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Fixed points, exceptional orbits, and homology of affine $\mathbb{C}^*$-surfaces

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0. Introduction

Mathematical objects with many automorphisms have a rich structure that makes them particularly interesting. As a nice example, we mention the case of smooth projective algebraic surfaces with an algebraic C*-action. Their structure has been described in [OrWa] by means of a weighted graph that reflects the fixed point set and the exceptional orbits of the action. In [FiKp], we have given a description of normal (not necessarily smooth) affine algebraic C*-surfaces using a weighted graph that represents some 'orbit space', the fixed point set, and the orbit data of the exceptional orbits.

In the present article, we study that class of surfaces from the viewpoint of algebraic topology. Our interest is the impact of a C*-action on local and global homological data: as an application of [FiKp] we calculate the homology and cohomology groups, both with compact and with closed supports. In the smooth case, most of the results have been obtained independently in [Ry]. A completion of that algebraic picture can be found in [FiKp₂], where the intersection homology is computed.

Let W denote a (connected) normal affine algebraic surface over the complex numbers and $\tau: \mathbb{C}^* \times W \to W$ an effective algebraic action of $\mathbb{C}^*$. Then W includes either precisely one *elliptic* fixed point of the action τ (i.e., an isolated fixed point adherent to every nearby orbit), or there is a curve of *parabolic* fixed points, or there is at most a finite number of *hyperbolic* fixed points (i.e., isolated fixed points that are not adherent to every nearby orbit). We call these the *elliptic*, the *parabolic*, and the *hyperbolic* cases, respectively. Thus actions without fixed points are included in the hyperbolic case. We set

$$F := W^{\mathbb{C}^*}, \quad W^* := W \backslash F, \quad Y(\tau) := \begin{cases} W^* /\!\!/ \mathbb{C}^* & \text{if } W \text{ is elliptic,} \\ W /\!\!/ \mathbb{C}^* & \text{otherwise,} \end{cases}$$

and we denote the canonical projection from W^*, resp. W in the nonelliptic case, onto $Y(\tau)$ by π.

   Not all of the (co-)homology groups really depend on the particular surface W under consideration; since affine algebraic varieties are Stein spaces, we only have to compute the homology groups for compact supports in dimensions 1 and 2 and for closed supports in dimensions 2 and 3 (see the beginning of section 1; for general coefficients we reduce the computation in 1.3, 2.6, and 2.18 in an explicit manner to integral coefficients). The central results about the non vanishing integral homology and cohomology as well as the Poincaré duality homomorphisms $P_j^{\varphi}(W, \mathbb{Z}) \colon H_{\varphi}^{4-j}(W, \mathbb{Z}) \to H_j^{\varphi}(W, \mathbb{Z})$ are described in the general structure table below, using Betti numbers and two torsion groups. These data then are made explicit in two additional tables.

j	$H_{\varphi}^{4-j}(W, \mathbb{Z})$	$P_j^{\varphi}(W, \mathbb{Z})$	$H_j^{\varphi}(W, \mathbb{Z})$	φ
1	$\mathbb{Z}^{b_3^{cld}} \oplus T_2^{cld}$	$\twoheadrightarrow$	$\mathbb{Z}^{b_1} \oplus T^2$	c
2	$\mathbb{Z}^{b_2^{cld}}$	$\rightarrow$	$\mathbb{Z}^{b_2}$	c
2	$\mathbb{Z}^{b_2} \oplus T^2$	$\hookrightarrow$	$\mathbb{Z}^{b_2^{cld}} \oplus T_2^{cld}$	cld
3	$\mathbb{Z}^{b_1}$	$\hookrightarrow$	$\mathbb{Z}^{b_3^{cld}}$	cld

(see 1.3 and 2.16). The Betti numbers b_j^{φ} of W (with $b_j := b_j^c$) and the torsion groups

$$T^2 := \text{Tors } H^2(W, \mathbb{Z}) \quad \text{and} \quad T_2^{cld} := \text{Tors } H_2^{cld}(W, \mathbb{Z})$$

are determined by data of the action τ. In order to describe them we have to distinguish the different types of $\mathbb{C}^*$-surfaces. If we set $\gamma := b_1(Y(\tau))$, then the Betti numbers are (see 3.1, 3.2, 3.7, and 3.9):

	b_1	b_2	b_2^{cld}	b_3^{cld}
(el)	0	0	γ	γ
(pa)	γ	0	0	γ
(hy), $F = 0$	$\gamma + 1$	γ	γ	$\gamma + 1$
(hy), $F \neq 0$	γ	$\gamma + |F| - 1$	$\gamma + |F| - 1$	γ

   The torsion groups reflect the structure of the exceptional orbits. Let $\{y_1, \ldots, y_r\}$ denote the set of critical values in $Y(\tau)$ of the quotient mapping π. Furthermore, let m_j be the multiplicity of the singular (or exceptional) fiber $\Phi_j := \pi^{-1}(y_j)$. According to [FiKp], every isolated fixed point w in W has a patching weight $l_w \in \mathbb{N}_{>0}$; we denote by l the greatest common divisor of all those patching weights l_w. For every prime number p we describe the p-Sylow subgroup $S_p(T)$ of a torsion group T separately. To that end we order the p-adic valuations $\mu_j := v_p(m_j)$ for the purposes of this introduction in such a way that

$\mu_1 \leqslant \cdots \leqslant \mu_r$ (for fixed p), and we set $\lambda := v_p(l)$. Then there is a decomposition

$$S_p(T_2^{cld}) \cong \bigoplus_{j=1}^{r-2} \mathbb{Z}_{p^{\mu_j}} \oplus R_p$$

where the missing direct factor R_p together with the group $S_p T^2$ can be read off from the following table:

	$S_p T^2$	R_p
(el)	0	$\mathbb{Z}_{p^{\lambda + \mu_r - 1}}$
(pa)	0	$\mathbb{Z}_{p^{\mu_r - 1}} \oplus \mathbb{Z}_{p^{\mu_r}}$
(hy), $F = 0$	$\displaystyle\bigoplus_{j=1}^{r-1} \mathbb{Z}_{p^{\mu_j}}$	$\mathbb{Z}_{p^{\mu_r - 1}}$
(hy), $F \neq 0$	$\displaystyle\bigoplus_{y_j \notin \pi(F)} \mathbb{Z}_{p^{\mu_j}}$	$\mathbb{Z}_{p^{\mu_r - 1}} \oplus \mathbb{Z}_{p^{\lambda + \mu_r}}$

In the first two sections of this article we recall some general facts about the homology of affine surfaces and list basic results about their geometry if they admit a nontrivial C*-action [cf. FiKp]. Section 3 is devoted to a systematic treatment of the homological results, including the computations in the parabolic case, which is the easiest one. Before we perform in sections 5 and 6 the explicit calculations for hyperbolic and elliptic actions, we discuss in section 4 the way back from algebraic topology to geometry:

For the investigation of the fixed points it is sufficient to calculate Betti numbers – in fact, the Euler characteristic e is sufficient in most cases; hence, one might restrict the attention to rational coefficients, which makes the computations definitely easier. But for the structure of the exceptional orbits it is precisely the torsion which reflects the geometry. For that reason we attach particular importance to integral coefficients. The case of general coefficients then is obtained by universal coefficient formulas.

For an affine C*-surface (W, τ) the type of the C*-action τ is not determined by the underlying affine algebraic surface W: the product surfaces $\mathbb{C}^* \times \mathbb{C}$ endowed with the diagonal actions $\tau_{a,b}$ (2.11) and the cyclic quotient surfaces of $(\mathbb{C}^2, \tau_{a,b})$ carry different types of actions. We shall see in 4.11 that these are the only examples. In almost all other cases the type of τ is even determined by the homology $H^\varphi(X, \mathbb{Z})$, $\varphi = c, cld$; for the few exceptions see 4.17 and 4.18. If one adds some information about the action τ like the Euler characteristic of the associated curve $Y(\tau)$, then that result can even be sharpened, see 4.21. We discuss questions of that kind in form of the flow chart given in 4.20.

From the homology tables of C*-surfaces one reads off some consequences. In particular, the number $\mathrm{fix}(W, \tau) := |\pi_0(W^{\mathbb{C}^*})|$ (also denoted by $\mathrm{fix}(\tau)$ or $\mathrm{fix}(W)$ if there is no risk of confusion) is a homotopy invariant, unless W is isomorphic to

$\mathbb{C}^* \times \mathbb{C}$, see 4.7. An analogous result does not hold for the number of exceptional orbits, as there the torsion group $T_2^{cld}(W)$ is involved. Since a direct factor $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ (for prime numbers p and q) might as well come from one exceptional orbit of order pq as from two exceptional orbits of orders p resp. q, one has to introduce the notion of a p-exceptional orbit. For $\mathbb{C}^*$-surfaces W *with* fixed points, the number of *closed* p-exceptional orbits is determined by the homotopy type of W.

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1. Generalities on the homology of affine surfaces

In this section we reduce the computation of the homology of a normal affine algebraic surface W to that of two global Betti numbers, a local Betti number, and two torsion modules.

For a principal ideal domain R of characteristic $q(R) \geqslant 0$ and an R-module M, we consider the homology $H_j^\varphi(W, M)$ and the cohomology $H_\varphi^j(W, M)$ for the families $\varphi = c$ of compact supports resp. $\varphi = cld$ of closed supports. As usual, we set $H_j(W, M) := H_j^c(W, M)$ and $H^j(W, M) := H_{cld}^j(W, M)$. Moreover, we denote by $_W\mathcal{H}_j M$ the j-th singular local homology sheaf of W with coefficients in M and by $\mathcal{H}_j(W, M)$ its global section space. By [AnNa, Th. 1] and Universal Coefficient Formulas, some of the (co-)homology modules are independent of the particular surface W and others are free:

$$H_j(W, M) = H_c^{4-j}(W, M) = H^j(W, M) = H_{4-j}^{cld}(W, M)$$

$$= \mathcal{H}_{4-j}(W, M) = 0 \quad \text{for } j = 3, 4$$

$$H_4^{cld}(W, M) \cong H^0(W, M) \cong H_0(W, M) \cong H_c^4(W, M) \cong \mathcal{H}_4(W, M) \cong M;$$

$$H_2(W, R), \ H_c^2(W, R), \ H^1(W, R), \ H_3^{cld}(W, R),$$

$$\text{and } \mathcal{H}_3(W, R) \text{ are free } R\text{-modules.}$$

As usual, $M * N$ denotes the module $\mathrm{Tor}^R(M, N)$. We consider the free part of the homology modules and their torsion part separately, using the notation

$$T_j^\varphi(R) := \mathrm{Tors}\, H_j^\varphi(W, R), \qquad T_\varphi^j(R) := \mathrm{Tors}\, H_\varphi^j(W, R);$$

$$b_j^\varphi(R) := \mathrm{rank}\, H_j^\varphi(W, R), \qquad b_\varphi^j(R) := \mathrm{rank}\, H_\varphi^j(W, R),$$

$$\beta(R) := \mathrm{rank}\, \mathcal{H}_2(W, R).$$

As normal surfaces have at most isolated singularities, all global and local

(co-)homology modules are related by the long exact Poincaré duality sequence [Kp, Satz 1.1]

$$\cdots \to \mathscr{H}_{j+1}(W) \to H_\varphi^{4-j}(W, R) \xrightarrow{P_j^\varphi} H_j^\varphi(W, R) \to \mathscr{H}_j(W) \to H_\varphi^{5-j}(W, R) \to \cdots.$$

$$(1.1)$$

Moreover, local Poincaré duality yields $\beta(R) = \mathrm{rank}\, \mathscr{H}_3(W, R)$. We shall omit the ring R if there is no ambiguity; again, we set $b_j := b_j^c$, $b^j := b_{cld}^j$, etc.

We intend to express all global homological data in terms of the usual singular cohomology with closed supports. The most complicated case is that of $T_2^{cld}(R)$, which, in a first step, can be replaced as follows: if Σ is a finite set in W that includes the singular locus $S(W)$, then relative Poincaré duality yields an isomorphism

$$H^2(W \setminus \Sigma, M) \to H_2^{cld}(W, \Sigma; M) \cong H_2^{cld}(W, M).$$

$$(1.2)$$

As a consequence, $T_2^{cld}(R)$ equals the torsion submodule of $H^2(W \setminus \Sigma, R)$ (independently of the set Σ).

1.3 THEOREM. *Assume that the kernel of the second Poincaré homomorphism $P_2^{cld}(W, R)$ is a torsion module. Then we obtain the following table for the Poincaré homomorphisms:*

j	$H_\varphi^{4-j}(W, M)$		$H_j^\varphi(W, M)$	φ
1	$M^{b^1+\beta} \oplus T_2^{cld} \otimes M$	$\twoheadrightarrow$	$M^{b^1} \oplus T^2 \otimes M$	c
2	$M^{b^2+\beta} \oplus T_2^{cld} * M$	$\to$	$M^{b^2} \oplus T^2 * M$	c
2	$M^{b^2} \oplus T^2 \otimes M$	$\to$	$M^{b^2+\beta} \oplus T_2^{cld} \otimes M$	cld
3	$M^{b^1} \oplus T^2 * M$	$\hookrightarrow$	$M^{b^1+\beta} \oplus T_2^{cld} * M$	cld

Proof. As an algebraic surface, W has finitely generated homology $H_*(W, R)$. Thus there are covariant Universal Coefficient Formulas

$$H_j^\varphi(W, M) \cong H_j^\varphi(W, R) \otimes M \oplus H_{j-1}^\varphi(W, R) * M,$$

$$(1.4)$$

$$H_\varphi^j(W, M) \cong H_\varphi^j(W, R) \otimes M \oplus H_\varphi^{j+1}(W, R) * M.$$

$$(1.5)$$

Hence, it suffices to compute $H_*^\varphi(W, R)$ and $H_\varphi^*(W, R)$. For $\{\varphi, \bar\varphi\} = \{c, cld\}$, a contravariant Universal Coefficient Formula yields $T_j^\varphi \cong T_{\bar\varphi}^{j+1}$. In combination with 1.2, that covers the torsion part. For the Betti numbers, we use the exact Poincaré duality sequence 1.1. As we consider only ranks, we may replace R by

its field of quotients $Q(R)$; hence, we may assume that R is even a field and that the homomorphism $P_2(W, R)$ is injective. Then 1.1 breaks up into two short exact sequences with coefficients in the field R, namely,

$$0 \to H^1(W) \to H^{cld}_3(W) \to \mathcal{H}_3(W) \to 0$$

$$\text{and } 0 \to H^2(W) \to H^{cld}_2(W) \to \mathcal{H}_2(W) \to 0.$$

Hence, $b^{cld}_2 = b^2 + \beta$ and $b^{cld}_3 = b^1 + \beta$. For the cohomological part we use an analogous argument, since, obviously, $b^j = b_j$.   □

In the next section we shall sharpen 1.3 for $\mathbb{C}^*$-surfaces.

2. Generalities on $\mathbb{C}^*$-surfaces

For the convenience of the reader, we recall some basic definitions and facts from [FiKp]. We let W denote an *affine* $\mathbb{C}^*$-*surface*, i.e., a (connected) normal complex affine algebraic surface endowed with an effective algebraic $\mathbb{C}^*$-action $\tau: \mathbb{C}^* \times W \to W$. Moreover, we denote by F the fixed point set of the action τ and by W^* its complement. A fixed point $x \in F$ is called

(el) *elliptic* if it has a neighborhood U such that every orbit that intersects U has x as a limit point;
(hy) *hyperbolic* if x is an isolated fixed point that is not elliptic;
(pa) *parabolic* if x is not an isolated fixed point.

2.1 DEFINITION. For a $\mathbb{C}^*$-surface W, we denote by $el(W)$ resp. by $hy(W)$ the number of elliptic resp. of hyperbolic fixed points, and by $pa(W)$ the number of connected components of the set of parabolic fixed points of W.

While for an affine $\mathbb{C}^*$-surface W the numbers $el(W)$ and $pa(W)$ are at most one, there is no restriction for $hy(W)$:

2.2 EXAMPLE. For every natural number $n \geqslant 0$ there is an affine surface W realizing $hy(W) = n$: If $p \in \mathbb{C}[z]$ is a polynomial with precisely n different zeros, then the algebraic variety $W := V(\mathbb{C}^3; p(z) - xy)$ with the action $t \cdot (x, y, z) = (tx, t^{-1}y, z)$ satisfies $hy(W) = n$. If z is an m-fold zero of p and $w = (0, 0, z) \in W$, then the local homology group $_W \mathcal{H}_{2,w} \mathbb{Z}$ is isomorphic to the cyclic group $\mathbb{Z}_m$.

Obviously, singular points are always fixed points. In contrast to the compact case, the fixed points in an affine $\mathbb{C}^*$-surface are always of the same type, and there are even actions without fixed points:

2.3 REMARK. Every $\mathbb{C}^*$-action τ on an affine $\mathbb{C}^*$-surface W belongs to exactly one of the following three classes:

(el) τ has precisely one elliptic and no other fixed point.

(pa) Every fixed point of τ is parabolic; in particular, $pa(W) = 1$.

(hy) τ is fixed point free or has finitely many hyperbolic fixed points.

We say that the action τ, or the pair (W, τ), or the affine $\mathbb{C}^*$-surface W is *elliptic, parabolic* or *hyperbolic*. Note that a given surface W may admit actions of different type:

2.4 EXAMPLE. Consider the affine quadric cone $W = V(\mathbb{C}^3; xy - z^2)$ with the $\mathbb{C}^*$-actions $t \cdot (x, y, z) := (t^a x, t^b y, t^c z)$ for $a, b, c \in \mathbb{Z}$ with $a + b = 2c$. If $ab > 0$, then 0 is an elliptic fixed point; if $ab < 0$, then 0 is a hyperbolic fixed point; if $ab = 0$, then the action is parabolic and F is isomorphic to a complex line.

An important tool for the investigation of affine $\mathbb{C}^*$-surfaces W is the algebraic mapping $\pi: W \to W/\!/\mathbb{C}^*$, where $W/\!/\mathbb{C}^*$ is the *algebraic* quotient [Kr, II.3.1], a smooth connected affine algebraic variety of dimension at most one, which parametrizes the *closed* orbits. The parabolic type is characterized by the condition dim $W/\!/\mathbb{C}^* = 1 = $ dim F, the hyperbolic type by dim $W/\!/\mathbb{C}^* = 1$ and dim $F = 0$. In the elliptic case, we have dim $W/\!/\mathbb{C}^* = 0$, so we replace the algebraic quotient by a more interesting object: for W^* instead of W the quotient $W^*/\!/\mathbb{C}^*$ exists as well, it is a compact curve, which coincides with the 'geometric' quotient $W^*/\mathbb{C}^*$. For an action τ on an affine $\mathbb{C}^*$-surface W, we set

$$Y := Y(\tau) := \begin{cases} W^*/\!/\mathbb{C}^* & \text{if } \tau \text{ is elliptic,} \\ W/\!/\mathbb{C}^* & \text{otherwise,} \end{cases} \tag{2.5}$$

and we let $\pi: W^* \to Y$ resp. $\pi: W \to Y$ denote the natural mapping. Then π realizes the $\mathbb{C}^*$-surface W^* resp. W as a semistable $\mathbb{C}^*$-surface over Y, see 6.1. For an elliptic action (or a hyperbolic action without fixed points) on W, the mapping π may be interpreted as a Seifert $\mathbb{C}^*$-bundle in the sense of [Ho]. Note that π is affine and that every open affine subset U of Y satisfies $U \cong \pi^{-1}(U)/\!/\mathbb{C}^*$. In order to investigate the properties of the fibers $\Phi_y := \pi^{-1}(y)$, it thus suffices to consider hyperbolic and parabolic *affine* $\mathbb{C}^*$-surfaces. After possibly composing the given action with the automorphism $t \mapsto t^{-1}$ of $\mathbb{C}^*$, we obtain the following types of fibers:

(el) $\Phi_y = O$,   (pa) $\Phi_y = \{x\} \cup O_+$,

(hy) $\Phi_y = O$ or $\Phi_y = O_- \cup \{x\} \cup O_+$,

where O, O_+, and O_- are orbits of positive dimension, and $\{x\}$ is the *source* of the orbit O_+ (i.e., $\lim_{t \to 0} tw = x$ for every $w \in O_+$), and the *sink* of the orbit O_- (i.e., $\lim_{t \to \infty} tw = x$ for every $w \in O_-$). We always shall assume that every fixed point of the action τ is the source of a nontrivial orbit.

2.6 REMARK. In the elliptic and the parabolic case, the morphism τ extends to a morphism $\bar{\tau}: \mathbb{C} \times W \to W$. The restriction of $\bar{\tau}$ to $[0, 1] \times W$ shows that F is a deformation retract of W, and the induced mapping $W \to F$, $x \mapsto \bar{\tau}(0, x)$, identifies F with the algebraic quotient $W /\!\!/ \mathbb{C}^*$.

An orbit $O = \mathbb{C}^* w$ of τ of positive dimension is of the form $\mathbb{C}^*/C_m \cong \mathbb{C}^*$, where $C_m := \{\eta \in \mathbb{C}; \eta^m = 1\} \subset \mathbb{C}^*$ is the isotropy group of the point w. If its *order m* is one, then $\mathbb{C}^* w$ is called a *principal orbit*, otherwise an *exceptional orbit*, of *(exceptional) order m > 1*. The exceptional orbits in an algebraic $\mathbb{C}^*$-surface are always finite in number. We want to parametrize the singular fibers of the mapping π. The set

$$A := \{y \in Y;\ \Phi_y \text{ is an exceptional orbit or its closure}\} =: \{y_1, \ldots, y_s\}$$

$$(2.7)$$

is the set of critical values of π in the nonhyperbolic case. In the hyperbolic case, the fibers containing fixed points w_j are singular as well. Then set $y_j = \pi(w_j)$ for $s + 1 \leqslant j \leqslant s + h$, so

$$B := A \cup \pi(F) =: \{y_1, \ldots, y_s, y_{s+1}, \ldots, y_{s+h}\} \tag{2.8}$$

is the set of critical values of π. For $j = 1, \ldots, s + h$ we call $\Phi_j := \Phi_{y_j}$ an *exceptional fiber* and y_j an *exceptional point*; their order or *multiplicity* m_j is the greatest common divisor of the orders of the one or two non-trivial orbits included in Φ_{y_j}. Note that $m_j = 1$ may occur for $j \geqslant s + 1$. For every point $y_j \in B$ we have

$$m_j = |\ker(\mathbb{C}^* \to \mathrm{Aut}(\Phi_{y_j}))|. \tag{2.9}$$

For the investigation of the torsion groups T in the homology, the p-Sylow subgroups $S_p(T)$ have to be calculated for every prime number p separately. To that end we shall consider the p-adic valuation

$$\mu_j := \mu_j(p) := v_p(m_j) \quad \text{for } j = 1, \ldots, s + h. \tag{2.10}$$

We usually arrange the index set, depending on p, in such a way that $\mu_1 \leqslant \cdots \leqslant \mu_s$. In the introduction we preferred the order $\mu_1 \leqslant \cdots \leqslant \mu_{s+h}$, which means that we had to mix up the elements of A and $\pi(F)$.

We need some details about the local structure in the *complex* topology near a nontrivial orbit O of order $m \geqslant 1$ in an affine $\mathbb{C}^*$-surface W. According to the analytic version of Luna's Slice Theorem ([Lu], cf. [BBSo, (0.2.1)] or [FiKp, 2.6]), the orbit O admits an open invariant neighborhood U isomorphic to a

complex $\mathbb{C}^*$-manifold $\mathbb{C}^* \times_{C_m} D$ (with the action induced from the first factor). Here $\eta \in C_m$ acts on the open unit disc D in $\mathbb{C}$ as $w \mapsto \eta^n w$ for some integer n relatively prime to m. If we denote with $\tau_{m,n}$ the standard $\mathbb{C}^*$-action

$$\tau_{m,n} \colon \mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2, \quad t \cdot (z, w) := (t^m z, t^n w) \quad \text{for } (m, n) = 1, \tag{2.11}$$

on $\mathbb{C}^2$ and on open invariant subspaces of $\mathbb{C}^2$, then we have

$$\mathbb{C}^* \times_{C_m} D \cong \{(z, w) \subset \mathbb{C}^* \times \mathbb{C}; \ |z^{-n} w^m| < 1\} \subset (\mathbb{C}^2, \tau_{m,n}). \tag{2.12}$$

The orbit O corresponds to Φ_0, and the mapping $\pi \colon U \to U /\!/ \mathbb{C}^*$ writes as $(z, w) \mapsto z^{-n} w^m$. Note that, in 2.12, we adopt the convention – to be kept up in the sequel – to write the fiber as the first factor and the basis as the second. If we disregard the $\mathbb{C}^*$-action, we can construct a homeomorphism

$$\mathbb{C}^* \times D \to U, \qquad (z, w) \mapsto (z, w|z|^{n/m}). \tag{2.13}$$

Hence, Φ_0 is a retract by deformation of U. The inclusion into U of an ordinary orbit $\Phi = \Phi_y = \{(z, w) \in U; \ z^{-n} w^m = y\}$ with $y \neq 0$ yields a natural homomorphism

$$H^1(\Phi_0, \mathbb{Z}) \cong H^1(U, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \cong H^1(\Phi_y, \mathbb{Z}), \tag{2.14}$$

since the "horizontal" projection $\Phi_y \to \Phi_0$, $(z, w) \mapsto (z, 0)$ is an m-fold covering.

Furthermore, we recall from [FiKp] the local structure of W near a nonelliptic fixed point w:

2.15 REMARK. (pa) A parabolic fixed point w is singular iff it is adherent to an exceptional orbit O. Then there is an open invariant neighborhood U of $\bar{O}$ isomorphic to $(\mathbb{C} \times D, \tau_{1,0})/C_{m,-n}$, where $(m, \bar{n})$ are the orbit data of O, i.e., m is the exceptional order and n describes the normal representation of the isotropy group C_m of O. In particular, U is contractible, and w is a cyclic quotient singularity with $\mathcal{H}_{2,w} \cong \mathbb{Z}_m$. For the details we refer to [FiKp, 3.2, 6.1, and 6.2].

(hy) In the hyperbolic case, w is the source of an orbit O_+ and the sink of an orbit O_-. If $(m_{+/-}, \bar{n}_{+/-})$ are the respective 'orbit data', then there is an open invariant neighborhood U of the exceptional fiber $\Phi_{\pi(w)} = O_- \cup \{w\} \cup O_+$ isomorphic to

$$U(m_+, \bar{n}_+, m_-, \bar{n}_-; l) = U_{m_-, \bar{n}_-} \cup \{w\} \cup U_{m_+, \bar{n}_+},$$

where the two open pieces $U_{m,\bar{n}}$ of type 2.12 are glued together in a way that is

controlled by the 'patching weight' $l \in \mathbb{N}_{>0}$ of the fixed point w. The open set U is contractible, and w is a cyclic quotient singularity with local homology $\mathscr{H}_{2,w} \cong \mathbb{Z}_{lm}$, where $m = \gcd(m_+, m_-)$ is the order of the fiber $\Phi_{\pi(w)}$. For details, we refer to [FiKp, 4.6, 6.2]. $\square$

For an affine $\mathbb{C}^*$-surface we can characterize the assumption of 1.3 and even describe precisely what happens in case of its failure. Again let $q = q(R)$ denote the characteristic:

2.16 REMARK. For an affine $\mathbb{C}^*$-surface W the Poincaré homomorphism $P_2^{cld}(W, R)$ is injective iff one of the following conditions holds:

(a) The characteristic q is zero;
(b) every hyperbolic fixed point w_j in W with $\mathscr{H}_{2,w_j}\mathbb{Z}_q \neq 0$ satisfies $\mu_j(q) \neq 0$;
(c) $hy(W) = 1$ and $\mu_j(q) = 0$ for $j = 1, \ldots, s + h$.

Proof. If W is not hyperbolic, then $F \cong W/\!\!/\mathbb{C}^*$ is a retract by deformation of W, by 2.6, so $H^2(W, R) = 0$. Hence, we may assume that W is hyperbolic. Then W has only cyclic quotient singularities [FiKp, 6.1] and hence is a rational homology manifold [KiBaKp, 5.E.1], so we have $\mathscr{H}_3(W, \mathbb{Z}) = 0$ and in particular $\mathscr{H}_3(W, R) \cong \mathscr{H}_2(W, \mathbb{Z}) * R$. Hence, if $\mathscr{H}_2(W, \mathbb{Z})$ has no q-torsion, then $\mathscr{H}_3(W, R)$ vanishes, so $P_2^{cld}(W, R)$ is injective, by 1.1. For $q \neq 0$ we may assume that $R = \mathbb{Z}_q$.

Then the exact sequence

$$H^2(W, \mathbb{Z}_q) \to H_2^{cld}(W, \mathbb{Z}_q) \to \mathscr{H}_2(W, \mathbb{Z}_q) \to 0$$

reads as

$$H^2(W, \mathbb{Z}) \otimes \mathbb{Z}_q \to H_2^{cld}(W, \mathbb{Z}) \otimes \mathbb{Z}_q \to \mathscr{H}_2(W, \mathbb{Z}) \otimes \mathbb{Z}_q \to 0$$

since $H^3(W, \mathbb{Z}) = H_1^{cld}(W, \mathbb{Z}) = \mathscr{H}_1(W, \mathbb{Z}) = 0$. Counting dimensions and using the results of 3.9, we obtain that $\mathrm{Ker}\, P_2^{cld}(W, \mathbb{Z}_q)$ is a vector space of dimension

$$k(q) = \begin{cases} hy(W) - 1 & \text{if } \lambda(q) > 0 \text{ and } \mu_j(q) = 0, \ 1 \leqslant j \leqslant s + h \\ |\{w_j \in F;\ \mathscr{H}_{2,w_j}\mathbb{Z}_q \neq 0,\ \mu_j(q) = 0\}| & \text{otherwise} \end{cases} \quad \square$$

2.17 COROLLARY. *If the action on W is hyperbolic and $q(R) \neq 0$, then*

$$b_2^{cld}(R) = b^2(R) + \beta(R) - k(q) \quad \text{and} \quad b_3^{cld}(R) = b^1(R) + \beta(R) - k(q). \quad \square$$

We finally reduce the computation to the case $R = \mathbb{Z}$:

2.18 REMARKS. (a) If $q(R) = 0$, then

$$b^j(R) = b^j(\mathbb{Z}), \ \beta(R) = \beta(\mathbb{Z}), \ T^2(R) = T^2(\mathbb{Z}) \otimes R, \ T_2^{cld}(R) = T_2^{cld}(\mathbb{Z}) \otimes R;$$

(b) If $q(R) \neq 0$ and M is an R-module with corank $\mathrm{cork}(M)$, then

$$b^1(R) = b^1(\mathbb{Z}) + \mathrm{cork}\, S_q(T^2(\mathbb{Z})), \qquad b^2(R) = b^2(\mathbb{Z}),$$

$$\beta(R) = \beta(\mathbb{Z}) + \mathrm{cork}\, S_q(\mathscr{H}_2(W, \mathbb{Z})),$$

$$T^2(R) = T_2^{cld}(R) = 0, \qquad H_j^{\varphi}(W, M) \cong H_j^{\varphi}(W, R) \otimes_R M,$$

$$H_{\varphi}^j(W, M) \cong H_{\varphi}^j(W, R) \otimes_R M.$$

(c) The module $T^2(R)$ is in a natural way a submodule of $T_2^{cld}(R)$.

Proof. The inclusion $T^2 \subset T_2^{cld}$ in c) is obvious in the situation of b), and it follows from the injectivity of $P_2^{cld}(W, R)$ otherwise. For the last line in b), we consider M as a vector space over the prime field $\mathbb{Z}_q \subset R$. Since there is no torsion in vector spaces, the covariant Universal Coefficient Formula yields that $H_{\varphi}^j(W, M) \cong H_{\varphi}^j(W, \mathbb{Z}_q) \otimes_{\mathbb{Z}_q} M$, which, by a change of rings, is isomorphic to $H_{\varphi}^j(W, R) \otimes_R M$. As a consequence of 1.5, we have $T^2(R) = 0$ for $q \neq 0$. The other statements follow in a similar manner. □

Hence, we are left with the following

PROBLEM. For integral coefficients, compute b^1, b^2, β, T^2, and T_2^{cld}.

3. Statement of the global results

In this section we describe the missing invariants for the integral (co-)homology of an affine $\mathbb{C}^*$-surface W. In the elliptic and the parabolic case, that can be done in terms of the homology of the algebraic curve Y introduced in 2.5, and of the local homology, which is known in many cases (e.g., [Ha], [KiBaKp], [OrWa], [Ra], . . .). For hyperbolic $\mathbb{C}^*$-surfaces, that is not sufficient; one has to know the structure of the exceptional orbits, too. Hence, we discuss in general the relation between that structure and the local homology.

3.1 PROPOSITION. *If the action τ is elliptic, then*

$$b^1 = b^2 = 0, \ T^2 = 0, \ and \ T_2^{cld} = \mathrm{Tors}\, \mathscr{H}_2(W).$$

Proof. By 2.6, we know that W is contractible. Hence, the reduced cohomology $\tilde{H}^j(W)$ vanishes. Moreover, 1.1 yields $T_2^{cld}(W) \cong \mathrm{Tors}\, \mathscr{H}_2(W)$. □

In the nonelliptic case, the affine algebraic curve $Y = Y(\tau)$ has a (unique) smooth compactification $\bar{Y}$; it is obtained from Y by adding a finite number, $l(Y)$, of points. Let $g(Y) := g(\bar{Y})$ denote the 'genus'. Then the integral

(co-)homology of Y is free of rank

$$b_1(Y) = \gamma := \gamma(Y) := 2g(Y) + l(Y) - 1, \quad \text{and} \quad b_2(Y) = 0.$$

In the parabolic case, the curve Y may be identified with the fixed point set F (cf. 2.6).

3.2 PROPOSITION. *If the action τ is parabolic, then*

$$b^1 = \gamma, \quad b^2 = \beta = 0, \quad T^2 = 0, \quad \text{and} \quad T_2^{cld} = \mathcal{H}_2(W).$$

Proof. As singular parabolic (or hyperbolic) fixed points are always cyclic quotient singularities ([KiBaKp, 5.E.1] or [FiKp, 6.1]), the surface W is a rational homology manifold. In particular, the global section space $\mathcal{H}_2(W, \mathbb{Z})$ of the second local homology is a torsion group, and $\mathcal{H}_3(W, \mathbb{Z})$ vanishes. Moreover, by 2.6, the curve $Y \cong F$ may be considered as a retract by deformation of the variety W. Thus we have

$$H^j(W) \cong H^j(Y) \cong \begin{cases} \mathbb{Z}^\gamma, & \text{for } j = 1, \\ 0, & \text{for } j \geqslant 2. \end{cases}$$

For the computation of T^2 we apply the exact Poincaré duality sequence 1.1

$$0 = H^2(W) \to H_2^{cld}(W) \to \mathcal{H}_2(W) \to H^3(W) = 0. \qquad \square$$

We now intend to express β and T_2^{cld} essentially in terms of the curve Y and the exceptional orders m_i. First let us assume that τ is *elliptic*. The surface W^*/C_m (that is the quotient with respect to the induced action of C_m for $m := \mathrm{lcm}(m_1, \ldots, m_s)$) is a principal $\mathbb{C}^*$-bundle over the compact curve $Y(\tau)$. As in [FiKp, 5.1], we introduce a 'patching weight' (the name has been chosen in analogy to the hyperbolic case) l_0 by

$$l_0 := l_0(W) := -c_1(W^*/C_m)([Y]).$$

For a fixed prime number p we set $\lambda := \lambda(p) := v_p(l_0)$.

3.3 THEOREM. *If the action τ is elliptic, then we have*

$$H_2^{cld}(W) \cong \mathcal{H}_2(W) \cong H^1(Y) \otimes \bigoplus_{p \text{ prime}} S_p \mathcal{H}_2(W),$$

so $\beta = b_1(Y) = 2g(Y)$. For fixed p and $\mu_1 \leqslant \cdots \leqslant \mu_s$ (cf. 2.10), the p-torsion

subgroup $S_p(T_2^{cld}) = S_p\mathscr{H}_2(W)$ is

$$S_p\mathscr{H}_2(W) \cong \mathbb{Z}_{p^{\lambda+\mu_s-1}} \oplus \bigoplus_{j=1}^{s-2} \mathbb{Z}_{p^{\mu_j}}.$$

Proof. The variety W/C_m is obtained from the line bundle associated to W^*/C_m by blowing down the zero-section. According to the theorem of Grauert-Mumford [Lau, 4.9], the Chern class of the bundle and thus the Chern number $c_1(W^*/C_m)[Y]$ is negative. Hence, according to 1.2, we may apply 6.2 with $S = W^*$ and $\zeta = 1$ (see also 6.5). □

Let us digress to an immediate application: Let $\bar{W}$ be a compactification (in the class of normal algebraic ℂ*-surfaces) of an elliptic ℂ*-surface (W, τ). After finitely many quadratic transformations and normalizations, we may assume that $\bar{W}\backslash W$ includes a parabolic fixed curve. Among those 'parabolic' compactifications, there exists a unique minimal element. It satisfies $\bar{W}\backslash W \cong Y(\tau)$. An arbitrary parabolic compactification then is obtained by successively blowing up fixed points (outside of W) of the minimal one.

3.4 COROLLARY. *Let (W, τ) be an elliptic* ℂ*-surface.*

(a) *If $\beta = 0$, then the minimal parabolic compactification is obtained from W by adding a projective line $\mathbb{P}_1$.*
(b) *If $\mathscr{H}_2(W)$ is torsion-free, then $l_0(W) = 1$ and the exceptional orders m_j are pairwise coprime.*

Proof. In case (a), we have $H^1(Y) = 0$ by 3.3, so $Y \cong \mathbb{P}_1$. In case (b), the number $\lambda(p)$ vanishes for every prime number p, so $c_1(W^*/C_m) = -1$ since c_1 is negative. □

For hypersurfaces in $\mathbb{C}^3$ with an elliptic ℂ*-action the results of [KiBaKp, p. 285/6] provide explicit numerical conditions for the vanishing of β and Tors $\mathscr{H}_2(W)$. Every such hypersurface is defined by a weighted homogeneous polynomial with positive weights. In order to give a flavor of that theory, we interpret it in one particular case:

3.5 EXAMPLE. Let W denote a normal weighted homogeneous surface in $\mathbb{C}^3$ that, up to multiplication of the monomials with nonzero complex numbers, is defined by a polynomial of the form

$$z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3} + \text{monomials of different multi-degree,}$$

for natural numbers a_j with $a_2 \geqslant 2$. Set $u := \gcd(a_2 - 1, a_3)$ and $d := a_3/u$. Then

$$\beta(W) = 0 \text{ iff } \begin{cases} \gcd(a_1, a_1 a_3) = \gcd(a_1, a_2) & \text{if } u = 1, \\ \gcd(a_1, da_2) = 1 & \text{if } u > 1, \end{cases}$$

$\text{Tors}_W \mathcal{H}_{2,0} = 0$ iff $\gcd(a_1, a_2) = 1$ and a_1 divides d.   □

Now we come to the *parabolic* case, where the computation of T_2^{cld} follows from 2.15:

3.6 THEOREM. *If the action τ is parabolic with exceptional orbits of orders m_1, ..., m_s, then*

$$T_2^{cld} \cong \mathcal{H}_2(W) \cong \bigoplus_{j=1}^{s} \mathbb{Z}_{m_j}.$$

□

Eventually we study the *hyperbolic* case. For a systematic treatment it is convenient to use the following notation:

$$\delta := \begin{cases} 0, & F \neq 0, \\ 1, & F = 0. \end{cases}$$

Let us begin with an action without fixed points (i.e., $\delta = 1$):

3.7 THEOREM. *If the action τ has no fixed point, then*

$$b^1 = \gamma + 1, \quad b^2 = \gamma, \quad \beta = 0, \quad and \quad T^2 \cong T_2^{cld} \cong \bigoplus_{p \text{ prime}} S_p T_2^{cld}$$

with the p-Sylow group $S_p T_2^{cld} \cong \bigoplus_{i=1}^{s-1} \mathbb{Z}_{p^{\mu_i}}$ for fixed p and the order $\mu_1 \leqslant \cdots \leqslant \mu_s$ (cf. 2.10).

Proof. Since W has no fixed points at all, it is necessarily a manifold. In particular, T_2^{cld} is isomorphic to T^2, which will be calculated together with the Betti numbers in 5.4 and 5.15 as the particular situation $h = 0$ and $\delta = 1$ of the general hyperbolic case.   □

In the absence of fixed points, the homology of W does not reveal the complete structure of exceptional orbits:

3.8 COROLLARY. *Let W be without fixed points. Then $S_p T^2$ has a direct factor $\mathbb{Z}_{p^\nu}$ iff the following condition holds: there exist two different orbits of the action τ such that the isotropy group of one orbit has $\mathbb{Z}_{p^\nu}$ as its p-Sylow group and the isotropy group of the other orbit includes $\mathbb{Z}_{p^\nu}$.*   □

The situation is different in the presence of fixed points: For $w_i \in F$, the order a_i of $\mathcal{H}_{2,w_i}$ is of the form $a_i = l_i m_i$, see 2.15(hy). For a fixed prime number p, we

introduce the notation

$$\lambda := \lambda(p) := v_p(\gcd(l_{s+1}, \ldots, l_{s+h})), \quad \text{and} \quad \mu_k = \max_{1 \le i \le s+h} \mu_i$$

with a suitably chosen index $k = k(p)$. Then we obtain

3.9 THEOREM. *If the action* τ *has at least one hyperbolic fixed point, then*

$$b^1 = \gamma, \qquad b^2 = \gamma + h - 1, \qquad \beta = 0,$$

$$T^2 \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{m_i}, \quad \text{and} \quad T_2^{cld} \cong \bigoplus_{p \text{ prime}} S_p T_2^{cld} \text{ with } S_p T_2^{cld} = \mathbb{Z}_{p^{\lambda + \mu_k}} \oplus \bigoplus_{k \ne i=1}^{s+h} \mathbb{Z}_{p^{\mu_i}}.$$

Proof. Since hyperbolic fixed points are regular points or cyclic quotient singularities [KiBaKp, 5.E.1], the number β vanishes. For the rest of the proof we refer to 5.4, 5.15, and 5.21. □

In [Ry] some of the homology groups above have been calculated independently by a different method: a detailed analysis of smooth affine $\mathbb{C}^*$-surfaces as invariant Zariski open parts of nonsingular projective $\mathbb{C}^*$-surfaces leads to the description of $H_{\cdot}^c(W, R)$ in the smooth case; if the surfaces are without fixed point, then there is the additional hypothesis that $R = \mathbb{Q}$.

4. From the cohomology to the type and the exceptional orbits

So far we have described the cohomology of an affine $\mathbb{C}^*$-surface (W, τ) in terms of the action τ. Our next step is to use the cohomology in the study of the following problem: Which types of actions does an affine surface W admit at all, and how far does the cohomology of W determine the number and the structure of the fixed points and of the exceptional orbits? To begin with, we collect the information on the type that can be read off immediately from the (global and local) Betti numbers and from the torsion group T^2 (cf. 3.2, 3.3, 3.7, and 3.9):

4.1 REMARK. Let (W, τ) be an affine $\mathbb{C}^*$-surface.

(el) If $\beta > 0$, then the action τ is elliptic.
(pa) If $b_1 > b_2 + 1$ (or equivalently, $e(W) < 0$), then the action τ is parabolic.
(hy) If $b_2 > 0$ or $T^2 \ne 0$, then the action τ is hyperbolic. □

We shall obtain more information about the remaining cases later. We now discuss some results on the number fix(W) of fixed point components. In that discussion, the topological Euler characteristic

$$e(W) = 1 - b_1(W) + b_2(W)$$

is particularly useful, as Hanspeter Kraft pointed out to us. We use the following formula:

4.2 PROPOSITION. *The values of the Euler characteristic of a (not necessarily affine) algebraic $\mathbb{C}^*$-surface X and of its fixed point set $F = X^{\mathbb{C}^*}$ coincide:*

$$e(X) = e(F).$$

Proof. For a normal affine $\mathbb{C}^*$-surface, that follows immediately from the results stated in the introduction. In the general case, we may use Mayer-Vietoris arguments, but it is more enlightening and natural to apply the 'additivity' and 'multiplicativity' properties of the Euler characteristic: The additivity yields $e(X) = e(X^* \cup F) = e(X^*) + e(F)$ (with $X^* := X \backslash F$). For any $\mathbb{C}^*$-surface W without fixed points, there is a decomposition $W = V \cup \bigcup O_j$ into a trivial $\mathbb{C}^*$-bundle $V = \mathbb{C}^* \times U$ and finitely many nontrivial orbits O_j. That yields

$$e(W) = e(\mathbb{C}^* \times U) + \sum e(O_j) = 0,$$

as $e(\mathbb{C}^* \times U) = e(\mathbb{C}^*) \cdot e(U) = 0$, thus proving our claim.   □

We explicitly note an immediate consequence of 2.6:

4.3 COROLLARY. *For a parabolic affine $\mathbb{C}^*$-surface (W, τ), we have*

$$e(W) = e(Y(\tau)) = 1 - \gamma.$$ □

In the next statements we carefully distinguish between properties of the action τ and those of the underlying variety W. Our first result on fix(τ) is an immediate consequence of the identity in 4.2 and of our previous results stated in section 3:

4.4 PROPOSITION. *For an affine $\mathbb{C}^*$-surface (W, τ), the following holds:*

(a) *If $e(W) < 0$, then τ is parabolic, so we have* fix$(\tau) = pa(W) = 1$.
(b) *If $e(W) = 0$, then either the action is fixed point free, or it is parabolic with $Y(\tau) \cong \mathbb{C}^*$.*
(c) *If $e(W) = 1$, then either τ is elliptic, or hyperbolic with one fixed point, or parabolic with $Y(\tau) \cong \mathbb{C}$.*
(d) *If $e(W) > 1$, then τ is hyperbolic, so we have* fix$(\tau) = hy(W) = e(W)$.
    In particular, the number fix(τ) *is determined by the homotopy type of the surface W in the following cases:*
(e) *If* fix$(\tau) \neq 0$, *in particular if $e(W) \neq 0$, then* fix$(\tau) = \max\{1, e(W)\}$.
(f) *If the action is not parabolic, then* fix$(\tau) = e(W)$.   □

4.5 REMARK. The quadratic cone (2.4) is an example of an affine surface with $e = 1$ on which all three types of actions can occur. Of course, that holds more generally for any cyclic quotient surface $W = (\mathbb{C}^2, \tau_{a,b})/C_{k,r}$. On the affine surface $\mathbb{C}^* \times \mathbb{C}$, with $e = 0$, both cases of (b) may occur: the action $\tau_{1,0}$ on $\mathbb{C}^* \times \mathbb{C}$ has no fixed point, while the action $\tau_{0,1}$ is parabolic with $F = \mathbb{C}^* \times 0$.

Nevertheless, there is the following result:

4.6 PROPOSITION. *The number* $\mathrm{fix}(\tau)$ *is determined by* $H^\varphi_\bullet(W, \mathbb{Z})$, $\varphi = c$ *and cld in the class of all affine* $\mathbb{C}$*-surfaces* (W, τ) *which are not isomorphic to* $(\mathbb{C}^* \times \mathbb{C}, \tau_{0,1})$. *Moreover, in the class of affine* $\mathbb{C}$*-surfaces with fixed points, the number* $\mathrm{fix}(\tau)$ *is determined by the homotopy type of* W.

Proof. Again by 4.4(e), we may restrict ourselves to the case of vanishing Euler characteristic. Let us assume that (V, σ) and (W, τ) are two $\mathbb{C}$*-surfaces with isomorphic homology, but with $\mathrm{fix}(\sigma) \neq \mathrm{fix}(\tau)$. By 4.4(b), we may assume that σ is parabolic and that τ has no fixed point. The flow chart 4.20 implies that $b_2 = 0 = T^2 = T_2^{cld}$ and $b_1 = 1$. Then 4.16 yields that $(V, \sigma) \cong (\mathbb{C}^* \times \mathbb{C}, \tau_{0,1})$, i.e. $\mathrm{fix}(\sigma) = 1$. Since the last case has been excluded by assumption, the number of fixed points is uniquely determined by the homology of the surface. $\square$

4.7 PROPOSITION. *The number* $\mathrm{fix}(\tau)$ *is a homotopy invariant in the class of all affine* $\mathbb{C}$*-surfaces* (W, τ) *where the underlying variety* W *is not isomorphic to* $\mathbb{C}^* \times \mathbb{C}$.

Proof. We may follow the argument of 4.6 with the exception of the claim that $T_2^{cld}(W)$ vanishes. Nevertheless, $\gamma(\sigma) = 1$, by 3.2, so $F(\sigma) \cong Y(\sigma) \cong \mathbb{C}^*$, and $\pi_1(W) \cong \pi_1(V) \cong \pi_1(Y(\sigma))$ is a free cyclic group. By 4.12, the surface W is isomorphic to $\mathbb{C}^* \times \mathbb{C}$, which is excluded by hypothesis. $\square$

We now discuss how to detect exceptional orbits (or fibers) from the homological data. If it were only for the investigation of fixed points, then we could have restricted the coefficients to the easier case of rational numbers. For the exceptional orbits, however, the torsion, and thus integer coefficients, are indispensable. Since, for coprime numbers p and q, a factor $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ might correspond to one as well as to two exceptional orbits, we introduce the following notion:

4.8 DEFINITION. Let p be a prime number. An exceptional orbit O_j or an exceptional fiber Φ_j in a $\mathbb{C}$*-surface is called p-exceptional if $\mu_j := v_p(m_j) > 0$.

For fixed p, we denote with $s(p)$ the number of those exceptional irreducible fibers $\Phi_1, \ldots, \Phi_s$ that are p-exceptional. Moreover, we let $h(p)$ denote the number of reducible fibers $\Phi_{y_{s+1}}, \ldots, \Phi_{y_{s+h}}$ that are p-exceptional. For the p-corank of the torsion group T_2^{cld} (i.e., the minimal number of generators of $S_p T_2^{cld}$), we use the abbreviation

$$cr(p) := \min\{n \in \mathbb{N}; \, \exists \mathbb{Z}^n \twoheadrightarrow S_p T_2^{cld}\}.$$

The enumeration of the (p-)exceptional orbits and fibers and the determination of their (p-) order is a simple but somewhat technical consequence of the results stated in section 3:

4.9 PROPOSITION. *The number and the (p-)orders of p-exceptional fibers in a $\mathbb{C}^*$-surface W can be estimated as follows:*

(el) $s(p) \leqslant cr(p) + 1$; *equality holds unless* $cr(p) = 0$, *or* $cr(p) = 1$ *and* $\lambda(p) \neq 0$. *For* $j \leqslant s(p) - 2$ *the numbers* $\mu_j(p)$ *are determined by* $S_p T_2^{cld}$; *if moreover* $\lambda(p)$ *is known, then also* $\mu_{s-1}(p)$ *is determined.*

(pa) $s(p) = cr(p)$, *and the numbers* $\mu_j(p)$ *are determined by* $S_p T_2^{cld}$.

(hy) $s(p) = cr(p) - h(p) + \delta$, *unless* $cr(p) \leqslant 1$, *whence* $s(p) + h(p) \leqslant 1$. *The numbers* $\mu_1(p), \ldots, \mu_{s-\delta}(p)$ *are determined by* $S_p T^2$, *while* $\mu_{s+1}(p), \ldots,$ $\mu_{s+h}(p)$ *are determined by* $S_p T_2^{cld}$ *possibly with the exception of a largest among them.* ☐

There is even a precise condition in terms of $s(p)$ for the failure of the above equalities: for an elliptic action that happens precisely in one of the following cases:

$s(p) = 0$ and $\lambda(p) = 0$, (then $cr(p) = 0$);

$s(p) = 0$ and $\lambda(p) > 0$, (then $cr(p) = 1$);

$s(p) = 1$ and $\lambda(p) > 0$, (then $cr(p) = 1$);

for a hyperbolic action that happens iff $s(p) = 0$, $\delta = 1$ (then $cr(p) = 0$), or if $s(p) = h(p) = \delta = 0 < \lambda(p)$ (then $cr(p) = 1$).

Using [FiKp, 1.16] it is not difficult to provide examples for the different cases of the above proposition. The number $s(p)$ is certainly not a homotopy invariant for an elliptic action, since, then the underlying surface is homotopically trivial, independent of the number of p-exceptional orbits. In the class of $\mathbb{C}^*$-surfaces with hyperbolic fixed points, the number $s(p)$ is a homotopy invariant, by 3.9. In general, for a given type of the action, the number $s(p)$ is determined by $e(W)$ and $H_2^{cld}(W, \mathbb{Z})$, with the exceptions indicated in 4.9. More details can be derived with the aid of 4.20.

We now want to discuss the structure of those affine $\mathbb{C}^*$-surfaces that are not covered in 4.1 or 4.6. In particular, we shall study the surfaces which admit different types of $\mathbb{C}^*$-actions. First let us recall the following results from sections 3 and 4 in [FiKp], where we denote with $C_{r,s}$ for $r \geqslant 1$ the cyclic group of matrices

$$C_{r,s} \cong \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^s \end{pmatrix} \in GL(2, \mathbb{C}); \ \zeta^r = 1 \right\}:$$

4.10 REMARK. The structure of parabolic or hyperbolic $\mathbb{C}^*$-surfaces with algebraic quotient $Y(\tau) = \mathbb{C}$ is uniquely determined by their exceptional fibers:

(pa) For at most one exceptional orbit, there is only the standard parabolic $\mathbb{C}^*$-surface $\bar{V}_{m,\bar{n}} \cong (\mathbb{C}^2, \tau_{1,0})/C_{m,-n}$ of [FiKp, 3.1], its orbit data are $(m, \bar{n})$, and the local homology in the corresponding singularity $\bar{0}$ is $\mathscr{H}_{2,\bar{0}} = \mathbb{Z}_m$. The general case of s exceptional orbits $\bar{V}_A = \bar{V}_S(m_1, \bar{n}_1; \ldots; m_s, \bar{n}_s)$, where $A \subset \mathbb{C}$ parametrizes the exceptional orbits O_i with data $(m_i, \bar{n}_i)$, is obtained by a gluing procedure as described in [FiKp, 3.5].

(hy) The surfaces with a fixed point free action are obtained as follows: For $s \geqslant 0$, a subset $A = \{y_1, \ldots, y_s\} \subset Y$, natural numbers $m_1, \ldots, m_s \in \mathbb{N}_{>1}$, units $\bar{n}_i \in \mathbb{Z}^*_{m_i}$, $m := \operatorname{lcm}(m_1, \ldots, m_s)$ for $s \geqslant 1$ and $m := 1$ otherwise, let $Z \to \mathbb{C}$ be a connected m-sheeted cyclic Galois covering with ramification points of order m_j over y_j. Then $C_m \subset \mathbb{C}^*$ acts as group of deck transformations on Z, and $\mathbb{C}^* \times_{C_m} Z$ (with the $\mathbb{C}^*$-action induced from the first factor) is a $\mathbb{C}^*$-surface without fixed points and with s closed exceptional orbits O_i of order m_i. For a suitable choice of Z even the prescribed orbit data $(m_i, \bar{n}_i)$ can be realized (cf. [FiKp, 2.9]). The hyperbolic surface with one fixed point $W = W(m_+, \bar{n}_+, m_-,$ $\bar{n}_-; l)$ of [FiKp, 4.2] has two distinguished orbits $O_{\pm}$ with orbit data $(m_+, \bar{n}_+)$ and $(m_-, \bar{n}_-)$, the local homology in the only possible singularity $\bar{0}$ is $\mathscr{H}_{2,\bar{0}} = \mathbb{Z}_r$ (with $r := l \cdot \gcd(m_+, m_-)$), and (W, τ) is a cyclic quotient $(\mathbb{C}^2, \tau_{a,b})/C_{c,d}$ (cf. [FiKp, 6.1]). The general case, where $A \subset \mathbb{C}$ parametrizes the closed exceptional orbits and $B \backslash A$ the fixed points, is again obtained by a gluing procedure, as described in [FiKp, 4.8].

4.11 PROPOSITION. *The only affine surfaces* $\mathbb{C}^*$*-surfaces* (W, τ) *where the type of* τ *is not determined by the affine surface* W *are the following:*

$e(W) = 0$:   *the product surfaces* $(\mathbb{C}^* \times \mathbb{C}, \tau_{a,b})$;

$e(W) \neq 0$:   *the cyclic quotient surfaces* $(\mathbb{C}^2, \tau_{a,b})/C_{k,r}$.

Proof. Let us first assume that the affine surface W admits an elliptic action τ and an additional nonelliptic action σ. Then we have $b_1 = 0 = b_2$, and W has at most one singular point; moreover, $Y(\sigma)$ is a complex line. Hence, 4.10 applies: If σ is parabolic, then there is at most one exceptional σ-orbit, so (W, σ) is isomorphic to some $(\mathbb{C}^2, \tau_{0,1})/C_{m,-n}$. If σ is hyperbolic, then there exists exactly one hyperbolic fixed point (by 4.4, as $e(W) = 1$) and no *closed* exceptional σ-orbit, as $T^2 = 0$. Hence, (W, σ) is of the form $W(m_+, \bar{n}_+, m_-, \bar{n}_-; l)$ $\cong (\mathbb{C}^2, \tau_{a,b})/C_{k,r}$ for suitable invariants a, b, k, and r.

If W is smooth, then (W, τ) is of the form $(\mathbb{C}^2, \tau_{a,b})$ for some $ab > 0$, by [FiKp, 6.1]; otherwise $F(\tau) = F(\sigma) = \operatorname{Sing}(W)$. In each case $\pi_1(W^*) = C_{k,r}$ is cyclic, and we may apply 4.12 to (W, τ).

Now let us assume that W admits both a parabolic action τ and a hyperbolic action σ. Then b_2 vanishes by 3.2, and that implies $\gamma(\sigma) = 0$ (i.e., $Y(\sigma) \cong \mathbb{C}$) by 3.7 and 3.9, and $h(\sigma) = e(W) \leqslant 1$. If the hyperbolic action σ has a fixed point (i.e., $h(\sigma) = 1$), then W has at most one singular point, no closed exceptional σ-orbit, and $b_1(W) = 0$. Hence, for the parabolic action τ, we get $Y(\tau) \cong \mathbb{C}$, and as there is at most one exceptional τ-fiber, we again obtain $(W, \tau) \cong \bar{V}_{m,\bar{n}} \cong (\mathbb{C}^2, \tau_{0,1})/C_{m,-n}$. As above, we conclude from 4.10 that (W, σ) is a cyclic quotient surface. If σ is fixed point free ($h = 0$), then the surface W is smooth with $b_1 = 1$. Hence, for the parabolic action τ, we have $Y(\tau) \cong \mathbb{C}^*$. As there are no exceptional fibers, we see that (W, τ) is analytically and even algebraically a globally trivial line bundle over the open rational curve $\mathbb{C}^*$ (cf. [FiKp, 3.2 and 1.13]). Hence, we obtain $(W, \tau) \cong (\mathbb{C}^* \times \mathbb{C}, \tau_{0,1})$. Since $\pi_1(W) = \mathbb{Z}$ and σ has no fixed point, Lemma 4.12 yields that $(W, \sigma) \cong (\mathbb{C}^* \times \mathbb{C}, \tau_{a,b})$. $\qquad\square$

4.12 LEMMA. *Let (W, τ) denote an affine $\mathbb{C}^*$-surface such that $\pi_1(W^*)$ is a cyclic group. Then*

(el) *If τ is elliptic, then (W, τ) is of the form $(\mathbb{C}^2, \tau_{a,b})/C_{k,r}$;*
(hy) *If τ has no fixed point, then (W, τ) is of the form $(\mathbb{C}^* \times \mathbb{C}, \tau_{a,b})$.*

Proof. (hy) According to [FiKp, 2.9] we fix a representation $W \cong \mathbb{C}^* \times_{C_m} Z$ with a branched Galois covering $Z \to Y$ with Galois group C_m. By the very definition of the C_m-action, the covering $\mathbb{C}^* \times Z \to \mathbb{C}^* \times_{C_m} Z$ is unbranched. As a consequence, the group $\pi_1(\mathbb{C}^* \times Z)$, as a subgroup of $\pi_1(W)$, is cyclic, which implies that $\pi_1(Z) = 0$. Thus Z is the complex plane. Finite subgroups of the automorphism group of $\mathbb{C}$ are rotation groups; hence, Y is the complex plane as well and the covering mapping $Z \to Y$ has exactly one branching point, i.e., $s = 1$. According to 4.10, (W, τ) then is isomorphic to some $(\mathbb{C}^* \times \mathbb{C}, \tau_{m,n}) \cong V_{m,\bar{n}}$.

(el) For an application of [FiKp, 6.1] it suffices to show that the number s of exceptional orbits is at most 2 and that $Y = Y(\tau)$ is rational. For an indirect proof let us assume that $s \geqslant 3$ or that Y is not rational. By 4.13 there exists a principal $\mathbb{C}^*$-bundle $\tilde{S} \to \tilde{Y}$ and a finite subgroup $\Gamma \subset \text{Aut}(\tilde{S})$ of $\mathbb{C}^*$-equivariant automorphisms that acts freely on $\tilde{S}$ such that the $\mathbb{C}^*$-surfaces W^* and $\tilde{S}/\Gamma$ are analytically isomorphic. Since subgroups and quotients of the cyclic group $\pi_1(W^*)$ are cyclic, $\pi_1(\tilde{S}) \subset \pi_1(W^*)$, $\Gamma \cong \pi_1(W^*)/\pi_1(\tilde{S})$ and also $\pi_1(\tilde{Y})$ are cyclic. In particular, the curve $\tilde{Y}$ is rational and the branched covering $\tilde{Y} \to Y$ has at most two ramification points. Thus Y itself is rational and $s \leqslant 2$, which is a contradiction. $\qquad\square$

4.13 PROPOSITION. *Let $\pi: S \to Y$ be a semistable $\mathbb{C}^*$-surface with a $\mathbb{C}^*$-action without fixed points over a connected smooth compact curve Y. Assume that one of the following conditions is true*

(a) *Y is not rational;*

(b) *the number s of exceptional orbits is at least* 3;
(c) $s = 2$ *and* $m_1 = m_2$.

Then there exists a principal $\mathbb{C}^*$*-bundle* $\tilde{\pi} \colon \tilde{S} \to \tilde{Y}$ *and a finite group* Γ *of* $\mathbb{C}^*$-*equivariant automorphisms of* $\tilde{S}$ *such that*

(1) Γ *acts freely on* $\tilde{S}$;
(2) Γ *induces an effective action on* $\tilde{Y}$;
(3) *there is an analytic isomorphism of* $\mathbb{C}^*$*-surfaces* $S \cong \tilde{S}/\Gamma$.

   Proof. We may assume that $s \geqslant 1$. Let $A = \{y_1, \dots, y_s\}$ be the set of points in Y such that the fibers $\pi^{-1}(y_i)$ are singular. Under the conditions (a)–(c) there exists a branched Galois covering $p \colon \tilde{Y} \to Y$ that is unramified over $Y_0 := Y \setminus A$ and such that every point in a fiber $p^{-1}(y_i)$ has ramification order m_i, the exceptional order of y_i. The existence of p is obvious for $Y \cong \mathbb{P}_1$, $s = 2$ and $m_1 = m_2$; in the remaining cases it follows from Fenchel's conjecture (see [BuNi] and [Fo]). Since the fibration p is analytically trivial above Y_0 ([FiKp, 2.8]), we look for a description of S of the following type

$$S \cong \left((\mathbb{C}^* \times Y_0) \cup \bigcup_{j=1,\dots,s} V_j \right) \Big/ \sim \tag{4.14}$$

with appropriate neighborhoods V_j of $\pi^{-1}(y_j)$. In an analogous manner we shall construct

$$\tilde{S} \cong \left((\mathbb{C}^* \times \tilde{Y}_0) \cup \bigcup_{j=1,\dots,s} \tilde{V}_j \right) \Big/ \sim \tag{4.15}$$

for $\tilde{Y}_0 := p^{-1}(Y_0)$. The construction of the $\tilde{V}_j$'s is local in nature. Hence, we may omit the index j in the notations. For $y \in A$ choose a sufficiently small disk D in Y centered at y. Then we fix a point $\tilde{y} \in p^{-1}(y)$. Let Γ denote the group of deck transformations of $p \colon \tilde{Y} \to Y$ and G the isotropy group $\Gamma_{\tilde{y}}$. According to Luna's slice theorem, for the connected component $\tilde{D}$ of $p^{-1}(D)$ containing $\tilde{y}$ there is a natural free action of G on $\mathbb{C}^* \times \tilde{D}$ extending the given action of $G \subset \Gamma$ on $\tilde{D} \subset \tilde{Y}$ such that

$$V := \pi^{-1}(D) \cong (\mathbb{C}^* \times \tilde{D})/G.$$

Since the fibration π is trivial over $D^* := D \setminus \{y\}$, there is a section

$$\sigma \colon D^* \to \pi^{-1}(D^*) \subset V$$

such that, in formula 4.14, for $z \in D^*$

$$\mathbb{C}^* \times Y_0 \supset \mathbb{C}^* \times D^* \ni (\lambda, y) \sim \lambda \sigma(y) \in V.$$

For the construction of $\tilde{S}$ we have to extend the G-variety $\mathbb{C}^* \times \tilde{D}$ to

$$\tilde{V} := \Gamma \times_G (\mathbb{C}^* \times \tilde{D})$$

with the natural action of Γ on the first component. The morphism $\sigma \circ p$ can be lifted so that this diagram commutes:

$$
\begin{array}{ccc}
\tilde{D}^* & \xrightarrow{\tilde{\sigma}} & \mathbb{C}^* \times \tilde{D} \\
\downarrow p & & \downarrow \\
D^* & \xrightarrow{\sigma} V \cong (\mathbb{C}^* \times \tilde{D})/G,
\end{array}
$$

where $\tilde{\sigma}$ is G-equivariant and has as second component the identity. Now every point $(\lambda, w) \in \mathbb{C}^* \times p^{-1}(D^*)$ is of the form $(\lambda, \gamma(\tilde{z}))$ for $\gamma \in \Gamma$ and $\tilde{z} \in \tilde{D}^*$. Then

$$\mathbb{C}^* \times \tilde{Y}_0 \supset \mathbb{C}^* \times p^{-1}(D^*) \ni (\lambda, w) = (\lambda, \gamma(\tilde{z})) \sim (\gamma, \lambda\tilde{\sigma}(\tilde{z})) \in \tilde{V}$$

prescribes how to glue $\tilde{V} = \tilde{V}_j$ into $\mathbb{C} \times \tilde{Y}_0$ in formula 4.15. It is not hard to check that $\tilde{S}$ and Γ satisfy the required conditions.   □

In [Pi] the case of an elliptic action on an affine surface W has been treated: for $S = W^*$ the corresponding principal $\mathbb{C}^*$-bundle $\tilde{S} \to \tilde{Y}$ and the group Γ have been constructed in such a way that W^* is isomorphic to $\tilde{S}/\Gamma$ as a complex surface. The reason not to discuss the involved $\mathbb{C}^*$-actions may have been that, for Y rational, $s = 2$ with $m_1 \neq m_2$ or $s = 1$ the induced action on $\tilde{S}/\Gamma$ is definitely not that of W^*. In that situation we may apply the fact that W then is a cyclic quotient of some $(\mathbb{C}^2, \tau_{a,b})$, see [FiKp, 6.1].

For the flow chart in 4.20, we still have to discuss surfaces W which share certain homological properties:

4.16 LEMMA. *Up to an algebraic isomorphism, every affine $\mathbb{C}^*$-surface (W, τ) with $b_1 = 1$, $b_2 = 0$, and $T_2^{cld} = 0$ is of one of the following types:*

(pa) $\mathbb{C}^* \times \mathbb{C}$ *with the parabolic action $\tau_{0,1}$ (and $Y(\tau) \cong \mathbb{C}^*$), or*
(hy) $\mathbb{C}^* \times_{C_m} Z$ *with the fixed point free $\mathbb{C}^*$-action induced from the first factor and with pairwise coprime exceptional orders $m_1, \ldots, m_s$ (and $Y(\tau) \cong \mathbb{C}$).*

Proof. By 4.4(b), the action is either parabolic, or it is fixed point free (in which case W is non-singular). In the parabolic case, there are at most cyclic quotient singularities which can be detected by their local homology and thus, according to 3.6, by T_2^{cld}. As that torsion group vanishes, W is a manifold in the parabolic case, too. Hence, 3.6 yields that τ has no exceptional orbits, so we are in the same situation as in the proof of 4.11. In the hyperbolic case, $b_2 = 0$ already implies $Y(\tau) \cong \mathbb{C}$, so we can apply the results recalled above (4.10, case (hy)).   □

Next we discuss the different cases where $b_1(W) = b_2(W) = 0$.

4.17 LEMMA. *Up to an algebraic isomorphism, every affine* $\mathbb{C}^*$*-surface* (W, τ) *with* $b_1 = b_2 = 0 = b_2^{cld}$, $T^2 = 0$, *and* T_2^{cld} *not cyclic (nonzero in particular), is of one of the following types:*

(el)   *The action* τ *is elliptic with* $Y(\tau) \cong \mathbb{P}_1$, *and there exist at least three exceptional orbits such that* $\gcd(m_i, m_j, m_k) \neq 1$.

(pa) W *is a parabolic* $\mathbb{C}^*$*-surface* $\bar{V}_A$ *(with* $Y(\tau) \cong \mathbb{C}$*) with at least two exceptional orbits such that* $\gcd(m_i, m_j) \neq 1$.

Proof. We divide the proof into two parts (a) and (b), where (a) will be useful also for the next lemmata:

(a) If $b_1 = b_2 = b_2^{cld} = 0$ and $T^2 = 0$, then 4.10 and the results of section 3 yield:

(el) $\beta = 0$, hence, $H^1(Y(\tau)) = 0$, i.e., $Y(\tau) \cong \mathbb{P}_1$;

(pa) $\gamma = 0$, i.e., $Y(\tau) \cong \mathbb{C}$, and thus (W, τ) is isomorphic to some $\bar{V}_A$;

(hy) $\gamma = 0$, i.e., $Y(\tau) \cong \mathbb{C}$, moreover $h = 1$, and there are no closed exceptional orbits, so (W, τ) is isomorphic to some surface $W(m_+, \bar{n}_+, m_-, \bar{n}_-; l)$. In particular, $T_2^{cld}(W)$ is then a cyclic group (by 3.9).

(b) The fact that T_2^{cld} is not a cyclic group is equivalent to the existence of at least one prime number p such that $\mathrm{cr}(p) \geq 2$. Certainly, by (hy) above, (W, τ) is not hyperbolic. The case (pa) of 4.9 implies that $s(p) = \mathrm{cr}(p) \geq 2$; in particular, $|A| \geq 2$, and p divides at least two exceptional orders. Finally let τ be elliptic. Again by 4.9, $s(p) = \mathrm{cr}(p) + 1 \geq 3$; hence, there exist three exceptional orders that are multiples of the number p. $\square$

The two following results can essentially be proved along the same lines:

4.18 LEMMA. *Up to an algebraic isomorphism, every affine* $\mathbb{C}^*$*-surface* (W, τ) *with* $b_1 = b_2 = 0 = b_2^{cld}$, $T^2 = 0$, *and* $T_2^{cld} \neq 0$ *cyclic, is of one of the following types:*

(el)   *an elliptic* $\mathbb{C}^*$*-surface with* $Y(\tau) \cong \mathbb{P}_1$ *where every three exceptional orbits satisfy* $\gcd(m_i, m_j, m_k) = 1$; *moreover,* $l_0(W) > 1$, *or there are at least two exceptional orbits satisfying* $\gcd(m_i, m_j) > 1$.

(pa)   *a parabolic* $\mathbb{C}^*$*-surface* $\bar{V}_A$ *(hence* $Y(\tau) \cong \mathbb{C}$*) with pairwise coprime exceptional orders, where at least one order* $m_i \neq 1$.

(hy)   *a hyperbolic* $\mathbb{C}^*$*-surface* $W(m_+, \bar{n}_+, m_-, \bar{n}_-; l)$ *(hence* $Y(\tau) \cong \mathbb{C}$*) which satisfies the condition* $l \cdot \gcd(m_+, m_-) \neq 1$.

The following remark extends [Ry, 2.10]. In terms of intersection homology it admits a more natural interpretation [FiKp$_2$].

4.19 PROPOSITION. *Up to an algebraic isomorphism, every affine* $\mathbb{C}^*$*-surface* (W, τ) *with* $b_1 = b_2 = 0 = b_2^{cld}$ *and* $T_2^{cld} = 0$ *is of the following type:*

(1) *the action τ is elliptic with at least three exceptional orbits, pairwise coprime exceptional orders $m_1, \ldots, m_s$, patching weight $l_0(W) = 1$, and $Y(\tau) \cong \mathbb{P}_1$; the surface W then is a homology manifold with $\pi_1(W^*)$ nonabelian, or*

(2) (W, τ) *is isomorphic to* $(\mathbb{C}^2, \tau_{a,b})$ *for suitable a and b (and thus elliptic for $ab > 0$, parabolic for $ab = 0$, and hyperbolic for $ab < 0$).*

For the proof we add the following: If the action is elliptic and there are at least three exceptional orbits, then the fundamental group $\pi_1(W^*)$ is not cyclic, according to 4.12. Thus it is not abelian, since $H_1(W^*)$ vanishes. For (2) one may apply [FiKp 6.1]. □

We now bring together all the results obtained about the type of a $\mathbb{C}^*$-action τ on an affine surface W; the verification of the flow chart below is easy with the homology descriptions given in section 3:

4.20 THEOREM. *Let e denote the Euler number of an affine $\mathbb{C}^*$-surface (W, τ). Then, for $e = 0, 1$, the flow chart below applies;*

> *$e < 0$, the action τ is parabolic.*
>
> *$e > 1$, the action τ is hyperbolic with precisely e fixed points.*

□
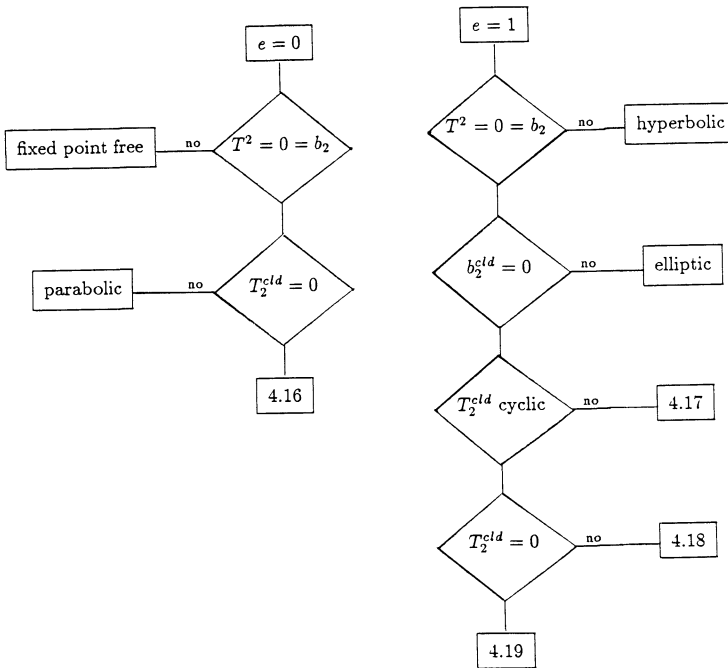
The surfaces $\mathbb{C}^2/C_{k,r}$ carry $\mathbb{C}^*$-actions of every type. Hence, those actions cannot be distinguished by homological data. But that is not the general situation:

4.21 COROLLARY. *In the class of affine* $\mathbb{C}^*$-surfaces (W, τ) *that are not isomorphic to a cyclic quotient of* $(\mathbb{C}^2, \tau_{a,b})$ *with* $ab < 0$, *the type of the action* τ *is uniquely determined by* $H_*(W, \mathbb{Z})$, $H_2^{cld}(W, \mathbb{Z})$ *and* $e(Y(\tau))$.

Proof. By 4.4 and the flow chart, the homology of W distinguishes the different types of the action τ, except for the situations described in 4.16–4.19. In those cases the number $e(Y)$ lies between 0 and 2, so the curve $Y(\tau)$ is isomorphic to $\mathbb{C}$, $\mathbb{C}^*$, or $\mathbb{P}_1$. For $e(Y) = 2$ the action certainly is elliptic. For $e(Y) = 0$ and W in one of the classes 4.16–4.19, the action is parabolic. Only in the situation of 4.18 there appear two $\mathbb{C}^*$-surfaces with the same orbit space $\mathbb{C}$ but with actions of different type. But one of them is isomorphic to some $W(m_+, \bar{n}_+, m_-, \bar{n}_-; l)$, which is a cyclic quotient of a surface $(\mathbb{C}^2, \tau_{a,b})$ for $ab < 0$, see 4.10.   □

We end this section with a more subtle question, which cannot be answered using singular (co)homology, so we come back to it in [FiKp$_2$]:

PROBLEM. Let (W, τ) be a hyperbolic $\mathbb{C}^*$-surface with $h \geqslant 1$ fixed points. For every fixed point and every prime number p, determine the p-adic valuation of the corresponding order a of the local homology, the associated patching weight l, and the order $m = \gcd(m^+, m^-)$ of the reducible fiber from global homological data of W.

5. The computation in the hyperbolic case

In the elliptic and the hyperbolic case, we still have to compute some Betti numbers and torsion groups. We begin with the hyperbolic case, since the techniques and results will be applied to the elliptic case, too. For the calculation of $b^1(W)$, we use the Leray spectral sequence associated to the algebraic quotient mapping $\pi: W \to Y$, since it is homologically proper (see [KrPeRa, 2.2]). As we want to localize the influence of the exceptional orbits and the fixed points on the homology, we also have to consider the restriction of π (again denoted by π) to an open subset $U := \pi^{-1}(X)$, where $X \subset Y$ is open and connected. We set

$$\delta := \delta(U) := \operatorname{rank} H^0(X, \pi_1 \mathbb{Z}) = \begin{cases} 1 & \text{iff } F \cap U = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

(where $\pi_j \mathbb{Z}$ denotes the j-th direct image sheaf on X of the constant sheaf $\mathbb{Z}$ on U, so $\pi_0 \mathbb{Z} \cong \mathbb{Z}$). The Leray spectral sequence induces an exact sequence

$$0 \to H^1(X, \pi_0 \mathbb{Z}) \to H^1(U, \mathbb{Z}) \to H^0(X, \pi_1 \mathbb{Z}) \to H^2(X, \pi_0 \mathbb{Z}) = 0, \tag{5.2}$$

consequently, we have

$$H^1(U) = H^1(X) \oplus \mathbb{Z}^{\delta(U)}. \tag{5.3}$$

As a particular case, we obtain the formula for the first Betti number of a hyperbolic $\mathbb{C}^*$-surface that appears in 3.7 and 3.9:

5.4 REMARK. For an affine hyperbolic $\mathbb{C}^*$-surface W with algebraic quotient Y, we have

$$b_1(W) := b_1(Y) + \delta(W). \qquad \qquad \square$$

For the missing data b_2, T^2, and T_2^{cld}, we have to determine $H^2(W)$ and $H^2(W^*) \cong H_2^{cld}(W)$, see 1.2. For nonempty F the induced mapping from $W^* = W \backslash F$ to Y is no longer homologically proper. So we have to apply a different technique for the computation. We calculate the groups under consideration together with the natural restriction homomorphism

$$\rho: H^2(W) \to H^2(W^*). \tag{5.5}$$

The idea is to use the relative homology with respect to a generic fiber (that approach is also basic for the computation of the intersection homology in [FiKp$_2$]). The homological properties of a generic fiber Φ_y (i.e., $y \in Y \backslash B$), are independent of y, so we just write Φ. Let V denote U or U^* (which in particular includes W and W^*). The exact sequence

$$\mathbb{Z} \cong H^1(\Phi) \xrightarrow{\ \partial\ } H^2(V, \Phi) \to H^2(V) \to H^2(\Phi) = 0 \tag{5.6}$$

yields $H^2(V) \cong H^2(V, \Phi)/\partial H^1(\Phi)$. Hence, we first describe $H^2(V, \Phi)$. In the notations of 2.8 and 2.13, we let the $D_i \subset X$ denote pairwise disjoint open disks, centered at the points $y_i \in B \cap X$. Moreover, we set

$$D_0 := X \backslash B, \qquad U_i := \pi^{-1}(D_i), \qquad U_i^* := U_i \backslash F. \tag{5.7}$$

If $X = Y$, then we also write W_j instead of U_j etc. For the remaining part of this section we fix an analytic trivialization $W_0 \cong \mathbb{C}^* \times D_0$, which exists, since $H^1(\mathcal{O}_0, \mathcal{D}^*) = 0$ for the (analytic) structure sheaf $\mathcal{O}$ of Y.

5.8 LEMMA. *There exists a natural commutative diagram*

$$
\begin{array}{ccccc}
H^2(W, \Phi) & \xrightarrow[\cong]{\varphi} & H^1(Y) \oplus \displaystyle\bigoplus_{i=1}^{s+h} H^2(W_i, \Phi) & \hookrightarrow & \displaystyle\bigoplus_{i=0}^{s+h} H^2(W_i, \Phi) \\
\downarrow & & \downarrow id \oplus \bigoplus \rho_i & & \downarrow \bigoplus \rho_i \\
H^2(W^*, \Phi) & \xrightarrow[\cong]{\varphi} & H^1(Y) \oplus \displaystyle\bigoplus_{i=1}^{s+h} H^2(W_i^*, \Phi) & \hookrightarrow & \displaystyle\bigoplus_{i=0}^{s+h} H^2(W_i^*, \Phi),
\end{array}
$$

where the composed horizontal mappings are induced by inclusions.

Proof. Step 1. Let ξ be the canonical generator of $H^1(\Phi)$. Then, for $x \in D_0$, there is an injective homomorphism

$$
\theta : H^1(X, x) \hookrightarrow H^1(D_0, x) \xrightarrow[\cong]{\xi \times} H^2(\Phi \times (D_0, x)) \cong H^2(U_0, \Phi) \tag{5.9}
$$

of free abelian groups. Thus we may consider $H^1(X) \cong H^1(X, x)$ as a subgroup of $H^2(U_0, \Phi)$, and for $X = Y$ we obtain existence and injectivity of the horizontal arrows on the right side, since $U_0 = U_0^*$.

Step 2. For a unified treatment, we let V_j denote U_j or U_j^*, and we let $r \leqslant s + h$ be the number of exceptional points of Y that lie in X. The statement of Lemma 5.8 then follows essentially from the special case $X = Y$, $V = W$ respectively W^* of the following

CLAIM. There exists a natural isomorphism

$$
\varphi : H^2(V, \Phi) \to H^1(X, x) \oplus \bigoplus_{j=1}^{r} H^2(V_j, \Phi) \tag{5.10}
$$

satisfying $(\theta \oplus \bigoplus_{j=1}^{r} id_{H^2(V_j, \Phi)}) \circ \varphi = (\iota_0, \ldots, \iota_r)$, where $\iota_j : H^2(V, \Phi) \to H^2(V_j, \Phi)$ denotes the homomorphism induced by the inclusion $V_j \subset V$.

Proof by induction on r: For $r = 0$, we have $V = U_0 \cong X \times \mathbb{C}^*$, so θ is bijective, and $\varphi := \theta^{-1}$ has the required properties. For the step '$r - 1 \Rightarrow r$' with $r \geqslant 1$, we set

$$
X' := X \backslash \{x_r\}, \quad D_r' := D_r \backslash \{x_r\}, \quad V' := \pi^{-1}(X'), \quad V_r' := V' \cap V_r \cong \mathbb{C}^* \times D_r'.
$$

There is a commutative diagram of Mayer-Vietoris sequences (for $x \in D_r'$ and $\Phi = \Phi_x$)

$$
\begin{array}{ccccc}
H^1(V, \Phi) \to H^1(V', \Phi) \oplus H^1(V_r, \Phi) & \xrightarrow{\sigma} & H^1(V_r', \Phi) \to \cdots \\
\uparrow & \uparrow & \uparrow \cong & \\
0 \to H^1(X, x) \xrightarrow{\alpha} H^1(X', x) \oplus H^1(D_r, x) & \xrightarrow{\chi} & H^1(D_r', x) \to 0
\end{array} \tag{5.11}
$$

which implies that σ is surjective. We now consider the following diagram, in which the top row is exact and continues the upper one of the above diagram, and the short exact middle row is induced from the lower one of 5.11 with $\beta = \alpha \oplus \text{id}$. We want to show that a mapping φ with the properties of the claim exists:

$$
\begin{array}{ccccc}
H^2(V, \Phi) & \xrightarrow{(\iota',\iota_r)} & H^2(V', \Phi) \oplus H^2(V_r, \Phi) & \xrightarrow{\iota'_r - \iota'_{rr}} & H^2(V'_r, \Phi) \to 0 \\
\downarrow{\varphi} & & \downarrow{\varphi' \oplus \text{id}} & & \uparrow{\xi\times} \\
H^1(X) \oplus \bigoplus_{j=1}^r H^2(V_j, \Phi) & \xhookrightarrow{\beta} & \left(H^1(X') \oplus \bigoplus_{j=1}^{r-1} H^2(V_j, \Phi) \right) \oplus H^2(V_r, \Phi) & \xrightarrow{\chi-0} & H^1(D'_r, x) \to 0 \\
\downarrow{\theta \oplus \text{id}} & & \downarrow{\theta' \oplus \text{id}} & & \downarrow{\xi\times} \\
\bigoplus_{j=0}^r H^2(V_j, \Phi) & = & \bigoplus_{j=0}^r H^2(V_j, \Phi) & \xrightarrow{\iota'_{r0} - \iota'_{rr}} & H^2(V'_r, \Phi) \to 0
\end{array}
$$

There we denote the inclusion mappings by

$$\iota': V' \xhookrightarrow{} V, \quad \iota_r: V_r \xhookrightarrow{} V, \quad \iota'_r: V'_r \xhookrightarrow{} V', \quad \iota'_{rj}: V'_r \xhookrightarrow{} V_j \text{ for } j = 0, r,$$

and we use the same symbols for the induced homomorphisms on the cohomology level. We want to show that $\tilde{\varphi} := (\varphi' \oplus \text{id}) \circ (\iota', \iota_r)$ is injective and that $\text{Im } \tilde{\varphi} = \text{Ker}(\chi - 0)$. If we can prove that the homomorphism ι'_{rr} is the zero-mapping, then the lower right corner of the diagram, and consequently the whole diagram (except possibly for the upper left corner) commutes. By induction hypothesis, the homomorphism $\varphi' \oplus \text{id}$ is an isomorphism, and (ι', ι_r) is injective since σ in diagram 5.11 is surjective, so $\tilde{\varphi}$ is injective. Moreover, the homomorphism $\xi \times \cdots : H^1(D'_r, x) \to H^2(V'_r, \Phi)$ is an isomorphism. From the commutativity of the diagram and the exactness of the upper and the middle row it then follows that $\tilde{\varphi}$ maps bijectively onto $\text{Ker}(\chi - 0) = \text{Im } \beta$, so $\varphi := \beta^{-1} \circ \tilde{\varphi}$ is well defined and has the properties of the claim. Thus the next step completes the proof of the claim and of 5.8:

Step 3. For $r \geqslant 1$ the homomorphism $\iota'_{rr}: H^2(V_r, \Phi) \to H^2(V'_r, \Phi)$ is the zero-mapping. For the proof let us note that the target group $H^2(V'_r, \Phi) \cong H^2(S^1 \times S^1, S^1 \times 1)$ is a free abelian group. We now start with the case that $V_r = U_r$; the case $V_r = U_r^*$ then will be an easy consequence. Let us first assume that U_r contains no fixed point, i.e. that $r \leqslant s$. Then $U_r = U_r^* \cong D_r \times \mathbb{C}^*$, by 2.13, so $H^2(U_r)$ vanishes. Moreover, the homomorphism σ in the exact sequence

$$0 \to H^1(U_r, \Phi) \to H^1(U_r) \xrightarrow{\sigma} H^1(\Phi) \to H^2(U_r, \Phi) \to H^2(U_r) = 0$$

is multiplication by the order m of the isotropy group of the exceptional orbit, by 2.14; hence,

$$H^1(U_r, \Phi) = 0 \quad \text{and} \quad H^2(U_r, \Phi) = \mathbb{Z}_m \text{ for } r \leqslant s, \tag{5.12}$$

and thus ι'_{rr} has to be the zero-mapping. Now let us assume that U_r includes a fixed point w. The exact cohomology sequence of the triple (U_r, U'_r, Φ) provides:

$$H^1(U_r, \Phi) \to H^1(U'_r, \Phi) \to H^2(U_r, U'_r) \to H^2(U_r, \Phi) \xrightarrow{\iota'_{rr}} H^2(U'_r, \Phi)$$
$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$
$$0 \qquad\qquad \mathbb{Z} \qquad\qquad \mathbb{Z}^2 \qquad\qquad \mathbb{Z}. \qquad\qquad\qquad\qquad (5.13)$$

The determination of those cohomology groups is immediate from the fact that U_r is contractible and thus $\tilde{H}^*(U_r) = 0$, and that $(U'_r, \Phi) \simeq (\Phi \times S^1, \Phi)$. As a consequence, the homomorphism i'_{rr} is the zero-mapping.

Finally we consider the case that $V_r = U_r^*$; we have to show that the homomorphism $H^2(U_r^*, \Phi) \to H^2(U'_r, \Phi)$ is the zero-mapping. In the commutative diagram

$$H^2(U_r, \Phi) \to H^2(U'_r, \Phi)$$
$$\downarrow \qquad\qquad \|$$
$$H^2(U_r^*, \Phi) \to H^2(U'_r, \Phi)$$

the vertical mapping has cokernel $H^3(U_r, U_r^*) \cong \mathscr{H}_{2,w}$, which is a cyclic torsion group ([FiKp, 6.2]). Its image in the free abelian group $H^2(U'_r, \Phi)$ necessarily vanishes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square\square$

Using the results just established we proceed to investigate the restriction homomorphism $\rho : H^2(W) \to H^2(W^*)$. To that end we combine the exact cohomology 'ladder' for $(W^*, \Phi) \subset (W, \Phi)$ and the isomorphism φ of 5.10 (for W and W^*) in the commutative diagram

$$\mathbb{Z} \cdot \xi = H^1(\Phi) \xrightarrow{\partial} H^2(W, \Phi) \cong H^1(X) \oplus \bigoplus_{i=1}^{s+h} H^2(W_i, \Phi) \twoheadrightarrow H^2(W)$$
$$\downarrow \cong \qquad\quad \downarrow \rho \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \rho$$
$$H^1(\Phi) \xrightarrow{\partial} H^2(W^*, \Phi) \cong H^1(X) \oplus \bigoplus_{i=1}^{s+h} H^2(W_i^*, \Phi) \twoheadrightarrow H^2(W^*).$$

Thus it suffices to study the relative groups and the homomorphisms

$$\partial_i : H^1(\Phi) \to H^2(W_i, \Phi) \quad \text{and} \quad \rho_i : H^2(W_i, \Phi) \to H^2(W_i^*, \Phi)$$

for $0 \leqslant i \leqslant s + h$. Let us start with the ∂_i's.

5.14 LEMMA. *The groups $H^2(W_i, \Phi)$ and the coboundary homomorphisms ∂_i are as follows*:

$i = 0$: $H^2(W_0, \Phi) \cong H^1(D_0) \cong \mathbb{Z}^{\gamma+s+h}$,   and $\partial_0 = 0$;

$1 \leqslant i \leqslant s$: $H^2(W_i, \Phi) = \mathbb{Z} \cdot \tau_i \cong \mathbb{Z}_{m_i}$,   and $\partial_i(\xi) = \tau_i$;

$s + 1 \leqslant i$: $H^2(W_i, \Phi) = \mathbb{Z} \cdot \tau_i \cong \mathbb{Z}$,   and $\partial_i(\xi) = \tau_i$.

Proof. For $i = 0$ the homomorphism $H^1(W_0) \cong H^1(\mathbb{C}^* \times D_0) \rightarrow H^1(\Phi)$ is surjective; hence, ∂_0 is the zero mapping. For $1 \leqslant i \leqslant s$ the fiber Φ_i is a closed exceptional orbit of order m_i. By 2.13 and 2.14, we know that $W_i \cong \mathbb{C}^* \times D_i$ and the inclusion $\Phi_i \subset W_i$ yields an exact sequence

$$H^1(W_i) \xrightarrow{\;\cdot m_i\;} H^1(\Phi) \xrightarrow{\;\partial_i\;} H^2(W_i, \Phi) \rightarrow H^2(W_i) = 0.$$

Accordingly, ∂_i maps ξ onto a generator τ_i of $H^2(W_i, \Phi) \cong \mathbb{Z}_{m_i}$. For $i \geqslant s + 1$ the exceptional point y_i lies in $\pi(F)$. From 2.15 (hy) we know that W_i then is contractible. Hence,

$$\tilde{H}^{j-1}(W_i^*) \cong H^j(W_i, W_i^*) \cong \begin{cases} 0, & j \leqslant 2 \\ \mathbb{Z}_{a_i}, & j = 3 \end{cases}$$

where $a_i = l_i \cdot m_i$, by 2.15 (hy). Thus the last commutative diagram applied to $W = W_i$, etc., simplifies to

$$
\begin{array}{ccccc}
0 \rightarrow \mathbb{Z} \cdot \xi & \xrightarrow{\partial_i} & H^2(W_i, \Phi) & \rightarrow & 0 \\
\| & & \downarrow \rho_i & & \\
0 \rightarrow \mathbb{Z} \cdot \xi & \rightarrow & H^2(W_i^*, \Phi) & \rightarrow \mathbb{Z}_{a_i} \rightarrow 0,
\end{array}
$$

so ∂_i is an isomorphism for $i \geqslant s + 1$. $\square$

The information thus obtained is sufficient to determine $H^2(W)$:

5.15 REMARK. The second Betti number of an affine surface W with a hyperbolic $\mathbb{C}^*$-action is

$$b^2 = \gamma + hy(W) - 1 + \delta(W).$$

Moreover, for a fixed prime number p and $\mu_1 \leqslant \cdots \leqslant \mu_s$ according to 2.10, we have

$$S_p T^2 = \bigoplus_{i=1}^{s-\delta} \mathbb{Z}_{p^{\mu_i}}.$$

Proof. From 5.8, 5.12, and 5.13 we obtain

$$H^2(W) \cong \left(H^1(Y) \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \oplus \mathbb{Z}^h \right)\bigg/ \mathbb{Z} \cdot (\partial_0 \xi, \ldots, \partial_{s+h} \xi).$$

By 5.14, that yields the description

$$H^2(W) \cong H^1(Y) \oplus \left(\bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \oplus \mathbb{Z}^h \right)\bigg/ \mathbb{Z} \cdot (\tau_1, \ldots, \tau_{s+h}). \qquad (5.16)$$

$\square$

Eventually we now come to the computation of $H^2(W^*)$ – which in particular determines $T_2^{cld} \cong \mathrm{Tors}\, H^2(W^*)$ – and of the restriction homomorphism ρ. As a preparation we first describe $H^2(W_i^*, \Phi)$ for $i \geqslant s + 1$:

5.17 LEMMA. *If* $y_i = \pi(w_i)$ *for some* $w_i \in F$ *and* m_i *is the order of* $\pi^{-1}(y_i)$, *then* $H^2(U_i^*, \Phi) \cong \mathbb{Z} \oplus \mathbb{Z}_{m_i}$.

Proof. For the sake of simplicity we abstain from writing the index i in the following proof. By 2.15(hy), the order m is a divisor of the order a of $\mathcal{H}_{2,w}$. Let O_- and O_+ be the non-trivial orbits in Φ_y with orders m_- resp. m_+; then $m = \gcd(m_-, m_+)$. Set $U_{+/-} := U^* \backslash O_{-/+}$ and $U_0 := U_+ \cap U_- = U \backslash \Phi_y$. First of all we consider the exact Mayer-Vietoris sequence

$$H^1(U_+, \Phi) \oplus H^1(U_-, \Phi) \to H^1(U_0, \Phi) \overset{\delta}{\to} H^2(U^*, \Phi)$$

$$\overset{\chi}{\to} H^2(U_+, \Phi) \oplus H^2(U_-, \Phi) \to H^2(U_0, \Phi)$$

Since (U_0, Φ) is homeomorphic to $(\mathbb{C}^* \times D^*, \mathbb{C}^* \times \text{point})$, it satisfies $H^j(U_0, \Phi) \cong \mathbb{Z}$ for $j = 1$ and 2; so, with 5.12, we obtain a short exact sequence:

$$0 \to \mathbb{Z} \overset{\delta}{\to} H^2(U^*, \Phi) \overset{\chi}{\to} \mathbb{Z}_{m_+} \oplus \mathbb{Z}_{m_-} \to 0. \qquad (5.18)$$

Hence, $H^2(U^*, \Phi)$ is of the form $\mathbb{Z} \oplus T$, where T is a torsion group. Next we use the exact sequence

$$0 = H^1(U_+, \Phi) \to H^2(U^*, U_+) \to H^2(U^*, \Phi) \overset{\sigma}{\to} H^2(U_+, \Phi) \cong \mathbb{Z}_{m_+},$$

associated to the triple (U^*, U_+, Φ). By excision, there is an isomorphism $H^2(U^*, U_+) \cong H^2(U_-, U_0)$. Since the pair (U_-, U_0) is homeomorphic to $(\mathbb{C}^* \times D, \mathbb{C}^* \times D^*) \simeq (S^1 \times D, S^1 \times S^1)$, the group $H^2(U^*, U_+)$ is free cyclic, and so is its image in $H^2(U^*, \Phi)$. Consequently, σ maps T injectively into $\mathbb{Z}_{m_+}$, so T is a cyclic group. In the same way T is a subgroup of $\mathbb{Z}_{m_-}$ and thus of $\mathbb{Z}_m$, since $m = \gcd(m_+, m_-)$. It follows from the exact sequence 5.18 that the cokernel of the mapping $\mathbb{Z} \cong H^1(U_0, \Phi) \overset{\delta}{\to} H^2(U^*, \Phi) \cong \mathbb{Z} \oplus T$ is the direct sum $T \oplus T'$ of

two finite cyclic groups and is isomorphic to $\mathbb{Z}_{m_+} \oplus \mathbb{Z}_{m_-}$. As T is a subgroup of $\mathbb{Z}_{\gcd(m_+,m_-)}$, the only possibility is $T \cong \mathbb{Z}_{\gcd(m_+,m_-)}$ and $T' \cong \mathbb{Z}_{\mathrm{lcm}(m_+,m_-)}$. $\square$

Before we can put together the local information about the surface W^*, we need an appropriate algebraic representation for the homomorphism $\mathbb{Z} \oplus \mathbb{Z}_m \cong H^2(U^*, \Phi) \to H^2(U^*) \cong \mathbb{Z}_a$:

5.19 LEMMA. *Let M be an abelian group of rank 1 such that $\mathrm{Tors}\, M$ is cyclic of order m. If $\psi: M \twoheadrightarrow \mathbb{Z}_a$ is injective when restricted to $\mathrm{Tors}\, M$, then, for $d := a/m$, there exists a commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z} \oplus \mathbb{Z}_m & \xrightarrow{\ \varphi\ } & \mathbb{Z}_a \\[2mm]
\Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \chi} \\[2mm]
M & \xrightarrow[\ \psi\ ]{} & \mathbb{Z}_a
\end{array}
$$

with $\varphi(\alpha,\ \beta) = \alpha - d\beta \bmod a$, and isomorphisms θ and χ; in particular, $\mathrm{Ker}(\psi\theta) = \mathbb{Z} \cdot (d, 1)$. $\square$

So far, 5.8, 5.14, and 5.17 yield:

$$
H^2(W^*,\ \Phi) \cong H^1(Y) \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \ \bigoplus_{i=s+1}^{s+h} (\mathbb{Z} \oplus \mathbb{Z}_{m_i}).
$$

If we use the transformations of 5.19, we obtain the desired description for the factor group $H^2(W^*) \cong H^2(W^*, \Phi)/H^1(\Phi)$: we set

$$
\Delta := \mathbb{Z} \cdot (\overbrace{1, \ldots, 1}^{s \text{ times}}, (l_{s+1}, 1), \ldots, (l_{s+h}, 1)) \subset \bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \oplus \bigoplus_{i=s+1}^{s+h} (\mathbb{Z} \oplus \mathbb{Z}_{m_i})
$$

for $l_i = a_i/m_i$, $a_i = |\mathscr{H}_{2,w_i}|$, $\{w_i\} := F \cap \Phi_i$ for $s + 1 \leqslant i \leqslant s + h$; then

$$
H^2(W^*) \cong H^1(Y) \oplus \left(\bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \oplus \bigoplus_{i=s+1}^{s+h} (\mathbb{Z} \oplus \mathbb{Z}_{m_i}) \right)\Big/ \Delta. \tag{5.20}
$$

Finally we can calculate the p-Sylow groups $S_p T_2^{cld} = T_2^{cld} \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ denotes the localization $(\mathbb{Z}\backslash(p))^{-1} \cdot \mathbb{Z}$. Then

$$
S_p T_2^{cld} = \mathrm{Tors}\left(\left(\bigoplus_{i=1}^{s+h} \mathbb{Z}_{p^{\mu_i}} \oplus \mathbb{Z}_{(p)}^h \right)\Big/ \Delta \right).
$$

By choosing an appropriate basis for the free module $\mathbb{Z}_{(p)}^h$, we may assume that the generator $\partial\xi$ of Δ is of the form $(1, \ldots, 1, 0, \ldots, 0, p^\lambda)$ with

$\lambda = \min_{s+1 \leqslant j \leqslant s+h} v_p(l_j)$; on the other hand, since $\mathbb{Z}(1, \dots, 1)$ is a direct factor of $\bigoplus_{i=1}^{s+h} \mathbb{Z}_{p^{\mu_i}}$, we may change the direct sum decomposition in such a way that (mixing up the elements of A and $\pi(F)$) $\mu_{s+h} = \max_{1 \leqslant i \leqslant s+h} \mu_i$ and $\partial \xi = (0, \dots, 0, 1, 0, \dots, 0, p^\lambda)$. Thus

$$S_p T_2^{cld} = (\mathbb{Z}_{p^{\mu_{s+h}}} \oplus \mathbb{Z}_{(p)})/(1, p^\lambda) \quad \oplus \quad \overset{s+h-1}{\underset{i=1}{\bigoplus}} \mathbb{Z}_{p^{\mu_i}}.$$

The kernel of the surjective homomorphism $\mathbb{Z}_{(p)} \to (\mathbb{Z}_{p^{\mu_{s+h}}} \oplus \mathbb{Z}_{(p)})/(1, p^\lambda)$, $1 \mapsto (1, 1)$ is isomorphic to $p^{\lambda+\mu_{s+h}} \mathbb{Z}_{(p)}$, so finally we obtain the description of the torsion group T_2^{cld}, the last piece of data missing in 3.7 and 3.9:

$$S_p T_2^{cld} \cong \mathbb{Z}_{p^{\lambda+\mu_{s+h}}} \oplus \overset{s+h-1}{\underset{i=1}{\bigoplus}} \mathbb{Z}_{p^{\mu_i}}. \tag{5.21}$$

6. The computation in the elliptic case

For an elliptic $\mathbb{C}^*$-surface (W, τ) it remains to compute $H_2^{cld}(W)$, see 1.3 and 3.1. The $\mathbb{C}^*$-invariant mapping $\pi: W^* \to Y(\tau)$ may be considered as a particular case of the following notion (see [FiKp, 1.4]):

6.1 DEFINITION. A semistable $\mathbb{C}^*$-surface over a smooth curve X is a (connected) normal algebraic $\mathbb{C}^*$-surface S together with an affine $\mathbb{C}^*$-invariant mapping $\pi: S \twoheadrightarrow X$ such that for every affine open subset $U \subset X$ the inverse image $\pi^{-1}(U)$ is a $\mathbb{C}^*$-surface with $U \cong \pi^{-1}(U)/\!\!/\mathbb{C}^*$.

Let us assume that S is without fixed points (then $S \to X$ is a Seifert $\mathbb{C}^*$-bundle in the sense of [Ho]) and that X is compact. If $m_1, \dots, m_s$ are the orders of the exceptional orbits and $m := \text{lcm}(m_1, \dots, m_s)$, then the quotient S/C_m has the structure of a principal $\mathbb{C}^*$-bundle over X. For the Chern number $c_1 := c_1(S/C_m)([X])$ and a prime number p we set

$$\lambda := \lambda(p) := v_p(c_1) \in \mathbb{N} \cup \{-\infty\}, \quad \text{and} \quad \zeta := \begin{cases} 0, & c_1 = 0, \\ 1, & \text{otherwise}. \end{cases}$$

If W is an elliptic $\mathbb{C}^*$-surface, then the patching weight of its fixed point w is $l_0 = -c_1(W^*/C_m)[X]$, so $\lambda = v_p(l_0)$.

6.2 THEOREM. *The integral cohomology of a semistable* $\mathbb{C}^*$*-surface S without fixed points over a smooth compact curve Y is given by*

$$H^j(S) \cong \begin{cases} \mathbb{Z}, & j = 0, 3 \\ H^1(Y) \oplus \mathbb{Z}^{1-\zeta}, & j = 1 \\ H^1(Y) \oplus \mathbb{Z}^{1-\zeta} \oplus T^2(S), & j = 2, \end{cases}$$

where, for fixed p and $\mu_1 \leqslant \cdots \leqslant \mu_s$, we have $S_p T^2(S) \cong \zeta \cdot \mathbb{Z}_{p^{\lambda+\mu_s-1}} \oplus \overset{s-2}{\underset{i=1}{\bigoplus}} \mathbb{Z}_{p^{\mu_i}}$.

Proof. The homology groups can be calculated by means of the techniques used in the previous sections (see [Fi, VI.6.2]). But it is shorter to write down a proof using classical results of Seifert about the fundamental group of Seifert fibrations ([Se], see also [Or]).

First of all S is homotopy equivalent to the Seifert S^1-bundle $S/\mathbb{R}_{>0}$ over X, which is an orientable compact real three-manifold. Hence, $H^3(S)$ is a free cyclic group. By Poincaré duality, we obtain

$$H^1(S) \cong H^1(S/\mathbb{R}_{>0}) \cong H_2(S/\mathbb{R}_{>0}) \cong H_2(S)$$

and thus $b_1(S) = b_2(S)$. Moreover, for $b_1 := b_1(S)$ and $T_1(S) := \operatorname{Tors} H_1(S)$, the Universal Coefficient Formulas show that

$$H^1(S) \cong \mathbb{Z}^{b_1} \quad \text{and} \quad H^2(S) \cong \mathbb{Z}^{b_1} \oplus T_1(S).$$

For every exceptional orbit $\Phi_i = \pi^{-1}(y_i)$ with orbit data $(m_i, \bar{n}_i)$, there is a unique natural number s_i such that

$$s_i n_i \equiv 1 \bmod m_i \quad \text{and} \quad 0 < s_i < m_i$$

(see also [FiKp, 5.3]). Denote by E the line bundle over Y obtained from the principal $\mathbb{C}^*$-bundle S/C_m by adding a zero section. Choose small closed pairwise disjoint disks D_i centered at y_i; moreover, let $\sigma_i \in \Gamma(D_i \setminus \{y_i\}, S)$ be holomorphic sections which induce meromorphic sections $\sigma_i/C_m \in \Gamma(D_i, \mathcal{M}(E))$ with a pole in y_i of order $s_i v_i$ where $v_i := m/m_i$.

For $Y_0 := \overline{Y \setminus \bigcup D_i}$ we define $\sigma \in \Gamma(\partial Y_0, S)$ by $\sigma|_{\partial D_i} = \sigma_i|_{\partial D_i}$. There is an obstruction number in $H^2(Y_0, \partial Y_0; \mathbb{Z}) \cong \mathbb{Z}$ against the extension of σ to a section in $\Gamma(Y_0, S)$; let b denote the negative of that integer. The orientation conventions of [Or] differ from ours, which originate from complex geometry; that explains the negative sign in our situation. The number b can be computed in the following way:

6.3 REMARK. Let $E \to Y$ be a line bundle over Y and $E' \to Y$ be the associated $\mathbb{C}^*$-bundle, which is obtained from E by removing the zero section N. Let $\varphi \in \mathscr{C}^\infty(Y_0, E)$ be a smooth section with isolated zeros y_j of order ε_j (defined as the intersection number of $\varphi(Y_0)$ and N at y_j) and without zeros on ∂Y_0. Then the obstruction to extending $\varphi|_{\partial Y_0} \in \mathscr{C}^\infty(\partial Y_0, E')$ to a section in $\mathscr{C}^\infty(Y_0, E')$ is the integer $\sum_j \varepsilon_j \in \mathbb{Z} \cong H^2(Y_0, \partial Y_0)$.

We use that remark in order to compare Seifert's invariant b with the Chern class $c_1(S/C_m)$. The obstruction to extending the section $\sigma/C_m \in \mathscr{C}^\infty(\partial Y_0, S/C_m)$

induced by σ to a section in $\mathscr{C}^\infty(Y_0, S/C_m)$ is the integer $-mb$. We replace

$$(\sigma_1/C_m, \ldots, \sigma_s/C_m) \in \Gamma \left(\bigcup_{i=1}^{s} D_i, \mathscr{M}(E) \right)$$

by a section $(\theta_1, \ldots, \theta_s) \in \mathscr{C}^\infty(\bigcup_{i=1}^{s} D_i, E)$ with zeros of order $-s_i v_i$ in the points y_i and nonzero elsewhere such that $\theta_i|_{\partial D_i} = (\sigma_i/C_m)|_{\partial D_i}$. Admitting an additional zero y_0 in the interior of Y_0 with an appropriate order v_0, there exists a smooth extension θ onto Y with zeros only in $\{y_0, \ldots, y_s\}$. Then the Chern number $c_1(E)$ is $v_0 + \sum_{i=1}^{s}(-s_i v_i)$. Moreover, since $-mb$ is the obstruction against the extension of σ/C_m, we obtain by the above remark $-mb = v_0 = c_1(E) + \sum_{i=1}^{s} s_i v_i$ and thus

$$l_0 = -c_1(S/C_m) = mb + \sum_{i=1}^{s} s_i v_i.$$

The homology group $H_1(S) \cong H_1(S/\mathbb{R}_{>0})$ is the fundamental group $\pi_1(S)$ abelianized. Hence, by [Or, 5.3], $H_1(S)$ is of the form $H_1(Y) \oplus L$ where the subgroup L is generated by the curves $q_i := \sigma_i(\partial D_i)$ for $i = 1, \ldots, s$ and $h: t \mapsto e^{2\pi i t} x_0$ for some $x_0 \in S \backslash \bigcup_{i=1}^{s} \Phi_i$. The relations between the generators of L are

$$m_i q_i + s_i h = 0 \quad \text{for } i = 1, \ldots, s \quad \text{and} \quad \sum_{i=1}^{s} q_i = bh. \tag{6.4}$$

With $\tau_i := \bar{1} \in \mathbb{Z}_{m_i}$ we consider the image G of the homomorphism

$$\mathbb{Z}^{s+1} \cong \bigoplus_{i=1}^{s} \mathbb{Z} \cdot q_i \oplus \mathbb{Z} \cdot h \to \bigoplus_{i=1}^{s} \mathbb{Z}_{m_i} \oplus \mathbb{Z},$$

$$q_i \mapsto (\tau_i, s_i v_i) \quad \text{for } i = 1, \ldots, s \quad \text{and} \quad h \mapsto (0, -m).$$

It is easy to see that G may be identified with the residue class group of $\bigoplus_{i=1}^{s} \mathbb{Z} \cdot q_i \oplus \mathbb{Z} \cdot h$ with respect to the first s relations of 6.4. Thus

$$L \cong G/\mathbb{Z}\left(\sum_{i=1}^{s} q_i - bh \right) \cong G/\mathbb{Z}\left(\tau_1, \ldots, \tau_s, \sum_{i=1}^{s} s_i v_i + mb \right) \cong G/\mathbb{Z}(\tau_1, \ldots, \tau_s, l_0).$$

Now we determine the p-subgroups of L, resp., $L \otimes \mathbb{Z}_{(p)}$ for a fixed prime number p. As usual we order the μ_i's so that $\mu_1 \leqslant \cdots \leqslant \mu_s$. Since we work over the localization $\mathbb{Z}_{(p)}$, we may assume that $\mu_1 > 0$, so every s_i is a unit in $\mathbb{Z}_{(p)}$. We set

$$\xi_0 := (0, \ldots, 0, \tau_s; s_s v_s)$$

$$\xi_i := (0, \ldots, \underbrace{\tau_i}_{i\text{th pos.}}, -s_i s_{i+1}^{-1} v_i/v_{i+1}\tau_{i+1}, 0, \ldots, 0; 0), \quad i = 1, \ldots, s-1.$$

For $i = 1, \ldots, s - 1$ the element ξ_i is of order p^{μ_i}, and

$$G \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)} \cdot \xi_0 \oplus \bigoplus_{i=1}^{s-1} \mathbb{Z}_{(p)} \xi_i.$$

Since $s_s v_s$ is a unit in $\mathbb{Z}_{(p)}$, the element

$$\xi := (\tau_1, \ldots, \tau_s, l_0) - l_0 (s_s v_s)^{-1} \xi_0$$

is of order $p^{\mu_s - 1}$, and

$$G \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)} \cdot \xi_0 \oplus \mathbb{Z}_{(p)} \cdot \xi \oplus \bigoplus_{i=1}^{s-2} \mathbb{Z}_{(p)} \xi_i.$$

As in the hyperbolic case, that description yields the claim of the theorem. $\square$
For further reference we add the following

6.5 REMARK. The $\mathbb{C}^*$-surface S without fixed points can be extended to a parabolic $\mathbb{C}^*$-surface $\bar{S} = S \cup F(\bar{S})$ by adding a fixed point w_x to every fiber Φ_x, which satisfies $w_x = \lim_{t \to 0} tw$ for an arbitrary point $w \in \Phi_x$ (see [FiKp, 3.4]). Then $\bar{S}/C_m \cong \overline{S/C_m}$ is the line bundle associated to the principal bundle S/C_m over X.

Now let W be an elliptic $\mathbb{C}^*$-surface and w_0 its fixed point. If we set $S := W^*$, then the inclusion $S \subset W$ extends to a proper morphism $\bar{S} \to W$ that maps the fixed point set $F(\bar{S}) = \bar{S} \setminus S$ to the point w_0. There exists a commutative diagram

$$
\begin{array}{ccc}
F & \hookrightarrow \bar{S} \to & W \\
\| & \downarrow & \downarrow \\
F/C_m & \hookrightarrow \bar{S}/C_m \to & W/C_m.
\end{array}
$$

Hence, W/C_m is obtained from the line bundle $\bar{S}/C_m$ by collapsing the zero-section F/C_m, and similarly, W is obtained from the 'Seifert line bundle' $\bar{S}$ by contracting F.

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