so that $d/2-r+d_1\geqslant 1$, i.e., $d/2+d_1\geqslant 1+r$ (in particular $d_1>\frac{2}{3}(r+1)$).

We are going to see first that d_1 cannot be odd. If it were, then

$$d_1 = 2v_F(2) + 1 < 1 + n + 2v_F(s_1) \quad \text{and} \quad v_F(s_1) \geqslant \frac{d_1+1}{2} - r.$$

From Lemma 3.5 we would have that $\psi(cN_1(\beta_1)-1))=1$ for all $\beta=1+ub$ such that $v_F(ub)\geqslant[(r+1)/2]$, i.e., $\psi(cs_1u+cu^2\Delta)=1$ for all u such that $v_F(u)\geqslant \frac{1}{2}[(r+1)/2]+n/2$. Let $u=ys_1/\Delta$. Then $\psi(cs_1^2/\Delta(y+y^2))=1$ for all y such that $v_F(y)=v_F(u)-n-v_F(s_1)$. But since $v_F(s_1)\geqslant d_1+1/2-r\geqslant\frac{1}{2}[(r+1)/2]-n/2$ we would have that $\psi(cs_1^2/\Delta(y+y^2))=1$ for all y in $\mathcal{O}$. We are in the case $p=2$ (because $d>1$) so that ψ has been taken with the property $\psi((y+y^2)=1$ for all y in $\mathcal{O}_F$. We get that $cs_1^2/\Delta\equiv 1(\mathrm{mod}\ P_F)$, i.e., $s_1^3-ss_1^2+\Delta\equiv 0(\mathrm{mod}\ P_F^{1-n})$ (because $c=s-s_1$). But

$$v_F(\Delta) = 1 - 2r$$

$$v_F(ss_1^2) \geqslant \frac{d}{2} - r + d_1 + 1 - 2r$$

$$v_F(s_1^3) \geqslant \tfrac{3}{2}(d_1 + 1) - 3r$$

Thus

$$v_F(\Delta) < v_F(ss_1^2) < v_F(s_1^3)(v_F(s) < v_F(s_1)).$$

It follows that

$$v_F(s_1^3 - ss_1^2 + \Delta) = {}^1 - 2r = -n,$$

a contradiction. Thus, d_1 is even and so $d_1=1+n+2v_F(s_1)$. As above we get that $cs_1^2/\Delta\equiv 1(\mathrm{mod}\ P_F)$, from which

$$d/2 - r + d_1 - 2r + 2r - 1 = 0,$$

i.e., $d/2+d_1=1+r$. Since

$$s_1^3 - ss_1^2 + \Delta \equiv 0(\mathrm{mod}\ P_F^{1-n}),$$

it follows that the congruence

$$x^3 - sx^2 + \Delta \equiv 0(\mathrm{mod}\ P_F^{1-n})$$

has a solution s_1 such that

$$\nu_F(s_1^3) = \tfrac{3}{2}d_1 - 3r > 1 - 2r$$

(because $d/2 + d_1 = 1+r$), so that $\nu_F(ss_1^2) = \nu_F(\Delta) = 1 - 2r$, i.e.,

$$\nu_F(s) + 2\nu_F(s_1) = 1-2r.$$

On the other hand if we make the change of variables $x=sz$ in $x^3-sx^2+\Delta=0$ we have that $z^3-z^2+\Delta/s^3=0$. But $\nu_F(\Delta/s^3)>0$ and so $z^3-z^2+\Delta/s^3 \equiv z^2(z-1)$ (mod P_F). By Hensel's lemma $x^3-sx^2+\Delta=0$ has a solution of valuation $\nu_F(s)$, from which $x^3-sx^2+\Delta \equiv 0(\mathrm{mod}\ P_F^{1-n})$ has one solution of valuation equal to the valuation of s and two other solutions, one of which is s_1, of valuation $\nu_F(s_1) > \nu_F(s) = d/2-r$.

(2) We now suppose that $\frac{2}{3}(r+1) \leqslant d$.

Let c be in $\mathfrak{C}_B$. Again there are several cases.

(i) By an argument similar to that of part (ii) above one may check that $d_1 < \frac{2}{3}(r+1)$ cannot occur.

(ii) $\frac{2}{3}(r+1) \leqslant d_1 < d$. We will see in fact that there is a nontrivial c if and only if the equation $x^3-sx^2+\Delta=0$ has a solution.

If there is a c then we have that

$$d_1 = 1 + n + 2\nu_F(s_1) \quad \text{and} \quad \nu_F(c) = \nu_F(s_1)$$

(since $\nu_F(s_1) < \nu_F(s)$). Also given that $U_F^d = N_1(U_{E_1}^{2d+1-d_1}) \subset N_1(U_{E_1}^{[(r+1)/2]})$ we get $d_1/2+d \geqslant 1+r$. As we know, $\psi(cus_1+cu^2\Delta)=1$ for all u such that $\nu_F(u) \geqslant \frac{1}{2}[(r+1)/2]+n/2$ and if we take $u=ys_1/\Delta$, then $\psi((cs_1^2/\Delta)(y+y^2))=1$ for all y such that

$$\nu_F(y) = \nu_F(u) - n - \nu_F(s_1) = \nu_F(u) - r + 1 - \frac{d_1}{2}.$$

But

$$\frac{1}{2}\left[\frac{r+1}{2}\right] + \frac{2r-1}{2} - r + 1 - \frac{d_1}{2} = \frac{1}{2} + \frac{1}{2}\left[\frac{r+1}{2}\right] - \frac{d_1}{2} < 0.$$

It follows that $\psi(cs_1^2/\Delta(y+y^2))=1$ for all y in $\mathcal{O}_F$, so that $cs_1^2/\Delta \equiv 1(\mathrm{mod}\ P_F)$, which implies in particular that $d_1/2-r+d_1-2r+2r-1=0$, i.e., $d_1=\frac{2}{3}(r+1)$ (note that $r+1$ is necessarily divisible by 3 in this case). The above congruence gives us as before $s_1^3-ss_1^2+\Delta \equiv 0(\mathrm{mod}\, P_F^{1-n})$. We notice that $v_F(s_1^3/\Delta)=\frac{3}{2}d_1-3r+2r-1=0$ and that $v_F(ss_1^2/\Delta)>0$ $(v_F(s)>v_F(s_1))$. Thus $s_1^3/\Delta+1 \equiv 0(\mathrm{mod}\, P)$. We look now at the equation $x^3-sx^2+\Delta=0$. Let $x=s_1z$; then $(s_1^3/\Delta)z^3-(ss_1^2/\Delta)z^2+1=0$, and the congruence $(s_1^3/\Delta)z^3+1 \equiv 0(\mathrm{mod}\, P_F)$ has the solution $z=1$. Furthermore the derivative of $(s_1^3/\Delta)z^3+1$ is $3z^2s_1^3/\Delta$ which has only 0 as a root, from which we see that we have distinct solutions mod P. Hensel's lemma applies and gives us a solution of $x^3-sx^2+\Delta=0$ of valuation $v(s_1)$ since this solution is of the form s_1v where v is a unit. We observe furthermore that if the above equation has all solutions in F, then all three of them have valuation $-n/3$.

Conversely, suppose that $x^3-sx^2+\Delta=0$ has a solution s_1. Then we are going to see that this gives us a nontrivial c. We observe first that if $v_F(s_1)>-n/3$, then $v_F(ss_1^2)=-n$, so that $v_F(s)<-n+\frac{2}{3}n=-n/3$. On the other hand $(d-1-n)/2 \leqslant v_F(s)$. Thus $(d-1-n)/2<-n/3$, i.e., $d<\frac{2}{3}(r+1)$, a contradiction. It follows that $v_F(s_1) \leqslant -n/3$. Consider the quadratic polynomial $x^2-s_1x+\Delta$. We observe that $v_F(s_1)<1-r$ is not possible, because if this is the case, then $v_F(s_1^3)<v_F(ss_1^2)$ $(v_F(s) \geqslant 1-r)$. Thus $v_F(s_1^3)=-n$ so that $-n/3=v_F(s_1)<1-r$ which is not possible since $n=2r-1$ is a nonnegative integer. It follows that $v_F(s_1) \geqslant 1-r$. If we let $x=z/\varpi^{1+n/2}$ in $x^2-s_1x+\Delta=0$ we get $z^2-s_1\varpi^{n+1/2}z+\varpi^{1+n/2}\Delta=0$, which is an Eisenstein polynomial. Thus a root of $x^2-s_1x+\Delta=0$ gives rise to a quadratic ramified extension E_1/F of different d_1. We observe that $v_F(s)<v_F(s_1)$ is not possible because if it were

$$d \leqslant 1+n+2v_F(s)<1+n+2v_F(s_1) \leqslant 1+n-\tfrac{2}{3}n=\tfrac{2}{3}(r+1),$$

a contradiction. It follows that $v_F(s_1) \leqslant v_F(s)$. We have also that $d_1 \leqslant 1+n+2v_F(s_1) \leqslant \frac{2}{3}(r+1)$, so $d_1/2 \leqslant (r+1)/3<[(r+1)/2]$ and $U_F^{[(r+1)/2]} \subset U_F^{d_1/2}$, whence $\psi(c_1(N_1(\beta_1)-1)=1$ where $c_1=c_{E_1}$. Furthermore,

$$v_F(2c_1)=v_F(2)+v_F(c_1) \geqslant \frac{d_1}{2}+1-d_1=1-\frac{d_1}{2} \geqslant \left[\frac{2-r}{2}\right].$$

Also, necessarily, $v_F(s_1)=-n/3$, because if $v_F(s_1)<-n/3$, then from the equation $x^3-sx^2+\Delta=0$ we have that $v_F(ss_1^2)=v_F(s_1^3)$, i.e., $v_F(s)=v_F(s_1)$ which implies $d \leqslant 1+n+2v_F(s_1) \leqslant \frac{2}{3}(r+1)$, a contradiction. Thus,

$$v_F(s_1)=-\frac{n}{3} \quad \text{and} \quad d_1=\tfrac{2}{3}(r+1) \quad (d_1<\tfrac{2}{3}(r+1)$$

would imply that $d_1<d$ so that $d_1=1+n+2v_F(s_1)=\frac{2}{3}(r+1)$). It follows that

$v_F(c+sc^2) \geqslant 1+r+[(1-r)/2]$. Finally,

$$v_F(c_1) = 1 - d_1 = -\frac{n}{3} < \left[\frac{r}{2}\right] + 1 - r.$$

Thus c_1 lies in $\mathfrak{C}_b$ but does not lie in $P_F^{[r/2]+1-r}$.

(iii) $\frac{2}{3}(r+1) \leqslant d_1 = d$. We are going to see first that $d = 2v_F(2)+1 = d_1$ is not possible. If it were, then $v_F(s_1) \geqslant (d_1+1)/2 - r$ and arguing as in (2)(ii) above we would have that $cs_1^2/\Delta \equiv 1(\operatorname{mod} P)$ which implies that

$$v_F(c) + d + 1 - 2r + 2r - 1 \leqslant 0,$$

i.e., $v_F(c) \leqslant -d$. But $U_F^d = N_1(U_{E_1}^{d_1+1}) \subset N_1(U_{E_1}^{[(r+1)/2]})$ so that $cP_F^d \subset \ker \psi$, a contradiction. It follows that d is even so that

$$d = 1 + n + 2v_F(s) = 1 + n + 2v_F(s_1) = d_1 \quad \text{and} \quad v_F(s) \leqslant v_F(c).$$

Using once more an argument similar to that of (2)(ii) we get $cs_1^2/\Delta \equiv 1(\operatorname{mod} P_F)$ from which $v_F(c) = 1-d$. We observe that $1-d$ cannot be greater than $d/2-r$. Then

$$v_F(c) = 1 - d \leqslant \frac{d}{2} - r = v_F(s).$$

Thus

$$v_F(c) = v_F(s) \quad \text{and} \quad 1 - d = \frac{d}{2} - r,$$

i.e., $d = \frac{2}{3}(r+1)$. On the other hand if we let

$$x = s_1 z \text{ in } x^3 - sx^2 + \Delta = 0$$

we get

$$\frac{s_1^3}{\Delta} z^3 - \frac{ss_1^2}{\Delta} z^2 + 1 = 0$$

which has, mod P_F, the solution $z = 1$, since $cs_1^2/\Delta \equiv 1(\operatorname{mod} P_F)$ and $c = s - s_1$. Furthermore, mod P, the polynomial $(s_1^3/\Delta)z^3 - (ss_1^2/\Delta)z^2 + 1$ has derivative $3vz^2$ for some unit v, so that 0 is the only root of the derivative. We now argue as in case (ii) to get that there is a c if and only if $x^3 - sx^2 + \Delta = 0$ has a solution.

(iv) $\frac{2}{3}(r+1) \leqslant d < d_1$. We are going to see that this final case cannot occur. If it did we would have $d = 1 + n + 2v_F(s)$. But then $v_F(s) < v_F(s_1)$. Thus

$v_F(s)=v_F(c)=d/2-r$. On the other hand $U_F^{d_1}=N_1(U_{E_1}^{d_1+1})\subset N_1(U_{E_1}^{[(r+1)/2]})$ from which $cP_F^{d_1}\subset \ker\psi$, so that $v_F(c)\geqslant 1-d_1$, i.e., $d/2-r\geqslant 1-d_1$. If d_1 is even, then $v_F(s_1)=d_1/2-r$ and we would have as before that $\psi((cs_1^2)/\Delta(y+y^2))=1$ for all y such that

$$v_F(y)=v_F(u)-n-v_F(s_1)=v_F(u)-r+1-\frac{d_1}{2},\qquad u=\frac{ys_1}{\Delta}.$$

But

$$r-\frac{1}{2}+\frac{1}{2}\left[\frac{r+1}{2}\right]-r+1-\frac{d_1}{2}=\frac{1}{2}+\frac{1}{2}\left[\frac{r+1}{2}\right]-\frac{d_1}{2}<0.$$

We have then $\psi((cs_1^2)/\Delta(y+y^2))=1$ for all y in $\mathcal{O}_F$. In case d_1 is odd, $2v_F(s_1)+1+n>d_1$ and we have also $\psi((cs_1^2)/\Delta(y+y^2))=1$ for all y in $\mathcal{O}$. In any case then we would have $cs_1^2/\Delta\equiv 1 (\mathrm{mod}\, P)$ which would imply $v_F(c)+v_F(s_1^2)+n=0$. Since $v_F(s_1)\geqslant d_1/2-r$ we would have $d/2-r+d_1-2r+2r-1\leqslant 0$, i.e., $d/2+d_1\leqslant 1+r$, a contradiction.

The above theorem has as a corollary

COROLLARY 3.7 (Criterion of Labesse–Langlands [L-L]). *An irreducible supercuspidal representation* π *of* G *remains irreducible upon restriction to* $\bar{G}$ *if and only if* π *is exceptional.*

Proof. This follows from Theorem 3.6 above and Theorem 4.2 of [K_2].

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