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0. Introduction

We work over an algebraically closed field k.

The moduli space $\mathfrak{M}_g$ for curves of genus $g \geqslant 2$ can be constructed by using Geometric Invariant Theory (GIT) to construct a quotient of either the Chow variety or the Hilbert Scheme which parameterizes n-canonically embedded curves in $\mathbf{P}^{(2n-1)(g-1)-1}$ for $n \geqslant 3$. The quotients of the Chow variety and the Hilbert scheme are complete. Thus the closure of $\mathfrak{M}_g$ in the quotient gives a compactification of $\mathfrak{M}_g$.

Mumford [Mu2] has shown using the Chow variety, and Gieseker [G] has shown using the Hilbert scheme that when $n \geqslant 5$ the associated compactification of $\mathfrak{M}_g$ is a moduli space for Mumford–Deligne stable curves. The aim of this paper is to show that when $g \geqslant 3$ and $n = 3$, the associated compactification of $\mathfrak{M}_g$ is a moduli space for a class of curves which we will call pseudo-stable.

DEFINITION. A reduced connected complete curve is *Mumford–Deligne stable* if

  (i) it has only ordinary double points as singularities; and
  (ii) every subcurve of genus 0 meets the rest of the curve at at least 3 points.

A reduced connected complete curve is *pseudo-stable* if

  (i) it has only ordinary double points and ordinary cusps as singularities;
  (ii) every subcurve of genus 1 meets the rest of the curve at at least 2 points; and
  (iii) every subcurve of genus 0 meets the rest of the curve at at least 3 points. ☐

Our method closely follows that of Mumford and Gieseker. We use 1-Parameter Subgroups (1-PS's) of $SL(N+1)$ to show that when $n = 3$ the Chow points of curves which are not pseudo-stable are unstable in the sense of GIT, and thus are not represented in the compactification of $\mathfrak{M}_g$. The singular curves are then shown to be represented in the compactification of $\mathfrak{M}_g$ by considering degenerations of smooth curves and the completeness of the quotient scheme.

In §1 we review Geometric Invariant Theory as it applies to the Chow variety. In §2 we present modified lemmas of Mumford and Gieseker to show that curves with bad singularities or multiple components have unstable Chow points. In §3 we show that curves with elliptic tails have unstable Chow points. In §4 we discuss the relationship between Mumford–Deligne stable curves and pseudo-stable curves. In §5 we use the previous results to construct a moduli space for pseudo-stable curves.

This work is a large part of the author's thesis written under David Gieseker. The author wishes to thank him again for his guidance and patience.

1. Chow stability

In this section we review Geometric Invariant Theory (GIT) as it applies to the Chow variety. More complete treatments can be found in [Mu1, Mu2 and Mo].

A non-negative Chow cycle X of dimension r in $\mathbf{P}^N$ is a formal sum $X = \Sigma\, a_i X_i$ where each X_i is a variety of dimension r in $\mathbf{P}^N$ and each a_i is a non-negative integer. The degree of X is $\Sigma\, a_i \deg X_i$. For each r, N and d there is a Chow variety which parameterizes non-negative Chow cycles in $\mathbf{P}^N$ of dimension r and degree d.

The natural action of $SL(N+1)$ on $\mathbf{P}^N$ determines a natural action on the Chow variety. GIT says that on the open subset of what are called semi-stable points, a quotient of the action by $SL(N+1)$ can be taken to get a projective scheme. On the open subset of what are called stable points, this quotient is an orbit space. We call a Chow cycle stable (resp. semi-stable) if its Chow point is stable (resp. semi-stable).

We will now consider a method for determining which Chow cycles are stable.

A *weighted flag* of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$ is a filtration $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)) = V_0 \supset \cdots \supset V_N$ where each V_i is a vector space of dimension $N+1-i$ and a set of integers $r_0 \geqslant r_1 \geqslant \cdots \geqslant r_N = 0$. Associated to each weighted flag is a 1-PS $\lambda(t)$ of $SL(N+1)$. If $k = \Sigma\, r_i$ and $X_0, \ldots, X_N$ are coordinates on $\mathbf{P}^N$ where $V_i = \mathrm{span}\{X_i, \ldots, X_N\}$, then $\lambda(t)X_i = t^{(N+1)r_i - k}X_i$. Note that we may choose a weighted flag by specifying the r_i's and the X_i's.

Let Φ_X and Φ_Y be the Chow forms of Chow cycles X and Y in $\mathbf{P}^N$, and let $\lambda(t)$ be the 1-PS of $SL(N+1)$ associated to a weighted flag F. Then

$$\Phi_{X+Y} = \Phi_X \Phi_Y \quad \text{and} \quad (\Phi_X)^{\lambda(t)}(\Phi_Y)^{\lambda(t)} = (\Phi_{X+Y})^{\lambda(t)}.$$

Further $\Phi_X = \Sigma_{i=a}^{i=b}\, \Phi_{X,i}$ where $(\Phi_X)^{\lambda(t)} = \Sigma_{i=a}^{i=b}\, t^i \Phi_{X,i}$. We call a the *F-weight of X*. Geometric Invariant Theory says that the point in the Chow variety corresponding to Φ_X is stable (resp. semi-stable) if the F-weight of X is <0 (resp. $\leqslant 0$) for every weighted flag of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$.

Note that the F-weight of the cycle $X + Y$ is the sum of the F-weights of the cycles X and Y.

If $f(n)$ is a polynomial of degree r, we denote by n.l.c. (f) (the normed linear coefficient of f) the integer e such that

$$f(n) = e \frac{n^r}{r!} + \text{lower terms.}$$

Let X be a variety in $\mathbf{P}^N$ of dimension r, and let F be a weighted flag of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$ as above. Let $\alpha: \tilde{X} \to X$ be a proper birational morphism of varieties. Let $X' = \tilde{X} \times \mathbf{A}^1$. Let $\mathscr{I}$ be the ideal sheaf of $\mathcal{O}_{X'}$ defined by

$$\mathscr{I} \cdot [\alpha^* \mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbf{A}^1}] = \text{subsheaf generated by } t^{r_i} \alpha^* X_i \quad (i = 1, \dots, n).$$

We denote

$$e_F(X) = \text{the n.l.c. of } X(\mathcal{O}_{X'}(n)/\mathscr{I}^n \mathcal{O}_{X'}(n))$$

which is a polynomial of degree $r+1$ for $n \gg 0$.

We note that Lemma 5.6 of [Mu2] shows that $e_F(X) = e_F(\tilde{X})$.

Mumford showed in [Mu2] that the F-weight of a variety X is

$$e_F(X) - \frac{r+1}{N+1} \deg(X) \sum r_i.$$

If a Chow cycle $Y = \Sigma a_i Y_i$ where the Y_i are varieties, we let $e_F(Y) = \Sigma a_i e_F(Y_i)$. We now have the following theorem.

THEOREM 1.1. *A Chow cycle is stable (resp. semi-stable) iff*

$$e_F(X) \leqslant \frac{r+1}{N+1} \deg(X) \sum r_i$$

for every weighted flag F of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$.

Proof. This is essentially Theorem 2.9 of [Mu2]. □

We now consider ways to estimate $e_F(X)$ for a reduced curve X in $\mathbf{P}^N$.

LEMMA 1.2. *Let X be an r-dimensional variety in $\mathbf{P}^N$. Let F be the weighted flag determined by $X_0, \dots, X_N$ and $r_0 \geqslant \cdots \geqslant r_N = 0$. Suppose $X_j, \dots, X_N$ vanish on X, and $r_0 = \cdots = r_{j-1}$. Then $e_F(X) = (r+1) r_0 \deg(X)$.*

Proof. We use the following result which is found in [Mo].

LEMMA 1.3. *Let R be the homogeneous coordinate ring of a variety X in* $\mathbf{P}^N$. *Let I be the ideal in* $R[t]$ *generated by* $\{X_i t^{r_i} \mid 0 \leqslant i \leqslant N\}$. *Then* $e_F(X) = n.l.c.$ $\dim_k(R[t]/I^m)_m$ *where* $R[t] = \bigoplus_{i=1}^{\infty} R_i[t]$ *is the grading for* $R[t]$.

Proof. This follows from combining Proposition 3.2 and Corollary 3.3 of [Mo]. □

Let $I = (t^{r_0} X_0, \ldots, t^{r_N} X_N)$ be an ideal of $R[t]$. Then $I = t^{r_0}(X_0, \ldots, X_N)$. Then $(R[t]/I^m)_m = R_m[t]/(t^{m r_0} R_m)$. So

$$\dim_k(R[t]/I^m)_m = r_0 m \dim_k R_m$$

$$= r_0 m \frac{\deg(X)}{r!} m^r + \text{lower terms}$$

$$= \frac{(r+1) r_0 \deg(X)}{(r+1)!} m^{r+1} + \text{lower terms}.$$

Thus $e_F(X) = (r+1) r_0 \deg(X)$ by Lemma 1.3. □

Let F be a weighted flag determined by $X_0, \ldots, X_N$ and $r_0 \geqslant \cdots \geqslant r_N = 0$. Let $\alpha: \tilde{X} \to X$ be the normalization of a reduced curve X in $\mathbf{P}^N$. Suppose there is an l such that $\alpha^* X_l$ does not vanish on $\tilde{X}$ and $r_l = 0$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_{\tilde{X} \times \mathbf{A}^1}$ defined by

$$\mathcal{I} \cdot [\alpha^* \mathcal{O} X(1) \otimes \mathcal{O}_{\mathbf{A}^1}] = \text{subsheaf generated by } t^{r_i} \alpha^* X_i \quad (i = 1, \ldots, n).$$

Note that $\mathcal{O}_{\tilde{X} \times \mathbf{A}^1} / \mathcal{I}$ has finite support, because $\alpha^* X_l$ does not vanish on $\tilde{X}$ and $r_l = 0$.

Let $P \in \tilde{X}$. We denote $e_F(\tilde{X})_P = n.l.c. \dim_k(\mathcal{O}_{X' \times \mathbf{A}^1, P \times \{0\}} / (\mathcal{I}_{P \times \{0\}})^m)$. Note that $e_F(X) = \Sigma_{p \in \tilde{X}} e_F(\tilde{X})_P$.

LEMMA 1.4. *In the above situation, suppose* $v(\alpha^* X_i) + r_i \geqslant a$ *for* $i = 0, \ldots, N$ *where v is the natural valuation on* $\mathcal{O}_{\tilde{X}, P}$. *Then* $e_F(\tilde{X})_p \geqslant a^2$.

Proof. Let s generate the maximal ideal of $\mathcal{O}_{\tilde{X}, P}$ Let

$$R = \mathcal{O}_{\tilde{X} \times \mathbf{A}^1, P \times \{0\}} = \mathcal{O}_{\tilde{X}, P}[t]_{(s,t)}.$$

Let

$$I = \mathcal{I}_{P \times \{0\}} = (t^{r_0} s^{v(X_0)}, \ldots, t^{r_N} s^{v(X_N)})$$

where $s^{v(X_i)} = 0$ if $v(X_i) = \infty$ Then

$$I^m = (\{s^b t^c \mid \text{there exist non-negative integers } n_0, \ldots, n_N \text{ such that } \Sigma n_i \geqslant m, \\ \Sigma n_i v(X_i) \leqslant b, \ \Sigma n_i r_i \leqslant c\}).$$

Thus

$\dim_k(R/I^m) \leqslant \sum_{c=0}^{ma} \min\{\sum n_i v(X_i) \mid \text{there exist non-negative integers}$
$n_0, \ldots, n_N \text{ such that } \sum n_i \geqslant m, \sum n_i r_i \leqslant c\}.$

We have

$v(X_i) + r_i \geqslant a$

$\sum n_i v(X_i) + \sum n_i r_i \geqslant a \sum n_i$

$\sum n_i v(X_i) \geqslant a \sum n_i - \sum n_i r_i$

So

$$\dim_k(R/I^m) \geqslant \sum_{c=0}^{ma} (am - c) \geqslant \frac{m^2}{2} a^2 + \text{lower terms}.$$

Finally $e_F(\tilde{X})_P = \text{n.l.c. } \dim_k(R/I^m) \geqslant a^2.$ $\square$

2. Unstable curves

In this section we consider genus $g \geqslant 3$ connected curves of degree $d = 6(g-1)$ in $\mathbf{P}^N$ where $N = 5(g-1)-1$. We show that if such a curve X is Chow semi-stable, then X must be reduced as a scheme and have only ordinary double points, ordinary cusps, and tachnodes which involve a line as singularities. We also show that if the Chow point of a curve X is in the closure of the set of Chow points of smooth curves Y such that $\mathcal{O}_Y(1) = \omega_Y^{\otimes 3}$, then $\mathcal{O}_X(1) = \omega_X^{\otimes 3}$.

The proofs in this section are elementary modifications of those of Gieseker [G] and Mumford [Mu2].

LEMMA 2.1. *If X has a triple point, then X is not Chow semi-stable.*

Proof. The proof of Proposition 3.1 in [Mu2] shows that such an X cannot be Chow semi-stable if $d/(N+1) < 3/2$. $\square$

LEMMA 2.2. *If X has a tachnode and a line does not pass through the tachnode, then X is not Chow semi-stable.*

Proof. This amounts to translating the proof of Lemma 5.8 in [G] which concerns the Hilbert scheme to the case of the Chow variety.

Suppose X has a tachnode at P which does not involve a line. Let $\tilde{X}$ be the normalization of X. Let Q and R be the inverse image of P in $\tilde{X}$. We can choose $X_0, \ldots, X_N$ in $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$ such that $X_1, \ldots, X_N$ vanish at P and $X_2, \ldots, X_N$ vanish to order $\geqslant 2$ at Q and R. Let $r_0 = 2$, $r_1 = 1$, and $r_i = 0$ for $i \geqslant 2$. The

weighted flag determined by the X_i's and the r_i's has $e_F(\tilde{X})_Q \geqslant 4$ and $e_F(\tilde{X})_R \geqslant 4$, so $e_F(X) \geqslant 8 \geqslant \frac{36}{5} = 2\frac{6}{5}3 = 2d/(N+1)\Sigma r_i$.   ◻

LEMMA 2.3. *If X has a cusp which is not ordinary, then X is not Chow semi-stable.*

Proof. Suppose X has a cusp which is not ordinary. Let $\tilde{X}$ be the normalization of X and let P be the inverse image of the cusp. We can choose $X_0, \ldots, X_N$ in $H^0(\mathbf{P}^N, \mathscr{O}_{\mathbf{P}^N}(1))$ so that $X_1, \ldots, X_N$ vanish to order $\geqslant 2$ at P and $X_2, \ldots, X_N$ vanish to order $\geqslant 4$ at P. Let $r_0 = 4$, $r_1 = 2$, and $r_i = 0$ for $i \geqslant 2$. Then $e_F(\tilde{X})_P \geqslant 16 \geqslant \frac{72}{5} = 2\frac{6}{5}6 = 2d/(N+1)\Sigma r_i$.   ◻

LEMMA 2.4. *If a connected Chow cycle X has any multiple components, then X is not Chow semi-stable.*

Proof. Suppose $X = Y + nC$ where Y is non-negative, C has no multiple components, $n \geqslant 2$, and Y and C have no common components. Choose $P \in Y \cap C$. Choose $X_0, \ldots, X_N$ in $H^0(\mathbf{P}^N, \mathscr{O}_{\mathbf{P}^N}(1))$ such that $X_1, \ldots, X_N$ vanish at P. Let $r_0 = 1$ and $r_i = 0$ for $i \geqslant 1$. Then

$$e_F(X) \geqslant e_F(Y)_P + ne_F(C)_P \geqslant 1 + 2 = 3 > 2\frac{6}{5}1 = 2\frac{d}{(N+1)}\Sigma r_i.$$

So X is not Chow semi-stable.

It remains to show the case where $X = nC$ where C is an irreducible curve and $n \geqslant 2$. Choose a smooth point $P \in C$. Choose $X_0, \ldots, X_N$ in $H^0(\mathbf{P}^N, \mathscr{O}_{\mathbf{P}^N}(1))$ such that $X_1, \ldots, X_N$ vanish at P and $X_2, \ldots, X_N$ vanish to order $\geqslant 2$ at P. Let $r_0 = 2$, $r_1 = 1$, and $r_i = 0$ for $i \geqslant 2$. Then

$$e_F(X) \geqslant n e_F(C)_P \geqslant 2 \cdot 4 > 36/5 = 2\frac{d}{(N+1)}\Sigma r_i.$$ ◻

LEMMA 2.5. *Let X be a connected curve in $\mathbf{P}^N$ which is Chow semi-stable. Let ω_X be the dualizing sheaf for X. Then*

  (i) *X is embedded by a complete linear system and $H^0(X, \mathscr{O}_X(1)) = 0$; and*
  (ii) *If $X = C_1 \cup C_2$ is a decomposition of X into two sets of components such that $\mathscr{W} = C_1 \cup C_2$ and $w = \#\mathscr{W}$ (counted with multiplicities), then*

$$|\deg(C_1) - 3 \deg_{C_1}(\omega_X)| \leqslant w/2.$$

Proof. By the previous lemmas we see that X must be generically reduced, and X is a local complete intersection. Thus ω_X is locally free.

The proof of this lemma follows almost word for word the proof of Proposition 5.5 in [Mu2] where 8/7 is replaced by 6/5. The only difference is in the proof that $H^1(C_1, \mathscr{O}_{C_1}(1)) = 0$ if C_1 is irreducible. Using 6/5 we can conclude

that if $H^1(C_1, \mathcal{O}_{C_1}(1)) \neq 0$, then $\deg(C_1)$ is $1, 2$ or 3 as opposed to 1 or 2 if $8/7$ is used. If $\deg(C_1) = 3$, then C_1 must be rational or elliptic of degree 3. It follows that $H^1(C_1, \mathcal{O}_{C_1}(1)) = 0$. □

LEMMA 2.6. *Suppose* $\mathcal{X} \to \mathrm{Spec}(k[[t]])$ *is a flat family of curves. Let* η *be the generic point of* $\mathrm{Spec}(k[[t]])$, *and let* 0 *be the special point. Suppose that* $\mathcal{X}$ *is embedded in* $\mathbf{P}^N \times \mathrm{Spec}(k[[t]])$ *so that* $\mathcal{X}_\eta$ *is smooth and* $\mathcal{O}_{\mathcal{X}_\eta}(1) = \omega_{\mathcal{X}_\eta}^{\otimes 3}$. *Suppose that* $\mathcal{X}_0$ *is Chow semi-stable. Then* $\mathcal{O}_{\mathcal{X}_0}(1) = \omega_{\mathcal{X}_0}^{\otimes 3}$.

Proof. Let D_i, $i = 1, \dots, m$, be the irreducible components of $\mathcal{X}_0$. It follows that $\mathcal{O}_{\mathcal{X}}(1) = \omega_{\mathcal{X}/k[[t]]}^{\otimes 3}(\Sigma\, a_i D_i)$. We may assume $a_i \geq 0$ for $i = 1, \dots, m$, and $\min a_i = 0$. Let $C_1 = \bigcup_{r_i = 0} D_i$, and let $C_2 = \mathcal{X}_0 - C_1$. Now

$$\#(C_1 \cap C_2) \leq \deg_{C_1}(\mathcal{O}_{\mathcal{X}_0}(\textstyle\sum a_i D_i))$$

$$= \deg C_1 - 3 \deg_{C_1}(\omega_{\mathcal{X}_0})$$

which contradicts (ii) of Lemma 2.5 if $C_2 \neq \varnothing$. □

Note that $\mathcal{X}_0$ in the above lemma cannot have a tachnode, because a tachnode would have a line L passing through it. It cannot be that

$$\deg(L) = 1 \quad \text{and} \quad \mathcal{O}_{\mathcal{X}_0}(1) = \omega_{\mathcal{X}_0}^{\otimes 3}.$$

3. Instability of elliptic tails

In this section we exhibit the weighted flag that shows that a curve X embedded by $\omega_X^{\otimes 3}$ cannot be Chow semi-stable and have an elliptic tail.

LEMMA 3.1. *Suppose* X *is embedded by* $\Gamma(X, \omega_X^{\otimes 3})$ *in* $\mathbf{P}^N$ ($N = 5(g-1)-1$), *and* X *has a subcurve* C_1 *of genus 1 which meets* $C_2 = X - C_1$ *at exactly one point. Then* X *is not Chow semi-stable.*

Proof. Suppose X satisfies the hypotheses of the lemma. From previous lemmas, we may assume that $P = C_1 \cap C_2$ is an ordinary double point. Hence $\mathcal{O}(1)|_{C_1} = \mathcal{O}_{C_1}(3P)$. We can choose $X_0, \dots, X_N$ in $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1))$ such that X_N and X_{N-1} vanish on C_2, and X_N vanishes to order 3 at P on C_1. Let $r_N = 0, r_{N-1} = 2$ and $r_i = 3$ for $i \leq N - 3$.

$$e_F(X) \geq e_F(C_2) + e_F(C_1)_P$$

$$= 2 \cdot 3 \cdot \deg(C_2) + 9 = 6[6(g-1)-3] + 9$$

$$= 36(g-1) - 9 > 36(g-1) - 48/5$$

$$= 2\tfrac{6}{5}[15(g-1)-4] = 2\tfrac{6}{5}[3(N+1)-4]$$

$$= 2\tfrac{6}{5}\textstyle\sum r_1.$$

4. Pseudo-stable curves

In this section we discuss the relationship between Mumford–Deligne stable (M-D stable) curves and pseudo-stable (p-stable) curves which were defined in the introduction.

LEMMA 4.1. *Let X be an M-D stable curve of genus $\geqslant 3$. Then there is a unique connected subcurve C of X such that*

(i) $\overline{X-C} = \cup C_i$ *where each C_i is connected and has genus 1;*

(ii) $\#(C_i \cap \overline{X-C_i}) = 1$ *and $C_i \cap C_j = \varnothing$; and*

(iii) *every connected genus 1 subcurve D of C satisfies $\#(D \cap \overline{X-D}) \geqslant 2$.*

Proof. To show existence choose a subcurve C of X minimal with respect to satisfying (i) and (ii). Suppose D is a connected genus 1 subcurve of C. If $D = C$ and $(D \cap \overline{X-D}) \leqslant 1$, then X has genes $\leqslant 2$ which violates the assumption that X has genus $\geqslant 3$. If $\#(D \cap \overline{C-D}) = 1$ and $\#(D \cap \overline{X-C}) = 0$, then $C - D$ satisfies (i) and (ii) which contradicts the minimality of C. Thus

$$\#((D \cap \overline{X-D}) = \#(D \cap \overline{C-D}) + \#(D \cap \overline{X-C}) \geqslant 2,$$

and C satisfies condition 3.

Suppose that C' also satisfies conditions (i), (ii) and (iii). Let Y be a connected component of $X - C$.

Claim. Y is irreducible.

Proof of Claim. Let Z be the irreducible component of Y which intersects C.

Suppose Z has genus 0. Then no connected component of $\overline{Y-Z}$ has genus 0, because such a connected component would have to intersect with Z three times which would imply that Y has genus $\geqslant 2$. Since Y has genus 1 and $\overline{Y-Z}$ has no connected components of genus 0, we must have $\overline{Y-Z}$ as genus 1 and $\#(Z \cap \overline{Y-Z}) = 1$. But then $\#(Z \cap \overline{X-Z}) = \#(Z \cap C) + \#(Z \cap \overline{Y-Z}) = 2$ implies that X is not M-D stable.

Thus Z must have genus 1 and $\overline{Y-Z}$ must be the union of connected components of genus 0, each of which intersects Z exactly once. Thus $\overline{Y-Z} = \varnothing$ since X is M-D stable. So the claim holds.

Now (iii) implies that $Y \subset \overline{X-C'}$. Since this is true for all connected components of $\overline{X-C}$, we have $C' \subset C$. The minimality of C with respect to conditions (i) and (ii) implies that $C' = C$.   □

LEMMA 4.2. *Let Y and Z be p-stable curves of the same genus $\geqslant 3$. Let $P_1, \ldots, P_n \in Y$ be points where Y has an ordinary cusp. Let $Q_1, \ldots, Q_m \in Z$ be*

points where Z has an ordinary cusp. Then there exists an M-D stable curve X of the same genus as Z and a morphism $f: X \to Z$ such that:

(i) *f is an isomorphism over $Z - \{Q_1, \ldots, Q_m\}$;*
(ii) *$f^{-1}(Q_i)$ is a connected genus 1 subcurve of X such that*

$$\# (f^{-1}(Q_i) \cap \overline{X - f^{-1}(Q_i)}) = 1 \quad \textit{for } i = 1, \ldots, m.$$

Furthermore, if there exists a morphism $g: X \to Y$ such that:

(i) *g is an isomorphism over $Y - \{P_1, \ldots, P_n\}$; and*
(ii) *$g^{-1}(P_i)$ is a connected genus 1 subcurve of X such that*

$$\# (g^{-1}(P_i) \cap \overline{X - g^{-1}(P_i)}) = 1 \quad \textit{for } i = 1, \ldots, n,$$

then Z is isomorphic to Y.

Proof. We can choose irreducible open neighborhoods U_i of Q_i in Z such that $U_i - \{Q_i\}$ is nonsingular for $i = 1, \ldots, m$. By replacing each U_i with the normalization of U_i, we get a curve C and a morphism $\pi: C \to Z$ such that $\pi^{-1}(Q_i)$ is a non-singular point of C, and π is an isomorphism over $Z - \{Q_1, \ldots, Q_m\}$. Let X be the curve obtained by attaching non-singular elliptic curves E_i to C at the points $\pi^{-1}(Q_i)$ such that each $E_i \cap C$ is an ordinary double point. Let $f: X \to Z$ be the morphism which sends each E_i to Q_i and such that $f|_C = \pi$.

Since the only singularities of Z which are not ordinary cusps are ordinary double points, the only singularities of X are ordinary double points. Suppose D is a connected genus 0 subcurve of X. Then $D \subset C$ and the genus of $\pi(D)$ is equal to the number of cusps in $\pi(D)$. Also

$$\# (D \cap \overline{X - D}) = \# (D \cap \overline{C - D}) + \# (D \cap \overline{X - C})$$

$$= \# (\pi(D) \cap \overline{Z - \pi(D)}) + \text{ the genus of } \pi(D)$$

$$\geqslant 3$$

because Z is p-stable and has genus $\geqslant 3$. Hence X is M-D stable.

Now suppose we have a morphism $g: X \to Y$ as in the second half of the lemma.

Claim. $\overline{f^{-1}(Z - \{Q_1, \ldots, Q_m\})} = \overline{g^{-1}(Y - \{P_1, \ldots, P_n\})}$.
Proof of Claim. We will show that both subcurves of X satisfy the conditions of Lemma 4.1.

Conditions (i) and (ii) follow immediately from the requirements for each $f^{-1}(Q_i)$ and each $g^{-1}(P_i)$.

Let D be a connected genus 1 subcurve of $\overline{f^{-1}(Z-\{Q_1,\ldots,Q_m\})}$. Let N be the number of cusps in $f(D)$. Let $g_{f(D)}$ be the genus of $f(D)$. Then $1+N=g_{f(D)}$. Also

$$\#(D\cap\overline{X-D})=\#(f(D)\cap\overline{Z-f(D)})+N.$$

We have $g_{f(D)}+\#(f(D)\cap\overline{Z-f(D)})\geqslant 3$, because Z is p-stable of genus $\geqslant 3$. Hence

$$1+N+\#(D\cap\overline{X-D})-N\geqslant 3$$

$$1+\#(D\cap\overline{X-D})\geqslant 3$$

$$\#(D\cap\overline{X-D})\geqslant 2$$

and $\overline{f^{-1}(Z-\{Q_1,\ldots,Q_m\})}$ satisfies conditions (iii) of Lemma 4.1. A similar argument shows that $\overline{g^{-1}(Y-\{P_1,\ldots,P_n\})}$ satisfies the same condition.

Let $C=\overline{f^{-1}(Z-\{Q_1,\ldots,Q_m\})}=\overline{g^{-1}(Y-\{P_1,\ldots,P_n\})}$. For any point $R\in C$ we have $g(R)$ is a cusp if and only if $R\in\overline{X-C}$. Thus both $\mathcal{O}_Y$ and $\mathcal{O}_Z$ can be identified with the same subsheaf of $\mathcal{O}_C$, so Y and Z are isomorphic.   □

For the rest of this section we take R to be a discrete valuation ring which contains its residue field k. We let K denote the quotient field of R. When X is an R-scheme, we use X_0 to denote the special fiber and X_η to denote the generic fiber.

LEMMA 4.3. *Let X be an integral projective R-scheme such that X_η is a non-singular curve and X_0 is generically reduced. Then X is normal iff X_0 is reduced.*

Proof. Note that every point $P\in X$ such that $\dim(\mathcal{O}_{X,P})=1$, $\mathcal{O}_{X,P}$ is regular. This is clear if $P\in X_\eta$, since X_η is an open non-singular subscheme of X. We have X is flat over $\mathrm{Spec}(R)$, because X is integral and $X_\eta\neq\varnothing$. Hence $\dim(X_0)=\dim(X_\eta)=1$. If $P\in X_0$ and $\dim(\mathcal{O}_{X,P})=1$, then $\bar{P}$ contains a closed point Q such that X_0 is smooth at Q, because X_0 is generically reduced. Thus $\mathcal{O}_{X,Q}$ is regular, because the ideal of X_0 in $\mathcal{O}_{X,Q}$ is principal. Hence $\mathcal{O}_{X,P}$ is the localization of a regular local ring, so $\mathcal{O}_{X,P}$ is regular.

It follows that X is normal iff for every point $P\in X$ such that $\dim(\mathcal{O}_{X,P})=2$, we have $\mathrm{depth}(\mathcal{O}_{X,P})=2$. If $\dim(\mathcal{O}_{X,P})=2$, then P is a closed point of X_0. We have $\mathrm{depth}(\mathcal{O}_{X,P})=\mathrm{depth}(\mathcal{O}_{X_0,P})+1$. Since X_0 is generically reduced, it follows that $\mathrm{depth}(\mathcal{O}_{X_0,P})=1$ iff X_0 is reduced.   □

LEMMA 4.4. *Let X and Y be projective R-schemes which are flat over R. Suppose X_0 and Y_0 are reduced curves. Suppose that there is an isomorphism of K*

schemes $\varphi: X_\eta \to Y_\eta$. Suppose that X_η and Y_η are non-singular connected curves. Then there is a projective R-scheme Z which is flat over R and R-morphisms $\pi_1: Z \to X$, $\pi_2: Z \to Y$ such that:

(i) $\pi_1|_{Z_\eta}$ is an isomorphism;

(ii) $\pi_2 \circ \pi_1|_{X_\eta^{-1}} = \varphi$;

(iii) If P is a generic point of Z_0, then either $\pi_1(P)$ is a generic point of X_0, or $\pi_2(P)$ is a generic point of Y_0;

(iv) Z_0 is reduced; and

(v) Z is flat over R.

Proof. Let W be the closure of the image of X_η in $X \times_R Y$ under the map $\mathrm{id} \times \varphi$, and give W the reduced induced subscheme structure.

Note that since $X_\eta \cong Y_\eta \cong W_\eta$ are integral, we have X and Y are integral by flatness. Since W is the closure of an irreducible subset, W is irreducible. Thus W is integral, since W is reduced. Note that W is projective over R, because X, Y and $X \times_R Y$ are.

Let Z be the normalization of W. Then Z is integral and projective, because W is so. Since both Z and W are integral, they are flat over $\mathrm{Spec}(R)$ proving assertion (v). Let $\psi: Z \to W$ be the natural map of Z to W. Let π_1 be ψ composed with the projection of W onto X, and let π_2 be ψ composed with the projection of W onto Y.

Suppose P is a generic point of Z_0, and suppose $\pi_1(P)$ is a closed point Q of X_0. Then $\psi(P) \in W_0$ is generic, because the morphism ψ is finite. Since $\psi(P)$ lies in $\{Q\} \times Y_0$, the projection of $\psi(P)$ onto Y must be a generic point of Y_0. Thus $\pi(P)$ is a generic point of Y_0, and assertion (iii) holds.

Since X and Y are normal by Lemma 4.3, Zariski's main theorem applies. Thus each π_i is an isomorphism over a neighborhood of a point P if $\pi_i^{-1}(P)$ is finite. This is true for all generic points of X_0 and Y_0, because Z is flat over R implies Z_0 has dimension 1. Assertion (iii) now implies that every generic point P of Z_0 is contained in an open set U of Z which is isomorphic to either $\pi_1(U)$ or $\pi_2(U)$. Thus Z_0 is generically reduced. So Z_0 is reduced by Lemma 4.3, and assertion (iv) holds.

We have W_η is non-singular, since it is isomorphic to X_η. Hence the morphism $\varphi: Z \to W$ is an isomorphism over W_η, and assertion (i) follows. Assertion (ii) follows immediately from the construction of Z and W.

LEMMA 4.5. Let X, Y, Z, π_1 and π_2 be as in Lemma 4.4. Suppose $P \in X_0$ is a closed point such that π_1 is not an isomorphism over any open neighborhood of P. Let $E = \pi_1^{-1}(P)$, and let $F = \overline{Z_0 - E}$. Let $C = \pi_2(E)$. Then the map $\pi_2: Z \to Y$ is an isomorphism over an open neighborhood of each point $Q \in C - \{\pi_2(Q') \mid Q' \in E \cap F\}$.

Proof. As mentioned in the proof of the previous lemma, X and Y are normal by Lemma 4.3, so Zariski's main theorem applies.

Suppose $Q \in C$, and π_2 is not an isomorphism over Q. Then $\pi_2^{-1}(Q)$ is a connected subcurve of Z_0. There are no common components of $\pi_2^{-1}(Q)$ and E by assertion (iii) of Lemma 4.4. Hence $\pi_2^{-1}(Q) \subset F$. Since $Q \in C = \pi_2(E)$, there is a point $Q' \in \pi_2^{-1}(Q) \cap E \subset F \cap E$, and the lemma holds. $\square$

LEMMA 4.6. *Suppose X, Y, Z, π_1 and π_2 are as in Lemma 4.4. Suppose $P \in X_0$ is a closed point, and U is an open neighborhood of $P \in X$ such that π_1 is an isomorphism over $U - \{P\}$. Let W be the R-scheme obtained by glueing $X - \{P\}$ and $\pi_1^{-1}(U)$ by the isomorphism*

$$\pi_1|_{\pi_1^{-1}(U - \{P\})}: \pi_1^{-1}(U - \{P\}) \to U - \{P\}.$$

Then W is projective over R.
   Proof. Our proof follows the method of the proof of Theorem II.7.17 of [H].

   Claim. There exists a coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ such that:

   (i) $W \cong P(\mathscr{S})$, where $\mathscr{S}$ is the graded $\mathscr{O}_X$-algebra $\mathscr{O}_X \oplus \mathscr{I} \oplus \mathscr{I}^2 \oplus \cdots$; and
   (ii) $\mathscr{O}_X/\mathscr{I}$ has support only at P.

   Proof of Claim. First note that we only need to describe $\mathscr{I}$ in a neighborhood of P, since we can extend $\mathscr{I}$ to all of X by $\mathscr{O}_X$. Note that we are free to shrink U as much as necessary as long as it remains an open neighborhood of P.
   We may assume U is affine. Let $S = \mathscr{O}(U)$ and let F be the quotient field of S. Then there is a graded ring $T = \bigoplus_{i=0}^{\infty} T_i$ such that: $T_0 = S$; T is finitely generated by T_1 as a T_0-algebra; and $\pi_1^{-1}(u) \cong \mathrm{Proj}(T)$. There is an S-morphism of T_1 into F, because Z and X are birational. There exists an $f \in S$ such that $f T_1 \in S$ when T_1 is considered as a submodule of F. Let $J = f T_1$. Then $T \cong S \oplus J \oplus J^2 \oplus \cdots$. Since $T_1 \neq 0$ and S is integral, there are at most finitely many height-1 prime ideals of S which contain J. Since X is normal, we may shrink U so that every height-1 prime ideal which contains J is principal. We can then find $x \in S$ so that $I = \{a \mid ax \in J\}$ is not contained in any height-1 prime ideal. Then $T \cong S \oplus I \oplus I^2 \oplus \cdots$, because I^n is isomorphic to J^n via multiplication by x^n. We can shrink U so that I is not contained in any maximal ideal of S which does not correspond to P. Now I induces the claimed ideal sheaf.
   Theorem 5.5.3 of [Gr] now implies that $W = P(\mathscr{S}')$ where $\mathscr{S}'$ is a coherent graded $\mathscr{O}_{\mathrm{Spec}(R)}$-algebra. Hence W is projective over R, because $\mathrm{Spec}(R)$ is affine. $\square$

LEMMA 4.7. *Let X, Y, Z, π_1 and π_2 be as in Lemma 4.4. Let P be a closed point of X_0 Suppose that either X_0 is non-singular at P, or P is an ordinary double point. Then if π_1 is not an isomorphism over any open neighborhood of P, $\pi_1^{-1}(P)$ is a curve of genus 0.*
   Proof. We can choose an open neighborhood U of P such that π_1 is an

isomorphism over $U - \{P\}$. Then the R-scheme W obtained by glueing $\pi_1^{-1}(U)$ and $X - \{P\}$ by π_1 is projective by Lemma 4.6. Let π be the morphism from W to X We have W is flat over R because W is integral as Z and X are. Hence X_0 and W_0 have the same genus, and $\pi|_{W_0}$ is and isomorphism over $X_0 - \{P\}$. The lemma now follows. □

LEMMA 4.8. *Let X, Y, Z, π_1 and π_2 be as in Lemma 4.4. Suppose that X_0 is an M-D stable curve and that Y_0 is a p-stable curve of genus $\geqslant 3$. Then:*

(i) $\pi_1 : Z \to X$ *is an isomorphism;*
(ii) $\pi_2 : Z \to Y$ *is an isomorphism over $Y - \{P \mid P \in Y_0 \text{ is a cusp}\}$.*
(iii) *If $P \in Y_0$ is a cusp, then $E = \pi_2^{-1}(P)$ is a connected curve of genus 1 and*
$$\#(E \cap \overline{Z_0 - E}) = 1.$$

Proof. Since X_0 is M-D stable, all points in X_0 are either non-singular or ordinary double points. Hence, if $P \in X_0$ is a closed point for which π_1 is not an isomorphism over any open neighborhood of P in X, $E = \pi_1^{-1}(P)$ is a connected curve of genus 0 by Lemma 4.7. Let $C = \pi_2(E)$, and let $F = \overline{Z_0 - E}$. Then π_2 is an isomorphism over an open neighborhood in Y of each point in $C - \{\pi_2(Q') \mid Q' \in E \cap F\}$ by Lemma 4.5. Since P is either non-singular or an ordinary double point, there are at most two points in $E \cap F$.

Let $D = \overline{Y_0 - C}$. Note that if $Q \in C \cap D$, then there is a point $Q' \in E \cap F$ such that $Q = \pi_2(Q')$. This is clear if π_2 is an isomorphism over a neighborhood of Q and follows from the above otherwise. Thus $\#(C \cap D) = 0$, 1 or 2. We will consider these three cases.

Case $\#(C \cap D) = 0$: Then we have $Y_0 = C$, since Y_0 is connected. Then there are at most two points in C which do not have any open neighborhoods in C over which $\pi_2|_E : E \to C$ is an isomorphism.

Suppose $\pi_2|_E$ is an isomorphism. Then $C = Y_0$ has genus 0, which contradicts Y_0 has genus $\geqslant 3$.

Suppose that there is exactly one point $Q \in C$ such that $\pi_2|_E$ is not an isomorphism over any open neighborhood of Q in C. Then Q must be either a cusp or an ordinary double point. If Q is a cusp, then $Y_0 = C$ has genus 1, which is a contradiction. If Q is an ordinary double point, then Q has two pre-image points in E. Then $C = Y_0$ has genus 1, which is a contradiction.

Suppose that there are two points Q_1, $Q_2 \in C$ which do not have open neighborhoods in C over which $\pi_2|_E$ is an isomorphism. Both Q_1 and Q_2 have at most one pre-image point in E, because $E \cap F$ contains at most two points and each $\pi_2^{-1}(Q_i) \subset F$. Thus Q_1 and Q_2 are cusps. In this case $Y_0 = C$ has genus 2 which is a contradiction.

Case $\#(C \cap D) = 1$. The morphism $\pi_1|_E : E \to C$ is an isomorphism over an open neighborhood of $C \cap D$ in C, because C is non-singular at this point. There is at most one point $Q \in C$ which does not have an open neighborhood in C over

which $\pi_2|_E$ is an isomorphism, because $E \cap F$ consists of at most two points, one of which gets mapped to $C \cap D$.

If $\pi_2|_E$ is an isomorphism with C, then C is a genus 0 subcurve of Y_0 such that $\#(C \cap \overline{Y_0 - C}) < 3$, and Y_0 is not p-stable.

Suppose there is exactly one point $Q \in C$ which does not have an open neighborhood in C over which $\pi_2|_E$ is and isomorphism. Then Q must be a cusp, because $\#(\pi^{-1}(Q) \cap E) \leqslant 1$. So C is a genus 1 subcurve of Y_0 with $\#(C \cap \overline{Y_0 - C}) < 2$, and Y_0 is not p-stable.

Case $\#(C \cap D) = 2$. Then $\pi_2|_E : E \to C$ must be an isomorphism over an open neighborhood of the points in $C \cap D$, because C is non-singular at these points. Then $\pi_2|_E$ must be an isomorphism, because of the restriction that $E \cap F$ contains at most two points. Hence C is a genus 0 subcurve of Y_0 such that $\#(C \cap \overline{Y_0 - C}) < 3$, and Y_0 is not p-stable.

We have now proved that there is no point $P \in X$ such that $\pi_1^{-1}(P)$ is infinite. Hence π_1 is an isomorphism, and assertion (i) holds.

It follows from (i) that Z_0 is an M-D stable curve. Let $Q \in Y_0$ be an ordinary double point or a closed point at which Y_0 is non-singular. Suppose there is no open neighborhood of Q over which π_2 is an isomorphism. Then $E = \pi_2^{-1}(Q)$ has genus 0 by Lemma 4.7. If Y_0 is non-singular at Q, then

$$\#(\overline{(\pi_2^{-1}(Y_0 - \{Q\})} \cap \pi_2^{-1}(Q)) = \#(\overline{Z_0 - \pi_2^{-1}(Q)} \cap \pi_2^{-1}(Q)) = 1.$$

This implies that Z_0 is not M-D stable. If Q is an ordinary double point, then

$$\#(\overline{(\pi_2^{-1}(Y_0 - \{Q\})} \cap \pi_2^{-1}(Q)) = \#(\overline{Z_0 - \pi_2^{-1}(Q)} \cap \pi_2^{-1}(Q)) \leqslant 2.$$

Again, this implies that Z_0 is not M-D stable. We have now proved assertion (ii).

If $P \in Y_0$ is a cusp, then π_2 cannot be an isomorphism over P, because Z_0 does not have any cusps. We can choose an open neighborhood U of P in Y such that π_2 is an isomorphism over $U - \{P\}$. Let W be the scheme obtained by glueing $\pi_2^{-1}(U)$ and $Y - \{P\}$ via the isomorphism

$$\pi_2|_{\pi_2^{-1}(U - \{P\})} : \pi_2^{-1}(U - \{P\}) \to U - \{P\}.$$

Then W is projective over R by Lemma 4.6. Let π be the morphism from W to Y. We have W is flat over R, W is integral as Y and Z are. Hence W_0 has the same genus as Y_0. We have that P has one pre-image point $Q \in \overline{W_0 - \pi_2^{-1}(P)}$, because a cusp has only one pre-image point in the normalization of a reduced curve. The curve $F = \pi^{-1}(U - \{P\})$ is non-singular at the pre-image of P, because Z_0 which contains only ordinary double points as singularities. Hence the genus of F is one less than the genus of Y_0. Then $\pi^{-1}(P)$ must have genus 1, because $\#(F \cap \pi^{-1}(P)) = 1$. Thus assertion (iii) holds.   □

5. The moduli space of pseudo-stable curves

In this section we will construct the moduli space of pseudo-stable curves, and show that it is complete.

THEOREM 5.1. *If C is a smooth curve embedded by a complete linear system of degree $d > 2g$, then C is Chow stable.*

Proof. This is Theorem 4.15 of [Mu2]. □

It follows that any smooth curve C of genus $g > 2$ embedded by $\Gamma(C, \omega_C^{\otimes 3})$ is Chow stable.

Let $g \geqslant 3$, $d = 6(g-1)$, and $N = 5(g-1)-1$. Let H be the Hilbert scheme parameterizing connected subcurves of $\mathbf{P}^N$ with Hilbert polynomial $P(n) = dn + 1 - g$. Let Ch be the Chow variety parameterizing 1-cycles of degree d in $\mathbf{P}^N$. Let $H_3 \subset H$ be the subscheme of 3-canonically embedded curves of genus g in $\mathbf{P}^N$. Let Ch_s be the open subset of Ch consisting of GIT-stable points under the action of $SL(N+1)$. Theorem 5.10 of [Mu1] says that there is a morphism $\Phi: H \to \text{Ch}$. Let $U_C = \Phi^{-1}(\text{Ch}_s) \cap \overline{H_3}$. Let $Z \to H$ be the universal curve over H, and for $h \in H$, let X_h denote the fiber over h of $Z \to H$.

LEMMA 5.2. *If $h \in U_C$, then X_h is p-stable.*

Proof. There is a morphism $\text{Spec}(k[[t]]) \to U_C$ such that the special point maps to h and $\mathscr{X} = Z \times_{U_C} \text{Spec}(k[[t]])$ has a smooth generic fiber over $\text{Spec}(k[[t]])$. Let $\mathscr{X}_\eta$ denote the generic fiber and $\mathscr{X}_0 = X_h$ denote the special fiber. Since $\mathscr{X}_0$ is Chow semi-stable, it follows from the lemmas in §2 that $\mathscr{X}_0$ has no triple points, no cusps which are not ordinary, no tachnodes which do not involve a line, and is generically reduced. Lemma 2.6 shows that $\mathcal{O}_{\mathscr{X}_0}(1) \cong \omega_{\mathscr{X}_0}^{\otimes 3}$. It follows that $\mathscr{X}_0$ does not contain any lines, so it does not have any tachnodes. Further $\mathscr{X}_0$ cannot have any rational subcurves that meet the rest of the $\mathscr{X}_0$ at 1 or 2 points. Lemma 3.1 show that $\mathscr{X}_0$ does not have any elliptic tails. Thus $\mathscr{X}_0$ must be p-stable. □

The map $\Phi|_{U_C}: U_C \to \text{Ch}$ is one-to-one and therefore quasi-affine. Now Proposition 1.8 of [Mu1] shows that the points in U_C are GIT-stable under the action of $SL(N+1)$ on the Hilbert scheme induced by $\Phi: H \to \text{Ch}$, because $U_C \subset \Phi^{-1}(\text{Ch}_s)$ and $\Phi|_{U_C}$ is quasi-affine.

Let Q be the GIT-quotient of U_C by the action of $SL(N+1)$. The following lemma shows that Q is a moduli space for p-stable curves.

LEMMA 5.3. *Suppose C is a p-stable curve. Then there is an $h \in U_C$ such that $X_h \cong C$.*

Proof. Let Y_0 be the M-D stable curve associated to C as in Lemma 4.2. There is a morphism $Y \to \text{Spec}(k[[t]])$ such that the generic fiber Y_η is smooth and the special fiber is Y_0. Embed Y_η in $\mathbf{P}^N_{k((t))}$ by $\Gamma(Y_\eta, \omega_{Y_\eta}^{\otimes 3})$ and let $\Psi(Y_\eta)$ denote its image there. Then Lemma 5.3 of [Mu2] says that by replacing $k[[t]]$ with some finite extension and choosing a suitable basis of $\Gamma(Y_\eta, \omega_{Y_\eta}^{\otimes 3})$ we may assume that the

closure X in $\mathbf{P}^N_{k[[t]]}$ of $\Psi(Y_\eta)$ satisfies

  (i) $X_\eta = Y_\eta$; and
  (ii) X_0 is Chow stable or Chow semi-stable with positive dimensional stabilizer.

The proof of the last lemma shows that X_0 must be p-stable. Lemma 4.8 says there is a Spec($k[[t]]$)-morphism $X \to Y$ with properties that lead us to conclude $X_0 \cong C$ by Lemma 4.2.

It is easy to see that any p-stable curve C with genus $\geqslant 3$ has only finitely many automorphisms. Each subcurve birational to $\mathbf{P}^1$ contains 3 points which are cusps or ordinary double points of C. Each subcurve birational to an elliptic curve contains at least 2 points which are cusps or ordinary double points of C. Thus the p-stable curve X_0 must be Chow stable and not just Chow semi-stable. Thus X_0 corresponds to a point in U_C. $\square$

We are now ready to prove

THEOREM 5.4. *The GIT-quotient Q of U_C is a complete moduli space for p-stable curves.*

Proof. It remains to show that Q is complete. Let Q' be the quotient of the semi-stable points of $\overline{U_C}$. Then GIT says that Q' is complete, and we have $Q \subset Q'$.

We will show $Q = Q'$. Let P be a point of Q'. Then there is a curve $Y \subset \mathbf{P}^N_{k[[t]]}$ such that Y_η is smooth with the $\omega^{\otimes 3}_{Y_\eta}$ embedding and Y_0 corresponds to $P \in Q'$. Thus Y induces a map Spec($k[[t]]$) $\to Q'$ where the special point is mapped to P. Lemma 5.3 of [Mu2] says that by replacing $k[[t]]$ with some finite extension and choosing a suitable embedding $\Psi \colon Y_\eta \to \mathbf{P}^N_{k((t))}$, we may assume that the closure X of $\Psi(Y_\eta)$ in $\mathbf{P}^N_{k[[t]]}$ is such that

  (i) $X_\eta = \Psi(Y_\eta)$ is embedded by $\Gamma(X_\eta, \omega^{\otimes 3}_{X_\eta})$; and
  (ii) X_0 is Chow stable.

Now Y and X induce the same mapping of Spec($k[[t]]$) to Q'. But X_0 corresponds to a point in Q. Thus $P \in Q$. $\square$

Acknowledgement

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