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Some abelian threefolds with nontrivial Griffiths group

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0. Introduction

Let E_1, E_2 and E_3 be elliptic curves defined over a global field $\mathscr{K}$. In this paper, I give a construction of a genus-three curve C on the product $X = E_1 \times E_2 \times E_3$, defined over $\mathscr{K}$. If ι is the inverse map for the group law on X, then the cycle $C - \iota C$ is homologically equivalent to zero over $\bar{\mathscr{K}}$, and (with certain restrictions on the E_i) I give an easily computible sufficient condition for $C - \iota C$ to have infinite order modulo algebraic equivalence. Moreover, I show that the condition is satisfied for an infinite class of examples. If $\mathscr{K}$ is a number field, however, the result hinges on the Purity Conjecture for etale cohomology.

The interest in these examples stems from a conjecture of Bloch [4] and Beilinson [2], which is similar to the Birch/Swinnerton–Dyer conjecture for curves. Let X be a complete smooth variety over a number field $\mathscr{K}$, and let $\mathrm{Griff}^2(X/\mathscr{K})$, the Griffiths group, be the group of cycles defined over $\mathscr{K}$, homologically trivial (over $\mathbb{C}$, for example), modulo cycles algebraically equivalent to zero over $\bar{\mathscr{K}}$. Then the conjecture is that $\mathrm{Griff}^2(X/\mathscr{K})$ has finite rank equal to the order of vanishing of a certain L-function. The evidence for this conjecture is rather sparse, largely because of the difficulty in producing examples of nontrivial cycles.

Results of Griffiths [10], Ceresa [5], B. Harris [11] and Clemens [6] show that the Griffiths group for cycles defined over $\mathbb{C}$, rather than $\mathscr{K}$, is often nonzero, and may not be finitely generated, even modulo torsion. However there are only a few examples where the Griffiths group of cycles defined over a number field is known to contain elements of infinite order. One source of examples is the following: Given an abelian variety X, let $\iota: X \to X$ be the inverse for the group law. If C is a curve on X, then C and $\iota(C)$ are homologically equivalent. Harris and Ceresa proved independently that for C a sufficiently general curve of genus $\geqslant 3$ over $\mathbb{C}$, and X its jacobian, the cycle $C - \iota C$ has infinite order modulo algebraic equivalence. The first example over a number field was the case $g = 3$, C the Fermat quartic. Using transcendental techniques, Harris [12] proved that $C - \iota C$ is not algebraically equivalent to zero (over $\mathbb{C}$). Using arithmetic methods, Bloch [3] showed further that $C - \iota C$ has infinite order in the Griffiths group. In fact, the jacobian of C is isogenous to a product $E \times E \times E$, where E is an elliptic curve with complex multiplication, and Bloch showed that the image of $C - \iota C$ gives a class of infinite order in the Griffiths group of $E \times E \times E$. More recently, Top [17] has applied Bloch's method to some other elliptic curves with complex multiplication. For some additional examples related to the Bloch–Beilinson conjecture, and some further references, see a recent paper by Schoen [15].

The criterion established here concerns a product of elliptic curves $E_1 \times E_2 \times E_3$ defined over a global field $\mathscr{K}$, and applies only to the case where for some prime $\mathfrak{p}$ of $\mathscr{K}$, E_1 has good reduction and E_2 and E_3 both have order 2 multiplicative reduction (their j-invariants have poles of order 2 at $\mathfrak{p}$). This assumption is made for ease of computation and is not crucial to the method. What is needed is that at least 2 of the elliptic curves have some kind of bad reduction at $\mathfrak{p}$.

To study cycles modulo algebraic equivalence, we need to use the Purity Conjecture for etale cohomology, which says that the local cohomology of a regular scheme with supports in a regular Cartier divisor is generated by the fundamental class of the divisor. The conjecture is known for schemes of equal characteristics, and for schemes of unequal characteristics and dimension at most 2, and for schemes of higher dimension which are smooth over a discrete valuation ring. Thus, the conjecture is valid in our case, if the global field $\mathscr{K}$ is a function field. For the case of a number field, however, we need to apply it to an arithmetic scheme of dimension 4, with bad reduction, a case for which the conjecture is not known. We take the point of view, suggested by Bloch in [4], that the Purity Conjecture is much less in doubt than the conjectures about cycles and L-functions, so it makes sense to accept Purity as a tool for investigating the other conjectures.

Let $\mathscr{K}$ be a global field, with ring of integers $\mathscr{R}$, and let $\mathfrak{p} \in \operatorname{spec} \mathscr{R}$ be a prime with residue characteristic different from 2. Let $\pi \in \mathscr{K} \cap \mathscr{R}_\mathfrak{p}$ be a uniformizing parameter for $\mathscr{R}_\mathfrak{p}$, and let $\alpha, \beta, \gamma \in \mathscr{K}$ be $\mathfrak{p}$-adic units which are all distinct mod $\mathfrak{p}$. Define elliptic curves over $\mathscr{K}$ by their affine equations:

$$E_1\colon y^2 = (x - \alpha)(x - \beta)(x - \gamma)$$

$$E_2\colon y^2 = x(x - \pi)(x - \alpha)(x - \beta) \text{ (normalize at } \infty)$$

$$E_3\colon y^2 = x(x - \pi)(x - \gamma).$$

Think of these as double covers of $\mathbb{P}^1$ by projecting onto the x-axis. Then the normalization of $C := E_1 \times_{\mathbb{P}^1} E_2$ is a curve of genus three, defined over $\mathscr{K}$, which embeds in $E_1 \times E_2 \times E_3$.

Let $\mathrm{CH}^2_{\mathrm{alg}}(E_1 \times E_2 \times E_3)$ be the subgroup of $\mathrm{CH}^2(E_1 \times E_2 \times E_3)$ consisting of cycles algebraically equivalent to zero. Let $\mathrm{CH}^2_{\mathrm{trans}}(E_1 \times E_2 \times E_3)$ be the subgroup of $\mathrm{CH}^2_{\mathrm{alg}}(E_1 \times E_2 \times E_3)$ generated by cycles $Z' - Z$ where Z is one of the "axes" $E_1 \times 0 \times 0, 0 \times E_2 \times 0, 0 \times 0 \times E_3$ (0 could be any $\mathscr{K}_{\mathfrak{p}}$-point of E_i), and Z' is a translate (hence the notation "trans") of Z by a $\mathscr{K}_{\mathfrak{p}}$-point of $E_1 \times E_2 \times E_3$.

For a prime $\mathfrak{q} \in \operatorname{spec} \mathscr{R}$, let $k(\mathfrak{q})$ denote the residue field $\mathscr{R}/\mathfrak{q}$, and $N(\mathfrak{q})$ the number of elements in $k(\mathfrak{q})$. The main results are the following:

THEOREM 1. *Let $e_+ \in E_1(\overline{k(\mathfrak{p})})$ be one of the two points with $x = 0$. If $16e_+ \neq 0$ in $E_1(\overline{k(\mathfrak{p})})$ (with the point at infinity as origin), then*

$$C - \iota C \notin \mathrm{CH}^2_{\mathrm{trans}}(E_1 \times E_2 \times E_3).$$

THEOREM 2. *Assume that the Purity Conjecture for étale cohomology holds for a regular model, X, of $E_1 \times E_2 \times E_3$ over $\mathscr{R}_{\mathfrak{p}}$. Suppose the order of $e_+ \in E_1(\overline{k(\mathfrak{p})})$ is divisible by an odd prime l with $\mathfrak{p} \nmid l$. Suppose further that there exist primes $\mathfrak{q}, \mathfrak{r} \in \operatorname{spec} \mathscr{R}$ satisfying*:

(i) *$\mathfrak{q} \nmid l$ and $\mathfrak{r} \nmid l$;*
(ii) *E_1, E_2, E_3 all have good reduction at $\mathfrak{q}$ and $\mathfrak{r}$;*
(iii) *For all eigenvalues λ_i of the geometric Frobenius map on $H^1(E_{i\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$, the products $\lambda_1\lambda_2\lambda_3$ are not divisible by $N(\mathfrak{q})$ in the ring of algebraic integers;*
(iv) *If $\mathfrak{f}$ is the geometric Frobenius map on $H^3(E_1 \times E_2 \times E_{3\overline{k(\mathfrak{r})}}, \mathbb{Q}_l)$ then $l \nmid \det(N(\mathfrak{r})^2 - \mathfrak{f})$.*

Then $C - \iota C$ has infinite order modulo algebraic equivalence.

Theorem 1 is proved in Sections 1–3. The proof uses intersection theory on a regular model, X, of $E_1 \times E_2 \times E_3$ over $\mathscr{R}_{\mathfrak{p}}$. In particular, X has bad reduction, with special fiber a union of components $Y_1, \ldots, Y_8$, and there is a specialization map

$$\sigma\colon \mathrm{CH}^2(X_{\mathscr{K}_{\mathfrak{p}}}) \to \Sigma \cong \mathrm{Pic}(E_{1k(\mathfrak{p})}) \otimes (\text{free abelian group}),$$

where Σ is a certain quotient of $\bigoplus_{i=1}^{8} \mathrm{CH}^2(Y_i)$. We show that if $4e_+ \neq 0$, then $\sigma(C - \iota C) \notin \sigma(\mathrm{CH}^2_{\mathrm{trans}}(X_{\mathscr{K}_{\mathfrak{q}}}))$.

The proof of Theorem 2 uses a map:

$$\varphi\colon \mathrm{CH}^2_{\mathrm{hom}}(X_{\mathscr{K}_{\mathfrak{p}}}) \to \mathrm{H}^1(\mathscr{K}_{\mathfrak{p}}, H^3(X_{\bar{\mathscr{K}}_{\mathfrak{p}}}, \mathbb{Z}_l(2))),$$

which is a sort of arithmetic analog of the Abel–Jacobi map into the intermediate jacobian. Here $\mathrm{CH}^2_{\mathrm{hom}}$ is the group of cycles homologically equivalent to zero, in the sense of l-adic cohomology. Condition (iii) of the theorem ensures that the images of $\mathrm{CH}^2_{\mathrm{trans}}(X_{\mathscr{K}_{\mathfrak{p}}})$ and $\mathrm{CH}^2_{\mathrm{alg}}(X_{\mathscr{K}_{\mathfrak{p}}})$ under φ are the same. This is proved in Section 4.

The Purity Conjecture is used to compare the specialization map σ with a similar map on étale cohomology, with the upshot being that Theorem 1 implies $\varphi(C-\iota C)\notin\varphi(\mathrm{CH}^2_{\mathrm{trans}}(X_{\mathscr{K}_\mathfrak{p}}))$, and so $C-\iota C$ is not algebraically equivalent to zero. This is the content of Section 5.

In Section 6, we discuss the condition for $C-\iota C$ to have infinite order. For cycles defined over $\mathscr{K}$, φ factors through $H^1(\mathscr{K}, H^3(X_{\bar{\mathscr{K}}}, \mathbb{Z}_l(2)))$, and condition (iv) above ensures that this group is torsion-free. It then follows that no multiple of $\varphi(C-\iota C)$ is in $\varphi(\mathrm{CH}^2_{\mathrm{trans}}(X_{\mathscr{K}_\mathfrak{p}}))$, and hence $C-\iota C$ has infinite order.

Since the conditions in Theorem 2 depend only on the reductions of the E_i modulo three primes $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$, any example that satisfies these conditions give rise to an infinite family of examples, by adding elements of $\mathscr{K}$ which are zero modulo $\mathfrak{pqr}$ to the coefficients $\alpha, \beta, \gamma, \pi$. One such family is exhibited in Section 7.

This work comprised my doctoral thesis at the University of Chicago. I would like to thank my advisor, Spencer Bloch, who suggested the problem, and contributed many crucial insights, as well as encouragement, toward the solution.

1. Regular models

With notation as above, let $K:=\mathscr{K}_\mathfrak{p}$, $R:=\mathscr{R}_\mathfrak{p}$ and $k:=k(\mathfrak{p})$. Let E_1, $\bar{E}_2$ and $\bar{E}_3$ be the closed subschemes of $\mathbb{P}^2_R$ given by the equations above. Then E_1 is smooth over R. Let E_2 and E_3 be the minimal regular models of $\bar{E}_2$ and $\bar{E}_3$. Each has special fiber of Kodaira type I_2, and is obtained by blowing up the node at $x=y=\pi=0$.

REMARK 1.1. It will be convenient to have the components of the special fibers of E_2 and E_3 defined over k. Since we are primarily interested in rational and algebraic equivalence over $\bar{\mathscr{K}}$, we may as well replace $\mathscr{K}$ by a finite extension $\mathscr{K}'$. If we take $\mathscr{K}'$ to be unramified over $\mathfrak{p}$, then the type of reduction of E_1, E_2 and E_3 will also be unaffected. We will henceforth tacitly assume this replacement whenever necessary. Thus any particular polynomial can be assumed to split over k. □

Take $\pi_i\colon E_i\to\mathbb{P}^1_R$, for $i=1,2,3$, to be projection onto the x-axis. Define $\bar{C}:=E_1\times_{\mathbb{P}^1_R}E_2$. The affine part of $\bar{C}$ is given by the two equations:

$$y_1^2=(x-\alpha)(x-\beta)(x-\gamma)$$
$$y_2^2=x(x-\pi)(x-\alpha)(x-\beta).$$

Note that this is singular along $x=\alpha$ and $x=\beta$. Let C be the blow-up of $\bar{C}$ along the closed subschemes (over R) given by $(x=\alpha, y_1=y_2=0)$ and $(x=\beta, y_1=y_2=0)$.

PROPOSITION 1.2.

(i) C *is regular, and* $C \to \mathrm{spec}(R)$ *is smooth in a neighborhood of the exceptional divisor.*
(ii) C_K *is a curve of genus* 3.
(iii) *The map*:

$$\bar{\varphi}_3\colon E_1 \times_{\mathbb{P}^1} \bar{E}_2 \backslash (\pi_1 \times \bar{\pi}_2)^{-1}(\{\alpha, \beta\}) \to \bar{E}_3 \quad (x, y_1, y_2) \mapsto \left(x, \frac{y_1 y_2}{(x-\alpha)(x-\beta)}\right)$$

extends to a morphism $\varphi_3\colon C \to E_3$

Proof. Part (i) is a straightforward but tedious exercise in blowing up arithmetic schemes; we omit the details. Part (ii) follows from the Riemann–Hurwitz formula, applied to the degree-two map $\varphi_1\colon C_K \to E_{1K}$. The singular points on $\bar{C}$ are nodes, from which it follows that φ_1 is ramified only at the four points of C where $x = 0$ and $x = \pi$. For (iii), note that the subscheme where φ_3 is not defined is precisely the center of the blow-up from $\bar{C}$ to C, and $z := y_1/y_2$ is a local parameter near the exceptional divisor. Then

$$\frac{y_1 y_2}{(x-\alpha)(x-\beta)} = \frac{z y_2^2}{(x-\alpha)(x-\beta)} = zx(x-\pi) \qquad \square$$

REMARK 1.3. This proposition implies that the exceptional divisor in C meets the special fiber, C_k, in a finite set of points. Since the cycle class of C in $(E_1 \times E_2 \times E_3)_k$ will be determined by intersecting with certain "test" divisors, which can be chosen to avoid any finite set of points, these points can be ignored, and we can work with $\bar{C}$ instead of C, when the time comes. $\square$

COROLLARY 1.4. *There is a proper map* $\varphi\colon C \to E_1 \times_R E_2 \times_R E_3$ *which is an embedding of the generic fiber.* $\square$

We need a regular model of $E_1 \times_R E_2 \times_R E_3$ on which to do intersection theory. Since E_1 is smooth, if we find a regular model E_{23} of $E_2 \times_R E_3$, then $E_1 \times_R E_{23}$ will be regular.

Introduce the following notation for the special fiber of either E_2 or E_3:

$A :=$ identity component ($=$strict transform of $(\bar{E}.)_k$)

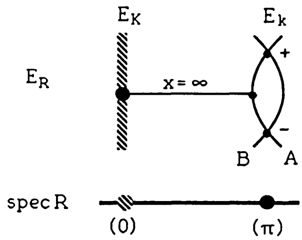
$B :=$ other component ($=$exceptional divisor of blow-up)

Note that $A \cong B \cong \mathbb{P}_k^1$ and $A \cap B$ consists of two points which we label $+$ and $-$:

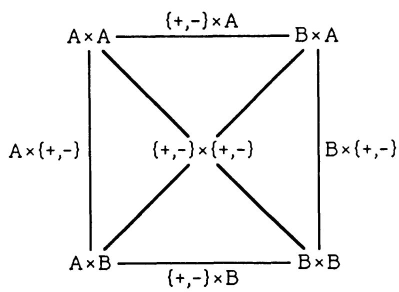
$A \cap B = \{+, -\}$

These two points correspond to the two choices of sign for the coordinate y/x in the blow-up, which satisfies $(y/x)^2 = \alpha\beta$ on E_2 and $(y/x)^2 = -\gamma$ on E_3. Hence the notation $+$ and $-$.

The geometry of E_2 and E_3 is illustrated in the following diagram:



Now consider the scheme $E_2 \times_R E_3$ Since E_2 and E_3 are regular, and one of them is always smooth over R away from the four points $\{+, -\} \times \{+, -\}$, these four points are the only singularities of $E_2 \times_R E_3$. The components of the special fiber, $(E_2 \times_R E_3)_k$, and their intersections are illustrated by the following incidence graph (edges represent intersection):



Let E_{23} be the blow-up of $E_2 \times_R E_3$ at the four points $\{\pm\} \times \{\pm\}$. The special fiber now has 8 components, which will be denoted as follows:

$$\begin{aligned} AA,\ AB,\ BA,\ BB \quad & := \text{strict transform of } A \times A, \text{ etc.} \\ & \cong \mathbb{P}^1_k \times \mathbb{P}^1_k \text{ with 4 points blown up} \\ Q_{++},\ Q_{+-},\ Q_{-+},\ Q_{--} & := \text{exceptional divisors over } \{+\} \times \{+\}, \text{ etc.} \end{aligned} \tag{1.1}$$

Also introduce the following notation:

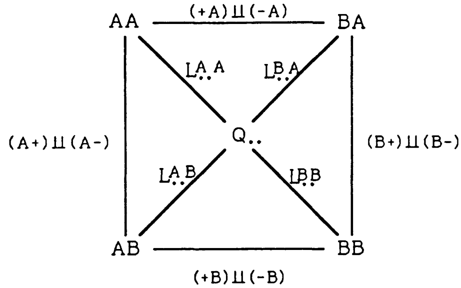
$A+,\ A-,\ B+,\ B- :=$ strict transform of $A \times \{+\}$, etc.

$+A,\ -A,\ +B,\ -B :=$ strict transform of $\{+\} \times A$, etc.

$$L^{Z_1Z_2}_{\varepsilon_1\varepsilon_2} := Z_1Z_2 \cap Q_{\varepsilon_1\varepsilon_2} \quad (Z_i = A \text{ or } B,\ \varepsilon_i = + \text{ or } -) \tag{1.2}$$

Note that $L^{Z_1Z_2}_{\pm\pm}$ are the exceptional $\mathbb{P}^1$'s in Z_1Z_2 = blow-up of $Z_1 \times Z_2$ at $\{\pm\} \times \{\pm\}$.

The incidence graph of the components of $(E_{23})_k$ then consists of the combination of four of the following type, one for each of the exceptional components $Q_{\pm\pm}$:



PROPOSITION 1.5.

(1) *E_{23} is regular.*

(2) *The closed fiber is the divisor*:

$$(\pi) = [AA] + [AB] + [BA] + [BB] + 2([Q_{++}] + [Q_{+-}] + [Q_{-+}] + [Q_{--}])$$

(3) *Each $Q_{..}$ is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Moreover, the L's are generators of* $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$, *with $L^{AA}_{..}$ and $L^{BB}_{..}$ in one ruling, and $L^{AB}_{..}$ and $L^{BA}_{..}$ in the other.*

Proof. All statements are local on $E_2 \times_R E_3$, so we can replace both $\bar{E}_2$ and $\bar{E}_3$ by the R-scheme $\bar{E}$ given by:

$$\bar{E}\colon y^2 = x(x - \pi)$$

Blowing-up at the ideal (x, y, π) gives a scheme:

$$E\colon \begin{cases} (y')^2 = 1 - p \\ xp = \pi \end{cases}$$

Here A is given by $p = 0$ and B by $x = 0$. Then $E_2 \times_R E_3$ is locally isomorphic to:

$$E \times_R E = \operatorname{spec} \frac{R[x_2, y_2, p_2, x_3, y_3, p_3]}{\begin{pmatrix} y_2^2 = 1 - p_2,\ y_3^2 = 1 - p_3 \\ x_2p_2 = x_3p_3 = \pi \end{pmatrix}}$$

The singular locus, $\{+, -\} \times \{+, -\} = (A \cap B) \times (A \cap B)$, is the subscheme corresponding to the ideal (x_2, p_2, x_3, p_3). Then E_{23} corresponds to the blow-up at this ideal. Introduce homogeneous coordinates X_2, P_2, X_3, P_3 on the blow-up, satisfying $X_2/P_2 = x_2/p_2$, $X_3/P_2 = x_3/p_2$ and $P_3/P_2 = p_3/p_2$. These then satisfy $X_2P_2 = X_3P_3$

The exceptional divisor, Q, is given by $x_2 = p_2 = x_3 = p_3 = \pi = 0$:

$$Q = \operatorname{proj} \frac{k[X_2, P_2, X_3, P_3]}{(X_2P_2 - X_3P_3)} \cong \mathbb{P}^1_k \times \mathbb{P}^1_k. \tag{$*$}$$

Since this is smooth, and it is a Cartier divisor on $E \times_R E$, (1) is proved. To see (3), note that the components of $(E_{23})_k$, other than Q, are given by:

$$AA: P_2 = P_3 = 0$$

$$AB: P_2 = X_3 = 0$$

$$BA: X_2 = P_3 = 0$$

$$BB: X_2 = X_3 = 0$$

which are equations of the desired rulings on the quadric ($*$).

In (2), the only point that is not obvious is the multiplicity of the Q's. To check this, we may localize to the open set where $P_2 \neq 0$. Then $x_2 = (X_2/P_2)p_2$, $p_3 = (P_3/P_2)p_2$ and $x_3 = (X_3/P_2)p_2$, so $p_2 = 0$ is the local defining equation for Q. Moreover, we have:

$$\pi = x_2p_2 = \left(\frac{X_2}{P_2}\right)p_2^2.$$

Therefore π vanishes to order 2 along Q. This proves (2). □

2. Specialization map

If V is any scheme over $\operatorname{spec} R$ such that the components of V_k are regularly embedded in V, we let $\tilde{V}_k$ denote the disjoint union of the components of V_k. Let $i\colon \tilde{V}_k \to V$ be the natural map (projection onto V_k followed by inclusion in V). Let $j\colon V_K \hookrightarrow V$ be the inclusion. Since the components of V_k are regularly embedded in V, there is a pull-back map i^* on Chow groups. Let $\rho := i^*i_*$. Then, for any $q \geqslant 1$, we get the following diagram, with exact row:

$$\begin{tikzcd} \mathrm{CH}^{q-1}(\tilde{V}_k) \arrow[r, "i_*"] \arrow[dr, "\rho"'] & \mathrm{CH}^q(V) \arrow[r, "j^*"] \arrow[d, "i^*"] & \mathrm{CH}^q(V_K) \arrow[r] & 0 \\ & \mathrm{CH}^q(\tilde{V}_k) & & \end{tikzcd}$$

Thus we get a specialization map:

$$\sigma\colon \mathrm{CH}^q(V_K) \to \Sigma^q(V) := \mathrm{coker}(\rho)$$

induced by the pull-back i^*.

Now let $X := E_1 \times_R E_{23}$. Then X is a regular scheme whose special fiber X_k, with its reduced structure, is a divisor with normal crossings. Therefore the preceding paragraph applies to X. X_k has eight components:

$$E_{1k} \times AA, \quad E_{1k} \times AB, \quad E_{1k} \times BA, \quad E_{1k} \times BB \cong E_{1k} \times \mathrm{Bl}_{4\mathrm{pts.}}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$$
$$E_{1k} \times Q_{++}, \quad E_{1k} \times Q_{+-}, \quad E_{1k} \times Q_{-+}, \quad E_{1k} \times Q_{--} \cong E_{1k} \times (\mathbb{P}^1_k \times \mathbb{P}^1_k).$$

These are all of the form $E_{1k} \times S$, with S a rational surface over k. Standard computations of the Chow ring of a blow-up show that

$$\mathrm{CH}^{\cdot}(E_{1k} \times S) \cong \mathrm{CH}^{\cdot}(E_{1k}) \otimes \mathrm{CH}^{\cdot}(S).$$

It then follows easily that

$$\Sigma^{\cdot}(X) = \mathrm{CH}^{\cdot}(E_{1k}) \otimes \Sigma^{\cdot}(E_{23}).$$

In particular,

$$\Sigma^2(X) = \mathrm{CH}^1(E_{1k}) \otimes \Sigma^1(E_{23}) \oplus \mathrm{CH}^0(E_{1k}) \otimes \Sigma^2(E_{23})$$

For cycles homologous to zero, the interesting part of this is the first direct summand. To compute this, we use the following description of the Chow groups of the components of E_{23k}:

$$\begin{aligned}\mathrm{CH}^1(Q_{..}) &= \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)\\ &= \mathbb{Z}\cdot[\{*\} \times \mathbb{P}^1] \oplus \mathbb{Z}\cdot[\mathbb{P}^1 \times \{*\}]\end{aligned}$$

(here $*$ denotes any point of $\mathbb{P}^1$) with rulings chosen so that:

$$[L^{AA}_{..}] = [L^{BB}_{..}] = [\mathbb{P}^1 \times *] \qquad [L^{AB}_{..}] = [L^{BA}_{..}] = [* \times \mathbb{P}^1],$$

and:

$$\begin{aligned}&\mathrm{CH}^1(Bl_{\{\pm\}\times\{\pm\}}(\mathbb{P}^1 \times \mathbb{P}^1))\\ &\quad = \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) \oplus \mathbb{Z}\cdot[L_{++}] \oplus \mathbb{Z}\cdot[L_{+-}] \oplus \mathbb{Z}\cdot[L_{-+}] \oplus \mathbb{Z}\cdot[L_{--}],\end{aligned}$$

where $\mathrm{CH}^1(\mathbb{P}^1\times\mathbb{P}^1)$ is identified with its image under pulling back along the blow-down map: $Bl_{\{\pm\}\times\{\pm\}}(\mathbb{P}^1\times\mathbb{P}^1)\to\mathbb{P}^1\times\mathbb{P}^1$. Recall that on the component $AA=Bl_{\{\pm\}\times\{\pm\}}(A\times A)$, we denote by $A+$ the strict transform of $A\times\{+\}$, etc. Then we have the following relations, for each Z_2, $Z_3=A$ or B and each $\varepsilon=+$ or $-$:

$$\begin{aligned}[][Z_2\varepsilon] &= [Z_2\times\{\varepsilon\}]-[L^{Z_2Z_3}_{+\varepsilon}]-[L^{Z_2Z_3}_{-\varepsilon}]\\ &= [\mathbb{P}^1\times *]-[L_{+\varepsilon}]-[L_{-\varepsilon}]\\ [\varepsilon Z_3] &= [*\times\mathbb{P}^1]-[L_{\varepsilon+}]-[L_{\varepsilon-}].\end{aligned}$$

With these conventions, the map $i^*i_*\colon \mathrm{CH}^0(\tilde{E}_{23k})\to\mathrm{CH}^1(\tilde{E}_{23k})$ is given by Table 1. Specifically, each row and column of Table 1 corresponds to a component of $(E_{23})_k$. The entry in row R and column C is the class of $R\cap C$ as a divisor on the component C. The self-intersections are computed using Proposition 1.5 and the fact that each component has zero intersection with the divisor (π). Then $\Sigma^1(E_{23})$ is the quotient of $\mathrm{CH}^1(\tilde{E}_{23k})$ by the row vectors of Table 1.

For computation, it is convenient to use a certain quotient of $\Sigma^1(E_{23})$: The action of $E_2(K)\times E_3(K)$ on $(E_2\times E_3)_K$ by translation can be extended to an action on E_{23}, which therefore induces an action on $\Sigma^1(E_{23})$, and we take the coinvariants. The action is discussed in [7]. However, since we will not need to use this geometric description, we just describe the quotient of $\Sigma^1(E_{23})$ directly.

First take $\Pi\cong\mathbb{Z}^8$ with basis $\{u_1,u_2,v_1,v_2,l_{++},l_{+-},l_{-+},l_{--}\}$, and define a surjective map $\mathrm{CH}^1(\tilde{E}_{23k})\to\Pi$ by:

$$\begin{aligned}
&[\mathbb{P}^1\times *]_{S_2S_3}\mapsto u_1\\
&[*\times\mathbb{P}^1]_{S_2S_3}\mapsto u_2 \quad \text{for } S_2,\, S_3=A \text{ or } B\\
&[L_{++}]_{AA},\,[L_{+-}]_{AB},\,[L_{-+}]_{BA},\,[L_{--}]_{BB}\mapsto l_{++}\\
&[L_{+-}]_{AA},\,[L_{++}]_{AB},\,[L_{--}]_{BA},\,[L_{-+}]_{BB}\mapsto l_{+-}\\
&[L_{-+}]_{AA},\,[L_{--}]_{AB},\,[L_{++}]_{BA},\,[L_{+-}]_{BB}\mapsto l_{-+}\\
&[L_{--}]_{AA},\,[L_{-+}]_{AB},\,[L_{+-}]_{BA},\,[L_{++}]_{BB}\mapsto l_{--}\\
&[\mathbb{P}^1\times *]_{Q_{++}},\,[*\times\mathbb{P}^1]_{Q_{+-}},\,[*\times\mathbb{P}^1]_{Q_{-+}},\,[\mathbb{P}^1\times *]_{Q_{--}}\mapsto v_1\\
&[*\times\mathbb{P}^1]_{Q_{++}},\,[\mathbb{P}^1\times *]_{Q_{+-}},\,[\mathbb{P}^1\times *]_{Q_{-+}},\,[*\times\mathbb{P}^1]_{Q_{--}}\mapsto v_2
\end{aligned}\tag{2.1}$$

Here the notation $[U]_V$ means the class of the divisor U on the component V of $\tilde{E}_{23k}$.

Table 1. Intersection of components of E_{23k}

	$CH^1(AA)$	$CH^1(AB)$	$CH^1(BA)$	$CH^1(BB)$	$CH^1(Q_{++})$	$CH^1(Q_{+-})$	$CH^1(Q_{-+})$	$CH^1(Q_{--})$
AA	$-2[\mathbb{P}^1\times *] -2[*\times\mathbb{P}^1]$	$2[\mathbb{P}^1\times *] -\Sigma[L_{..}]$	$2[*\times\mathbb{P}^1] -\Sigma[L_{..}]$	0	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$
AB	$2[\mathbb{P}^1\times *] -\Sigma[L_{..}]$	$-2[\mathbb{P}^1\times *] -2[*\times\mathbb{P}^1]$	0	$2[*\times\mathbb{P}^1] -\Sigma[L_{..}]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$
BA	$2[*\times\mathbb{P}^1] -\Sigma[L_{..}]$	0	$-2[\mathbb{P}^1\times *] -2[*\times\mathbb{P}^1]$	$2[\mathbb{P}^1\times *] -\Sigma[L_{..}]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$	$[*\times\mathbb{P}^1]$
BB	0	$2[*\times\mathbb{P}^1] -\Sigma[L_{..}]$	$2[\mathbb{P}^1\times *] -\Sigma[L_{..}]$	$-2[\mathbb{P}^1\times *] -2[*\times\mathbb{P}^1]$	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$	$[\mathbb{P}^1\times *]$
Q_{++}	$[L_{++}]$	$[L_{++}]$	$[L_{++}]$	$[L_{++}]$	$-[\mathbb{P}^1\times *] -[*\times\mathbb{P}^1]$	0	0	0
Q_{+-}	$[L_{+-}]$	$[L_{+-}]$	$[L_{+-}]$	$[L_{+-}]$	0	$-[\mathbb{P}^1\times *] -[*\times\mathbb{P}^1]$	0	0
Q_{-+}	$[L_{-+}]$	$[L_{-+}]$	$[L_{-+}]$	$[L_{-+}]$	0	0	$-[\mathbb{P}^1\times *] -[*\times\mathbb{P}^1]$	0
Q_{--}	$[L_{--}]$	$[L_{--}]$	$[L_{--}]$	$[L_{--}]$	0	0	0	$-[\mathbb{P}^1\times *] -[*\times\mathbb{P}^1]$

We denote by $\bar{\Sigma}(E_{23})$ the push-out in the diagram:

$$\begin{tikzcd}
\mathrm{CH}^1(\tilde{E}_{23k}) \arrow[r] \arrow[d] & \Pi \arrow[d, dotted] \\
\Sigma^1(E_{23}) \arrow[r, dotted] & \bar{\Sigma}(E_{23})
\end{tikzcd}$$

Thus $\bar{\Sigma}(E_{23})$ has generators u_1, u_2, v_1, v_2, l_{++}, l_{+-}, l_{-+}, l_{--}, with relations obtained by projecting the rows of Table 1 by the map given in (2.1). This is easily seen to give just the one relation:

$$0 = l_{++} + l_{+-} + l_{-+} + l_{--} - v_1 - v_2.$$

We write $\bar{\Sigma} := \mathrm{CH}^1(E_{1k}) \otimes \bar{\Sigma}(E_{23})$. By composing with the specialization map σ, we obtain the following:

PROPOSITION 2.1. *There is a specialization map*

$$\bar{\sigma}: \mathrm{CH}^2(X_K) \to \bar{\Sigma} \cong \mathrm{CH}^1(E_{1k}) \otimes \mathbb{Z}^7$$

Taking generators u_1, u_2, v_1, v_2, l_{++}, l_{+-}, l_{-+}, l_{--} for $\mathbb{Z}^7$, with the relation:

$$0 = l_{++} + l_{+-} + l_{-+} + l_{--} - v_1 - v_2,$$

then $\bar{\sigma}$ is given by taking closure in X_R and intersecting with components of X_k, followed by the map (2.1) *tensored with the identity on* $\mathrm{CH}^1(E_{1k})$. □

Let ι_i be the involution on E_{iK} given by $(x, y) \mapsto (x, -y)$, and let $\iota := \iota_1 \times \iota_2 \times \iota_3$ on X_K. This is the inverse for the group law, for a suitable choice of origin on the E_i.

PROPOSITION 2.2. *There is an automorphism ι^* of $\bar{\Sigma}$ such that the following diagram commutes*:

$$\begin{tikzcd}
\mathrm{CH}^2(X_K) \arrow[r, "\bar{\sigma}"] \arrow[d, "\iota^*"'] & \bar{\Sigma} \arrow[d, "\iota^*"] \\
\mathrm{CH}^2(X_K) \arrow[r, "\bar{\sigma}"] & \bar{\Sigma}
\end{tikzcd}$$

Moreover, $\iota^* = \iota_1^* \otimes \iota_{23}^*$, *where* ι_{23}^*: $\bar{\Sigma}(E_{23}) \to \bar{\Sigma}(E_{23})$ *fixes* u_1, u_2, v_1, v_2, *interchanges* l_{++} *and* l_{--}, *and interchanges* l_{+-} *and* l_{-+} *(notation as in Proposition* 2.1).

Proof. Each ι_i acts on the smooth (over R) points of E_i, because that is the Neron model. For $i = 1$, this is all of E_1 For $i = 2$ or 3, ι_i acts by -1 on the group of components E_i/E_i°, which in our case is $\mathbb{Z}/2\mathbb{Z}$, so the components are fixed. Each component is isomorphic to $\mathbb{P}^1$, and the map is $t \mapsto 1/t$. It is straightforward to show that this map extends to all of E_i. It is then clear that the singular points $+$ and $-$ of the fiber are interchanged. It follows that $\iota_2 \times \iota_3$ on $E_2 \times E_3$ lifts to the blow-up E_{23}, since the center of the blow-up is its own scheme-theoretic inverse image under $\iota_2 \times \iota_3$ The lifted map stabilizes the components AA, AB, BA, BB and interchanges Q_{++} with Q_{--} and Q_{+-} with Q_{-+}. Thus for Y one of the components AA, AB, BA, BB, ι^* acts on $\mathrm{CH}^1(Y) = \mathrm{CH}^1(\mathrm{bl}_{4\mathrm{pts}}(\mathbb{P}^1 \times \mathbb{P}^1))$ as the identity on $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ and by permuting the exceptional lines. Specifically, $[L_{++}]$ and $[L_{--}]$ are switched, as are $[L_{+-}]$ and $[L_{-+}]$. The action on $\mathrm{CH}^1(Q_{++}) = \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ is to take $[L^{AA}] = [\mathbb{P}^1 \times *]_{Q_{++}}$ to $[L^{AA}] = [\mathbb{P}^1 \times *]_{Q_{--}}$, and $[L^{AB}] = [* \times \mathbb{P}^1]_{Q_{++}}$ to $[L^{AB}] = [* \times \mathbb{P}^1]_{Q_{--}}$. Referring to (2.1), this gives the proposition. □

3. Specialization of $C - \iota C$

Recall that $\bar{C} = E_1 \times_{\mathbb{P}^1_R} E_2$ and C is its normalization. There is a map $\varphi: C \to E_1 \times_R E_2 \times_R E_3$. Let $\pi_i: E_i \to \mathbb{P}^1$ be the degree 2 maps given by projecting onto the x-axis. Notice that $\varphi: C \to E_1 \times E_2 \times E_3$ is compatible with all the π_i, so φ factors through $E_1 \times_{\mathbb{P}^1} E_2 \times_{\mathbb{P}^1} E_3$. In fact C is one of the two components of the scheme $E_1 \times_{\mathbb{P}^1_R} E_2 \times_{\mathbb{P}^1_R} E$.

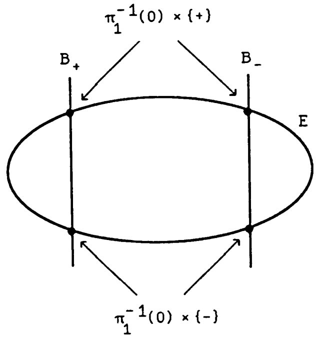
The following is straightforward.

PROPOSITION 3.1. *The special fiber* C_k *has* 3 *components,* E, B_+, *and* B_-, *with*:

$$E = \text{normalization of } E_{1k} \times_{\mathbb{P}^1} A$$

$$B_+ \amalg B_- = \pi_1^{-1}(0) \times_k B,$$

where A *and* B *are the components of* E_{2k}. *All three components have multiplicity* 1 *in* C_k. E *is an elliptic curve isogenous to* E_{1k}, *and* $B_+ \cong B_- \cong \mathbb{P}^1_k$. *They intersect as in the following diagram*:



□

To avoid confusion in what follows, I will denote the components of E_{2k} by A_2, B_2, and those of E_{3k} by A_3, B_3.

PROPOSITION 3.2. *$\varphi\colon C_k \to E_{1k} \times E_{2k} \times E_{3k}$ is determined by the following:*

$$\begin{tikzcd}
& E_{1k} \arrow[dr, "\text{branched over } \{\alpha,\beta,\gamma,\infty\}", "\pi_1"'] & \\
E \arrow[ur, "\text{unramified}", "\varphi_1"'] \arrow[r, "\text{branched over } \pi_2^{-1}(\{\gamma,\infty\})", "\varphi_2"'] \arrow[dr, "\varphi_3", "\text{branched over } \pi_3^{-1}(\{\alpha,\beta\})"'] & A_2 \arrow[r, "\text{branched over } \{\alpha,\beta\}", "\pi_2"'] & \mathbb{P}^1_k \\
& A_3 \arrow[ur, "\pi_3", "\text{branched over } \{\gamma,\infty\}"'] &
\end{tikzcd}$$

(All maps above have degree 2)

$$\begin{tikzcd}
& \{e_+\} \subset E_{1k} \\
B_+ \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2", "\cong"'] \arrow[dr, "\varphi_3", "\cong"'] & B_2 \\
& B_3
\end{tikzcd}
\qquad
\begin{tikzcd}
& \{e_-\} \subset E_{1k} \\
B_- \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2", "\cong"'] \arrow[dr, "\varphi_3", "\cong"'] & B_2 \\
& B_3
\end{tikzcd}$$

Proof. The only thing left to check is the information about φ_3. Since $\pi_3 \circ \varphi_3 = \pi_1 \circ \varphi_1$, the finiteness and ramification data for $\varphi_3|_E$ are clear. In particular, $\pi_3 \circ \varphi_3|_E$ finite implies that $\varphi_3|_E$ maps into A_3.

Similarly, $\pi_3 \circ \varphi_3|_{B_\pm} = \pi_1 \circ \varphi_1|_{B_\pm} = \{0\}$ (constant map) implies that $\varphi_3|_{B_\pm}$

maps into B_3. To see that these are isomorphisms, recall that φ_3 is induced by the (rational) map $E_1 \times_{\mathbb{P}^1_R} \bar{E}_2 \to \bar{E}_3$, given by the algebra homomorphism:

$$\begin{array}{ccc} \dfrac{R[x,y]}{(y^2 = x(x-\pi)(x-\gamma))} & \xrightarrow{\quad\quad\quad\quad} & \dfrac{R[x,y_1,y_2]_{((x-\alpha)(x-\beta))}}{\begin{pmatrix} y_1^2 = (x-\alpha)(x-\beta)(x-\gamma) \\ y_2^2 = x(x-\pi)(x-\alpha)(x-\beta) \end{pmatrix}} \\ & \begin{array}{l} x \mapsto x \\ y \mapsto \dfrac{y_1 y_2}{(x-\alpha)(x-\beta)} \end{array} & \\ \bar{E}_3 & \dashleftarrow & E_1 \times_{\mathbb{P}^1} \bar{E}_2 \end{array}$$

After blowing up the ideal (x, y, π) in $\bar{E}_3$ and (x, y_2, π) in $E_1 \times_{\mathbb{P}^1} \bar{E}_2$, and restricting to exceptional divisors, the map becomes (writing X, Y, P for x, y, π as elements of $S^1(I)$, where I is the ideal):

$$\begin{array}{ccc} \dfrac{k[X,Y,P]}{(Y^2 = -\gamma X(X-P))} & \longrightarrow & \dfrac{k[X,Y,P]}{(Y^2 = \alpha\beta X(X-P))} \\ & \begin{array}{l} X \mapsto X \\ Y \mapsto \dfrac{y_1}{\alpha\beta} Y \\ P \mapsto P \end{array} & \end{array} \tag{3.1}$$

where $y_1^2 = -\alpha\beta\gamma$, the two choices of y_1 corresponding to B_+ and B_-. This map is clearly an isomorphism. □

CONVENTION 3.3. Recall that $\pi_1^{-1}(0) = \{e_+, e_-\}$ and that we are writing $\{+, -\}$ for both $A_2 \cap B_2$ and $A_3 \cap B_3$. Proposition 3.2 implies that $\varphi_3(\{e_+\} \times \{+, -\}) = \{+, -\}$ Fix the labeling of the points $+$ and $-$ so that:

$$\varphi_3(e_+, +) = +.$$

From (3.1) it then follows that $\varphi_3(e_{\varepsilon_1}, \varepsilon_2) = \varepsilon_1\varepsilon_2$ for $\varepsilon_1, \varepsilon_2 = +$ or $-$. □

Notice that

$$\begin{aligned} \varphi^{-1}(E_{1k} \times \{\pm\} \times \{\pm\}) &= \pi_1^{-1}(0) \times \pi_2^{-1}(0) \\ &= \{e_\pm\} \times \{\pm\} \end{aligned}$$

Therefore φ lifts to a map $\varphi' \colon \mathrm{Bl}_{\{e_\pm\} \times \{\pm\}}(C) \to X$.

LEMMA 3.4. *Let $C_{\varepsilon_1,\varepsilon_2}$ be the exceptional divisor over $(e_{\varepsilon_1}, \varepsilon_2)$ in $C' := Bl_{\{e_\pm\} \times \{\pm\}}(C)$. Then*:

$$\varphi'(C_{\varepsilon_1,\varepsilon_2}) \subset \{e_{\varepsilon_1}\} \times Q_{\varepsilon_2(\varepsilon_1\varepsilon_2)},$$

and $\varphi'(C_{\varepsilon_1,\varepsilon_2})$ does not meet $E_{1k}\times A_2B_3$ or $E_{1k}\times B_2A_3$, but meets $E_{1k}\times A_2A_3$ and $E_{1k}\times B_2B_3$ transversally in one point each.

Proof. The first statement is clear from Convention 3.3. The rest is easily checked from the equations of the pertinent schemes. □

This lemma, together with Proposition 3.2, implies that each component of $\varphi'(C_k')$ is contained in exactly one component of X_k. Thus φ' induces a map $\tilde{\varphi}\colon \tilde{C}_k' \to \tilde{X}_k$ of the normalizations (=disjoint union of the components). Then $\bar{\sigma}([C_K])$ can be computed using the following commutative diagram:

$$\begin{tikzcd}
{[C']} \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] \arrow[dd, bend right=40] & {[C']} \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & {[C_K]} \arrow[d, phantom, sloped, "\in"] \\
\mathrm{CH}^0(C') \arrow[r, "\varphi'_*"] \arrow[d] & \mathrm{CH}^2(X) \arrow[r, "j^*"] \arrow[d] & \mathrm{CH}^2(X_K) \arrow[d, "\bar{\sigma}"] \\
{[C_k'] \in \mathrm{CH}^0(\tilde{C}_k')} \arrow[r, "\tilde{\varphi}_*"] & \mathrm{CH}^2(\tilde{X}_k) \arrow[r] & \bar{\Sigma}
\end{tikzcd}$$

and computing the image of $\tilde{\varphi}_*[C_k']$.

Let E, B_+ and B_- denote also the strict transforms of the components of C_k, in C_k'.

PROPOSITION 3.5. *The map $\tilde{\varphi}_*\colon \mathrm{CH}^0(\tilde{C}_k')\to\mathrm{CH}^2(\tilde{X}_k)$ is given by:*

(i) $[E]\mapsto 2[E_{1k}\times * \times *]+([e_+]+[e_-])\otimes([\mathbb{P}^1\times *]+[*\times\mathbb{P}^1])$
$\quad -[e_+]\otimes([L_{++}]+[L_{--}])$
$\quad -[e_-]\otimes([L_{+-}]+[L_{-+}])$
in $\mathrm{CH}^2(E_{1k}\times AA)$

(ii) $[B_+]\mapsto [e_+]\otimes([\mathbb{P}^1\times *]+[*\times\mathbb{P}^1]-[L_{++}]-[L_{--}])$
$[B_-]\mapsto [e_-]\otimes([\mathbb{P}^1\times *]+[*\times\mathbb{P}^1]-[L_{+-}]-[L_{-+}])$
in $\mathrm{CH}^2(E_{1k}\times BB)$

(iii) $[C_{\varepsilon_1,\varepsilon_2}]\mapsto [e_{\varepsilon_1}]\otimes[*\times\mathbb{P}^1]$ *in* $\mathrm{CH}^2(E_{1k}\times Q_{\varepsilon_2,\varepsilon_1\varepsilon_2})$.

Proof. $\tilde{\varphi}_*([E])$ is the strict transform of $\varphi_*([E])\in\mathrm{CH}^2(E_{1k}\times A_2\times A_3)$. Clearly, $\varphi|_E$ has degree 1 since already $C\to E_1\times_{\mathbb{P}^1}E_2$ has degree 1. Therefore $\varphi_*([E])=[\varphi(E)]$.

Suppose

$$[\varphi(E)] = n[E_{1k}\times * \times *]+[D_1\times A_2\times *]+[D_2\times * \times A_3],$$

with D_1 and D_2 divisors on E_{1k}. Let $\mathrm{pr}_1\colon E_{1k} \times A_2 \times A_3 \to E_{1k}$ be the projection. Then:

$$n = \deg([\varphi(E)] \cdot [* \times A_2 \times A_3]) = 2,$$

since $\deg(\varphi\colon E \to E_{1k}) = 2$

$$D_1 = \mathrm{pr}_{1*}([\varphi(E)] \cdot [E_{1k} \times \{+\} \times A_3]) = (e_+) + (e_-)$$
$$D_2 = \mathrm{pr}_{1*}([\varphi(E)] \cdot [E_{1k} \times A_2 \times \{+\}]) = (e_+) + (e_-).$$

Since

$$\begin{aligned}[\varphi'(E)] &= \mathrm{bl}^*[\varphi(E)] - [\{e_+\} \times L_{++}] - [\{e_+\} \times L_{--}] \\ &\quad - [\{e_-\} \times L_{+-}] - [\{e_-\} \times L_{-+}],\end{aligned}$$

where $\mathrm{bl}\colon \tilde{X}_k \to E_{1k} \times \tilde{E}_{2k} \times \tilde{E}_{3k}$ is the blow-down map, this proves (i).

Since the compositions:

$$\begin{tikzcd}[column sep=small] & & B_2 \cong \mathbb{P}^1 \\ B_+ \arrow[r, "\varphi"] & \{e_+\} \times B_2 \times B_3 \arrow[ur] \arrow[dr] & \\ & & B_3 \cong \mathbb{P}^1 \end{tikzcd}$$

are isomorphisms, it is clear that:

$$\varphi_*([B_+]) = [\{e_+\} \times \mathbb{P}^1 \times *] + [\{e_+\} \times * \times \mathbb{P}^1]$$

in $\mathrm{CH}^2(E_{1k} \times B_2 \times B_3)$. Similarly,

$$\varphi_*([B_-]) = [\{e_-\} \times \mathbb{P}^1 \times *] + [\{e_-\} \times * \times \mathbb{P}^1]$$

Now using:

$$[\varphi'(B_+)] = \mathrm{bl}^*[\varphi(B_+)] - [\{e_+\} \times L_{++}] - [\{e_+\} \times L_{--}]$$
$$[\varphi'(B_-)] = \mathrm{bl}^*[\varphi(B_-)] - [\{e_-\} \times L_{+-}] - [\{e_-\} \times L_{-+}],$$

we get (ii).

It follows from the proof of Lemma 3.4 that $\deg(\varphi'|_{C_{\varepsilon_1,\varepsilon_2}}) = 1$, so $\varphi'_*([C_{\varepsilon_1,\varepsilon_2}]) = [\varphi'(C_{\varepsilon_1,\varepsilon_2})]$. Suppose:

$$[\varphi'(C_{\varepsilon_1,\varepsilon_2})] = n[E_{1k} \times * \times *] + [D_1 \times \mathbb{P}^1 \times *] + [D_2 \times * \times \mathbb{P}^1]$$

in $\mathrm{CH}^2(E_{1k} \times Q_{\varepsilon_2,\varepsilon_1\varepsilon_2})$. Take $p \in E_{1k}$ with

$$p \notin \{e_{\varepsilon_1}\} \cup \mathrm{supp}(D_1) \cup \mathrm{supp}(D_2).$$

Then it is clear that:

$$\begin{aligned} n &= \deg([\varphi'(C_{\varepsilon_1,\varepsilon_2})] \cdot [p \times \mathbb{P}^1 \times \mathbb{P}^1]) \\ &= 0. \end{aligned}$$

Let $\mathrm{pr}_1 : E_{1k} \times Q_{\varepsilon_2,\varepsilon_1\varepsilon_2} \to E_{1k}$ be the projection. Then:

$$\begin{aligned} D_1 &= \mathrm{pr}_{1*}([\varphi'(C_{\varepsilon_1,\varepsilon_2})] \cdot [E_{1k} \times * \times \mathbb{P}^1]) \\ &= 0 \end{aligned}$$

by Lemma 3.4 and the fact that $[L^{AB}] = [L^{BA}] = [* \times \mathbb{P}^1]$ in $\mathrm{CH}^1(Q)$. Also,

$$\begin{aligned} D_2 &= \mathrm{pr}_{1*}([\varphi'(C_{\varepsilon_1,\varepsilon_2})]) \cdot [E_{1k} \times \mathbb{P}^1 \times *]) \\ &= \mathrm{pr}_{1*}([\varphi'(C_{\varepsilon_1,\varepsilon_2})]) \cdot [E_{1k} \times L^{AA}]) \\ &= [e_{\varepsilon_1}] \end{aligned}$$

by the lemma and the fact that $\mathrm{pr}_1 \circ \varphi'(C_{\varepsilon_1,\varepsilon_2}) = e_{\varepsilon_1}$. This proves (iii). □

Since $(e_{\varepsilon_1}, \varepsilon_2) \in C_k$ is a double point of the special fiber of C, but is regular on C, the exceptional divisor $C_{\varepsilon_1,\varepsilon_2}$ has multiplicity 2 in $[C'_k]$. Thus we have:

$$[C'_k] = [E] + [B_+] + [B_-] + 2\Sigma[C_{\varepsilon_1,\varepsilon_2}].$$

Applying Proposition 3.5 and the description of the map $\mathrm{CH}^1(\tilde{X}_k) \to \bar{\Sigma}$ given in Proposition 2.1, we get:

PROPOSITION 3.6. *The specialization of* $[C_K] \in \mathrm{CH}^2(X_K)$ *in* $\bar{\Sigma}$ *is given by*:

$$\begin{aligned} \bar{\sigma}([C_K]) = 2&([e_+] + [e_-]) \otimes (u_1 + u_2) \\ &+ 2[e_+] \otimes (l_{++} + l_{--}) \\ &+ (4[e_+] - 2[e_-]) \otimes (l_{+-} + l_{-+}) \\ &- 4([e_+] - [e_-]) \otimes v_1. \end{aligned}$$

□

Now applying the action of the inversion map ι^* on $\bar{\Sigma}$, as given in Proposition 2.2, we get:

THEOREM 3.7.

$$\bar{\sigma}([C_K]-[\iota C_K])=2([e_+]-[e_-])\otimes(l_{++}+l_{--}+3(l_{+-}+l_{-+})-4v_1)$$

in $\bar{\Sigma}$. *In particular, this is nonzero if and only if* $2([e_+]-[e_-])\neq 0$ *in* $\mathrm{CH}^1(E_{1k})$. □

If we take the group structure on E_1 with the point at infinity as the identity then $e_+=-e_-$. Thus if we identify $E_1(k)$ with $\mathrm{Pic}^0(E_{1k})$ in the usual way, then:

$$2([e_+]-[e_-])=4e_+\in E_1(k).$$

COROLLARY 3.8. *Suppose* $16e_+\neq 0$. *Let* $\mathrm{CH}^2_{\mathrm{trans}}(X_K)\subset \mathrm{CH}^2_{\mathrm{hom}}(X_K)$ *be the subgroup generated by cycles of the following forms*:

$$[E_{1K}\times\{x_2\}\times\{x_3\}]-[E_{1K}\times\{x_2'\}\times\{x'\}]$$
$$[\{x_1\}\times E_{2K}\times\{x_3\}]-[\{x_1'\}\times E_{2K}\times\{x_3'\}]$$
$$[\{x_1\}\times\{x_2\}\times E_{3K}]-[\{x_1'\}\times\{x_2'\}\times E_{3K}]$$

with $x_i,\ x_i'\in E_i(K)$. *Then* $\bar{\sigma}([C_K]-[\iota C_K])\notin\bar{\sigma}(\mathrm{CH}^2_{\mathrm{trans}}(X_K))$.

Proof. If x is a K-point of E_i for $i=2$ or 3, then its closure (over R) will meet the special fiber at a smooth point, that is, it will not pass through either of the points $+$ or $-$ in E_{ik}. This implies that the closure of any of the above cycles will not meet the components Q_{++}, Q_{+-}, Q_{-+}, or Q_{--} of X_k. Therefore, for any $\Psi\in\mathrm{CH}^2_{\mathrm{trans}}(X_K)$, the coefficient of v_1 in $\bar{\sigma}(\Psi)$ is zero. But the coefficient of v_1 in $\bar{\sigma}([C_K]-[\iota C_K])$ is $-16e_+\neq 0$. □

REMARK. If C_K is hyperelliptic, then ιC_K is a translate of C_K, so one might expect that $\sigma([C_K]-[\iota C_K])=0$. It can be shown that C_K is hyperelliptic if and only if $\alpha\beta=(\alpha+\beta-\pi)\gamma$. This condition does in fact imply that e_+ is a point of order 4 on E_{1k}

4. Algebraic equivalence

We study the map

$$\varphi\colon \mathrm{CH}^2_{\mathrm{hom}}(X_K)\to H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2))),$$

introduced in [3]. See [14] for a precise definition of this map. It arises by combining the cycle class map $\mathrm{CH}^2(X_K) \to H^4(X_K, \mathbb{Z}_l(2))$ with projection onto one of the graded pieces in the Hochschild–Serre spectral sequence:

$$E_2^{p,4-p} = H^p(K, H^{4-p}(X_{\bar{K}}, \mathbb{Z}_l(2))) \Rightarrow H^4(X_K, \mathbb{Z}_l(2)).$$

LEMMA 4.1. *Let V be a variety of dimension d over K, and $\Gamma \in \mathrm{CH}^2(V \times X_K)$ a correspondence defined over K. Let*

$$\mathrm{CH}^2_\Gamma(X_K) := \Gamma_* \mathrm{CH}^2_{\mathrm{hom}}(V) \subset \mathrm{CH}^2_{\mathrm{hom}}(X_K)$$

and

$$M_\Gamma := \Gamma_* H^{2d-1}(V_{\bar{K}}, \mathbb{Z}_l(d)) \subset H^3(X_{\bar{K}}, \mathbb{Z}_l(2)).$$

Then φ restricted to $\mathrm{CH}^2_\Gamma(X_K)$ factors:

$$\begin{tikzcd} \mathrm{CH}^2_\Gamma(X_K) \arrow[rr, "\varphi"] \arrow[dr] & & H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2))) \\ & H^1(K, M_\Gamma) \arrow[ur] & \end{tikzcd}$$

Proof. This follows from functoriality of the usual cycle class, and the resulting commutative square:

$$\begin{tikzcd} \mathrm{CH}^d_{\mathrm{hom}}(V) \arrow[r, "\Gamma_*"] \arrow[d, "\varphi"'] & \mathrm{CH}^2_\Gamma(X_K) \arrow[d, "\varphi"] \\ H^1(K, H^{2d-1}(V_{\bar{K}}, \mathbb{Z}_l(d))) \arrow[r, "H^1(K,\Gamma_*)"] & H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2))) \end{tikzcd}$$

□

DEFINITION 4.2. Let S be the set of all pairs (V, Γ), where V is a variety over K, and $\Gamma \in \mathrm{CH}^2(V \times X_K)$ is a correspondence defined over K. Then define $\mathrm{CH}^2_{\mathrm{alg}}(X_K)$ to be the subgroup of $\mathrm{CH}^2(X_K)$ generated by all $\Gamma_* \mathrm{CH}^{\dim V}_{\mathrm{hom}}(V)$, as (V, Γ) ranges over S. Similarly, define M_{alg} to be the subgroup of $H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$ generated by all $\Gamma_* H^{2\dim V - 1}(V_{\bar{K}}, \mathbb{Z}_l(\dim V))$, as (V, Γ) ranges over S. □

REMARK. Note that $\mathrm{CH}^2_{\mathrm{alg}}(X_K)$ may be smaller than the subgroup of cycles defined over K which are algebraically equivalent to zero over $\bar{K}$. However, these two groups will agree up to torsion, so the smaller group will suffice for our purposes. This question is discussed further in Section 6. □

COROLLARY 4.3. *The map* φ *restricted to* $\mathrm{CH}^2_{\mathrm{alg}}(X_K)$ *factors through* $H^1(K, M_{\mathrm{alg}})$. □

Recall that $\mathrm{CH}^2_{\mathrm{trans}}(X_K)$ is generated by differences of translates of an "axis" of the product $E_1 \times E_2 \times E_3$. This can also be described as the image of a certain correspondence from a variety $A \cong X^2$ to X. Specifically, let $A_1 = E_2 \times E_3$, $A_2 = E_1 \times E_3$, $A_3 = E_1 \times E_2$, and take $A = A_1 \times A_2 \times A_3$. Since $X \cong E_i \times A_i$, there are isomorphisms:

$$\tau_1 : E_1 \times (A_1 \times A_1) \times A_2 \times A_3 \to A \times X$$

$$\tau_2 : E_2 \times (A_2 \times A_2) \times A_1 \times A_3 \to A \times X$$

$$\tau_3 : E_3 \times (A_3 \times A_3) \times A_1 \times A_2 \to A \times X$$

Consider the cycle:

$$\Gamma := \tau_1(E_1 \times \Delta_{A_1} \times A_2 \times A_3) + \tau_2(E_2 \times \Delta_{A_2} \times A_1 \times A_3) + \tau_3(E_3 \times \Delta_{A_3} \times A_1 \times A_2).$$

Here Δ_V denotes the diagonal in $V \times V$. If (a_i, b_i) is a point of A_i, then:

$$\Gamma_*(a_1, b_1, a_2, b_2, a_3, b_3) = E_1 \times a_1 \times b_1 + a_2 \times E_2 \times b_2 + a_3 \times b_3 \times E_3.$$

Hence if we let $T \subset \mathrm{CH}^6_{\mathrm{hom}}(A)$ be the subgroup of cycles supported on K-points, then $\Gamma_*(T) = \mathrm{CH}^2_{\mathrm{trans}}(X_K)$.

LEMMA 4.4. *With* A *and* Γ *as above, the map*:

$$\Gamma_* : H^{11}(A_{\bar{K}}, \mathbb{Z}_l(6)) \to H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$$

is injective, and its image is $\bigoplus_{i \neq j} H^1(E_i, \mathbb{Z}_l(1)) \otimes H^2(E_j, \mathbb{Z}_l(1))$ *in the Künneth decomposition for* $H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$.

Proof. That the image is as stated is a straightforward computation. Injectivity then follows by comparing ranks. □

PROPOSITION 4.5. *Suppose*

$$M_{\mathrm{alg}} = \bigoplus_{i \neq j} H^1(E_i, \mathbb{Z}_l(1)) \otimes H^2(E_j, \mathbb{Z}_l(1)).$$

Then

$$\varphi(\mathrm{CH}^2_{\mathrm{trans}}(X_K)) = \varphi(\mathrm{CH}^2_{\mathrm{alg}}(X_K)) = H^1(K, M_{\mathrm{alg}}).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
T & \xrightarrow[\Gamma_*]{} & \mathrm{CH}^2_{\mathrm{trans}}(X_K) \\
\cap & & \cap \\
\mathrm{CH}^6_{\mathrm{hom}}(A) & \xrightarrow[\Gamma_*]{} & \mathrm{CH}^2_{\mathrm{alg}}(X_K) \\
\Big\downarrow \varphi & & \Big\downarrow \varphi \\
H^1(K, H^{11}(A_{\bar{K}}, \mathbb{Z}_l(6))) & \xrightarrow[\Gamma_*]{\cong} & H^1(K, M_{\mathrm{alg}}) \to H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2)))
\end{array}$$

where, as above, T is the subgroup of cycles supported on $A(K)$. From this, we see that it suffices to show that the map $T \to H^1(K, H^{11}(A_{\bar{K}}, \mathbb{Z}_l(6)))$ is surjective. If E is one of the elliptic factors of A, then the inclusion $i: E \hookrightarrow A$ (with 0 in the other components) induces $i_*: \mathrm{Pic}^0(E_K) \to T \subset \mathrm{CH}^6_{\mathrm{hom}}(A_K)$ and also $i_*: H^1(K, H^1(E_{\bar{K}}, \mathbb{Z}_l(1))) \to H^1(K, H^{11}(A_{\bar{K}}, \mathbb{Z}_l(6)))$. But $H^1(K, H^{11}(A_{\bar{K}}, \mathbb{Z}_l(6)))$ is the direct sum of the images of these i_*, as E ranges over all the factors of A. Thus it suffices to show $\varphi: \mathrm{Pic}^0(E_K) \to H^1(K, H^1(E_{\bar{K}}, \mathbb{Z}_l(1)))$ surjective. φ comes by inverse limit from a map $\mathrm{Pic}^0(E_K) \to H^1(K, H^1(E_{\bar{K}}, \boldsymbol{\mu}_{l^n}))$. We have $\mathrm{Pic}^0(E_K) = E(K)$ and $H^1(E_{\bar{K}}, \boldsymbol{\mu}_{l^n}) = {}_{l^n}E(\bar{K})$ (= kernel of multiplication by l^n), and φ can be identified with the connecting homomorphism in the Galois cohomology of the sequence:

$$0 \to {}_{l^n}E(\bar{K}) \to E(\bar{K}) \xrightarrow{l^n} E(\bar{K}) \to 0.$$

Specifically, the long exact sequence for Galois cohomology gives:

$$0 \to E(K)/l^nE(K) \to H^1(K, {}_{l^n}E(\bar{K})) \to {}_{l^n}H^1(K, E(\bar{K})) \to 0.$$

Since K is a local field, it is well-known that $E(K) \cong R \oplus F$, where R is the ring of integers in K, and F is a finite group. Therefore, since $l \in R^\times$, we get $E(K)/l^nE(K) \cong F[l]$ (:= l-primary part of F) for n sufficiently large.

By [16], $H^1(K, E(\bar{K})) \cong \mathrm{Hom}(E(K), \mathbb{Q}/\mathbb{Z})$, so:

$$\begin{aligned}
{}_{l^n}H^1(K, E(\bar{K})) &\cong \mathrm{Hom}(R \oplus F, \mathbb{Z}/(l^n)) \\
&\cong \mathrm{Hom}(F[l], \mathbb{Z}/(l^n)) \\
&\cong F[l]
\end{aligned}$$

for n sufficiently large. Therefore the exact sequence above gives the following short exact sequence of inverse systems:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \downarrow\cdot l & & \downarrow\cdot l & & \\
0 & \to & F[l] & \to & H^1(K, {}_{l^{n+1}}E(\bar{K})) & \to & F[l] & \to & 0 \\
 & & \| & & \downarrow\cdot l & & \downarrow\cdot l & & \\
0 & \to & F[l] & \to & H^1(K, {}_{l^n}E(\bar{K})) & \to & F[l] & \to & 0 \\
 & & \| & & \downarrow\cdot l & & \downarrow\cdot l & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

for $n \gg 0$. Since the maps in the inverse system on the left are surjective, we get a short exact sequence of inverse limits:

$$0 \to F[l] \to \varprojlim_n H^1(K, {}_{l^n}E(\bar{K})) \to T_l(F[l]) \to 0.$$

Since $F[l]$ is finite, no element (except zero) is infinitely divisible by l, so $T_l(F[l]) = 0$. This completes the proof. □

Recall that $\mathscr{K}$ is our global field, with ring of integers $\mathscr{R}$. We show that condition (iii) of Theorem 2 implies that the situation is indeed as described above.

LEMMA 4.6. *Let*

$$M_{\mathscr{K}} \subset H^3(X_{\bar{\mathscr{K}}}, \mathbb{Z}_l(2)) = H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$$

be the $\mathbb{Z}_l$-submodule generated by images $\Gamma_* H^{2d-1}(V_{\bar{K}}, \mathbb{Z}_l(d))$ with Γ and V defined over $\mathscr{K}$ and $d = \dim V$. If $M_{\mathscr{K}}$ is a direct summand of $H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$, then $M_{\mathscr{K}} = M_{\mathrm{alg}}$.

Proof. Clearly $M_{\mathscr{K}} \subset M_{\mathrm{alg}}$. In fact, $M_{\mathrm{alg}}/M_{\mathscr{K}}$ is finite, since any correspondence over K can be specialized to give a correspondence defined over a finite extension of $\mathscr{K}$. Then the result follows from the observation that $H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$ is torsion-free. □

PROPOSITION 4.7. *Suppose there is a prime $\mathfrak{q} \in \operatorname{spec} \mathscr{R}$ such that $\mathfrak{q} \nmid l$, X has good reduction at $\mathfrak{q}$, and none of the products $\lambda_1\lambda_2\lambda_3$ is divisible by $N(\mathfrak{q})$ in the ring of algebraic integers. Here each λ_i is an eigenvalue of the geometric Frobenius map on $H^1(E_{i\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$. Then*

$$M_{\mathrm{alg}} = \bigoplus_{i \neq j} H^2(E_{i\bar{K}}) \otimes H^1(E_{j\bar{K}}),$$

and hence

$$\varphi(\mathrm{CH}^2_{\mathrm{trans}}(X_K)) = \varphi(\mathrm{CH}^2_{\mathrm{alg}}(X_K)) = H^1(K, M_{\mathrm{alg}}). \qquad \square$$

Proof. Since X has good reduction at $\mathfrak{q}$, we have $H^3(X_{\bar{K}}) = H^3(X_{\bar{\mathscr{K}}}) = H^3(X_{\overline{k(\mathfrak{q})}})$. Moreover, M_{alg} is generated by images of correspondences over $k(\mathfrak{q})$, because a correspondence $\Gamma\colon V_{\mathscr{K}} \rightsquigarrow X_{\mathscr{K}}$ can be extended over $\mathscr{R}$, and then reduced mod $\mathfrak{q}$ to give a correspondence over $k(\mathfrak{q})$. This works even if V has bad reduction, because the transposed correspondence $\Gamma'\colon X_{k(\mathfrak{q})} \rightsquigarrow V_{k(\mathfrak{q})}$ always induces a map $\Gamma'_*\colon H^3(X) \to H^1(V)$, since by the smoothness of X, the direct image $H^n(V \times X_{\overline{k(\mathfrak{q})}}) \to H^{n-6}(V_{\overline{k(\mathfrak{q})}})$ can be defined as zero on $H^{n-i}(V) \otimes H^i(X)$ for $i \neq 0$, and by the trace map on $H^6(X_{\overline{k(\mathfrak{q})}})$ for $i = 6$; then $\Gamma_*\colon H^1(V_{\overline{k(\mathfrak{q})}})^* \to H^3(X_{\overline{k(\mathfrak{q})}})$ can be taken to be the dual of Γ'_*.

Given a correspondence $\Gamma \in \mathrm{CH}^2(V \times X_{k(\mathfrak{q})})$, the map

$$H^{2d-1}(V_{\overline{k(\mathfrak{q})}}, \mathbb{Q}_l(d)) \to H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Q}_l(2))$$

is $\mathrm{Gal}(\overline{k(\mathfrak{q})}/k(\mathfrak{q}))$-invariant, where $d = \dim(V)$. Thus if $\mathfrak{f}_g$ is the geometric Frobenius map on either V or X, then the map

$$\Gamma_*\colon H^{2d-1}(V_{\overline{k(\mathfrak{q})}}, \mathbb{Q}_l) \to H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$$

satisfies $\mathfrak{f}_g \circ \Gamma_* = N(\mathfrak{q})^{2-d} \cdot \Gamma_* \circ \mathfrak{f}_g$. We can always take V to be a (possibly singular) curve, so we find that the eigenvalues of $\mathfrak{f}_g$ on

$$M_{\mathrm{alg}}(-2) \subset H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$$

must be divisible by $N(\mathfrak{q})$ in the ring of algebraic integers.

The conclusion of the proposition is equivalent to

$$M_{\mathrm{alg}} \cap H^1(E_{1\bar{K}}) \otimes H^1(E_{2\bar{K}}) \otimes H^1(E_{3\bar{K}}) = 0.$$

Therefore it suffices to find a good-reduction prime $\mathfrak{q}$ for which none of the eigenvalues of the geometric Frobenius automorphism on

$$H^1(E_{1\overline{k(\mathfrak{q})}}, \mathbb{Q}_l) \otimes H^1(E_{2\overline{k(\mathfrak{q})}}, \mathbb{Q}_l) \otimes H^1(E_{3\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$$

is divisible by $N(\mathfrak{q})$. These eigenvalues are products $\lambda_1\lambda_2\lambda_3$ of eigenvalues on $H^1(E_{i\overline{k(\mathfrak{q})}}, \mathbb{Q}_l)$, for $i = 1, 2, 3$, so the proposition is proved. $\square$

5. Specialization in étale cohomology

In this section, we let $Y := (X_k)_{\mathrm{red}}$ be the special fiber with its reduced structure and $U := X_K$ be the generic fiber. Note that $Y \subset X$ is a divisor with normal crossings. Let $Y_1, \ldots, Y_8$ be the components of Y.

Given a prime number $l \neq \mathrm{char}(k)$, there is a long exact sequence in étale cohomology:

$$\cdots \to H^i_Y(X, \mathbb{Z}_l(2)) \to H^i(X, \mathbb{Z}_l(2)) \to H^i(U, \mathbb{Z}_l(2)) \to H^{i+1}_Y(X, \mathbb{Z}_l(2)) \to \cdots$$

Denote

$$H^4(U, \mathbb{Z}_l(2))_0 := \ker(H^4(U, \mathbb{Z}_l(2)) \to H^5_Y(X, \mathbb{Z}_l(2))).$$

Then there is the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^4_Y(X, \mathbb{Z}_l(2)) \arrow[r] \arrow[d, equal] & H^4(X, \mathbb{Z}_l(2)) \arrow[r] \arrow[d] & H^4(U, \mathbb{Z}_l(2))_0 \arrow[r] \arrow[d] & 0 \\
H^4_Y(X, \mathbb{Z}_l(2)) \arrow[r] & \bigoplus_{i=1}^{8} H^4(Y_i, \mathbb{Z}_l(2)) \arrow[r] & \Sigma_h \arrow[r] & 0
\end{tikzcd}$$

Here the vertical map in the middle is defined by pulling back to each component, and Σ_h is defined to be the cokernel of the first map in the bottom row.

As in Section 1, we write Σ for $\mathrm{coker}(\mathrm{CH}^1(Y) \to \bigoplus_i \mathrm{CH}^2(Y_i))$.

PROPOSITION 5.1. *There is a commutative cube*:

$$\begin{tikzcd}
& \mathrm{CH}^2(X)\otimes\mathbb{Z}_\ell \arrow[rr] \arrow[dl, "\mathrm{cl}_X"'] \arrow[dd] & & \mathrm{CH}^2(U)\otimes\mathbb{Z}_\ell \arrow[dl, "\mathrm{cl}_U"'] \arrow[dd, "\sigma"] \\
H^4(X,\mathbb{Z}_\ell(2)) \arrow[rr] \arrow[dd] & & H^4(U,\mathbb{Z}_\ell(2))_0 \arrow[dd, "\sigma"] & \\
& \bigoplus_i \mathrm{CH}^2(Y_i)\otimes\mathbb{Z}_\ell \arrow[rr] \arrow[dl, "\oplus \mathrm{cl}_{Y_i}"] & & \Sigma\otimes\mathbb{Z}_\ell \arrow[dl, "\mathrm{cl}_\Sigma"] \\
\bigoplus_i H^4(Y_i,\mathbb{Z}_\ell(2)) \arrow[rr] & & \Sigma_h &
\end{tikzcd}$$

Proof. The front and back faces commute by definition. The cycle class maps can be defined using the K-theoretical description of the Chow groups [4]. Or, since we are going to assume purity for étale cohomology in any case, one can define these maps as in [9]. Then the top and left faces commute by functoriality of the cycle classes, except that we must check that $\mathrm{cl}_U\colon \mathrm{CH}^2(U)\to H^4(U,\mathbb{Z}_l(2))$ really does factor through $H^4(U,\mathbb{Z}_l(2))_0$. This follows from surjectivity of $\mathrm{CH}^2(X)\to\mathrm{CH}^2(U)$ and exactness of the sequence:

$$H^4(X,\mathbb{Z}_l(2))\to H^4(U,\mathbb{Z}_l(2))\to H^5_Y(X,\mathbb{Z}_l(2)).$$

Commutativity of the top face implies that:

$$\ker(\mathrm{CH}^2(X)\to\mathrm{CH}^2(U))\to\ker(H^4(X,\mathbb{Z}_l(2))\to H^4(U,\mathbb{Z}_l(2))).$$

Now the existence of cl_Σ making the cube commute follows, once we note that, by exactness of the localization sequences for CH^2 and H^4, the following two sequences are exact:

$$\ker(\mathrm{CH}^2(X)\to\mathrm{CH}^2(U))\to\bigoplus_i \mathrm{CH}^2(Y_i)\to\Sigma\to 0,$$

$$\ker(H^4(X,\mathbb{Z}_l(2))\to H^4(U,\mathbb{Z}_l(2)))\to\bigoplus_i H^4(Y_i,\mathbb{Z}_l(2))\to\Sigma_h\to 0. \qquad \square$$

To study the map cl_Σ, we first investigate $\otimes\,\mathrm{cl}_{Y_i}$. Recall that each $Y_i\cong E_{1k}\times S_i$ for some rational surface S_i over k.

PROPOSITION 5.2. *Let k be a finite field and E a smooth projective curve over k. Let $l\neq\mathrm{char}(k)$ be a prime. Then*

$$\mathrm{Pic}^0(E)\otimes\mathbb{Z}_l\cong H^1(k,\mathrm{Pic}^0(E)\otimes\mathbb{Z}_l\cong H^1(k,H^1(E_{\bar{k}},\mathbb{Z}_l(1)).$$

Proof. From the Kummer sequence $0\to\mu_{l^n}\to\mathbb{G}_m\xrightarrow{l^n}\mathbb{G}_m\to 0$, it follows that

$$H^1(E_{\bar{k}},\mu_{l^n})\cong {}_{l^n}H^1(E_{\bar{k}},\mathbb{G}_m)\cong {}_{l^n}\mathrm{Pic}(E_{\bar{k}})={}_{l^n}\mathrm{Pic}^0(E_{\bar{k}}).$$

Taking inverse limits, one finds that $H^1(E_{\bar{k}},\mathbb{Z}_l(1))\cong T_l\,\mathrm{Pic}^0(E_{\bar{k}})$ and this is an isomorphism of $\mathrm{Gal}(\bar{k}/k)$-modules. Now consider the short exact sequence of Galois modules:

$$0\to T_l\,\mathrm{Pic}^0(E_{\bar{k}})\xrightarrow{\quad l^n\quad} T_l\,\mathrm{Pic}^0(E_{\bar{k}})\to {}_{l^n}\mathrm{Pic}^0(E_{\bar{k}})\to 0.$$

Taking cohomology, we get an exact sequence

$$T_l\,\mathrm{Pic}^0(E_k)\to {}_{l^n}\mathrm{Pic}^0(E_k)\to {}_{l^n}H^1(k,T_l\,\mathrm{Pic}^0(E_{\bar{k}}))\to 0.$$

Now $\mathrm{Pic}^0(E_k)=\{k\text{-points of }\mathrm{Pic}^0(E_{\bar{k}})\}$ is a finite group (since k is finite);

hence $T_l \operatorname{Pic}^0(E_k)=0$ and ${}_{l^n}\operatorname{Pic}^0(E_k)=\operatorname{Pic}^0(E_k)\otimes \mathbb{Z}_l$ for n sufficiently large. So all that remains is to show that $H^1(k, H^1(E_{\bar{k}}, \mathbb{Z}_l(1)))$ is torsion. Write $H^1(E_{\bar{k}}, \mathbb{Z}_l(1)) = M \oplus$ torsion, where M is a free $\mathbb{Z}_l$-module ($\cong \mathbb{Z}_l^{2g}$ if g = genus of $E_{\bar{k}}$). Then it suffices to show that $H^1(k, M)$ is torsion. From the exact sequence:

$$H^0(k, M \otimes \mathbb{Q}_l/\mathbb{Z}_l) \to H^1(k, M) \to H^1(k, M \otimes \mathbb{Q}_l)$$

and the fact that $H^0(k, M \otimes \mathbb{Q}_l/\mathbb{Z}_l)$ is torsion, we are reduced to showing that $H^1(k, M \otimes \mathbb{Q}_l) = 0$. Now since k is finite,

$$H^1(k, M \otimes \mathbb{Q}_l) \cong M \otimes \mathbb{Q}_l/(1 - F)M \otimes \mathbb{Q}_l$$

where $F \in \operatorname{Gal}(\bar{k}/k)$ is the Frobenius automorphism. But $M \otimes \mathbb{Q}_l \cong H^1(E_{\bar{k}}, \mathbb{Q}_l(1))$, so all eigenvalues of F on $M \otimes \mathbb{Q}_l$ have absolute value $q^{1/2} \neq 1$ (here $q := \#(k)$). Therefore $(1-F)$ is injective, hence also surjective. □

PROPOSITION 5.3. *Let k be a finite field, $l \neq \operatorname{char}(k)$ a prime. Let E be a complete smooth curve over k, such that $E(k) \neq \varnothing$ and $H^r(E_{\bar{k}}, \mathbb{Z}_l)$ is torsion-free, for all r. Let S be a complete smooth surface over k which is k-birational to $\mathbb{P}^2$, such that* $\operatorname{Gal}(\bar{k}/k)$ *acts trivially on* $\operatorname{CH}^1(S_{\bar{k}})$. *Then for each r, the cycle class map* $\operatorname{cl}: \operatorname{CH}^r(E \times S) \otimes \mathbb{Z}_l \to H^{2r}(E \times S, \mathbb{Z}_l(r))$ *is an isomorphism.*

Proof. We have

$$\operatorname{CH}^r(E \times S) \cong \bigoplus_{s+t=r} \operatorname{CH}^s(E) \otimes \operatorname{CH}^t(S).$$

Since $H^{\cdot}(E_{\bar{k}}, \mathbb{Z}_l)$ and $H^{\cdot}(S_{\bar{k}}, \mathbb{Z}_l)$ have no torsion and $H^t(S_{\bar{k}}, \mathbb{Z}_l) = 0$ if t is odd, we have also,

$$H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r)) \cong \bigoplus_{s+t=r} H^{2s}(E_{\bar{k}}, \mathbb{Z}_l(s)) \otimes H^{2t}(S_{\bar{k}}, \mathbb{Z}_l(t)).$$

The map $\operatorname{CH}^r(E \times S) \to H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))$ is compatible with these decompositions. Furthermore,

$$\operatorname{CH}^t(S) \otimes \mathbb{Z}_l \to H^{2t}(S_{\bar{k}}, \mathbb{Z}_l(t)) \quad \text{and} \quad \operatorname{CH}^0(E) \otimes \mathbb{Z}_l \to H^0(E_{\bar{k}}, \mathbb{Z}_l)$$

are isomorphisms for all t and $\operatorname{CH}^1(E) \otimes \mathbb{Z}_l \to H^2(E_{\bar{k}}, \mathbb{Z}_l(1))$ is surjective since E has a k-point. This proves that $\operatorname{CH}^r(E \times S) \otimes \mathbb{Z}_l \to H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))$ is surjective and therefore the Galois action on $H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))$ is trivial. Now,

$$\begin{aligned}\ker(\operatorname{CH}^r(E \times S) \to H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))) &= \operatorname{CH}^r_{\mathrm{hom}}(E \times S)\\ &= \operatorname{Pic}^0(E) \otimes \operatorname{CH}^{r-1}(S)\end{aligned}$$

From the Hochschild–Serre spectral sequence, using $\mathrm{cd}_l(k) = 1$, we get that

$$\ker(H^{2r}(E \times S, \mathbb{Z}_l(r)) \to H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))) = H^1(k, H^{2r-1}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))).$$

Putting all this together gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to \mathrm{Pic}^0(E) \otimes \mathrm{CH}^{r-1}(S) & \to & \mathrm{CH}^r(E \times S) & \to & H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r)) & \to & 0 \\ \downarrow & & \downarrow & & \| & & \\ 0 \to H^1(k, H^{2r-1}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))) & \to & H^{2r}(E \times S, \mathbb{Z}_l(r)) & \to & H^0(k, H^{2r}(E \times S_{\bar{k}}, \mathbb{Z}_l(r-1)) & \to & 0 \end{array}$$

Now

$$H^{2r-1}(E \times S_{\bar{k}}, \mathbb{Z}_l(r)) \cong H^1(E_{\bar{k}}, \mathbb{Z}_l(1)) \otimes H^{2r-2}(S_{\bar{k}}, \mathbb{Z}_l(r-1)),$$

and $H^{2r-2}(S_{\bar{k}}, \mathbb{Z}_l(r-1)) \cong \mathrm{CH}^{r-1}(S) \otimes \mathbb{Z}_l$ is a free $\mathbb{Z}_l$-module with trivial Galois action. Therefore

$$H^1(k, H^{2r-1}(E \times S_{\bar{k}}, \mathbb{Z}_l(r)) \cong H^1(k, H^1(E_{\bar{k}}, \mathbb{Z}_l(1))) \otimes \mathrm{CH}^{r-1}(S).$$

From Proposition 5.2 it now follows that:

$$\mathrm{Pic}^0(E) \otimes \mathrm{CH}^{r-1}(S) \otimes \mathbb{Z}_l \xrightarrow{\cong} H^1(k, H^{2r-1}(E \times S_{\bar{k}}, \mathbb{Z}_l(r))),$$

which proves the proposition. □

COROLLARY 5.4. *The cycle class map*:

$$\bigoplus_i \mathrm{cl}_i \colon \bigoplus_i \mathrm{CH}^2(Y_i) \to \bigoplus_i H^4(Y_i, \mathbb{Z}_l(2))$$

is an isomorphism.

Proof. Each $Y_i \cong E_{1k} \times S_i$, where S_i is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_{4\mathrm{pts.}}(\mathbb{P}^1 \times \mathbb{P}^1)$. Since the branch points of π_1 were chosen to be defined over R, E_{1k} has k-points. Since the four points being blown up on $\mathbb{P}^1 \times \mathbb{P}^1$ are defined over k, the Galois action on $\mathrm{CH}^1(S_i)$ is trivial. Therefore Proposition 5.3 applies. □

To study $H^4_Y(X, \mathbb{Z}_l(2))$, we use the spectral sequence:

$$E_2^{p,q} = H^p(Y, \mathbf{H}^q_Y(X, \mathbb{Z}_l(2))) \Rightarrow H^{p+q}_Y(X, \mathbb{Z}_l(2)), \tag{5.1}$$

where $\mathbf{H}^q_Y$ denotes the local cohomology sheaf, with support in Y. For $q \geqslant 2$, this is isomorphic (after a shift in degrees) to the Leray spectral sequence for the inclusion map $j \colon U \hookrightarrow X$. To analyze this spectral sequence, we need the following conjecture of Grothendieck's, which is known for schemes of equal characteristics and for arbitrary schemes of relative dimension one over a discrete valuation ring [8], but not for schemes of unequal characteristics of higher dimension. Thus the results of this section are unqualified if $\mathscr{K}$ is a function field, but rely on a conjecture if $\mathscr{K}$ is a number field.

CONJECTURE 5.5 [Purity]. *Let V be a regular scheme, $D \subset V$ a regular Cartier divisor, $j\colon (V \backslash D) \hookrightarrow V$ the inclusion map and $l \notin \mathrm{char}(V)$ a prime. Then $R^q j_*(\mathbb{Z}_l) = 0$ for $q \geqslant 2$.* □

The following is well known (see [1] and [9]):

PROPOSITION 5.6. *Let V be a regular scheme and assume $D \subset V$ is a Cartier divisor with normal crossings, with components $D_1, \ldots, D_s$ and $j\colon (V \backslash D) \hookrightarrow V$ the inclusion. Then*:

(i) *The adjunction map $\mathbb{Z}_l \to j_* j^*(\mathbb{Z}_l)$ is an isomorphism.*
(ii) $R^1 j_*(\mathbb{Z}_l) \cong \bigoplus_{i=1}^s \mathbb{Z}_l(-1)_{D_i}$.
(iii) *Assuming purity, the cup product gives an isomorphism*

$$\wedge^q R^1 j_*(\mathbb{Z}_l) \xrightarrow{\cong} R^q j_*(\mathbb{Z}_l).$$ □

COROLLARY 5.7. *Let $\{Y_i^{(q)}\}_i$ be the components of all intersections of q distinct components of Y. Then*

$$\mathbf{H}_Y^q(X, \mathbb{Z}_l(2)) \cong \begin{cases} 0 & \text{if } q = 0 \text{ or } 1 \\ \bigoplus_i \mathbb{Z}_l(1)_{Y_i} & \text{if } q = 2 \\ \bigoplus_i \mathbb{Z}_{l\,Y_i^{(2)}} & \text{if } q = 3 \\ \bigoplus_i \mathbb{Z}_l(-1)_{Y_i^{(3)}} & \text{if } q = 4 \\ 0 & \text{if } q \geqslant 5 \end{cases}$$

Proof. For $q = 0$ or 1, see [13]. For $q \geqslant 2$, there is an isomorphism $R^{q-1} j_*(j^* \mathscr{F}) \xrightarrow{\cong} \mathbf{H}_Y^q(X, \mathscr{F})$ for any sheaf $\mathscr{F}$. Taking $\mathscr{F} = \mathbb{Z}_l(2)$, we get:

$$\begin{aligned} \mathbf{H}_Y^q(X, \mathbb{Z}_l(2)) &\cong R^{q-1} j_*(\mathbb{Z}_l(2)) \\ &\cong R^{q-1} j_*(\mathbb{Z}_l) \otimes \mathbb{Z}_l(2) \\ &\cong \wedge^{q-1}\left(\bigoplus_i \mathbb{Z}_l(-1)_{Y_i}\right) \otimes \mathbb{Z}_l(2) \\ &\cong \bigoplus_{i_1 < \cdots < i_{q-1}} (\mathbb{Z}_l(-1)_{Y_{i_1}} \otimes \cdots \otimes \mathbb{Z}_l(-1)_{Y_{i_{q-1}}}) \otimes \mathbb{Z}_l(2) \\ &\cong \bigoplus_{i_1 < \cdots < i_{q-1}} \mathbb{Z}_l(3-q)_{Y_{i_1} \cap \cdots \cap Y_{i_{q-1}}}. \end{aligned}$$

But for every $i_1 < \cdots < i_{q-1}$, the components of $Y_{i_1} \cap \cdots \cap Y_{i_{q-1}}$ are disjoint. Therefore

$$\bigoplus \mathbb{Z}_l(3-q)_{Y_{i_1} \cap \cdots \cap Y_{i_{q-1}}} \cong \bigoplus_i \mathbb{Z}_l(3-q)_{Y_i^{(q-1)}}.$$

Note also that for any $i_1 < i_2 < i_3 < i_4$, we have $Y_{i_1} \cap Y_{i_2} \cap Y_{i_3} \cap Y_{i_4} = \varnothing$. This gives the stated result. □

We now analyze the spectral sequence (5.1). For $p+q=4$, we have the following:

$$E_2^{0,4} = H^0\left(Y, \bigoplus_i \mathbb{Z}_l(-1)_{Y_i^{(3)}}\right) = \bigoplus_i H^0(Y_i^{(3)}, \mathbb{Z}_l(-1)) = 0$$

$$E_2^{1,3} = H^1\left(Y, \bigoplus_i \mathbb{Z}_{l\,Y_i^{(2)}}\right) = \bigoplus_i H^1(Y_i^{(2)}, \mathbb{Z}_l)$$

$$E_2^{2,2} = H^2\left(Y, \bigoplus_i \mathbb{Z}_l(1)_{Y_i}\right) = \bigoplus_i H^2(Y_i, \mathbb{Z}_l(1))$$

$$E_2^{3,1} = E_2^{4,0} = 0$$

Therefore we get a short exact sequence:

$$0 \to E_\infty^{2,2} \to H_Y^4(X, \mathbb{Z}_l(2)) \to E_\infty^{1,3} \to 0.$$

For $E_2^{1,3}$, the only nonzero differential is $d_2^{1,3}\colon E_2^{1,3} \to E_2^{3,2}$. Now

$$E_2^{1,3} = \bigoplus H^1(Y_i^{(2)}, \mathbb{Z}_l) \quad \text{and} \quad E_2^{3,2} = \bigoplus H^3(Y_i, \mathbb{Z}_l(1)).$$

From the Hochschild–Serre spectral sequence, we have:

$$0 \to H^0(k, H^1((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)) \to H^1(Y_i^{(2)}, \mathbb{Z}_l) \to H^1(k, H^0((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)) \to 0.$$

Now $H^0(k, H^1((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)) = 0$. Since all the $Y_i^{(2)}$ are defined over k, the Galois action on $H^0((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)$ is trivial. Therefore:

$$\begin{aligned} H^1(k, H^0((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)) &= H^0((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l)_{\mathrm{Gal}(\bar{k}/k)} \\ &= H^0((Y_i^{(2)})_{\bar{k}}, \mathbb{Z}_l) \\ &= H^0(Y_i^{(2)}, \mathbb{Z}_l) \\ &= \mathrm{CH}^0(Y_i^{(2)}) \otimes \mathbb{Z}_l. \end{aligned}$$

Thus $E_2^{1,3} \cong \bigoplus_i \mathrm{CH}^0(Y_i^{(2)}) \otimes \mathbb{Z}_l$.

Similarly,

$$H^3(Y_i, \mathbb{Z}_l(1)) \cong H^1(k, H^2((Y_i)_{\bar{k}}, \mathbb{Z}_l(1))).$$

By Proposition 5.3, $H^2((Y_i)_{\bar{k}}, \mathbb{Z}_l(1))$ is generated by cycles defined over k. Therefore the Galois action is again trivial, and so

$$\begin{aligned} H^1(k, H^2((Y_i)_{\bar{k}}, \mathbb{Z}_l(1))) &\cong H^2((Y_i)_{\bar{k}}, \mathbb{Z}_l(1)) \\ &\cong \mathrm{NS}(Y_i) \otimes \mathbb{Z}_l. \end{aligned}$$

Thus $E_2^{3,2} \cong \bigoplus_i \mathrm{NS}(Y_i) \otimes \mathbb{Z}_l$.

PROPOSITION 5.8. *The following diagram commutes*:

$$\begin{array}{ccc} E_2^{1,3} & \xrightarrow{-d_2} & E_2^{3,2} \\ \wr\Vert & & \wr\Vert \\ \bigoplus_i \mathrm{CH}^0(Y_i^{(2)}) \otimes \mathbb{Z}_l & \longrightarrow & \bigoplus_i \mathrm{NS}^1(Y_i) \otimes \mathbb{Z}_l \\ [Y_{i_0}^{(2)}] & \longmapsto & \sum_{Y_j \supset Y_{i_0}^{(2)}} [Y_{i_0}^{(2)}]_{Y_j} \end{array}$$

Specifically, if $Y_{i_0}^{(2)} \subset Y_{j_1} \cap Y_{j_2}$, *then*:

$$[Y_{i_0}^{(2)}] \mapsto [Y_{i_0}^{(2)}]_{Y_{j_1}} + [Y_{i_0}^{(2)}]_{Y_{j_2}}.$$

Proof. Let $\tilde{E}^{p,q}$ be the Leray spectral sequence of the inclusion map $j\colon U \hookrightarrow X$. There is a homomorphism of spectral sequences $\tilde{E}_r^{p,q-1} \to E_r^{p,q}$ which is an isomorphism for $r=2$ and $q \geqslant 2$. Thus it suffices to compute the differential on $\tilde{E}_2^{1,2}$. This can be deduced from the following lemma, applied to the pairs $(V, D) = (X, Y_i)$ and $(V, D) = (Y_i, Y_i \cap Y_j)$:

LEMMA 5.9. *Let V be a regular scheme and assume that the Purity Conjecture holds on V. Let $D \subset V$ be a regular irreducible Cartier divisor, with $j\colon V \backslash D \hookrightarrow V$ the inclusion. Let $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$ and let $E_r^{p,q} \Rightarrow H^{p+q}(U, \Lambda(1))$ be the Leray spectral sequence for j. Then the map*:

$$\begin{array}{ccc} E_2^{p,1} & \xrightarrow{d_2} & E_2^{p+2,0} \\ \Vert & & \Vert \\ H^p(V, R^1 j_* \Lambda(1)) & & H^{p+2}(V, R^0 j_* \Lambda(1)) \\ \Vert & & \Vert \\ H^p(D, \Lambda) & \longrightarrow & H^{p+2}(V, \Lambda(1)) \end{array}$$

is minus the Gysin map.

Proof. Take an injective resolution $\Lambda(1) \to \mathscr{I}^\cdot$. There is a short exact sequence of complexes of sheaves on V:

$$0 \to \mathbf{\Gamma}_Y(\mathscr{I}^\cdot) \to \mathscr{I}^\cdot \to j_* j^*(\mathscr{I}^\cdot) \to 0.$$

Taking natural resolutions of these complexes and applying $\Gamma(V, \cdot)$, we get a short exact sequence of bicomplexes:

$$0 \to K'^{\cdot\cdot} \to K^{\cdot\cdot} \to K''^{\cdot\cdot} \to 0. \tag{5.2}$$

These bicomplexes give rise to spectral sequences:

$$E_2'^{p,q} = H^p(D, \mathbf{H}_D^q(V, \Lambda(1))) \Rightarrow H_D^{p+q}(V, \Lambda(1))$$

$$E_2^{p,q} = H^p(V, \mathbf{ext}_V^q(\mathbb{Z}, \Lambda(1))) = \begin{cases} H^p(V, \Lambda(1)) & \text{if } q=0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

$$E_2''^{p,q} = H^p(V, R^q j_* j^* \Lambda(1)) \Rightarrow H^{p+q}(V \backslash D, j^*\Lambda(1)).$$

By purity, E_2' is degenerate, with $E_2'^{p,q} = 0$ if $q \neq 2$. Also, $E_2''^{p,q} = 0$ unless $q = 0$ or $q = 1$. Taking cohomology of (5.2) with respect to the second differential (d_{II}) gives a connecting map $H_{\mathrm{II}}^1(K'') \to H_{\mathrm{II}}^2(K')$, which induces a homomorphism

$$E_2''^{p,1} \cong H_{\mathrm{I}}^p H_{\mathrm{II}}^1(K'') \to H_{\mathrm{I}}^p H_{\mathrm{II}}^2(K') \cong E_2'^{p,2},$$

which we denote by ∂. Then ∂ is an isomorphism, by the degeneracy of $E_2^{p,q}$.

The Gysin map is defined as the composition

$$E_2'^{p,2} \cong H^{p+2}(K'^\cdot) \to H^{p+2}(K^\cdot) \cong E_2^{p+2,0}.$$

Thus we need to show that the composition

$$E_2''^{p,1} \cong E_2'^{p,2} \cong H^{p+2}(K'^\cdot) \to H^{p+2}(K^\cdot) \cong E_2^{p+2,0} \cong E_2''^{p+2,0}$$

is equal to $-d_2$.

Take $\alpha \in Z_2''^{p,1}$ representing a class $[\alpha] \in E_2''^{p,1}$. That is,

$$\alpha \in F^p K''^{p+1} = K''^{p,1} \oplus K''^{p+1,0} \quad \text{and} \quad d(\alpha) \in F^{p+2} K''^{p+2} = K''^{p+2,0},$$

where $d = d_{\mathrm{I}} + d_{\mathrm{II}}$. This means $\alpha = \alpha^{p,1} + \alpha^{p+1,0}$, with $\alpha^{i,j} \in K''^{i,j}$ satisfying:

$$\begin{aligned} 0 &= d_{\mathrm{II}}(\alpha^{p,1}) \\ 0 &= d_{\mathrm{I}}(\alpha^{p,1}) + d_{\mathrm{II}}(\alpha^{p+1,0}). \end{aligned} \tag{5.3}$$

First we compute $d_2([\alpha])$. It is represented by $d(\alpha) \in Z_2''^{p+2,0}$. But (5.3) implies $d(\alpha) = d_{\mathrm{I}}(\alpha^{p+1,0})$. Thus $d_{\mathrm{I}}(d(\alpha)) = 0$. This then gives also $d_{\mathrm{II}}(d(\alpha)) = 0$. Therefore $d(\alpha) = d_{\mathrm{I}}(\alpha^{p+1,0})$ also represents a class in $H_{\mathrm{I}}^{p+2} H_{\mathrm{II}}^{0}(K'')$. If we make the identification $H_{\mathrm{I}}^{p+2} H_{\mathrm{II}}^{0}(K'') \cong E_2''^{p+2,0}$, then this class corresponds to $d_2([\alpha])$.

To compute the Gysin map, first note that under the identification $E_2''^{p,1} \cong H_{\mathrm{I}}^{p} H_{\mathrm{II}}^{1}(K'')$, the class $[\alpha]$ is represented by $\alpha^{p,1}$. Choose $\beta^{i,j} \in K^{i,j}$ lifting $\alpha^{i,j}$. Under the connecting homomorphism $H_{\mathrm{I}}^{p} H_{\mathrm{II}}^{1}(K'') \to H_{\mathrm{I}}^{p} H_{\mathrm{II}}^{2}(K')$, we have $[\alpha^{p,1}] \mapsto [d_{\mathrm{II}}(\beta^{p,1})]$. Now it is easy to check that $d(\beta^{p,1}) + d_{\mathrm{II}}(\beta^{p+1,0}) \in Z_2'^{p,q}$ represents $[d_{\mathrm{II}}(\beta^{p,1})]$ under the identification $H_{\mathrm{I}}^{p} H_{\mathrm{II}}^{2}(K') \cong E_2'^{p,2}$. We need to find the image of this class in $H^{p+2}(K'^{\cdot}) \cong E_2'^{p,2}$.

The degeneracy of E' gives $E_2'^{p,2} \cong E_\infty'^{p,2}$, which is equivalent to

$$Z_2'^{p,2} + F^{p+1} K'^{p+2} = Z_\infty'^{p,2} + F^{p+1} K'^{p+2}.$$

Thus there exists $\gamma \in F^{p+1} K'^{p+2}$ such that $d(\beta^{p,1}) + d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma \in Z_\infty'^{p,2}$. That is:

$$d(d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma) = 0, \tag{5.4}$$

and so $d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma$ represents the desired class in $H^{p+2}(K'^{\cdot})$. We use the same notation also for the class in $H^{p+2}(K^{\cdot})$.

We need to find the image of this class under the isomorphism $H^{p+2}(K^{\cdot}) = F^{p+2} H^{p+2}(K^{\cdot}) \cong E_\infty^{p+2,0}$. We have $E_\infty^{p+1,1} = 0$, which means that $Z_\infty^{p+1,1} = F^{p+1} d(K^{p+1}) + Z_\infty^{p+2,0}$. Thus $d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma = d(\delta) + \varepsilon$, for some $\delta \in K^{p+1}$ and $\varepsilon \in Z_\infty^{p+2,0}$. Then $[\varepsilon]$ will be the image of $[d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma]$ in $E_\infty^{p+2,0}$.

Write $\gamma = \gamma^{p+1,1} + \gamma^{p+2,0}$, with $\gamma^{i,j} \in K'^{i,j}$. From (5.4), we get:

$$\begin{aligned}
0 &= d_{\mathrm{II}}(\gamma^{p+1,1}) \\
0 &= d_{\mathrm{I}}(d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1}) + d_{\mathrm{II}}(\gamma^{p+2,0}) \\
0 &= d_{\mathrm{I}}(\gamma^{p+2,0})
\end{aligned}$$

This implies:

$$\begin{aligned}
&d_{\mathrm{II}}(d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1}) = 0 \\
&d_{\mathrm{I}}(d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1}) = d_{\mathrm{II}}(-\gamma^{p+2,0}).
\end{aligned}$$

Therefore $d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1}$ represents a class in $H_{\mathrm{I}}^{p+1} H_{\mathrm{II}}^{1}(K^{\cdot\cdot})$. But this is isomorphic to $E_2^{p+1,1} = 0$. Thus there exist $\zeta^{p,1} \in \ker(d_{\mathrm{II}}) \cap K^{p,1}$ and $\eta^{p+1,0} \in K^{p+1,0}$ such that

$$\begin{aligned}
d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1} &= d_{\mathrm{I}}(\zeta^{p,1}) + d_{\mathrm{II}}(\eta^{p+1,0}) \\
&= d(\zeta^{p,1}) + d_{\mathrm{II}}(\eta^{p+1,0})
\end{aligned}$$

From this we get the following relations in $H^{p+2}(K^{\cdot})$:

$$\begin{aligned}[d_{\mathrm{II}}(\beta^{p+1,0}) + \gamma^{p+1,1} + \gamma^{p+2,0}] &= [\gamma^{p+2,0} + d_{\mathrm{II}}(\eta^{p+1,0})] \\ &= [\gamma^{p+2,0} - d_{\mathrm{I}}(\eta^{p+1,0})]\end{aligned}$$

But now $\gamma^{p+2,0} - d_{\mathrm{I}}(\eta^{p+1,0}) \in Z_{\infty}^{p+2,0}$, so we can take this for ε above. Finally, if we denote the map $K \to K''$ by π and the Gysin map by g, then:

$$\begin{aligned}g([\alpha]) &= [\pi(\gamma^{p+2,0} - d_{\mathrm{I}}(\eta^{p+1,0}))] \\ &= [-d_{\mathrm{I}}(\pi(\eta^{p+1,0}))].\end{aligned}$$

Now let $a^{p,1} := \alpha^{p,1} + \pi(\zeta^{p,1})$ and $a^{p+1,0} := \pi(\eta^{p+1,0})$ and define $a := a^{p,1} + a^{p+1,0} \in F^pK''^{p+1}$. Then:

$$\begin{aligned}d_{\mathrm{II}}a^{p,1} &= d_{\mathrm{II}}\alpha^{p,1} + \pi(d_{\mathrm{II}}\zeta^{p,1}) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}d_{\mathrm{I}}a^{p,1} + d_{\mathrm{II}}a^{p+1,0} &= d_{\mathrm{I}}\alpha^{p,1} + \pi(d_{\mathrm{I}}\zeta^{p,1} + d_{\mathrm{II}}\eta^{p+1,0}) \\ &= -d_{\mathrm{II}}\alpha^{p+1,0} + \pi(d_{\mathrm{II}}\beta^{p+1,0} + \gamma^{p+1,1}) \\ &= -d_{\mathrm{II}}\alpha^{p+1,0} + d_{\mathrm{II}}\alpha^{p+1,0} \\ &= 0.\end{aligned}$$

Therefore $a \in Z_2''^{p,1}$. Recall that then $a^{p,1}$ represents $[a] \in H_{\mathrm{I}}^pH_{\mathrm{II}}^1(K'')$. Since $\beta^{p,1} + \zeta^{p,1} \in K^{p,1}$ is a lifting of $a^{p,1}$, the connecting homomorphism $\partial: H_{\mathrm{I}}^pH_{\mathrm{II}}^1(K'') \to H_{\mathrm{I}}^pH_{\mathrm{II}}^2(K')$ maps $[a]$ to:

$$\begin{aligned}\partial([a]) &= [d_{\mathrm{II}}(\beta^{p,1} + \zeta^{p,1})] \\ &= [d_{\mathrm{II}}(\beta^{p,1})] \\ &= \partial([\alpha]).\end{aligned}$$

But we know that ∂ is an isomorphism. Therefore $[a] = [\alpha]$ in $E_2''^{p,1}$. Then we get

$$d_2([\alpha]) = d_2([a]) = [d_{\mathrm{I}}(\pi(\eta^{p+1,0}))] = -g([\alpha]). \qquad \square$$

PROPOSITION 5.10. $E_{\infty}^{1,3} = 0$.

Proof. We will show that the map described in Proposition 5.8 is injective. Let $\{Z_i\}$ be the components of E_{23k} and $\{Z_i^{(2)}\}$ the components of pairwise intersections of the Z_i's, so that $Y_i = E_{1k} \times Z_i$ and $Y_i^{(2)} = E_{1k} \times Z_i^{(2)}$. Then:

$$\mathrm{CH}^0(Y_i^{(2)}) \cong \mathrm{CH}^0(E_{1k}) \otimes \mathrm{CH}^0(Z_i^{(2)}),$$
$$\mathrm{NS}(Y_i) \cong \mathrm{CH}^0(E_{1k}) \otimes \mathrm{NS}(Z_i) \oplus \mathrm{NS}(E_{1k}) \otimes \mathrm{CH}^0(Z_i)$$

and the map described above clearly decomposes as $id_{E_1} \otimes \delta$, where

$$\delta\colon \bigoplus_i \mathrm{CH}^0(Z_i^{(2)}) \to \bigoplus_i \mathrm{NS}(Z_i)$$

maps a component $Z_i^{(2)}$ of $Z_r \cap Z_s$ to the class $[Z_i^{(2)}]_{Z_r} + [Z_i^{(2)}]_{Z_s}$. It is therefore enough to show that δ is injective. Recall that the components of E_{23k} are:

$$\{AA, AB, BA, BB, Q_{++}, Q_{+-}, Q_{-+}, Q_{--}\}.$$

Writing $S, S' \in \{A, B\}$ and $\varepsilon, \varepsilon' \in \{+, -\}$, the intersections are:

$$SA \cap SB = S+ \cup S-$$
$$AS \cap BS = +S \cup -S$$
$$SS' \cap Q_{\varepsilon,\varepsilon'} = L^{SS'}_{\varepsilon,\varepsilon'},$$

and all other intersections are empty. Recall also that $S+$ is the strict transform of $S \times \{+\}$ in $\mathrm{Bl}_{\{\pm\}\times\{\pm\}}(S \times S')$. With our usual identification $S \times S' \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$, we have $S \times \{+\} = \mathbb{P}^1 \times *$, a line passing through the two points $(+, +)$ and $(-, +)$ of the center of the blow-up. A similar analysis holds for $+S, S-$ and $-S$ The result is:

$$[S\varepsilon]_{SS'} = [\mathbb{P}^1 \times *]_{SS'} - [L_{+,\varepsilon}]_{SS'} - [L_{-,\varepsilon}]_{SS'}$$
$$[\varepsilon S']_{SS'} = [* \times \mathbb{P}^1]_{SS'} - [L_{\varepsilon,+}]_{SS'} - [L_{\varepsilon,-}]_{SS'}$$

Take an element $\zeta \in \bigoplus_i \mathrm{CH}^0(Z_i^{(2)})$,

$$\zeta = \sum_{S,\varepsilon} (a_{S,\varepsilon}[S\varepsilon] + b_{\varepsilon,S}[\varepsilon S]) + \sum_{S,S',\varepsilon,\varepsilon'} c_{S,S',\varepsilon,\varepsilon'}[L^{SS'}_{\varepsilon,\varepsilon'}].$$

Then,

$$\begin{aligned}\delta(\zeta) &= \sum_{S,\varepsilon} a_{S,\varepsilon}([S\varepsilon]_{SA} + [S\varepsilon]_{SB}) + \sum_{S',\varepsilon} b_{\varepsilon,S'}([\varepsilon S']_{AS'} + [\varepsilon S']_{BS'})\\ &\quad + \sum_{S,S',\varepsilon,\varepsilon'} c_{S,S',\varepsilon,\varepsilon'}([L^{SS'}]_{Q_{\varepsilon,\varepsilon'}} + [L_{\varepsilon,\varepsilon'}]_{SS'})\\ &= \sum_{S,S'} ((a_{S,+} + a_{S,-})[\mathbb{P}^1 \times *]_{SS'} + (b_{+,S'} + b_{-,S'})[* \times \mathbb{P}^1]_{SS'})\\ &\quad + \sum_{S,S',\varepsilon,\varepsilon'} ((c_{S,S',\varepsilon,\varepsilon'} - a_{S,\varepsilon'} - b_{\varepsilon,S'})[L_{\varepsilon,\varepsilon'}]_{SS'} + c_{S,S',\varepsilon,\varepsilon'}[L^{SS'}]_{Q_{\varepsilon,\varepsilon'}})\end{aligned}$$

Now suppose $\delta(\zeta)=0$. Then we get, for all S, S', ε, ε':

$$\begin{aligned} a_{S,+} + a_{S,-} &= 0 \\ b_{+,S'} + b_{-,S'} &= 0 \\ c_{S,S',\varepsilon,\varepsilon'} - a_{S,\varepsilon'} - b_{\varepsilon,S'} &= 0 \\ c_{S,S',\varepsilon,\varepsilon'} &= 0. \end{aligned}$$

From this we get, for all S, S', ε, ε':

$$a_{S,\varepsilon'} = -b_{\varepsilon,S'}.$$

This implies, in particular, that $a_{S,\varepsilon}=a_{S',\varepsilon'}$ and $b_{\varepsilon,S}=b_{\varepsilon',S'}$ for all $S, S', \varepsilon, \varepsilon'$. Then the first two relations give $a_{S,\varepsilon}=b_{S,\varepsilon}=0$, for all S and ε □

From this proposition and the remarks above, we get:

COROLLARY 5.11. $E_\infty^{2,2} \cong H_Y^4(X, \mathbb{Z}_l(2))$. □

To calculate $E_\infty^{2,2}$, note that the only nonzero differential is $d_2^{0,3}\colon E_2^{0,3} \to E_2^{2,2}$, so $E_2^{2,2} \twoheadrightarrow E_\infty^{2,2}$. By Proposition 5.3, we have

$$E_2^{2,2} = \bigoplus H^2(Y_i, \mathbb{Z}_l(1)) \cong \bigoplus_i \mathrm{CH}^1(Y_i) \otimes \mathbb{Z}_l$$

From the definition of the cycle class map in [9], one sees that the following diagram commutes:

$$\begin{tikzcd} \bigoplus_i \mathrm{CH}^1(Y_i) \otimes \mathbb{Z}_l \arrow[r, "\bigoplus (\mathrm{incl})_*"] \arrow[d, two heads] & \mathrm{CH}^2(X) \otimes \mathbb{Z}_l \arrow[d, "\mathrm{cl}_X"] \\ E_\infty^{2,2} \cong H_Y^4(X, \mathbb{Z}_l(2)) \arrow[r] & H^4(X, \mathbb{Z}_l(2)) \end{tikzcd}$$

Recall that Σ_h is defined to be the cokernel of the composition:

$$H_Y^4(X, \mathbb{Z}_l(2)) \to H^4(X, \mathbb{Z}_l(2)) \to \bigoplus H^4(Y_i, \mathbb{Z}_l(2)).$$

Then we get the following.

PROPOSITION 5.12. *The cycle class map*:

$$\mathrm{cl}\colon \bigoplus \mathrm{CH}^2(Y_i) \otimes \mathbb{Z}_l \to \bigoplus H^4(Y_i, \mathbb{Z}_l(2))$$

induces an isomorphism:

$$\mathrm{cl}_\Sigma : \Sigma \otimes \mathbb{Z}_l \xrightarrow{\cong} \Sigma_h.$$

Proof. By the remarks above, there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\bigoplus_i \mathrm{CH}^1(Y_i) \otimes \mathbb{Z}_l & \longrightarrow & \bigoplus_i \mathrm{CH}^2(Y_i) \otimes \mathbb{Z}_l & \longrightarrow & \Sigma \otimes \mathbb{Z}_l & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}_\Sigma} & & \\
E_\infty^{2,2} & \longrightarrow & \bigoplus_i H^4(Y_i, \mathbb{Z}_l(2)) & \longrightarrow & \Sigma_h & \longrightarrow & 0
\end{array}$$

Since the left-most vertical map is surjective and cl is an isomorphism by Proposition 5.4, we find that cl_Σ is an isomorphism. □

Let C, e_+ and $\mathrm{CH}^2_{\mathrm{trans}}$ be as in Section 3 and let

$$\varphi : \mathrm{CH}^2_{\mathrm{hom}}(X_K) \to H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2)))$$

be the map discussed in Section 4.

THEOREM 5.13. *If there exists an odd prime l dividing the order of e_+ in $E_1(k)$, then $\varphi(C - \iota C) \notin \varphi(\mathrm{CH}^2_{\mathrm{trans}}(X_K))$.*

Proof. Let $W = \mathrm{CH}^2_{\mathrm{trans}}(X_K) \otimes \mathbb{Z}_l$. Since ι^* stabilizes $\mathrm{CH}^2_{\mathrm{trans}}(X_K)$, we can decompose W as $W_+ \oplus W_-$, where W_+ and W_- are the $+1$ and -1 eigenspaces for ι^*. If $\Psi \in W_+$, then $\iota^*\varphi(\Psi) = \varphi(\iota^*\Psi) = \varphi(\Psi)$. But $\iota^* = -1$ on $H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2)))$ and l is odd, so we get $\varphi(W_+) = 0$. So it is enough to show that $\varphi(C - \iota C) \notin \varphi(W_-)$. So assume $\Psi \in W_-$ We must show that $\varphi([C - \iota C] - \Psi) \neq 0$; that is,

$$\mathrm{cl}([C - \iota C] - \Psi) \overset{?}{\notin} \ker(H^4(X_K, \mathbb{Z}_l(2)) \to H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2))))$$

From the Hochschild–Serre spectral sequence and the fact that $\mathrm{cd}_l(K) = 2$, we have:

$$\begin{aligned}
&\ker(H^4(X_K, \mathbb{Z}_l(2)) \to H^1(K, H^3(X_{\bar{K}}, \mathbb{Z}_l(2)))) \\
&\quad \cong H^2(K, H^2(X_{\bar{K}}, \mathbb{Z}_l(2))),
\end{aligned}$$

and hence ι^* acts trivially on this kernel. On the other hand,

$$\iota^*([C - \iota C] - \Psi) = -([C - \iota C] - \Psi),$$

so it suffices to show that $\mathrm{cl}_X([C-\iota C]-\Psi)\neq 0$. But

$$\sigma\circ \mathrm{cl}_X([C-\iota C]-\Psi)=\mathrm{cl}_\Sigma\circ\sigma([C-\iota C]-\Psi)\neq 0$$

since cl_Σ is an isomorphism and $\sigma([C-\iota C]-\Psi)\neq 0$ by Corollary 3.8. □

6. Infinite order

Let $\mathscr{K}$ be a global field, with ring of integers $\mathscr{R}$ and let X be an integral scheme which is flat and proper over $\mathscr{R}$, with smooth generic fiber.

PROPOSITION 6.1. *Suppose there is a prime* $\mathfrak{q}\in\operatorname{spec}\mathscr{R}$ *not dividing* l, *such that* X *has good reduction at* $\mathfrak{q}$ *and* $l\nmid\det(N(\mathfrak{q})^2-\mathfrak{f}_g)$, *where* $\mathfrak{f}_g$ *is the geometric Frobenius map on* $H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Z}_l)$. *Then* $H^1(\mathscr{K},H^3(X_{\bar{\mathscr{K}}},\mathbb{Z}_l(2)))$ *is torsion-free.*

Proof. From the short exact sequence of $\mathrm{Gal}(\bar{\mathscr{K}}/\mathscr{K})$-modules:

$$0\to H^3(X_{\bar{\mathscr{K}}},\mathbb{Z}_l(2))\to H^3(X_{\bar{\mathscr{K}}},\mathbb{Q}_l(2))\to H^3(X_{\bar{\mathscr{K}}},\mathbb{Q}_l/\mathbb{Z}_l(2))\to 0$$

we find

$$H^0(\mathscr{K},H^3(X_{\bar{\mathscr{K}}},\mathbb{Q}_l/\mathbb{Z}_l(2)))\twoheadrightarrow H^1(\mathscr{K},H^3(X_{\bar{\mathscr{K}}},\mathbb{Z}_l(2)))_{\text{torsion}}.$$

Therefore it suffices to prove that

$$H^3(X_{\bar{\mathscr{K}}},\mathbb{Q}_l/\mathbb{Z}_l(2))^{\mathrm{Gal}(\bar{\mathscr{K}}/\mathscr{K})}=0.$$

Since X has good reduction at $\mathfrak{q}$, we have:

$$H^3(X_{\bar{\mathscr{K}}},\mathbb{Q}_l/\mathbb{Z}_l(2))^{\mathrm{Gal}(\bar{\mathscr{K}}/\mathscr{K})}\subset H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Q}_l/\mathbb{Z}_l(2))^{\mathrm{Gal}(\overline{k(\mathfrak{q})}/k)}$$

so it suffices to show that

$$H^0(k(\mathfrak{q}),H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Q}_l/\mathbb{Z}_l(2)))=0.$$

Since $k(\mathfrak{q})$ is finite, the Weil conjectures give

$$H^0(k(\mathfrak{q}),H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Q}_l(2)))=H^1(k(\mathfrak{q}),H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Q}_l(2)))=0,$$

from which it follows that:

$$H^0(k(\mathfrak{q}),H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Q}_l/\mathbb{Z}_l(2)))\cong H^1(k(\mathfrak{q}),H^3(X_{\overline{k(\mathfrak{q})}},\mathbb{Z}_l(2))).$$

Now if $\mathfrak{f}_a(2)$ is the arithmetic Frobenius automorphism acting on $H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Z}_l(2))$, then to show $H^1(k(\mathfrak{q}), H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Z}_l(2))) = 0$, it suffices to show that $\det(\mathfrak{f}_a(2) - 1) \in \mathbb{Z}_l^\times$. If $\mathfrak{f}_g$ is the geometric Frobenius automorphism acting on $H^3(X_{\overline{k(\mathfrak{q})}}, \mathbb{Z}_l)$ (note the untwisting), then $\mathfrak{f}_a(2) = N(\mathfrak{q})^2 \cdot \mathfrak{f}_g^{-1}$ Since $\det(\mathfrak{f}_g) \in \mathbb{Z}_l^\times$, it suffices to check whether $\det(N(\mathfrak{q})^2 - \mathfrak{f}_g) \in \mathbb{Z}_l^\times$. □

Proof of Theorem 2. Denote

$$H^3(X_{\overline{\mathscr{K}_\mathfrak{p}}}, \mathbb{Z}_l(2)) = H^3(X_{\bar{\mathscr{K}}}, \mathbb{Z}_l(2))$$

simply by H. Write φ for the map $\mathrm{CH}^2_{\mathrm{hom}}(X_{\mathscr{K}_\mathfrak{p}}) \to H^1(\mathscr{K}_\mathfrak{p}, H)$ and write $\varphi_{\mathscr{K}}$ for the corresponding map on $\mathrm{CH}^2_{\mathrm{hom}}(X_{\mathscr{K}})$ The galois group $\mathrm{Gal}(\overline{\mathscr{K}_\mathfrak{p}}/\mathscr{K}_\mathfrak{p})$ is naturally contained in Gal $(\bar{\mathscr{K}}/\mathscr{K})$ as the decomposition group of a prime over $\mathfrak{p}$. The inclusion map induces a morphism of the respective Hochschild–Serre spectral sequences. Then functoriality of the cycle-class map gives a commutative square:

$$\begin{array}{ccc}
\mathrm{CH}^2_{\mathrm{hom}}(X_{\mathscr{K}}) & \xrightarrow{\varphi_{\mathscr{K}}} & H^1(\mathscr{K}, H) \\
\downarrow & & \downarrow \\
\mathrm{CH}^2_{\mathrm{hom}}(X_{\overline{\mathscr{K}_\mathfrak{p}}}) & \xrightarrow{\varphi} & H^1(\mathscr{K}_\mathfrak{p}, H)
\end{array}$$

Now suppose some multiple $m \cdot (C - \iota C)$ were algebraically equivalent to zero over $\bar{\mathscr{K}}$. Then this would be true already over a finite extension $\mathscr{L}$ of $\mathscr{K}$. By pushing everything down to $X_{\mathscr{K}}$, we would get that $n \cdot (C - \iota C)$ is algebraically equivalent to zero over $\mathscr{K}$, where $n = m \cdot [\mathscr{L} : \mathscr{K}]$. Then $n \cdot \varphi_{\mathscr{K}}(C - \iota C) \in H^1(K, M_{\mathscr{K}})$, with $M_{\mathscr{K}}$ as in Section 4. But by Lemma 4.6 and Proposition 4.7, $M_{\mathscr{K}} = M_{\mathrm{alg}}$ is a direct summand of H. Since $H^1(\mathscr{K}, H)$ is torsion-free by the above proposition, this implies that already $\varphi_{\mathscr{K}}(C - \iota C) \in H^1(\mathscr{K}, M_{\mathrm{alg}})$. But then also $\varphi_K(C - \iota C) \in H^1(K, M_{\mathrm{alg}})$ by commutativity of (6.1). Again by Proposition 4.7, we get $\varphi_K(C - \iota C) \in \varphi_K(\mathrm{CH}^2_{\mathrm{trans}}(X_K))$. But this contradicts Theorem 5.13. Therefore no multiple of $C - \iota C$ is algebraically equivalent to zero over $\bar{\mathscr{K}}$ □

7. A family of examples

THEOREM 7.1. *Let $K = \mathbb{Q}$ and let a, b, c, d be arbitrary integers. Let E_1, E_2, E_3 be as in the Introduction, with*

$$\alpha = 1 + 23 \cdot 29 \cdot 13 \cdot a$$
$$\beta = 2 + 23 \cdot 29 \cdot 13 \cdot b$$
$$\gamma = 3 + 23 \cdot 29 \cdot 13 \cdot c$$
$$\pi = 23(1 + 23 \cdot 29 \cdot 13 \cdot d).$$

Then Theorem 2 applies, with $l=3$, $\mathfrak{p}=23$, $\mathfrak{q}=29$ and $\mathfrak{r}=13$. Thus, assuming the Purity Conjecture holds, the cycle $C-\iota C\in \mathrm{CH}^2_{\mathrm{hom}}(E_1\times E_2\times E_3)$, defined over $\mathbb{Q}$, has infinite order modulo algebraic equivalence over $\bar{\mathbb{Q}}$

Proof. It is clear that α, β, γ are 23-adic units which are distinct mod 23 and that π is a uniformizing parameter. It is easy to check that the point $e_+\in E_1(\overline{\mathbb{F}_{23}})$ given by $x=0$, $y=\sqrt{-\alpha\beta\gamma}$ has order $l=3$. Moreover,

$$\begin{aligned}&\alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha-\pi)(\beta-\pi)(\gamma-\pi)\\&\qquad\equiv 2^5\cdot 3^2\cdot 5\cdot 7\cdot 11 \pmod{29\cdot 13},\end{aligned}$$

so X has good reduction at both 29 and 13. Thus it remains to verify hypotheses (iii) and (iv).

By counting points, one finds the following eigenvalues for the geometric frobenius map on $H^1(E_{i\overline{\mathbb{F}_{29}}}, \mathbb{Z}_l)$:

$$\begin{aligned}E_1&: \lambda_1 = -5\pm 2\sqrt{-1}\\E_2&: \lambda_2 = -3\pm 2\sqrt{-5}\\E_3&: \lambda_3 = -1\pm 2\sqrt{-7}\end{aligned}$$

These generate an extension of degree 8 over $\mathbb{Q}$. The coefficient of $\sqrt{-35}$ in $\lambda_1\lambda_2\lambda_3$ is ± 8, which is not divisible by 29. Thus (iii) is satisfied.

The eigenvalues of Frobenius on $H^1(E_{i\overline{\mathbb{F}_{13}}}, \mathbb{Z}_l)$ are as follows:

$$\begin{aligned}E_1&: \lambda_1 = 3\pm 2\sqrt{-1}\\E_2&: \lambda_2 = -3\pm 2\sqrt{-1}\\E_3&: \lambda_3 = 3\pm 2\sqrt{-1}.\end{aligned}$$

From the Kunneth decomposition, the eigenvalues of Frobenius on $H^3(E_1\times E_2\times E_{3\overline{\mathbb{F}_{13}}}, \mathbb{Z}_l)$ are of the forms $13\lambda_i$ (on $H^1(E_i)\otimes H^0(E_j)\otimes H^2(E_k)$), and $\lambda_1\lambda_2\lambda_3$ (on $H^1(E_1)\otimes H^1(E_2)\otimes H^1(E_3)$). Thus the determinant of $\mathfrak{f}_g-13^2$ is the product of the six factors $13\lambda_i-13^2$ and the eight factors $\lambda_1\lambda_2\lambda_3-13^2$ (with all possible choices of $\pm$ in the λ_i). Since all the λ_i are in $\mathbb{Z}[\sqrt{-1}]$ and 3 is prime in $\mathbb{Z}[\sqrt{-1}]$, then 3 divides $\det(\mathfrak{f}_g-13^2)$ if and only if it divides one of the factors. But it is easily seen that this is not the case. Thus (iv) also holds. □

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