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A regulator map for surfaces with an isolated singularity

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Introduction

We denote by X a projective surface with an isolated singularity x_0 and $p: S \to X$ the desingularization of X. We assume that the exceptional divisor $C = p^{-1}(x_0)$ is irreducible and non-singular. The example we have in mind is the cone over a non-singular curve C.

Our aim is to construct a map $\rho: H^1(X, \mathscr{K}_{2X})_h \to ?$, which extends to this case the regulator map $r: H^1(S, \mathscr{K}_{2S})_h \to (F^1H^2(S, \mathbb{C}))^*/H_2(S, \mathbb{Z})(1)$. Here $H^1(S, \mathscr{K}_{2S})_h$ is the subgroup of $H^1(S, \mathscr{K}_{2S})$ of 'homologically trivial classes' (the definition is recalled below) and F^1 is the Hodge filtration. This regulator map r is a generalization of the Picard map

$$\mathrm{Pic}^0(Z) = H^1(Z, \mathscr{K}_{1Z})_h \to (F^dH^{2d-1}(Z, \mathbb{C}))^*/H_{2d-1}(Z, \mathbb{Z})$$

where Z is a non-singular variety of dimension d. The regulator map for non-singular varieties has been defined in vast generality by Beilinson, who has built on previous work of Bloch. We shall use the regulator map r only for $H^1(S, \mathscr{K}_{2S})$; we have learned about it by reading [8], our construction of ρ is motivated by the construction of r given there.

The point we want to make is that the map ρ is useful in detecting information about the singularity and that it shows the need for a generalization of the Hodge theory that can provide information also in the 'unibranch case', by this we mean here a singularity for which the ordinary cohomology of the singular variety embeds in the cohomology of the desingularization. More precisely, as already happens with the Picard group of a cuspidal curve, $H^1(X, \mathscr{K}_{2X})$ contains some data which is not detected by the mixed-Hodge structure on the cohomology of X but which is detected by computing ρ by means of integration of a certain type of differentials of second kind. We recover in this way a result of Srinivas, see [10], to the effect that there is a copy of the additive group $\mathbb{C}$ in the

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kernel of the map from K_1 of the ordinary quadratic cone to K_1 of the desingularization.

1. The space H of forms of the second kind

Our aim here is to produce a convenient cohomological space which is described in terms of differential forms on S having poles along C and which has similar properties in the case of the singular variety X to the properties which $F^1H^2(S, \mathbb{C})$ has for S. It turns out that a certain subspace H of the cohomology space $H^1(S, \Omega_S^{\wedge 1}(2C))$, to be described below, is the right object.

We denote $\Omega_S^1(mC)$ the sheaf of meromorphic forms on S with pole on C of order at most m, and $\Omega_S^{\wedge 1}(mC)$ the subsheaf of closed forms.

By Poincare's lemma for holomorphic differentials the sequence

$$0 \to \mathbb{C} \to \mathcal{O}_S \to \Omega_S^{\wedge 1} \to 0 \text{ is exact.}$$

The Poincare's residue map (see [3]) from forms on S to forms on C gives the exact sequence

$$0 \to \Omega_S^{\wedge 1} \to \Omega_S^{\wedge 1}(1C) \to \mathbb{C}_C \to 0.$$

The sequence

$$0 \to \mathcal{Q} \to \Omega_S^{\wedge 1}(2C) \to \mathbb{C}_C \to 0$$

is obtained by considering the differential map $\mathrm{d}\colon \mathcal{O}_S(C) \to \Omega_S^{\wedge 1}(2C)$, the sheaf $\mathcal{Q}$ being the image sheaf $\mathrm{d}(\mathcal{O}_S(C))$:

$$0 \to \mathbb{C}_S \to \mathcal{O}_S(C) \to \mathcal{Q} \to 0.$$

By the proof of [3, (10.19)] the differential map

$$\mathrm{d}\colon \mathcal{O}_S(C) \to \Omega_S^{\wedge 1}(2C) \text{ has cokernel } \mathbb{C}_C$$

$$\mathcal{O}_S(C) \to \Omega_S^{\wedge 1}(2C) \to \mathbb{C}_C \to 0.$$

Our sheaves fit in the following diagram, whose first column is exact, because of the second diagram below.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & & \hat{\Omega}_S^1 & \to & \hat{\Omega}_S^1(1C) & \to \mathbb{C}_C \to 0 \\
 & & \downarrow & & \downarrow & \| \\
0 \to & & \mathcal{Q} & \to & \hat{\Omega}_S^1(2C) & \to \mathbb{C}_C \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_C(C) & = & \mathcal{O}_C(C) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
0 \to & \mathbb{C}_S & \to & \mathcal{O}_S & \to & \hat{\Omega}_S^1 & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0 \to & \mathbb{C}_S & \to & \mathcal{O}_S(C) & \to & \mathcal{Q} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_C(C) & = & \mathcal{O}_C(C) & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

From the first diagram we recover this exact sequence

$$0 \to \hat{\Omega}_S^1 \to \hat{\Omega}_S^1(2C) \to \mathcal{O}_C(C) \oplus \mathbb{C}_C \to 0$$

The associated sequence of cohomology is

$$\begin{aligned}
\cdots H^m(S, \hat{\Omega}_S^1) &\to H^m(S, \hat{\Omega}_S^1(2C)) \\
&\to H^m(C, \mathcal{O}_C(C)) \oplus H^m(C, \mathbb{C}_C)) \\
&\to H^{m+1}(S, \hat{\Omega}_S^1) \to H^{m+1}(S, \hat{\Omega}_S^1(2C)) \\
&\to H^{m+1}(C, \mathcal{O}_C(C)) \oplus H^{m+1}(C, \mathbb{C}_C))
\end{aligned}$$

We recall, cf. [3], that in general for any smooth projective variety S: $H^m(S, \hat{\Omega}_S^1) \approx F^1 H^{m+1}(S, \mathbb{C}) \subset H^{m+1}(S, \mathbb{C})$, the main reason for this being that the map

$$\mathrm{d}: H^m(S, \mathcal{O}_S) \to H^m(S, \hat{\Omega}_S^1) \text{ is } 0.$$

In our case the inclusion $H^m(S, \hat{\Omega}_S^1) \subset H^{m+1}(S, \mathbb{C})$ factors through

$H^m(S, \Omega_S^{\wedge 1}) \to H^m(S, \mathscr{Q})$, which is therefore injective. It follows that $H^{m-1}(S, \mathscr{Q}) \to H^{m-1}(C, \mathcal{O}_C(C))$ is surjective, hence also $H^{m-1}(S, \Omega_S^{\wedge 1}(2C)) \to H^{m-1}(C, \mathcal{O}_C(C))$ is surjective.

Since S is projective then $H^0(C, \mathbb{C}_C) \to H^1(S, \Omega_S^{\wedge 1})$ is an injective map, in fact the space $H^0(C, \mathbb{C}_C)$ is sent to the space generated by the class of C in $H^1(S, \Omega_S^{\wedge 1}) = F^1H^2(S, \mathbb{C}) \subset H^2(S, \mathbb{C})$.

Collecting these facts we find the exact sequence

$$\begin{aligned} 0 \leftarrow H^0(C, \mathbb{C}_C) &\to H^1(S, \Omega_S^{\wedge 1}) \to H^1(S, \Omega_S^{\wedge 1}(2C)) \\ &\to H^1(C, \mathcal{O}_C(C)) \oplus H^1(C, \mathbb{C}_C) \to H^2(S, \Omega_S^{\wedge 1}) \to \cdots \end{aligned}$$

We have an isomorphism $H^2(S, \Omega_S^{\wedge 1}) \approx H^3(S, \mathbb{C})$, because S is a projective surface. Let P be the kernel of $H^1(C, \mathbb{C}_C) \to H^3(S, \mathbb{C})$. The following sequence is exact

$$\begin{aligned} 0 \to H^0(C, \mathbb{C}_C) &\to H^1(S, \Omega_S^{\wedge 1}) \to H^1(S, \Omega_S^{\wedge 1}(2C)) \\ &\to H^1(C, \mathcal{O}_C(C)) \oplus P \to 0 \end{aligned}$$

We define H to be the kernel of $H^1(S, \Omega_S^{\wedge 1}(2C)) \to P$. Note the sequence

$$0 \to (H^1(S, \Omega_S^{\wedge 1})/H^0(C, \mathbb{C}_C)) \to H \to H^1(C, \mathcal{O}_C(C)) \to 0 \tag{1.1}$$

We shall call H the space of forms of second kind, in analogy with the definition used in [4]. Indeed we will show that the cohomology space $H^1(S, \Omega_S^{\wedge 1}(2C))$ can be described in term of classes of certain C^∞ forms with poles along C, and then the elements of H are represented by the forms with zero residue, because H is by definition the kernel of the residue map $H^1(S, \Omega_S^{\wedge 1}(2C)) \to H^1(C, \mathbb{C}_C)$. We shall use the space H as the generalization to the singular case of the Hodge space $H^1(S, \Omega_S^{\wedge 1}) = F^1H^2(S, \mathbb{C})$. More precisely we use H in the same way as $H^1(S, \Omega_S^{\wedge 1})$ is used in the construction of the regulator map in the non-singular case, i.e. we show that a quotient of the dual space H^* can be used as the range of the desired map ρ. We will see in the case of the ordinary quadratic singularity that the map ρ can be used to locate non-trivial elements in the kernel of the map from K_1 of a singular surface to the K_1 group of the desingularization.

The space H of second kind differentials plays the same role for the K_1 groups of a surface with an isolated singularity as the vector space of Rosenlicht differentials plays for the Picard group of a curve with an ordinary cusp singularity. In both cases the vector spaces are described by means of forms in the desingularization which have poles of second kind along the exceptional locus; integration with such forms allows to detect elements in the group which are killed in the desingularization. In the case of the Picard variety of the cuspidal curve the space of differentials involved is the space of meromorphic differentials with at most a pole of order 2 along the distinguished point in the desingularization. In the following section we describe H in a similar way, by means of a Dolbeault-like theorem.

2. Dolbeault cohomology

For the sake of generality we deal in this part with a non-singular variety M of dimension n with a distinguished non-singular divisor D. $\Omega^q(k)$ or $\Omega^q(kD)$ denotes the sheaf of meromorphic q-forms on M which have poles on D of total order $\leqslant k$, i.e. the stalk of $\Omega^q(k)$ at a point is the vector space of forms $\varphi = \sum_{0 \leqslant v \leqslant k} f^{-v} \varphi_v$, where φ_v is a holomorphic q-form and f is a local defining equation for D. For each $k \geqslant 1$ there is a complex of sheaves $\Omega^*(k)$, defined as

$$\Omega^0(k) \xrightarrow{d} \Omega^1(k+1) \xrightarrow{d} \Omega^2(k+2) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(k+n) \to 0$$

where $\Omega^0(k) = \mathcal{O}(kD)$.

It follows from Section 10 of [3] that the complexes $\Omega^*(k)$, $k \geqslant 1$, are all quasi-isomorphic. Let the cohomology sheaves of the complex $\Omega^*(k)$ be $\mathscr{L}^q(k) = \Omega^{\wedge q}(k+q)/d\Omega^{q-1}(k+q-1)$, then (i) $\mathscr{L}^0(k)$ is the constant sheaf $\mathbb{C}_M$, (ii) $\mathscr{L}^1(k)$ is isomorphic to the constant sheaf $\mathbb{C}_D$, (iii) $\mathscr{L}^q(k) = 0$, $q \geqslant 2$.

The same result holds for the complex of the sheaves $\Omega^q(*)$, which are the sheaves of meromorphic forms with poles of arbitrary order on D.

Motivated by the isomorphism $(+) F^1H^2(M, \mathbb{C}) \cong H^1(M, \Omega^{\wedge 1})$ between the Hodge filtration and the cohomology of the sheaf of closed regular forms, we are interested in the cohomology spaces $H^1(M, \Omega^{\wedge 1}(*))$ and $H^1(M, \Omega^{\wedge 1}(k))$.

The proof of the isomorphism $(+)$ uses the fine resolution

$$0 \to \Omega^{\wedge 1} \to \mathscr{A}^{1,0} \xrightarrow{d} \mathscr{A}^{2,1} \xrightarrow{d} \mathscr{A}^{3,2} \xrightarrow{d} \cdots$$

where we denote by $\mathscr{A}^{q,m}$ the sheaf on M of C^∞ forms of type $(q,0) + \cdots + (q-m, m)$ and $\mathscr{A}^{(a,b)}$ the space of forms of type (a,b), so that $\mathscr{A}^{q,m} = \bigoplus \mathscr{A}^{(a,b)}$, with $a + b = q$ and $0 \leqslant b \leqslant m$.

We consider the analogous sequences

$$0 \to \Omega^{\wedge 1}(k) \to \mathscr{A}^{1,0}(k) \xrightarrow{d} \mathscr{A}^{2,1}(k+1) \xrightarrow{d} \mathscr{A}^{3,2}(k+2) \xrightarrow{d} \cdots$$

and

$$(*) \quad 0 \to \Omega^{\wedge 1}(*) \to \mathscr{A}^{1,0}(*) \xrightarrow{d} \mathscr{A}^{2,1}(*) \xrightarrow{d} \mathscr{A}^{3,2}(*) \xrightarrow{d} \cdots$$

where we denote by $\mathscr{A}^{q,m}(k+q)$ the sheaves of q differential forms φ of type $(q,0) + \cdots + (q-m, m)$, which are C^∞ on $M - D$ and which have the local property that $f^{(q+k)}\varphi \in \mathscr{A}^{q,m}(M)$ if f is a local defining analytic equation for D. $\mathscr{A}^{q,m}(*)$ is the limit of the sheaves $\mathscr{A}^{q,m}(k)$. Note that the sheaves $\mathscr{A}^{q,m}(*)$ are acyclic, because they admit partition of unity.

We denote $A^{q,m}(*)$ the space of global sections of $\mathscr{A}^{q,m}(*)$, we denote $Z^{q,m}(*)$ the subspace of the closed forms and $B^{q,m}(*)$ the subspace of the exact forms. We shall show that the complex $(*)$ is exact, whence

(2.1) PROPOSITION. $H^1(M, \Omega^{\wedge 1}(*)) = Z^{2,1}(*)/B^{2,1}(*)$.

In other words an element of $H^1(M, \Omega^{\wedge 1}(*))$ is represented by a closed C^∞ 2-form of type $(2, 0) + (1, 1)$ with a pole of arbitrary order along D, 0 is represented by the forms of this type which are the differential of a C^∞ 1-form of type $(1, 0)$ a pole of arbitrary order along D.

Proof. We look at the following complex of sheaves on M

$$(**) \quad \mathscr{A}^0(*) \xrightarrow{d} \mathscr{A}^1(*) \xrightarrow{d} \mathscr{A}^2(*) \xrightarrow{d} \cdots$$

Using lemma (8.7) of [3], we see that this complex is quasi-isomorphic to the complex $\mathscr{A}^*\langle \log D \rangle$

$$\cdots \xrightarrow{d} \mathscr{A}^{q-1}\langle \log D \rangle \xrightarrow{d} \mathscr{A}^{q}\langle \log D \rangle \xrightarrow{d} \mathscr{A}^{q+1}\langle \log D \rangle \xrightarrow{d} \cdots$$

where $\mathscr{A}^q\langle \log D \rangle$ is the sheaf of C^∞ q-forms with log poles along D, cf. [4]. Now the cohomology sheaves of this complex are in degree 0 $\mathbb{C}_M$, in degree 1 $\mathbb{C}_D$ and otherwise they are 0, cf. [5].

We need to consider another complex of sheaves

$$(***) \quad \mathscr{A}^{(0,0)}(*) \xrightarrow{\bar\partial} \mathscr{A}^{(0,1)}(*) \xrightarrow{\bar\partial} \mathscr{A}^{(0,2)}(*) \xrightarrow{\bar\partial} \mathscr{A}^{(0,3)}(*) \rightarrow$$

where $\mathscr{A}^{(0,p)}(*)$ denotes the sheaf of C^∞ p-forms of type $(0, p)$ with a pole of arbitrary order along D, and $\bar\partial$ is as usual. This complex is a resolution of the sheaf $\mathscr{O}(*) = \Omega^0(*)$ of meromorphic functions with arbitrary pole along D. The complexes give a diagram with exact columns:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathscr{A}^{1,0}(*) & \rightarrow & \mathscr{A}^{2,1}(*) & \rightarrow & \mathscr{A}^{3,2}(*) & \rightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathscr{A}^{1}(*) & \rightarrow & \mathscr{A}^{2}(*) & \rightarrow & \mathscr{A}^{3}(*) & \rightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathscr{A}^{(0,1)}(*) & \rightarrow & \mathscr{A}^{(0,2)}(*) & \rightarrow & \mathscr{A}^{(0,3)}(*) & \rightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

This is an exact sequence of the three horizontal complexes, hence the cohomology sheaves of the three complexes fit in a long sequence. Now we know that the second and the third complex are exact but for the first cohomology sheaf, which are the kernels $\mathscr{S}$ of $\mathscr{A}^1(*) \to \mathscr{A}^2(*)$ and $\mathscr{T}$ of $\mathscr{A}^{(0,1)}(*) \to \mathscr{A}^{(0,2)}(*)$. By exactness in (***), one has $\mathscr{T} = \bar{\partial}(\mathscr{A}^0(*))$. The first complex in the diagram is therefore a resolution of the kernel K of the surjection $\mathscr{S} \to \bar{\partial}(\mathscr{A}^0(*))$. We want to show that the kernel K is $\Omega^{\wedge 1}(*)$. Since $\Omega^*(*D)$ and $\mathscr{A}^*(*)$ are both quasi-isomorphic to $\Omega^*\langle \log D\rangle$, we have a map from the first exact sequence below to the second one. This map is the identity on $\mathbb{C}_D$.

$$0 \to d\mathcal{O}(*D) \to \Omega^{\wedge 1}(*) \to \mathbb{C}_D \to 0$$

$$0 \to d(\mathscr{A}^0(*)) \to \mathscr{S} \to \mathbb{C}_D \to 0.$$

In order to prove $K = \Omega^{\wedge 1}(*)$ one has only to check that the inclusion $d\mathcal{O}(*D) \subset (d(\mathscr{A}^0(*)) \cap \mathscr{A}^{10}(*))$ is in fact an equality. Locally this amounts to show that if $\omega = d(gf^{-n})$ is of type $(1, 0)$ then g is analytic, but this is clear since we are saying $\bar{\partial} g = 0$. We have proved that the following complex is exact

$$0 \to \Omega^{\wedge 1}(*) \to \mathscr{A}^{1,0}(*) \xrightarrow{d} \mathscr{A}^{2,1}(*) \xrightarrow{d} \mathscr{A}^{3,2}(*) \to \cdots$$

the proposition follows, because the sheaves $\mathscr{A}^{q,q-1}(*)$ are acyclic.

The analogous complex for forms with finite order poles

$$0 \to \Omega^{\wedge 1}(k) \to \mathscr{A}^{1,0}(k) \xrightarrow{d} \mathscr{A}^{2,1}(k+1) \xrightarrow{d} \mathscr{A}^{3,2}(k+2) \xrightarrow{d} \cdots$$

is not exact in general. The proof given for the complex of forms with arbitrary pole cannot be used here, because the third sequence in the diagram is not exact any longer, since $\bar{\partial}$ does not increase the order of the pole. On the other hand by using the exactness of

$$0 \to \Omega^{\wedge 1}(k) \to \mathscr{A}^{1,0}(k) \to d(\mathscr{A}^{1,0}(k)) \to 0,$$

we see that $H^1(M, \Omega^{\wedge 1}(k)$ is represented by the space of global C^∞ 2-forms of type $(2, 0) + (1, 1)$ which are locally the differential of a form of type $(1, 0)$ which has a pole along D of order k at most, modulo exact ones. That is

(2.2) PROPOSITION

$$H^1(M, \Omega^{\wedge 1}(2)) = H^0(M, d(\mathscr{A}^{1,0}(2)))/dA^{1,0}(2)$$

In fact something more is true, it is possible to choose forms with a pole of order at most two as representative for classes in $H^1(M, \Omega^{\wedge 1}(2))$. Let

$\omega \in H^0(M, d(\mathscr{A}^{1,0}(2)))$ represent the class $[\omega]$ in $H^1(M, \Omega^{\wedge 1}(2))$, let U_i be a cover of M, such that on each U_i $\omega = \omega_i = d\sigma_i$, where $\sigma_i \in \mathscr{A}^{1,0}(2)(U_i)$, i.e. σ_i is of type $(1,0)$ with a pole of order at most 2 along D; let g_i be a partition of unity associated with the covering $\{U_i\}$. We define $\alpha = \sum g_i\sigma_i$, so that α is in $A^{1,0}(2)$. Then the form $\omega' = \omega - d\alpha = -\sum (dg_i)\sigma_i$ has a pole of order 2 at most along D.

We come back to the case of the surface S with a distinguished divisor C. Let ω represent an element in the vector space $H \subset H^1(S, \Omega^{\wedge 1}(2C))$. We may assume that ω has a pole of order at most 2 along C. Since $[\omega] \in H$, by definition the class determined by ω in $H^2(S - C, \mathbb{C})$ maps to 0 in $H^1(C, \mathbb{C})$, therefore it is the restriction of a class from $H^2(S, \mathbb{C})$. Let β be a $\mathscr{C}^\infty$ form on S which represents this class. Let $\varphi = \omega - \beta$, then φ is closed, indeed exact on $S - C$, and, locally, $f^2\varphi$ is smooth on S. An easy argument along the lines of [3, lemma 8.7, lemma 8.9] shows that $\varphi = \mathrm{d}\eta' + \psi$, where ψ and η' have both poles of order 1 at most along C. Further ψ is exact in the full De-Rham complex of $S - C$, because φ is exact. Now ψ belongs to log De-Rham complex of $S - C$, which is quasi-isomorphic to the full De-Rham complex, therefore ψ is exact also in the log De-Rham complex, i.e. ψ is the differential of a form with at most a pole of order one. We conclude $\varphi = \mathrm{d}\eta$, where η has a first order pole at most. We have shown

(2.3) PROPOSITION. *A class in the space H may be represented by a closed form ω of type* $(2,0) + (1,1)$ *such that*: (i) *ω is $\mathscr{C}^\infty$ on $S - C$ and it has a pole of order at most* 2 *along C*, (ii) $\omega = \beta + \mathrm{d}\eta$, *where β is a smooth closed form on S and η is $\mathscr{C}^\infty$ on $S - C$ and it has a pole of order* 1 *at most along C.*

From the sequence

$$0 \to \Omega_S^{\wedge 1} \to \Omega_S^{\wedge 1}(2C) \to \mathcal{O}_C(C) \oplus \mathbb{C}_C \to 0$$

we have induced maps from the acyclic resolution of $\Omega_S^{\wedge 1}(2C)$ to the Dolbeault resolution of $\mathcal{O}_C(C)$. The map from H to $H^1(C, \mathcal{O}_C(C))$ can be described on representative elements of the type given in the proposition in the following way.

By changing further ω in its class in H, we may assume that η is of type (0, 1) and then $\bar{\partial}\eta$ has no poles on C. If locally $\eta = \eta'/f$, then $\bar{\partial}\eta' = f\gamma$, where γ is a $\mathscr{C}^\infty$ form and f is a local analytic equation for C. The restriction of η' to C is $\bar{\partial}$ closed, and the transition functions for η' are the same as for the local equations f of C. This means that the restrictions to C of the local forms η' glue to give a 1-cocycle in the Dolbeault resolution of the sheaf $\mathcal{O}_C(C)$. We will sometime refer to this 1-cocycle as the second residue form of ω and write it as $\mathrm{SR}(\omega)$, in fact

(2.4) LEMMA. *The image of class(ω) in $H^1(C, \mathcal{O}_C(C))$ is represented by* $\mathrm{SR}(\omega)$.

There should be no risk of confusion of the second residue SR with the usual notion of the 'topological residue' R: $H^2(S - C, \mathbb{C}_S) \to H^1(C, \mathbb{C}_C)$, because the definition just given applies only to representative forms for elements of H, which by definition have zero topological residue.

Proof of (2.4). We have $\omega = \beta + d\eta$, where ω is closed of type $(2,0)+(1,1)$ and $\eta \in \mathscr{A}^{(0,1)}(\mathcal{O}_S(C))$. We write $\beta = \beta' + \beta''$, with β' of type $(2,0)+(1,1)$ and β'' of type $(0,2)$. Then $0 = \bar\partial\eta + \beta''$, therefore β'' is $\bar\partial$ closed. Since β'' is smooth it represents a Dolbeault class for $H^2(S, \mathcal{O}_S)$.

Let a be the map $H^1(S, \mathcal{O}_S(C)) \to H^1(C, \mathcal{O}_C(C))$, let b be the map $H \to H^1(C, \mathcal{O}_C(C))$, let δ be the map $H^1(S, \mathcal{O}_S(C)) \to H$ induced by $H^1(S, \mathcal{O}_S(C)) \to H^1(S, \Omega^{\wedge 1}(2C))$. Since $\mathcal{O}_S(C)) \to \mathcal{O}_C(C)$ factors through $\mathcal{O}_S(C)) \to Q$, $Q \to \Omega^{\wedge 1}(2C)$, $\Omega^{\wedge 1}(2C) \to \mathcal{O}_C(C)$, it follows by functoriality that $a = b\delta$. By our description δ is induced by the differential map $\mathrm{d}\colon \mathcal{O}_S(C) \to \Omega^{\wedge 1}(2C)$, hence $\delta(\mathrm{class}(\alpha)) = \mathrm{class}(\mathrm{d}\alpha)$, where α is a $\bar\partial$ closed form which represents a class in $H^1(S, \mathcal{O}_S(C))$.

We assume for a moment that $H^2(S, \mathcal{O}_S) = 0$. In this case $\mathrm{class}(\beta'') = 0$, so that there is a form $\varphi \in \mathscr{A}^{(0,1)}(\mathcal{O}_S)$ with $\beta'' = \bar\partial\varphi$. We can write $\omega = (\beta' - \partial\varphi) + d(\varphi + \eta)$; since ω is closed also $(\beta' - \partial\varphi)$ is closed and therefore $(\beta' - \partial\varphi)$ represents a class in $H^1(S, \Omega^{\wedge 1})$. On the other hand $\varphi + \eta \in \mathscr{A}^{(0,1)}(\mathcal{O}_S(C))$ and it is $\bar\partial$ closed, so $\varphi + \eta$ represents a class in $H^1(S, \mathcal{O}_S(C))$. Locally $\eta = \eta'/f$, hence $\varphi + \eta = (\mathrm{f}\varphi + \eta')/f$; since f is a local equation for C both $(f\varphi + \eta')$ and η' restrict to the same form on C, this form being $\mathrm{SR}(\omega)$. Direct computation of the arrows induced from the Dolbeault resolution of $\mathcal{O}_S(C)$ to the one of $\mathcal{O}_C(C)$ shows $a(\mathrm{class}(\varphi+\eta)) = \mathrm{class}(\mathrm{SR}(\omega))$. Therefore $\mathrm{class}(\mathrm{SR}(\omega)) = b(\mathrm{class}(\omega))$, because of: (i) $b(\mathrm{class}(\omega)) = b(\mathrm{class}(d(\varphi + \eta)))$, since $(\omega - d(\varphi + \eta)) = (\beta' - \partial\varphi)$ comes from $H^1(S, \Omega^{\wedge 1})$, (ii) $b(\mathrm{class}(d(\varphi + \eta))) = b\delta(\mathit{class}(\varphi + \eta)) = a(\mathrm{class}(\varphi + \eta))$.

If $H^2(S, \mathcal{O}_S) \neq 0$, still it follows from (III.11.2) of [6] and GAGA that there is a analytic neighborhood S^+ of C with $H^2(S^+, \mathcal{O}_{S^+}) = 0$. The same argument of before applies.

(2.5) Let $K(S)$ be the image of $H_2(S, \mathbb{Z})(1)$ in $H^1(S, \Omega^{\wedge 1})^*$ and let $K(X)$ be the image of $H_2(S - C, \mathbb{Z})(1)$ in H^*. We shall denote $B(S) := H^1(S, \Omega^{\wedge 1})^*/K(S)$. The group $B(S)$ is the range of the regulator map of Levine, $r\colon H^1(S, \mathscr{K}_{2S})_h \to B(S)$. Similarly we define $B(X) := H^*/K(X)$.

LEMMA. $0 \to H^1(C, \mathcal{O}_C(C))^* \to B(X) \to B(S)$ is exact.

Proof. The map $K(X) \to K(S)$ is injective, because H is by definition the space of forms with zero residue on C. Recalling sequence (1.1) and the snake lemma, we see that our statement is equivalent to the injectivity of $K(S)/K(X) \to H_0(C, \mathbb{C})$. There is a long exact sequence $H_m(S) \to H_{m-2}(C) \to H_{m-1}(S - C) \to H_{m-1}(S)$, from which it follows that $K(S)/K(X)$ is isomorphic with a subgroup of $H_0(C, \mathbb{Z})$, hence $K(S)/K(X)$ maps to $H_0(C, \mathbb{C})$ injectively.

(2.6) REMARK. In the rest of the paper we shall assume that the map

$H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is injective. This happens for instance in the case of cones. By duality it is true in general that the kernel of $H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is isomorphic to the cokernel of $H^2(S, \mathbb{Z}) \to H^2(C, \mathbb{Z})$, which is in any case a finite quotient of $\mathbb{Z}$, of order m say. The hypothesis we are making is only for the sake of simplicity of notations; we find convenient to go on using the groups $B(X)$ and $B(S)$ as the range of the regulator maps. It will be apparent from the arguments in part 3 below that, if $H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is not injective, then we may replace $B(X)$ by its quotient $H^*/(1/m)H_2(S - C, \mathbb{Z}(1)$ and similarly for $B(S)$ (here $(1/m)H_2(S - C, \mathbb{Z})(1)$ means the subgroup of $H_2(S - C, \mathbb{C})$ of the elements z such that mz is $2\pi i$ an integral class). The reason is due to the fact that in the definition of the regulator map ρ, there are choices of a certain 2-chain Δ in $S - C$ which must have as boundary a certain 1-cycle, the choice being given only up to 2-cycles. In the case of 1-cycles which may be in the kernel of the map $H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ such a Δ exists with rational coefficients of denominator m, and we must take into account this ambiguity in the choice of Δ.

3. The regulator map ρ

We shall define below a pairing $\rho(\xi, \alpha, \Delta(\xi))$ between a cycle ξ representing a class in $H^1(X, \mathscr{K}_{2X})_h$ and a form of second kind α representing an element of H. As it is indicated in the notation, the pairing depends for each ξ on the choice of a certain 2-chain Δ, in a manner which is clarified below; the pairing is well defined only up to $2\pi i$-integral periods of the forms in H, since Δ may be changed by adding integral 2-cycles on $S - C$ and ρ is computed by means of integration. It turns out that, except for the periods, the definition of the pairing does not depend on the representants chosen for cohomology classes in $H^1(X, \mathscr{K}_{2X})_h$ and H, so the pairing gives a map

(3.1) $\rho: H^1(X, \mathscr{K}_{2X})_h \to H^*/H_2(S - C, \mathbb{Z})(1).$

Our definition is based on the definition given in [8] for the regulator map r in the non-singular case,

$$r: H^1(S, \mathscr{K}_{2S})_h \to (F^1H^2(S))^*/H_2(S, \mathbb{Z})(1).$$

We have been told that the construction of r is originally due to Bloch.

For brevity's sake, we call R the local ring of X at the singular point, i.e. $R = \mathcal{O}_{X,x_0}$. We write X^{*m} for the set of irreducible closed subvarieties of codimension m in X which do not contain the singular point x_0.

From [2], one knows that $H^1(X, \mathscr{K}_{2X})$ is the cohomology of the complex:

$$(G(X)) \quad K_2(R) \xrightarrow{T} \bigoplus_{x \in X^{*1}} \mathbb{C}(x)^* \xrightarrow{\text{div}} \bigoplus_{x \in X^{*2}} \mathbb{Z}_{(x)}$$

where T is the tame symbol and div is the divisor map associating to a rational function its divisor. We recall that $H^1(S, \mathscr{K}_{2S})$ is the cohomology of the Gersten complex:

$$(G(S)) \quad K_2(k(S)) \xrightarrow{T} \bigoplus_{x \in S^1} \mathbb{C}(x)^* \xrightarrow{\text{div}} \bigoplus_{x \in S^2} \mathbb{Z}_{(x)}.$$

Let $Z^{1,1}(X)$ denote the kernel of the map div in $G(X)$. We define a cycle map $\gamma_1^1 \colon Z^{1,1}(X) \to H_1(S - C, \mathbb{Z})$ in the same way as the cycle map $\gamma_1^1 \colon Z^{1,1}(S) \to H_1(S, \mathbb{Z})$ is defined in [8].

Let σ be the real positive axis inside of $\mathbb{CP}^1$, oriented so that $\partial\sigma = (0) - (\infty)$. If D is a codimension 1 subvariety in X^{*1} and f is a non-constant rational function on D, let $\mu \colon D' \to D$ be a resolution of the singularities of D, so that f determines a morphism $f \colon D' \to \mathbb{CP}^1$. The chain on D, $\gamma(f) := \mu_*(f^{-1}(\sigma))$, is independent of the choice of μ, and it has boundary $\text{div}(f)$. If f is constant, set $\gamma(f) = 0$. If $\xi = \Sigma(D_i, f_i)$ is in $Z^{1,1}(X)$, then the chain $\sum \gamma(f_i)$ on D has zero boundary; we set $\gamma(\xi) = \gamma_1^1(\xi)$ to be the homology class in $H_1(S - C, \mathbb{Z}) = H_1(X - \{x_0\}, \mathbb{Z})$ of the 1-cycle $(i_D)_*(\sum \gamma(f_i))$.

(3.2) REMARK. If $\xi = \sum(D_i, f_i)$ is in $Z^{1,1}(S)$ then $\gamma(\xi)$ is always a torsion class in $H_1(S, \mathbb{Z})$. In order to prove this it is enough to show that the pairing $\int_{\gamma(\xi)} \omega = 0$ for any holomorphic one form ω, because $(i_D)_*(\Sigma\gamma(f_i))$ is a real cycle. In fact by Hodge $H^1(S, \mathbb{C}) = H^{(10)} \oplus H^{(01)}$ and $H^{(01)}$ is conjugate to $H^{(01)}$, which is space of the holomorphic 1-forms. Now $\int_{\gamma(\xi)} \omega = \sum \int_{\gamma(f_i)} \omega$, and the integral $\int_{\gamma(f_i)} \omega$ is 0, because it is equal to the integral along σ in $\mathbb{CP}^1$ of the (zero) holomorphic 1-form which is the trace via f_i of the restriction of ω to D_i.

(3.3) We define $Z^{1,1}(X)_h$ to be the kernel of γ_1^1. If ξ belongs to the image under the tame symbol $T(K_2(R))$ in the complex GR. then $\xi \in Z^{1,1}(X)_h$. Since the group $K_2(R)$ is generated by the symbols $\{f, g\}$, f and g being invertible elements of R, it is enough to prove that $\xi \in Z^{1,1}(X)_h$ for $\xi = T\{f, g\}$. The rational functions f and g define a rational map $h \colon X \to Q$, where Q is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Since f and g are regular and invertible at the point x_0, then h is regular at x_0. In this situation ξ is the pull back from the quadric Q of the element $\zeta = T\{t, u\} \in Z^{1,1}(Q)_h$, here t and u are the natural rational functions on the two factors. It is simple to see that one can choose a chain $\Delta(\zeta)$, associated with ζ, such that $\Delta(\zeta)$ avoids the point p_0 which is the image of x_0 via h. In this case the pull back to S of $\Delta(\zeta)$ is a 2-chain on $S - C$ with boundary $\gamma(\xi)$. We define

$$H^1(X, \mathscr{K}_{2X})_h := Z^{1,1}(X)_h / T(K_2(R)).$$

(3.4) Let $\xi = \sum (D_i, f_i)$ be an element in $Z^{1,1}(X)_h$. We define the pairing $\rho(\xi, \alpha, \Delta(\xi))$, which depends on the choice of a chain $\Delta(\xi)$ having boundary $\gamma(\xi)$, where $\gamma(\xi)$ is the loop described above.

Given a curve D on S with rational function f, we take the resolution $\mu: D' \to D$. On $D' - f^{-1}(\sigma)$ we can define the single valued logarithm $\log(f)$ to be the pullback by f of the principal branch of the logarithm on $\mathbb{CP}^1 - (\sigma)$. If f is constant, let $\log(f)$ be the value of the principal branch of the logarithm at f. Let $I(f)$ denote the function $H^0(S, d\mathscr{A}^{1,0}(2C)) \to \mathbb{C}$ defined by setting $I(f)(\alpha) = \int_{D'} \log(f) \cdot \mu^*(\alpha)$. Note that $\mu^*(\alpha)$ is smooth on D', because D is supported in $X - \{x_0\} = S - C$.

We define

$$\rho(\xi, \alpha, \Delta(\xi)) := \sum I(h_i)(\alpha) + (2\pi i) \int_{\Delta(\xi)} \alpha.$$

(3.5) We begin by showing that $\rho(\xi, \alpha, \Delta(\xi))$ depends only on the cohomology class of α, i.e. if $\varepsilon \in A^{10}(1C)$, then $\rho(\xi, d(\varepsilon), \Delta(\xi)) = 0$.

By Stokes theorem $\int_\Delta d(\varepsilon) = \int_{\Sigma\gamma(f_i)} \varepsilon$ and also on each curve D_i

$$\begin{aligned} I(f)(d(\varepsilon)) &= \int_{D'} \log(f) \mu^*(d\varepsilon) \\ &= -\int_{D'} (df/f) \wedge \mu^*(\varepsilon) + \int_{h^{-1}(\sigma)+} \log(f) \mu^*(\varepsilon) - \int_{h^{-1}(\sigma)-} \log(f) \mu^*(\varepsilon) \end{aligned}$$

where $+$ and $-$ refer to the two normal directions of $f_i^{-1}(\sigma)$ in D_i'. Since the limiting values of $\log(f)$ in the two integrals differ by $2\pi i$

$$I(f)(d(\varepsilon)) = -(2\pi i) \int_{\gamma(f)} \varepsilon,$$

because $-\int (df/f) \wedge \mu^*(\varepsilon)$ vanishes by reasons of type.

Therefore $\rho(\xi, *, \Delta(\xi))$: $H \to \mathbb{C}$ is well defined. For a different choice $\Delta^+(\xi)$ for Δ, $\rho(\xi, *, \Delta(\xi)) - \rho(\xi, *, \Delta^+(\xi)) = (2\pi i) \int_{\Delta(\xi) - \Delta^+(\xi)}$, so $\rho(\xi, *) = \rho(\xi, *, \Delta)$ gives a uniquely determined element in $B(X) := H^*/(2\pi i)(H_2(S - E, \mathbb{Z}))$.

(3.6) We now show that if $\xi \in T(K_2(R))$ then $\rho(\xi)$ is the trivial element in $B(X)$. Therefore ρ gives a well defined map $H^1(X, \mathscr{K}_{2X})_h \to B(X)$, because the complex $(G.)$ is an acyclic resolution of the sheaf $\mathscr{K}_{2X}$.

By linearity it is enough to show that $\rho(T\{f, g\}) = 0$, where f and g are invertible in R. In this case we have a rational map from X to the quadric, $h: X \to Q$. The map h is regular at x_0, and it lifts to $F: (S, C) \to (Q, p_0)$, where F is regular along C. Without restriction for our considerations we may as well assume that h is regular on all of X, in fact we may blow up X and S so to resolve the indeterminacy of h and compute the required integrals on the resulting surface.

If $F(S)$ is a point, clearly f and g are constant functions so that the tame symbol $T\{f, g\}$ is zero and the regulator pairing is trivial.

If $F(S)$ is a curve G' on Q the vanishing of the regulator pairing is a consequence of the product formula, cf. n. 4, §1, III of [9]. The map $S - G'$ factors through the normalization G of G'. On G we have two rational functions t and u such that $f = F^*t$ and $g = F^*u$. The product formula says:

$$\prod_{p\in G} \{[(-1)^{\mathrm{ord}_p(u)\mathrm{ord}_p(t)}(t^{\mathrm{ord}_p(u)})(u^{-\mathrm{ord}_p(t)})](p)\} = 1.$$

By definition of the tame symbol,

$$\xi = T(\{f, g\}) = \sum (F^{-1}(p), (-1)^{\mathrm{ord}_p(u)\mathrm{ord}_p(t)}[(t^{\mathrm{ord}_p(u)})(u^{-\mathrm{ord}_p(t)})](p)).$$

Here $F^{-1}(p)$ denotes the fiber over p of F, and we use an additive notation to the effect that if $F^{-1}(p) = nA + mB$ then $(nA + mB, c) = n(A, c) + m(B, c) = (A, c^n) + (B, c^m)$. For this choice of ξ the chain Δ can be taken to be zero, since for each curve which appears in ξ, the associated function is a constant and γ is therefore zero. Next remark we need is that all the fibers of F that are contained in $S - C$ are in fact homologically equivalent in $S - C$, because the points of $G - F(C)$ are homologically equivalent. Let now ω be a form which represents an element in H. The pairing is defined as

$$\rho(\xi, \omega) = \sum_p \log[(-1)^{\mathrm{ord}_p(u)\mathrm{ord}_p(t)}(t^{\mathrm{ord}_p(u)})(u^{-\mathrm{ord}_p(t)})(p)] \int_{F^{-1}(p)} \omega.$$

Since the fibers are equivalent, all the integrals are the same. Using the product formula we conclude $\rho(\xi, \omega) = 0$.

Now we deal with the case when h is generically finite. We start with a general fact. Let $S^0(\varepsilon) := S - N(\varepsilon)$ be the complement of a tubular neighborhood $N(\varepsilon)$ of radius ε of the support of $T(\{f, g\})$.

(3.7) LEMMA. *Let ω be a C^∞ closed 2-form on a nonsingular surface S, then for any two rational functions f and g of S*

$$2\pi i \rho(T(\{f, g\}), \omega, \Delta) = \lim_{\varepsilon\to 0} \int_{S^0(\varepsilon)} (df/f) \wedge (dg/g) \wedge \omega,$$

where Δ is conveniently chosen.

Proof. Without restriction we may assume that the map $F: S \to Q$ induced by the functions f and g is regular everywhere, in particular f and g are regular maps from S to $\mathbb{P}^1$. We set $\Gamma(f) = f^{-1}(\sigma)$, where σ is the real semipositive line on $\mathbb{P}^1$; $\Gamma(f)$ is a 3-chain on S and $\log f$ is a well defined function on $S - \Gamma(f)$. In the same way we define $\Gamma(g)$. Note that the boundary $\partial\Gamma(f) = D(f)$, the divisors associated with f, similarly $\partial\Gamma(g) = D(g)$. We assume for the moment that $D(f)$ and $D(g)$ have no common component. Because of this hypothesis we can take $\Delta = \Gamma(f) \cap \Gamma(g)$; by the same reason the restrictions of f to $D(g)$ and of g to $D(f)$ are well defined invertible rational functions. Using additive notations, one has $T(\{f, g\}) = (D(g), f) + (Df), g^{-1}) = (D(g), f) - (D(f), g)$. Therefore:

$$\lim_{\varepsilon \to 0} \int_{S^0(\varepsilon)} \left(\frac{\mathrm{d}f}{f}\right) \wedge \left(\frac{\mathrm{d}g}{g}\right) \wedge \omega$$

$$= \lim_{\varepsilon \to 0} \left[\int_{\partial N(\varepsilon) \cap [S - \Gamma(f)]} \log f \left(\frac{\mathrm{d}g}{g}\right) \wedge \omega - 2\pi i \int_{\Gamma(f)} \left(\frac{\mathrm{d}g}{g}\right) \wedge \omega \right]$$

$$= 2\pi i \left\{ \int_{D(g)} (\log f)\omega + \int_{\partial\Gamma(f)} - (\log g)\omega + 2\pi i \int_{\Gamma(f) \cap \Gamma(g)} \omega \right\}$$

$$= 2\pi i \rho(\xi, \omega, \Delta).$$

If $D(f)$ and $D(g)$ have a common component the result still follows from the previous computation, using additivity and this lemma:

LEMMA (Levine). *The group $K_2(k(S))$ is generated by symbols $\{f, g\}$, with* $\mathrm{div}(f)$ *and* $\mathrm{div}(g)$ *having no common component.*

Proof. We assume that S is embedded in a projective space with homogeneous coordinates $y_0, y_1, \ldots, y_N$. According to lemma 2.2 in [7] $K_2(k(S))$ is generated by lemma symbols $\{a, b^{-1}\}$, with a and b represented by polynomials in the affine ring $\mathbb{C}[(y_1/y_0), \ldots, (y_N/y_0)]$ and with $\mathrm{div}(a)$ and $\mathrm{div}(b)$ reduced and having no common components, but for the divisor $\mathrm{div}(y_0)$, the intersection of S with the hyperplane $y_0 = 0$. Up to a change of coordinates, we may assume that $\mathrm{div}(y_0)$ is irreducible and reduced. Therefore we can write $a = A/(y_0)^m$ and $b = B/(y_0)^n$, where: (i) A and B are homogeneous polynomials in the y's of degree m and n respectively, (ii) on S there is no divisor which is a common component to any two of $\mathrm{div}(A)$, $\mathrm{div}(B)$ and $\mathrm{div}(y_0)$. Let M and N be linear forms in the y's, general enough so that on S $\mathrm{div}(M)$ and $\mathrm{div}(N)$ have no component in common with each other or with any of $\mathrm{div}(A)$, $\mathrm{div}(B)$ and $\mathrm{div}(y_0)$.

In $K_2(k(S))$ we have

$$\{a, b^{-1}\} = \{(AM^{-m})(My_0^{-1})^m, (B^{-1}N^n)(N^{-1}y_0)^n\},$$

this is the product of the following four symbols

$$\{(AM^{-m}), (B^{-1}N^n)\}, \qquad \{(AM^{-m}), (N^{-1}y_0)^n\},$$
$$\{(My_0^{-1})^m, (B^{-1}N^n)\}, \qquad \{(My_0^{-1})^m, (N^{-1}y_0)^n\}.$$

The first three symbols are clearly of the type $\{f, g\}$, with $\mathrm{div}(f)$ and $\mathrm{div}(g)$ having no common component on S.

To deal with the fourth symbol we recall that it is a power of $\{(My_0^{-1}), (N^{-1}y_0)\}$, and that in $K_2(k(S))$:

$$\begin{aligned}&\{(My_0^{-1}), (N^{-1}y_0)\}\\ &\quad = \{(My_0^{-1}), (N^{-1}y_0)\}\{(My_0^{-1}), (1-(My_0^{-1}))\}\\ &\quad = \{(My_0^{-1}), (y_0 - M)(N^{-1})\}.\end{aligned}$$

By our hypotheses $\mathrm{div}(My_0^{-1})$ and $\mathrm{div}((y_0 - M)(N^{-1}))$ have no common component. This concludes the proof of the lemma.

(3.8) We denote X_{x_0} the scheme associated with the local ring of X at x_0, and denote Z the fiber $f^{-1}(X_{x_0})$ in S. We consider an element $a \in H^0(Z, \mathscr{K}_{2Z})$, then $T(a) \in Z^{1,1}(X)$, because the support of $T(a)$ does not meet C, since it does not intersect Z. In fact $T(a) \in Z^{1,1}(X)_h$, because $T^{(a)} \in Z^{1,1}(S)_h$ and we have assumed that the map $H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is injective. We have commented in remark (2.6) on what variations can be used if this map is not injective. By the Gersten resolution a is given by an element in the group K_2 of the function field $k(S)$, so that a is represented as a product of Steinberg symbols $\{f_j, g_j\}$. Associated to a, we have the d log forms $(\mathrm{d}f_j/f_j) \wedge (\mathrm{d}g_j/g_j)$, they add to give a form $\alpha := \mathrm{d}\log a$. Because of Matsumoto theorem d log is a well defined morphism from $K_2(k(S))$ to the space of meromorphic 2-forms on S, cf. [1]. Under this map the support of the tame symbol of an element contains the polar locus of the corresponding d log form. Therefore in the case of the element a the poles of α are contained in $S - C$.

For any form ν which is locally of the type $\nu = \mu \wedge (\mathrm{d}e/e)$, where e is a local analytic equation for C and μ is smooth, we set $R(\nu)$ to be the residue form of ν on C, $R(\nu)$ is defined locally by the restriction of the form μ. By our hypotheses α is a rational 2-form which is regular near C, if η has simple poles along C then $\alpha \wedge \eta$ is of the preceding type and $R(\alpha \wedge \eta)$ is defined.

We use below the notations $S(\varepsilon) = S - \{N(\varepsilon) \cup M(\varepsilon)\}$, where $N(\varepsilon)$ and $M(\varepsilon)$ are neighborhoods of radius ε of the polar locus of α and of C respectively.

(3.9) LEMMA. If η is smooth on $S - C$ and it has only simple poles along C, then

$$2\pi i\rho(T(a),\mathrm{d}\eta,\Delta) + 2\pi \mathrm{i} \int_C R(\alpha \wedge \eta) = \left\{\lim_{\varepsilon\to 0} \int_{S(\varepsilon)} \alpha \wedge \mathrm{d}\eta\right\}.$$

Proof. From Stokes theorem:

$$\lim_{\varepsilon\to 0} \int_{S(\varepsilon)} \alpha \wedge \mathrm{d}\eta = \lim_{\varepsilon\to 0} \int_{\partial N(\varepsilon)} \alpha \wedge \eta + \lim_{\varepsilon\to 0} \int_{\partial M(\varepsilon)} \alpha \wedge \eta$$
$$= \lim_{\varepsilon\to 0} \int_{\partial N(\varepsilon)} \alpha \wedge \eta + 2\pi i \int_C R(\alpha \wedge \eta).$$

Let (D_j, h_j) be one of the summands in $T(a)$. There are two cases to consider, according to whether h_j is constant or not. If h_j is constant so is $\log(h_j)$ and $\int_{D_j} \log(h_j)\,\mathrm{d}\eta = 0$, because $\log(h_j)\,\mathrm{d}\eta$ is exact on D_j. In other words in this case the contribution of (D_j, h_j) to $\rho(T(a), \mathrm{d}\eta, \Delta)$ is zero. If h_j is not constant then D_j is a component of the polar locus of α. The boundary $\partial N(\varepsilon)$ is a 3-cycle, and it is the union of the closure of the normal S^1 bundles over the smooth part of the components in the support of the polar locus of α. By definition α is representable near D_j as the sum of a regular closed 2-form α_r with a 2-form of the type $\varphi \wedge (\mathrm{d}\delta/\delta)$, where δ is a local equation for D_j and $\varphi = (\mathrm{d}h_j^+)/h_j^+ = \mathrm{d}\log(h_j^+)$, h_j^+ being a rational function without zeroes or poles along D_j such that it restricts to h_j on D_j. Denoting $N_j(\varepsilon)$ the disc bundle of radius ε around D_j,

$$\lim_{\varepsilon\to 0} \int_{\partial N_j(\varepsilon)} \alpha \wedge \eta = -2\pi i \int_{D_j} (\mathrm{d}\log(h_j)) \wedge \eta$$
$$= 2\pi i \left[\int_{D_j} \log(h_j) \wedge \mathrm{d}\eta + 2\pi i \int_{\gamma_j} \eta\right],$$

where, as before, γ_j is the path on D_j which is the pull back of the real positive axis σ on $\mathbb{P}^1$ by means of h_j. The weighted sum of the γ_j is the boundary of Δ, adding everything we get

$$\lim_{\varepsilon\to 0} \int_{\partial N(\varepsilon)} \alpha \wedge \eta = 2\pi i \left[\sum \int_{D_j} \log(h_j) \wedge \mathrm{d}\eta + 2\pi i \int_\Delta \mathrm{d}\eta\right]$$
$$= 2\pi i\rho(T(a), \mathrm{d}\eta, \Delta).$$

(3.10) We consider a representative form ω for an element of H as it is described in (2.3), so that $\omega = \beta + \mathrm{d}\eta$, where β is a smooth and closed 2-form on S, and η is as it is in (3.9) above. We take a as before, writing again $\alpha = \mathrm{d}\log a$. From the preceding results, using additivity and type, we obtain

THEOREM.

$$0 = \lim_{\varepsilon \to 0} \int_{S(\varepsilon)} \alpha \wedge \omega = 2\pi i \left[\rho(T(a), \omega, \Delta) + \int_C R(\alpha \wedge \eta) \right].$$

When f and g are invertible regular elements in the local ring at x_0, f and g restrict to constant functions along C on S. Then C is contracted via the map $F = (f, g)$ from S to $\mathbb{P}^1 \times \mathbb{P}^1$, therefore C is a component of the zero divisor associated to the Jacobian determinant $(\mathrm{d}f/f) \wedge (\mathrm{d}g/g)$. It follows $R((\mathrm{d}f/f) \wedge (\mathrm{d}g/g) \wedge \eta) = 0$ on C, so that $\int_C R((df/f) \wedge (\mathrm{d}g/g) \wedge \eta) = 0$.

(3.11) THEOREM. If $a \in K_2(\mathcal{O}_{x_0})$, then there is a $\Delta(T(a))$ with

$$\rho(T(a), \omega, \Delta(T(a))) = 0.$$

4. A question of duality

We are motivated by the quest for a way to compute the kernel in

$$0 \to \mathrm{KER} \to H^1(X, \mathscr{K}_{2X}) \to H^1(S, \mathscr{K}_{2S}) \to \mathrm{COK} \to 0,$$

and by the analogy with the case of the Picard group of a curve with an ordinary cusp. Our considerations descend from the wish to better understand the results of Srinivas [10].

There is a sequence of sheaves in the Zariski topology of X

$$0 \to \mathscr{F} \to \mathscr{K}_{2X} \to f_* \mathscr{K}_{2S} \to \mathscr{A} \to 0$$

here $\mathscr{A}$ and $\mathscr{F}$ are skyscraper supported at the point x_0, hence the following is exact

$$\begin{aligned} &H^0(X, \mathscr{K}_{2X}) \to H^0(X, f_* \mathscr{K}_{2S}) \to H^0(X, \mathscr{A}) \\ &\quad \to H^1(X, \mathscr{K}_{2X}) \to H^1(X, f_* \mathscr{K}_{2S}) \to 0 \end{aligned}$$

Now $H^0(X, f_* \mathscr{K}_{2S}) = H^0(S, \mathscr{K}_{2S})$ and $0 \to H^1(X, f_* \mathscr{K}_{2S}) \to H^1(S, \mathscr{K}_{2S})$ is an inclusion (because of the Leray spectral sequence), therefore this is exact

$$H^0(X, \mathscr{K}_{2X}) \to \mathscr{H}^0(S, \mathscr{K}_{2S}) \to H^0(X, \mathscr{A}) \to H^1(X, \mathscr{K}_{2X}) \to H^1(S, \mathscr{K}_{2S}).$$

The analogous sequence for the case of the local scheme X_{x_0} gives $H^0(X, \mathscr{A}) = H^0(Z, \mathscr{K}_{2Z})/K_2(X_{x_0})$, here Z is the schematic fiber of f over X_{x_0}. We conclude that there is an isomorphism

$$\mathrm{KER} \sim H^0(Z, \mathscr{K}_{2Z})/[K_2(X_{x_0}) + H^0(S, \mathscr{K}_{2S})].$$

The group $T(H^0(Z, \mathscr{K}_{2Z}))$ is contained in $Z^{1,1}(X)_h$, cf. (3.3), because it is contained in $Z^{1,1}(S)_h$ and the map $H_1(S - C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is injective by the hypothesis in (2.6). It follows that KER is a subgroup of $H^1(X, \mathscr{K}_{2X})_h$ and therefore ρ induces a map $H^0(Z, \mathscr{K}_{2Z}) \to B(X)$. The regulator map r of Levine vanishes on tame symbols from $K_2(k(S))$ (cf. also (3.7)), hence it vanishes on $H^0(Z, \mathscr{K}_{2Z})$, i.e. the map $H^0(Z, \mathscr{K}_{2Z}) \to B(X) \to B(S)$ is zero. It follows from (2.5) that ρ gives a map from $H^0(Z, \mathscr{K}_{2Z})$ to $(H^1(C, \mathscr{O}_C(C)))^*$. By Serre duality $(H^1(C, \mathscr{O}_C(C)))^* = H^0(C, \mathscr{O}_C(\omega_C(-C)))$, hence we have 'regulator' maps reg: $H^0(Z, \mathscr{K}_{2Z}) \to H^0(C, \mathscr{O}_C(\omega_C(-C)))$ and ρ: KER $\to H^0(C, \mathscr{O}_C(\omega_C(-C)))$.

On the other hand, using the d log map$\{f, g\} \to (\mathrm{d}f/f) \wedge (\mathrm{d}g/g)$, and the fact that K_2 of a local rings is generated by symbols, one has a map $H^0(Z, \mathscr{K}_{2Z}) \to H^0(Z^{\mathrm{an}}, \Omega_Z^2)$, cf. [1]. Combining this map with the adjunction map $H^0(Z^{\mathrm{an}}, \Omega_Z^2) \to H^0(C, \mathscr{O}_C(\omega_C(-C)))$, one obtains another morphism adj: $H^0(Z, \mathscr{K}_{2Z}) \to H^0(C, \mathscr{O}_C(\omega_C) - C)))$. It is simple to see that under this map $K_2(X_{x_0})$ goes to zero (as we have noted before the Jacobian of a map vanishes along a divisor which is contracted). Also $H^0(S, \mathscr{K}_{2S})$ vanishes under d log:

(4.1) PROPOSITION. *$H^0(S, \mathscr{K}_{2S}) \to H^0(S^{\mathrm{an}}, \Omega_S^2)$ is the zero map.*

Proof. To begin we recall that in $H^2(S, \mathbb{C})$ we have $H^0(S^{\mathrm{an}}, \Omega_S^2) \cap (2\pi i)^2 H^2(S, \mathbb{Z})_{\mathbb{Q}} = 0$, because one group is invariant under complex conjugation while the other is conjugate to $H^2(S, \mathscr{O}_S)$. In order to prove the proposition we show that for ξ in $H^0(S, \mathscr{K}_{2S})$ d log(ξ) belongs to $(2\pi i)^2 H^2(S, \mathbb{Z})$. It is enough to prove that d log(ξ) has periods which are multiple of $(2\pi i)^2$ along integral 2-cycles which are supported on the complement of a convenient, possibly reducible, divisor D. Indeed (i) the cokernel of the map $H_2(S - D, \mathbb{Z}) \to H_2(S, \mathbb{Z})$ is the image of the intersection map $H_2(S, \mathbb{Z}) \to \oplus H_0(D_i)$, (ii) since S is projective, it follows from Hodge theory and Poincare duality that if a topological cycle has intersection multiplicity a_i with the divisor D_i there is an algebraic cycle with the same intersection multiplicity a_i with D_i, (iii) d log(ξ) vanishes along algebraic cycles because of type. Taking D large enough we reduce the computation to the case when d log ξ is of the type $\mathrm{d}f/f \wedge \mathrm{d}g/g$ and D contains the supports of the divisors of f and g. In this situation $\mathrm{d}f/f \wedge \mathrm{d}g/g$ is the pull back to $S - D$ of the class $\mathrm{d}t/t \wedge \mathrm{d}u/u$ on $\mathbb{C}^* \times \mathbb{C}^*$, where t and u are obvious parameters. Clearly $\mathrm{d}t/t \wedge \mathrm{d}u/u$ is a class in $(2\pi i)^2 H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})$, and therefore the pull back $\mathrm{d}f/f \wedge \mathrm{d}g/g$ is of the required type.

Therefore we have a second map, which we shall call adjunction or, briefly, adj: KER $\to H^0(C, \mathscr{O}_C(\omega_C(-C)))$.

For the case of cones this second map is equivalent to the one used by Srinivas in his paper [10].

(4.2) THEOREM. *The two maps ρ and* adj *coincide.*

Let $\xi \in H^0(Z, \mathscr{K}_{2Z})$ represent an element in KER. Thus there are rational functions f_i and g_i on S such that $\xi = \sum T(\{f_i, g_i\}) = \sum(D_j, h_j)$, where we write only those indexes j for which h_j is not 1, and C does not meet any of the curves D_j, which form the support of ξ. We write $\alpha = \sum(\mathrm{d}f_i/f_i) \wedge (\mathrm{d}g_i/g_i)$, because of our hypothesis α is regular near C.

To prove the theorem we take an element φ in the dual space $H^1(C, \mathcal{O}_C(C))$ and we prove that reg(ξ) and adj(ξ) operate in the same way on φ.

Under the surjection $H \to H^1(C, \mathcal{O}_C(C))$, φ is the image of the class in H represented by a form ω, so that the pairing of φ with reg(ξ) is by definition $\rho(\xi, \omega, \Delta(\xi))$. The theorem amounts to the equality $\rho(\xi, \omega, \Delta(\xi)) = \langle \mathrm{adj}(\xi), \varphi \rangle$, where $\langle \mathrm{adj}(\xi), \varphi \rangle$ is the pairing on C which comes from Serre duality. We know from the theorem in (3.10) that for some $\Delta(\xi)$, $\rho(\xi, \omega, \Delta(\xi)) = -\int_C R(\alpha \wedge \eta)$. In order to evaluate $R(\alpha \wedge \eta)$, we write locally near C $\alpha = \mu \wedge \mathrm{d}e$, where μ is a holomorphic 1-form and e is an equation for C. We recall that $\eta = \eta'/e$, where η' is C^∞. Locally the residue on C is $R(\alpha \wedge \eta) = R(\mu \wedge \mathrm{d}e \wedge \eta'/e) = -(\mu \wedge \eta')_{|C}$.

The image of ξ via the adjunction map adj: $H^0(Z, \mathscr{K}_{2Z}) \to H^0(C, \mathcal{O}_C(\omega_C(-C))$ is given locally by the restriction of μ to C. Further, since φ is the element in $H^1(C, \mathcal{O}_C(C))$ which is the image of ω, we known from (2.4) above that φ is represented by η'. The pairing $\langle \mathrm{adj}(\xi), \varphi \rangle$, which comes from the duality of $H^0(C, \mathcal{O}_C(\omega_C(-C))$ with $H^1(C, \mathcal{O}_C(C))$, is computed by integration:

$$\begin{aligned}\langle \mathrm{adj}(\xi), \varphi \rangle &= \int_C \mathrm{adj}(\xi) \wedge \varphi = \int_C \mathrm{adj}(\xi) \wedge \eta' \\ &= \int_C (\mu \wedge \eta')_{|C} = -\int_C R(\alpha \wedge \eta) = \rho(\xi, \omega).\end{aligned}$$

This concludes the proof that the two maps are the same.

5. Rational forms and the cone

Our aim here is to compute as explicitly as possible the map ρ in one example. We shall in this way recover a result of Srinivas [10, §4]. The example we deal with is the one which has motivated our choice of the space H in Section 1.

We denote by X the ordinary cone in the complex projective space $\mathbb{P}_3$, we take the equation of X to be: $\xi_1\xi_2 - \xi_3^2 = 0$ where ξ_i, $i = 0, \ldots, 3$, are the homogeneous coordinates of $\mathbb{P}_3$. There is one (up to a constant multiple) rational 3-form on $\mathbb{P}_3$ with a pole of order two along X, namely

$$\varphi := (\textstyle\sum_{i=0,\dots,3}(-1)^i \xi_i \, d\xi_0 \cdots (d\xi_i)^\wedge \cdots d\xi_3)/(\xi_1\xi_2 - \xi_3^2)^2.$$

Let $\mathbb{P}_+$ be the blow-up of $\mathbb{P}_3$ along the vertex of the cone, let S be the proper transform of X, let E be the exceptional divisor on $\mathbb{P}_+$ and let $C = E \cap S$ be the exceptional line in S. The rational form φ lifts to a form φ_+ on $\mathbb{P}_+$, which happens to have a second order pole along E and along S. Following the pattern of a computation in [3] we shall define a residue image $R(\varphi)$ in $H^1(S, \Omega_S^{\wedge 1}(2C))$, and compute its class by means of a representative Cech cocycle. We recall that in this example $H = H^1(S, \Omega_S^{\wedge 1}(2C))$.

Consider the following sheaf sequences

$$0 \to \Omega_{\mathbb{P}_+}^{\wedge 2}(S + 2E) \to {}^*\Omega_{\mathbb{P}_+}^2(S + 2E) \xrightarrow{d} \Omega_{\mathbb{P}_+}^3(2S + 2E) \to 0$$

$$0 \to \Omega_{\mathbb{P}_+}^{\wedge 2}(2E) \to \Omega_{\mathbb{P}_+}^{\wedge 2}(S + 2E) \xrightarrow{R} \Omega_S^{\wedge 1}(2C) \to 0 \tag{5.1}$$

which are defined in a similar way to what Griffiths does in loc. cit. More precisely: $\Omega_{\mathbb{P}_+}^3(2S + 2E)$ is the sheaf on $\mathbb{P}_+$ of meromorphic 3 forms with a second order pole along $S + E$, ${}^*\Omega_{\mathbb{P}_+}^2(S + 2E)$ is the sheaf on $\mathbb{P}_+$ of meromorphic 2 forms θ with a first order pole along S and a second order pole along E such that $d\theta$ has on E only a second order pole; $\Omega_{\mathbb{P}_+}^{\wedge 2}(S+2E)$ is the subsheaf of closed forms; the map $R: \Omega_{\mathbb{P}_+}^{\wedge 2}(S + 2E) \to \Omega_S^{\wedge 1}(2C)$ is defined by $R(\alpha \, dh/h) = \alpha_{|S}$, the restriction of α to S, here h is a local equation for E. We define the residue map $R: H^0(\mathbb{P}_3, \Omega_{\mathbb{P}_3}^3(2X)) \to H^1(S, \Omega_S^{\wedge 1}(2C))$
to be the composition of the following morphisms

$$H^0(\mathbb{P}_3, \Omega_{\mathbb{P}_3}^3(2X)) \to H^0(\mathbb{P}_+, \Omega_{\mathbb{P}_+}^3(2S + 2E)) \to H^1(\mathbb{P}_+, \Omega^{\wedge 2}_{\mathbb{P}_+}(S + 2E))$$

and

$$H^1(\mathbb{P}_+, \Omega^{\wedge 2}_{\mathbb{P}_+}(S + 2E)) \to H^1(S, \Omega_S^{\wedge 1}(2C)).$$

We explain now the procedure we use to compute $R(\varphi)$ as an element in the Cech cohomology.

The surface S is a $\mathbb{P}_1$ bundle over $\mathbb{P}_1$, C being a section. We decompose $\mathbb{P}_1$ in two affine lines and lift this decomposition to S, calling A and B the two open sets. We note that both A and B are in fact trivial $\mathbb{P}_1$ bundles over the affine line. Using sequences as we did in Section 1 one computes that $H^1(A, \Omega^{\wedge 1}(2C))$ and $H^1(B, \Omega^{\wedge 1}(2C))$ are both 0, therefore the following is exact:

$$H^0(A, \Omega^{\wedge 1}(2C)) \oplus H^0(B, \Omega^{\wedge 1}(2C)) \to H^0(A \cap B, \Omega^{\wedge 1}(2C))$$
$$\to H^1(S, \Omega^{\wedge 1}(2C)) \to 0.$$

In other words an element in $H^1(S, \Omega^{\wedge 1}(2C))$ is represented by a global section $s_{A,B}$ in $H^0(A \cap B, \Omega^{\wedge 1}(2C))$. Note that $s_{A,B}$ is a Cech-cocycle for the open cover $\{A, B\}$.

The following fact, which is standard, will be used for the computation of $R(\varphi)$. If $\mathscr{U} = \{A, B, U_j\}_{j\in J}$ is another open cover of S, which contains also A and B, and if $\{s_{V,U}\}$ is a Cech-1-cocycle associated with this cover (meaning that the $s_{V,U}$ are sections in $H^0(V \cap U, \Omega^{\wedge 1}(2C))$, where U and V vary in $\mathscr{U}$) then, under the above hypotheses, the image of $\{s_{V,U}\}$ in $H^1(S, \Omega^{\wedge 1}(2C))$ is exactly the class determined by $s_{A,B}$.

(5.2) Here we compute the class of $R(\varphi)$ as Cech-cocycle. We use affine coordinates $w_i = \xi_i(\xi_0)^{-1}$. The form φ is therefore the meromorphic form

$$\varphi = (\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}w_3)/(w_1w_2 - (w_3)^2)^2.$$

The blow up of the origin in $\mathbb{A}^3$ is described by

$$(w_1, w_2, w_3, \tau_1, \tau_2, \tau_3) \to (w_1, w_2, w_3)$$

with relations $w_i\tau_j = w_j\tau_i$, here τ_i are homogeneous coordinates.

We need an open cover $\mathscr{D} = \{D_i\}$ for $\mathbb{P}_+$, where we require that on each open set D_i the form $\varphi = \mathrm{d}\theta_i$, θ_i being a rational form with second order pole on E and first order pole along S. We take D_1 (resp. D_2) to be the open set $\tau_1 \neq 0$ (resp. $\tau_2 \neq 0$) on $\mathbb{P}_+$; the actual choice of the other D's is not important in practice, but it can be done because sequence (5.1) is exact. On $D_i \cap D_j$ the difference $\theta_{ij} := \theta_i - \theta_j$ is a section of $\Omega^{\wedge 2}_{\mathbb{P}_+}(S + 2E)$; the family $\{\theta_{ij}\}$ is a 1-cocycle for $\Omega^{\wedge 2}_{\mathbb{P}_+}(S + 2E)$. It turns out that $A := D_1 \cap S$ and $B := D_2 \cap S$ are open sets with the properties required above, so that the residue of θ_{12} will give the image $R(\varphi)$.

On D_1 the coordinates are $t_2 = \tau_2(\tau_1)^{-1}$, $t_3 = \tau_3(\tau_1)^{-1}$ and $u = w_1$; so $w_2 = ut_2$, $w_3 = ut_3$.

The form $\varphi = (\mathrm{d}u\,\mathrm{d}t_2\,\mathrm{d}t_3)/u^2\,(t_2 - (t_3)^2)^2$, hence on D_1 $\varphi = \mathrm{d}(\mathrm{d}u\,\mathrm{d}t_3/u^2(t_2 - (t_3)^2))$ i.e. $\varphi = \mathrm{d}\theta_1$, where $\theta_1 = \mathrm{d}u\,\mathrm{d}t_3/u^2(t_2 - (t_3)^2)$.

Similarly on D_2 the coordinates are $s_1 = \tau_1(\tau_2)^{-1}$, $s_3 = \tau_3(\tau_2)^{-1}$ and $v = w_2$; so $w_1 = vs_1$, $w_3 = vs_3$. The form φ is on D_2 $\varphi = (\mathrm{d}s_1\,\mathrm{d}v\,\mathrm{d}s_3)/v^2(s_1 - (s_3)^2)^2$ and therefore $\varphi = \mathrm{d}(-\mathrm{d}v\,\mathrm{d}s_3/v^2(s_1 - (s_3)^2))$, i.e. $\varphi = \mathrm{d}\theta_2$, where $\theta_2 = -\mathrm{d}v\,\mathrm{d}s_3/v^2(s_1 - (s_3)^2)$.

On $D_1 \cap D_2$ we get

$$\begin{aligned}\theta_1 - \theta_2 &= (\mathrm{d}w_1\,\mathrm{d}(w_3/w_1)/w_1^2(w_2/w_1 - (w_3/w_1)^2)) \\ &\quad + (\mathrm{d}w_2\,\mathrm{d}(w_3/w_2)/w_2^2(w_1/w_2 - (w_3/w_2)^2)) \\ &= [\mathrm{d}w_1\,\mathrm{d}w_3/w_1(w_2w_1 - (w_3)^2)] + [\mathrm{d}w_2\,\mathrm{d}w_3/w_2(w_1w_2 - (w_3)^2)] \\ &= \mathrm{d}(w_1w_2 - (w_3)^2)\,\mathrm{d}w_3/w_1w_2(w_1w_2 - (w_3)^2).\end{aligned}$$

Since $(w_1w_2 - (w_3)^2)$ is a local equation for S, we get on $S \cap D_1 \cap D_2 = A \cap B$

$$R(\theta_1 - \theta_2) = \mathrm{d}w_3/w_1w_2 = \mathrm{d}w_3/(w_3)^2 = \mathrm{d}(-(w_3)^{-1}).$$

(5.3) Here we compute the class of $R(\varphi)$ as Dolbeault-cocycle. To compute explicitly we need to relate the Cech cohomology with the 'Dolbeault' cohomology for $H^1(S, \Omega^{\wedge 1}(2C))$, in other words we want to express the class of a cocycle $s_{A,B}$ by means of C^∞ forms of the type explained in Section 2. This is done in the following way.

We use a C^∞ partition of unity associated with the cover $\{A, B\}$, i.e. we suppose given two C^∞ functions $\alpha: S \to \mathbb{C}$ and $\beta: S \to \mathbb{C}$, with $\alpha + \beta = 1$, and with support $(\alpha) \subset A$, support$(\beta) \subset B$.

Let $s_{A,B}$ be a section in $H^0(A \cap B, \Omega_S^{\wedge 1}(2C))$, then the image of $s_{A,B}$ in $H^1(S, \Omega_S^{\wedge 1}(2C))$ is also given by considering the global form ω, say, defined by $\omega = (\mathrm{d}\beta)(s_{A,B})$ on A and by $\omega = (-\mathrm{d}\alpha)(s_{A,B})$ on B. In our case we have that $R(\varphi)$ is represented by $(\mathrm{d}\beta) \wedge \mathrm{d}(-(w_3)^{-1}) = \mathrm{d}(\beta \mathrm{d}(-(w_3)^{-1}) = \omega|_A$ on A and by $(-\mathrm{d}\alpha) \wedge \mathrm{d}(-(w_3)^{-1}) = -\mathrm{d}(\alpha(\mathrm{d}(-(w_3)^{-1}))) = \omega|_B$ on B; in particular $R(\varphi)$ is exact both on A and B.

(5.4) Here we compute the regulator pairing of $R(\varphi)$ with an interesting element in $H^1(X, \mathscr{K}_2)$.

The w's are rational functions on S, the associated divisors are $(w_1)_0 = C + 2L_1$, where L_1 is the line $\{\xi_1 = 0 = \xi_3\}$, and $(w_1)_\infty = C_\infty$, where C_∞ is the intersection of X with the plane $\{\xi_0 = 0\}$. Similarly $(w_2)_0 = C + 2L_2$, where L_2 is the line $\{\xi_2 = 0 = \xi_3\}$, and $(w_2)_\infty = C_\infty$. Since on S $w_1w_2 = (w_3)^2$, then $(w_3)_0 = L_1 + L_2 + C$, $(w_3)_\infty = C_\infty$. Note that A is the complement of L_1 in S and B is the complement of L_2, i.e. $A = S - L_1$, $B = S - L_2$.

For any complex number $s = t^{-1}$, we denote C_t the divisor on S which is the pull back to S of the divisor $s\xi_3 = \xi_0$ on the cone X. The Steinberg symbol $\{w_1/w_3,\ 1 - sw_3\} \in K_2(k(S))$ has image under the tame symbol the cocycle $x_s := (C_t, w_1/w_3) + (C_\infty, w_3/w_1)$. By the Gersten resolution x_s represents the zero element in $H^1(S, \mathscr{K}_{2S})$. Since its support does not intersect the exceptional divisor C, x_s represents an element in $H^1(X, \mathscr{K}_{2X})$, cf. [2], which is therefore in the kernel of the map $H^1(X, \mathscr{K}_{2X}) \to H^1(S, \mathscr{K}_{2S})$. We compute in a moment that the pairing $\rho(x_s, R(\varphi)) = (2\pi i)s$, hence each x_s gives a non-zero element in the kernel.

We consider first the case $s = 1$. In this case one can see explicitly that x_s is in $Z^{1,1}(X)_h$. Take the 1-chain γ to be the sum of: (i) the real component of C_1 which contains the point $w_1 = 1$, $w_2 = 1$, $w_3 = 1$, oriented from $w_1 = 0$ to $w_1 = \infty$, with: (ii) the 1-chain on C_∞ which is the projection of the previous component but taken with opposite orientation. The real positive part of the cone which stretches from $w_3 = 1$ to $w_3 = \infty$ is a 2-chain Δ with $\partial\Delta = \gamma$.

By definition, see (3.4), the value of $\rho(x_1)(R(\varphi))$ is the sum of three integrals, one computed along Δ, and the other two computed along C_1 and C_∞. In our case those last two integrals vanish, because on C_1 and on $C_\infty R(\varphi)$ is the zero form; indeed we see from the description in (5.3) that $R(\varphi)$ is locally either $(d\beta) \wedge d(-(w_3)^{-1})$ or $(-d\alpha) \wedge d(-(w_3)^{-1})$, now on $C_1 w_3 = 1$ identically and on $C_\infty (w_3)^{-1} = 0$. To compute the integral of $R(\varphi)$ along Δ, we cut Δ in two parts by drawing on S the real semi-line $L(1)(w_1)^{1/2} = (w_2)^{1/2} = \tau$, where τ varies from $\tau = 1$ to $\tau = \infty$. Note that one part of Δ is contained in A and the other is contained in B, so we may use Stokes formula on each part. Keeping in account orientations and the remarks above we have:

$$(2\pi i)\int_\Delta R(\varphi) = (2\pi i)\left[\int_{L(1)} \beta d(-(w_3)^{-1}) + \int_{L(1)} \alpha d(-(w_3)^{-1})\right]$$
$$= (2\pi i)\int_{L(1)} d(-(w_3)^{-1}) = (2\pi i).$$

In case of an arbitrary s, we note that under the transformation $w_1 = tz_1$, $w_2 = tz_2$, $w_3 = tz_3$ the cone is mapped to the (same) cone of equation $0 = z_1z_2 - (z_3)^2$, the form $\varphi = (dw_1\, dw_2\, dw_3)/(w_1w_2 - (w_3)^2)^2$ becomes $s(dz_1\, dz_2\, dz_3)/(z_1z_2 - (z_3)^2)^2$, the curve C_t is mapped to the curve $z_3 = 1$ and the curve C_∞ is mapped to the curve $z_3 = \infty$. Using the computation just given, we find $\rho(x_s, R(\varphi)) = (2\pi i)s$.

Let $B(S)$ and $B(X)$ be the groups defined in (2.5). We have exact sequences

$$0 \to (H^1(C, \mathcal{O}_C(C))^* \to B(X) \to B(S) \to H^0(C, \mathbb{C}_C)^*/\mathbb{Z} \to 0$$
$$0 \to \mathrm{KER} \to H^1(X, \mathcal{K}_{2X}) \to H^1(S, \mathcal{K}_{2S}) \to \mathrm{COK} \to 0.$$

In our example $H^1(X, \mathcal{K}_{2X})_h = H^1(X, \mathcal{K}_{2X})$ and $H^1(S, \mathcal{K}_{2S})_h = H^1(S, \mathcal{K}_{2S})$. The regulator maps induce a morphism from the second sequence to the first. The class of $R(\varphi)$ maps to a non-zero element of $H^1(C, \mathcal{O}_C(C)) \cong \mathbb{C}$. It follows from our computation of the pairings $\rho(x_s, R(\varphi))$ that $\rho(\mathrm{KER}) = (H^1(C, \mathcal{O}_C(C))^* \cong \mathbb{C}$.

The analogy with the cuspidal curve case is almost complete, still missing is the inversion theorem, which in this case would say that $\mathrm{KER} \approx (H^1(C, \mathcal{O}_C(C))^* \cong \mathbb{C}$. We have no information in this direction.

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Note. The results of our paper have been deeply generalized by H. Esnault in: A regulator map for singular varieties, *Math. Ann.*