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1. Introduction

1.1. The classical notion of a cusp form f in the upper half-plane leads first to the concept of a cusp form on the adele group of $GL(2)$ over $\mathbb{Q}$, and thence to the idea of an automorphic cuspidal representation π_f of the adele group of $GL(2)$. We recall that the adele group of $GL(2)$ is the restricted product of the local groups $GL(2, \mathbb{Q}_p)$ where p is a place of $\mathbb{Q}$. If p is infinite then $\mathbb{Q}_p$ is the real field $\mathbb{R}$; if p is finite then Q_p is the p-adic field. The unitary representation π_f may be expressed as $\otimes\pi_p$ with one local representation π_p for each local group $GL(2, \mathbb{Q}_p)$. It is in this way that the unitary representation theory of p-adic groups such as $GL(2, \mathbb{Q}_p)$ enters into the modern theory of automorphic forms [6, 18].

A classical problem in C^*-algebra theory is to determine the structure of C^*-algebras constructed from groups. In this article we show how unitary representation theory can be used to elucidate the structure of the reduced C^*-algebra $C_r^*(G)$ when G is a p-adic Chevalley group.

The algebra $C_r^*(G)$ is defined as follows. We choose a left-invariant Haar measure on G, and form the Hilbert space $L^2(G)$. The left regular representation λ of $L^1(G)$ on $L^2(G)$ is given by

$$(\lambda(f))(h) = f * h$$

where $f \in L^1(G)$, $h \in L^2(G)$ and $*$ denotes convolution. The C^*-algebra generated by the image of λ is the reduced C^*-algebra $C_r^*(G)$.

1.2. Before describing our results, we turn back to the case of linear connected reductive Lie groups. The examples to bear in mind are the identity component of the general linear group $GL(n, \mathbb{R})$ and the special linear group $SL(n, \mathbb{R})$.

Wassermann [17] has determined the structure of $C_r^*(G)$, up to stable isomorphism, as a direct sum of component C^*-algebras. Each component is the crossed product of an abelian C^*-algebra by a finite group. The abelian C^*-algebra is $C_0(\hat{A}/W'(\tau))$ with $\hat{A}$ the unitary dual of the split component A of a Levi factor M. The finite group is the Knapp–Stein R-group $R(\tau)$ of a representation τ in the discrete series of the ${}^0M$ subgroup of a Levi factor M. The groups $W'(\tau)$

and $R(\tau)$ are related by $W(\tau)=W'(\tau).R(\tau)$ (semidirect product) where $W(\tau)$ is the isotropy subgroup of the appropriate Weyl group $W(M)=N_G(M)/M$. The R-group $R(\tau)$ controls the reducibility of the induced representation $i_{GM}(\tau)$, obtained by first extending τ across a parabolic subgroup P with Levi factor M and then unitarily inducing from P to G. In fact the commuting algebra of $i_{GM}(\tau)$ is isomorphic to the group algebra of the finite group $R(\tau)$. Each component C^*-algebra is of the form $C_0(\hat{A}/W'(\tau)) \rtimes R(\tau)$ with τ in the discrete series of ${}^0M$. Reducibility of the tempered representation $i_{GM}(\tau)$ will prevent the component C^*-algebra from being abelian. Irreducibility of $i_{GM}(\tau)$ will result in an abelian C^*-algebra $C_0(\hat{A}/W(\tau))$. The classic example of this phenomenon is provided by $G=SL(2,\mathbb{R})$. Here the ${}^0M$ subgroup with M minimal is the 2-element group $\mathbb{Z}/2$. The trivial representation of $\mathbb{Z}/2$ gives rise to the C^*-algebra $C_0(\mathbb{R})$ which corresponds to the even principal series of $SL(2,\mathbb{R})$; the non-trivial representation of $\mathbb{Z}/2$ gives rise to the crossed product $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2$ which corresponds to the odd principal series. The crossed product arises because the induced representation $i_{GM}(\tau)$ has 2 irreducible components when τ is the non-trivial representation of $\mathbb{Z}/2$. This is the unique reducible representation in the unitary principal series of $SL(2,\mathbb{R})$.

1.3. We now turn to the case of reductive p-adic groups. The theory of the R-group and the normalization of standard intertwining operators has now reached a rather advanced stage of development [19]. There is, in the background, the optimism of the Lefschetz principle as formulated by Harish–Chandra, which says that whatever is true for real reductive groups is also true for p-adic groups. One might hope to obtain a structure theorem for the reduced C^*-algebra of reductive p-adic groups modelled on Wassermann's result.

In order to focus on the difficulty, we shall consider p-adic Chevalley groups. We recall that a p-adic Chevalley group is semisimple. We shall also restrict ourselves to the case of minimal Levi sugroups M. In this case a minimal Levi subgroup is a maximal torus T in G, and we have the basic decomposition

$$T={}^0T.A$$

in the notation of Steinberg [16]. The group A is a finitely generated free abelian group whose rank is the parabolic rank of G. Its unitary dual $\hat{A}$ is a compact torus of dimension equal to this rank. The Weyl group of G, namely $N_G(T)/T$, acts on the compact torus $\hat{A}$. Here one can pinpoint the drastic difference from the case of real Lie groups. In the case of real Lie groups, the Weyl group acts linearly on the real vector space $\hat{A}$. In the p-adic case, $\hat{A}$ is a compact torus. The Weyl group acts as automorphisms of $\hat{A}$, but it cannot of course act linearly. It is true that each isotropy subgroup acts linearly on the tangent space at each point of $\hat{A}$, but this does not seem to help. The non-linearity of the action of the Weyl group means that we cannot adapt the proof of Wassermann to the p-adic case.

To compensate for non-linearity, we have to supply a sufficient condition on the representation τ. The representation τ is in the discrete series of ${}^0T$; but since ${}^0T$ is compact, this means any unitary character of ${}^0T$.

1.4. We now describe this condition. We consider the root system underlying the p-adic Chevalley group G. Let α be a root and $\alpha^\vee$ be a co-root. Define τ_α by the equation

$$\tau_\alpha(u) = \tau(\alpha^\vee(u))$$

for all p-adic units u. Our condition is that

(*) $\tau_\alpha \neq 1$ for all $\alpha > 0$ and $W(\tau)$ is abelian.

In this case we can recover a component C^*-algebra of the form

$$C(\hat{A}) \rtimes R(\tau)$$

exactly as in the real case. The condition (*) is the exact analogue of an essential representation in the discrete series of ${}^0M$, as in [17].

We illustrate this condition in the case of the special linear group $SL(n)$. The standard maximal torus of $SL(n)$ comprises all elements of the form $\mathrm{diag}(x_1, x_2, \ldots, x_n)$ with $x_1 x_2 \cdots x_n = 1$. The subgroup ${}^0T$ comprises all elements of the form $\mathrm{diag}(u_1, u_2, \ldots, u_n)$ with $u_1 u_2 \cdots u_n = 1$ and the u_j are all p-adic units. A unitary character τ of ${}^0T$ is necessarily of the form

$$\tau: \mathrm{diag}(u_1, \ldots, u_n) \mapsto \chi_1(u_1) \cdots \chi_n(u_n)$$

with each χ_j a unitary character of the group of p-adic units. The unitary character τ has projective coordinates $(\chi_1 : \cdots : \chi_n)$. The condition (*) is then that all coordinates $\chi_1, \ldots, \chi_n$ are distinct.

The unitary dual of the group U of p-adic units may be identified with the group of all p^mth roots of 1 for $m = 1, 2, 3, \ldots$. So each character χ_j has prime power order.

The situation is sufficiently well illustrated by the p-adic group SL(10). Let χ be a unitary character of $F^\times$ of order 5. Such characters will exist if and only if $p = 5$ or $p \equiv 1 \bmod 5$. The arithmetic of the field enters at this point. Let the unitary character τ have co-ordinates

$$(1 : \chi : \chi^2 : \chi^3 : \chi^4 : \mu : \mu\chi : \mu\chi^2 : \mu\chi^3 : \mu\chi^4)$$

where μ is another character independent of χ. The Weyl group of $SL(10)$ is the symmetric group S_{10} and the isotropy subgroup $W(\tau)$ is the cyclic group $\mathbb{Z}/5$. But condition (*) is satisfied and so the character τ contributes a component C^*-algebra

$$C(\hat{A}) \rtimes \mathbb{Z}/5$$

where $\hat{A}$ is a 4-dimensional compact torus. This torus may be identified with

$\mathbb{T}^5/\mathbb{T}$ with the group $\mathbb{Z}/5$ acting by cyclic permutation. In this case the induced representation $i_{GM}(\tau)$ is a unitary representation with 5 distinct irreducible components. Once again, the reducibility prevents the component C^*-algebra from being abelian.

The condition (*) is conspicuously not met by the unramified unitary principal series of the p-adic group $SL(2)$. For here the character τ has coordinates $(1:1)$.

1.5. One might hope for a rigid relationship between the arithmetic of the local field F and the structure of the component C^*-algebras of the reduced C^*-algebra. There is one central example in which this is definitely true. For consider the p-adic Chevalley group $SL(l)$ with l prime and l dividing $p-1$. In this case there are $l-1$ component C^*-algebras stably isomorphic to the crossed product

$$C(\mathbb{T}^l/\mathbb{T}) \rtimes \mathbb{Z}/l$$

where $\mathbb{T}$ is the unit circle and $\mathbb{Z}/l$ acts by cyclic permutation. The l fixed points have an arithmetic significance. Each fixed point σ is associated to a cyclic of order l totally ramified extension field, and all such extension fields are accounted for in this way. We recall that an extension field E of F is *totally ramified* if the valuation on E satisfies

$$\mathrm{val}(E^\times) = (1/l)\mathbb{Z}, \qquad \mathrm{val}(F^\times) = \mathbb{Z}$$

where val denotes valuation. Since $i_{GM}(\sigma)$ has l distinct irreducible components, there is at this point a rigid relationship between the arithmetic of the field, the structure of the crossed product $C(\mathbb{T}^l/\mathbb{T}) \rtimes \mathbb{Z}/l$, and the reducibility of the tempered representation $i_{GM}(\sigma)$.

1.6. In Section 2 of this article, we unravel the proof in Wassermann [17, p. 560] and then adapt it to the p-adic case. The main difference, as we have already explained, is to replace the real vector space $\hat{A}$ by a compact torus S, to replace the linear action of the appropriate Weyl group on $\hat{A}$ by a non-linear action on S, and then to compensate for this non-linearity. We must emphasize that this section is modelled closely on Wassermann's proof.

To convert the standard intertwining operators into a unitary 1-cocycle involves solving a difficult normalization problem. Shahidi has recently proved a conjecture of Langlands on normalization of intertwining operators by means of local Langlands root numbers and L-functions, at least when the group is quasi-split and the inducing representation is generic [19, Theorem 7.9]. In the special case of the minimal unitary principal series of a p-adic Chevalley group G, the normalization was dealt with by Keys [7]. Fix a Borel subgroup B of G. Keys' method involves Fourier analysis on the unipotent radical of the Borel subgroup opposed to B, and a suitable version of the gamma function. This converts the standard intertwining operators into a unitary 1-cocycle. This, taken in conjunction with condition (*), allows us to apply the C^*-algebra results of Section 2. The resulting C^*-algebras are crossed products of the form

$$C(S) \rtimes R(\tau).$$

In Section 4, we compute all R-groups which can arise in this context. This section depends on Key's classification of R-groups [7].

In Section 5, we illustrate the arithmetic aspect in the case of the special linear group $SL(l)$. This section hinges on the Artin reciprocity law in local class field theory.

2. On certain fixed-point algebras

2.1. Let A be a C^*-algebra, let Γ be a finite group, and let (A, Γ, α) be a C^*-dynamical system. Let $C(\Gamma, A)$ be the linear space of all maps from Γ to A. We shall denote the crossed product of A by Γ as $A \rtimes_\alpha \Gamma$.

Define p by $p(t) = I$ for all t in Γ. Then p is a projection in the multiplier algebra $M(A \rtimes_\alpha \Gamma)$. Let A^α be the fixed-point algebra of A. We embed A^α in $A \rtimes_\alpha \Gamma$ by sending x to the constant function whose value is x. The image of this embedding is precisely $p(A \rtimes_\alpha \Gamma)p$ so we have

$$A^\alpha \cong p(A \rtimes_\alpha \Gamma)p.$$

2.2. Suppose that $t \mapsto u_t$ is a map from Γ into the unitary group of $M(A)$ such that

$$u_{st} = u_s \alpha_s(u_t)$$

and let $\beta_t = (\mathrm{Ad}\, u_t)\alpha_t$. The map $t \mapsto u_t$ is a unitary 1-cocycle and the C^*-dynamical systems (A, Γ, α) and (A, Γ, β) are exterior equivalent [10, p. 357]. The map Φ defined by $\Phi(y)(t) = y(t)u_t^*$ secures an isomorphism of crossed products

$$\Phi : A \rtimes_\alpha \Gamma \cong A \rtimes_\beta \Gamma.$$

Let q be the map from Γ into $M(A)$ defined by $q(t) = u_t$. Then $\Phi(q)(t) = q(t)u_t^* = u_t u_t^* = I$. We therefore have

$$\begin{aligned} A^\beta &\cong \Phi(q)(A \rtimes_\beta \Gamma)\Phi(q) \\ &\cong q(A \rtimes_\alpha \Gamma)q. \end{aligned}$$

The fixed-point algebra with respect to β is isomorphic to a corner of the crossed product with respect to α.

2.3. An element b in a C^*-algebra A is *strictly positive* if $\varphi(b) > 0$ for every state φ of A. Let e be a projection in the multiplier algebra of A and let $\hat{A}$ be the unitary dual of A. The *support* $\mathrm{supp}(e)$ is defined as the set of all π in $\hat{A}$ such that

$\pi(e) \neq 0$. A *corner* of A is a hereditary subalgebra of A of the form eAe where e is a projection in $M(A)$. A corner is *full* if it is not contained in any proper closed 2-sided ideal of A. Two C^*-algebras A and B are *stably isomorphic* if $A \otimes k \cong B \otimes k$ where k is the standard C^*-algebra of compact operators. These ideas are intimately related and we shall need the following result.

2.4. LEMMA. *Let A be a C^*-algebra with a strictly positive element. Let e and f be projections in $M(A)$ with the same support. Then eAe and fAf are stably isomorphic.*

Proof. Let I be the closed 2-sided ideal of A generated by eAe and fAf. One first checks by methods of Dixmier [5, Section 3.2] that eAe is a full corner of I. Now I is a C^*-algebra with a strictly positive element and each of eAe and fAf is a full corner of I. Therefore eAe and fAf are stably isomorphic, by results in [1, Corollary 2.9) and [2].

2.5. From now on, S will denote a compact torus. We shall regard S as a compact abelian Lie group. We shall exploit the fact that Pontryagin duality works just as well for S as it does for a real finite-dimensional vector space, as in [17, p. 560].

2.6. We shall suppose that the finite group Γ acts as automorphisms of S. The unitary dual $\hat{S}$ is a finitely generated free abelian group. The unitary dual of $\hat{S}$ is canonically isomorphic to S by Pontryagin duality. The action of Γ on S determines an action of Γ on $\hat{S}$, and an action α of Γ on $C(S)$. We have the basic Fourier transform

$$C(S) \rtimes_\alpha \Gamma \cong C^*(\hat{S} \rtimes \Gamma)$$

where $\hat{S} \rtimes \Gamma$ denotes semidirect product.

Since each Γ-orbit in S is finite, and S is a complete separable metric space, we can use the Mackey machine. This gives us a prescription for the unitary dual of the countable discrete group $\hat{S} \rtimes \Gamma$. Let $y \in S$ and let Γ_y be the stabilizer of y. Form the semidirect product $\hat{S} \rtimes \Gamma_y$. Let σ be an irreducible representation of Γ_y. Then $y \otimes \sigma$ is an irreducible representation of $\hat{S} \rtimes \Gamma_y$. Induce from $\hat{S} \rtimes \Gamma_y$ to $\hat{S} \rtimes \Gamma$ to obtain the irreducible representation $\pi_{y,\sigma}$. We shall regard $\pi_{y,\sigma}$ as an irreducible representation of the crossed product $C(S) \rtimes_\alpha \Gamma$.

Fix φ in $C(S)$ and t_0 in Γ. Define F in $C(S) \rtimes_\alpha \Gamma$ by setting $F(t_0) = \varphi$ and $F(t) = 0$ if $t \neq t_0$. The character of σ will be denoted χ_σ.

2.7. LEMMA. *The trace of $\pi_{y,\sigma}(F)$ is given by*

$$|\Gamma|^{-1} \sum \varphi(r.y)\chi_\sigma(r^{-1}sr).$$

The summation is over all r in R and all $r^{-1}sr$ in Γ_y where R is a set of coset representatives of Γ/Γ_y.

Proof. The proof is a tedious computation using the Fourier transform identification $C(S) \rtimes_\alpha \Gamma \cong C^*(\hat{S} \rtimes \Gamma)$ and the standard formula for the induced character as in Serre [14].

2.8. Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. The action α of Γ on $C(S)$ is extended to an action α on $C(S) \otimes M_n(\mathbb{C})$ as follows: $\alpha_t(\varphi \otimes T) = (\alpha_t(\varphi)) \otimes T$. We then have

$$(C(S) \rtimes_\alpha \Gamma) \otimes M_n(\mathbb{C}) = (C(S) \otimes M_n(\mathbb{C}) \rtimes_\alpha \Gamma.$$

The representation $\pi_{y,\sigma}$ extends uniquely to an irreducible representation of $(C(S) \rtimes_\alpha \Gamma) \otimes M_n(\mathbb{C})$ which we continue to denote by $\pi_{y,\sigma}$. If now $\varphi \in C(S, M_n(\mathbb{C}))$, $t_0 \in \Gamma$ and F is defined as $F(t_0) = \varphi$, $F(t) = 0 (t \neq t_0)$ then we have similarly

$$\mathrm{Trace}(\pi_{y,\sigma})(F) = |\Gamma|^{-1} \sum \mathrm{Tr}\, \varphi(ry) \chi_\sigma(r^{-1}sr).$$

Let $u_t \in C(S, U(n))$ be a unitary 1-cocycle, with $t \in \Gamma$. Let $q(u)$ be the map from Γ into $C(S) \otimes M_n(\mathbb{C})$ defined by $q(u)(t) = u_t$ for all t in Γ. Then $q(u)$ is a projection in $(C(S) \otimes M_n(\mathbb{C})) \rtimes_\alpha \Gamma$. A routine computation based on 2.7 will now yield the trace-multiplicity formula

$$\mathrm{Trace}(\pi_{y,\sigma}) q(u) = \langle \psi_y, \bar{\sigma} \rangle$$

where $\psi_y(t) = u_t(y)$ for all t in the isotropy subgroup Γ_y and $\bar{\sigma}$ is the conjugate of the representation σ. That is to say, the rank of the projection $\pi_{y,\sigma}(q(u))$ is equal to the multiplicity with which $\bar{\sigma}$ occurs in ψ_y.

2.9. From now on, we shall take our C^*-algebra A to be $C(S) \otimes k(H)$. Our unitary 1-cocycle $t \to u_t$ will be a map from Γ into the unitary group $C(S, U(H))$. Define

$$v_t(x) = u_t(0) \qquad (x \in S, t \in \Gamma).$$

Then $t \to v_t$ is a unitary 1-cocycle. In fact, $t \to v_t$ is a unitary representation of Γ. Define, as in (2.2)

$$q(u)(t) = u_t, \quad q(v)(t) = v_t.$$

We shall suppose that there exists an increasing sequence (e_n) of finite-rank projections in $L(H)$ which converges strongly to I and commutes with each u_t. Let $\psi_y^n(t)$ be the compression of $u_t(y)$:

$$\psi_y^n(t) = e_n u_t(y) e_n$$

for each natural number n, each y in S, each t in the isotropy subgroup Γ_y. Suppose further that for each y in S there exists N such that ψ_y^n and $\psi_0^n \mid \Gamma_y$ are *quasi-equivalent* whenever $n > N$. This means that ψ_y^n and $\psi_0^n \mid \Gamma_y$, as unitary representations of the isotropy subgroup Γ_y, have the same irreducible components (though not necessarily with the same multiplicity). So our condition is

(*) $\psi_0^n \mid \Gamma_y$ is quasi-equivalent to $\psi_y^n (n > N)$.

2.10. LEMMA. *If* (*) *holds then the projections* $q(u)$ *and* $q(v)$ *have the same support.*

Proof. Let $q_n(u)$ and $q_n(v)$ be the projections in $M((C(V) \otimes k) \rtimes_\alpha \Gamma)$ defined by

$$q_n(u)(t) = e_n u_t e_n,\ q_n(v) = e_n v_t e_n$$

for all t in Γ. Note that $q_n(u))$ is an increasing sequence of projections. Since $e_n \to I$ strongly, it follows that $\pi(q_n(u)) \to \pi(q(u))$ strongly for any representation π of $(C(V) \otimes k) \rtimes \Gamma$. Therefore

$$\begin{aligned}\pi_{y,\sigma}(q(u)) \neq 0 &\Leftrightarrow \mathrm{Trace}(\pi_{y,\sigma}(q(u)) \neq 0\\ &\Leftrightarrow \mathrm{Trace}\ \pi_{y,\sigma}(q_n(u)) \neq 0 \quad (n > N)\\ &\Leftrightarrow \langle \psi_y^n, \bar{\sigma} \rangle \neq 0 \text{ by trace-multiplicity}\\ &\Leftrightarrow \langle \psi_0^n, \bar{\sigma} \rangle \neq 0 \text{ by } (*)\\ &\Leftrightarrow \mathrm{Trace}\ \pi_{y,\sigma}(q_n(v)) \neq 0 \text{ by trace-multiplicity}\\ &\Leftrightarrow \mathrm{Trace}\ \pi_{y,\sigma}(q(v)) \neq 0\\ &\Leftrightarrow \pi_{y,\sigma}(q(v)) \neq 0.\end{aligned}$$

Therefore $\mathrm{supp}(q(u)) = \mathrm{supp}(q(v))$.

2.11. LEMMA. *Let* $A = C(S) \otimes k$ *and let* α *be the action of* Γ *given by* $\alpha_t(\varphi \otimes T) = \alpha_t(\varphi) \otimes T$. *Let* $t \to u_t$ *be a unitary* 1*-cocycle with* u_t *in* $C(S, U(H))$ *and let*

$$\beta_t = (\mathrm{Ad}\ u_t)\alpha_t$$

$$\gamma_t = (\mathrm{Ad}\ v_t)\alpha_t.$$

If (*) *holds then* A^β *is stably equivalent to* A^γ.

Proof. Note first that (A, Γ, α), (A, Γ, β) and (A, Γ, γ) are exterior equivalent C^*-dynamical systems. By (2.2) we have

$$A^\beta \cong q(u)(A \rtimes_\alpha \Gamma)q(u)$$

and

$$A^\gamma \cong q(v)(A \rtimes_\alpha \Gamma)q(v).$$

Since $C(S)$ is a unital C^*-algebra, A and $A \rtimes_\alpha \Gamma$ each contains a strictly positive element. Therefore, by (2.4) and (2.10), A^β and A^γ are stably isomorphic.

2.12. Now $t \to v_t$ is a unitary representation $\Gamma \to U(H)$. Let Γ' comprise all t in Γ such that v_t is scalar, so that Γ' is a normal subgroup of Γ. We shall suppose that

(**) Γ admits a complementary subgroup Γ_1 such that Γ is a semidirect product $\Gamma' \rtimes \Gamma_1$.

(***) The unitary representation of Γ_1 given by $t \to v_t$ is quasi-equivalent to the left regular representation λ of Γ_1 on $\mathbb{C}(\Gamma_1)$.

2.13. THEOREM. *Let (A, Γ, β) be the C^*-dynamical system under discussion. If conditions (*), (**) and (***) hold then the fixed point algebra A^β is stably isomorphic to the crossed product $C(S/\Gamma') \rtimes \Gamma_1$.*

Proof. We have

$$\gamma_t = (\mathrm{Ad}\ v_t)\alpha_t = \alpha_t.$$

whenever t lies in the normal subgroup Γ'. Now

$$A^\gamma \cong (A^\gamma_{\Gamma'})^\gamma_{\Gamma_1} \quad \text{by (**)}$$

$$\cong (C(S/\Gamma') \otimes k)^\gamma_{\Gamma_1}.$$

where subscripts Γ_1 (resp. Γ') signify restrictions of γ to Γ_1 (resp. Γ'). We have

$$\gamma_r = \tilde{\alpha}_r \otimes \mathrm{Ad}\ v_r$$

where $\tilde{\alpha}_r$ is the action induced on the quotient space S/Γ' and $r \in \Gamma_1$. By Takai duality we have

$$C(S/\Gamma') \rtimes_{\tilde{\alpha}} \Gamma_1 = (C(S/\Gamma') \otimes \mathrm{End}(\mathbb{C}\Gamma_1))^\delta$$

where, for all r in Γ_1,

$$\delta_r = \tilde{\alpha}_r \otimes \mathrm{Ad}\ \lambda_r.$$

By (***) it is now clear that A^γ is stably isomorphic to $C(S/\Gamma') \rtimes \Gamma_1$.

The theorem now follows from Lemma (2.11).

3. Minimal unitary principal series for p-adic Chevalley groups

3.1. Throughout this section, $\mathbb{Q}_p$ denotes a p-adic field. Let

$$\mathcal{O} = \{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}$$

$$\mathcal{O}^\times = \{x \in \mathbb{Q}_p^\times : |x|_p = 1\}$$

Now

$$\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathcal{O}^\times \quad \text{and} \quad \mathcal{O}^\times \cong \langle \varepsilon \rangle \times \mathcal{O}_1,$$

where $\mathcal{O}_1 = \{1 + x : |x|_p < 1\}$ and $\langle \varepsilon \rangle \cong \mathbb{Z}/(p-1)$ as in [13].

Let L be a complex semi-simple Lie algebra and Δ be its root system. Throughout this section, G denotes a p-adic Chevalley group determined by L. The notations $x_\alpha(t)$, $h_\alpha(t)$, $w_\alpha(t)$ are defined as in [16]. Let $X_\alpha = \{x_\alpha(t) : t \in \mathbb{Q}_p\}$. Then G is generated by all X_α, $\alpha \in \Delta$. Let T be the subgroup of G generated by all $h_\alpha(t)$. Then T is a maximal torus of G. Denote the Weyl group $N_G(T)/T$ by W. Let

U be the subgroup of G generated by all X_α, where α is a positive root. Then we have the following Levi decomposition [16]:

Let B be the group generated by T and U, then $B = T.U$ (semi-direct product) and U is normal in B. Moreover B is a Borel subgroup of G. Let K be the group generated by $\{x_\alpha(t): \alpha \in \Delta, t \in \mathcal{O}\}$. Then K is a good maximal compact subgroup of G. We have the Iwasawa decomposition $G = KB = KTU$ (non-uniquely). Let ${}^0T$ be the group generated by $\{h_\alpha(t): \alpha \in \Delta, t \in \mathcal{O}^\times\}$ and let A be the group generated by $\{h_\alpha(p^n): \alpha \in \Delta, n \in \mathbb{Z}\}$. Then we have the direct product decomposition

$$T = {}^0T.A.$$

In fact, the decomposition $T = {}^0T.A$ is not canonical (in contrast to real groups) and depends on the choice of uniformizer that is here taken to be p. For an arbitrary p-adic field, there is no such canonical uniformizer. In fact, we only have

$$0 \to {}^0T \to T \to A \to 0$$

with no canonical splitting.

Let $\Psi(T) = \{\varphi \in \mathrm{Hom}(T, U(1)): \varphi({}^0T) = 1\}$. Note that $\Psi(T) \cong \hat{A}$. Let $X(T) = \mathrm{Hom}(T, \mathbb{Q}_p^\times)$. Then $X(T)$ is a free abelian group of finite rank.

3.2. LEMMA. *The map which sends (ξ, λ) in $X(T) \times \mathbb{R}$ to the character*

$$x \mapsto \exp(-2\pi i\lambda \text{ val } \xi(x))$$

of T induces a homomorphism of $X(T) \otimes \mathbb{R}$ onto $\Psi(T)$, whose kernel is $X(T) \otimes \mathbb{Z}$.

Proof. Steinberg [16] and Rodier [12]. From the above lemma, we see that $\Psi(T)$ is a compact torus.

Now we define an action of the Weyl group W on $\hat{T}$. If $\chi \in \hat{T}$, $w \in W$, $w.\chi$ is defined by

$$(w.\chi)(m) = \chi(x_w^{-1} m x_w), \quad \text{for } m \in T,$$

where x_w is any representative of w in $N_G(T)$.

3.3. The groups B, U are defined as in (3.1). Let (σ, V) be an irreducible unitary representation of B. Let $H(\sigma)$ be the space of all smooth functions $f: G \to \mathbb{C}$ such that

$$f(gmu) = \delta_B^{-1/2}\sigma^{-1}(m)f(u) \quad \text{for all } g \in G,\ m \in B,\ u \in U,$$

where δ_B is the modular function of B. The factor $\delta_B^{-1/2}$ is used so that unitary representations induce to unitary representations. Define the induced representation $\mathrm{Ind}_B^G\sigma$ to be left translation in $H(\sigma)$. Note that $\mathrm{Ind}_B^G\sigma$ is an admissible unitary representation of G. Let τ be a unitary character of ${}^0T$ and λ a unitary character of A. Define

$$\tau_0(mau) = \tau(m)$$

$$\lambda_0(mau) = \lambda(a),\ m \in {}^0T,\ a \in A,\ u \in U.$$

Then τ_0, λ_0 are unitary characters of B. By abuse of notation, we also denote τ_0 and λ_0 by τ and λ respectively. Note that $\mathrm{Ind}_B^G\tau$ and $\mathrm{Ind}_B^G(\lambda\tau)$ can be realized on the same Hilbert space for any $\lambda \in \Psi(T)$, see Silberger's book [15].

3.4. In this section, the results were proved by Keys [7]. Suppose σ is a unitary character of T. Define the standard intertwining operators

$$A(\bar{w}, \sigma)f(g) = \int f(gu\bar{w})\,\mathrm{d}u$$

where the integration is over $U \cap w\bar{U}w^{-1}$, $\bar{w} \in N_G(T)$ is a coset representative for $w \in W$ and $f \in H(\sigma)$. We normalize $A(\bar{w}, \sigma)$ by the gamma function $\Gamma_w(\sigma)$. To define $\Gamma_w(\sigma)$, proceed as follows. Fix a non-trivial additive character χ of F. A gamma function $\Gamma(\lambda)$ is associated to each non-trivial quasi-character λ of $F^\times$. Define $\Gamma(\lambda)$ as P.V. $\int \overline{\chi(x)\lambda(x)}|x|^{-1}\,\mathrm{d}x$. In case λ is unramified, this formula must be understood in terms of analytic continuation, for details see Taibleson [20, p.48]. Suppose now that α is a simple root, and let w be the basic reflection w_α. Then $\Gamma_w(\sigma)$ is defined as $\Gamma(\sigma_\alpha)$ where σ_α is the unitary character of $F^\times$ given by $\sigma_\alpha = \sigma o h_\alpha$. If w is a product of basic reflections, then $\Gamma_w(\sigma)$ is defined by an iterative formula which follows the 1-cocycle relation, as in Keys [7, p. 360]. Let

$$\mathfrak{a}(\bar{w}, \sigma) = (1/\Gamma_w(\sigma))A(\bar{w}, \sigma).$$

By analytic continuation, $\mathfrak{a}(\bar{w}, \sigma)$ is defined for all unitary characters σ. With suitable choice of coset representative $\bar{w}$, the cocycle relation

$$\mathfrak{a}(w_1w_2, \sigma) = \mathfrak{a}(w_1, w_2\sigma)\mathfrak{a}(w_2, \sigma)$$

holds for all w_1, w_2 in W.

3.5. Let τ and λ be defined as in (3.3). By the Peter-Weyl theorem we have the orthogonal direct sum over all ρ in $\hat{K}$

$$(\mathrm{Ind}_B^G\tau)\,|\,K \cong \bigoplus n_\rho.\rho$$

Let V_ρ be the isotypic component of class ρ of $(\mathrm{Ind}_B^G\tau)\,|\,K$. Then

$$H(\tau) \cong \bigoplus_{\rho\in\hat{K}} V_\rho \tag{3.5.1}$$

Since $(\mathrm{Ind}_B^G\tau, H(\tau))$ is an admissible representation, then [3] we have $\dim V_\rho < \infty$ for all ρ in $\hat{K}$. Let $W_\tau = \{w \in W : w.\tau = \tau\}$. By [15, 5.2.1] we have

$$\mathrm{Ind}_B^G(\lambda\tau)\,|\,K \cong \mathrm{Ind}_B^G\tau\,|\,K \quad \text{for all } \lambda \text{ in } \Psi(T).$$

Therefore $\mathfrak{a}(w, \lambda\tau)$ belongs to the commuting algebra of $\mathrm{Ind}_B^G\tau \mid K$, for all w in W_τ and λ in $\Psi(T)$. Then we have

$$\mathfrak{a}(w, \lambda\tau)V_\rho = V_\rho \quad \text{for all } \rho \in \hat{K},\ w \in W_\tau,\ \lambda \in \Psi(T).$$

3.6. Let $W_{\lambda,\tau} = \{w \in W : w\tau = \tau \text{ and } w\lambda = \lambda\}$. Define $u_w(\lambda) = \mathfrak{a}(w^{-1}, \lambda\tau)^{-1}$, $w \in W_\tau$, $\lambda \in \Psi(T)$. Then $u_w \in C(\Psi(T), U(H(\tau)))$ is a 1-cocycle and $u_{(.)}(\lambda)$ is a representation of $W_{\lambda,\tau}$ where $U(H(\tau))$ is the set of all unitary operators on $H(\tau)$. Let $0 \in \Psi(T)$ be the trivial character of T.

3.7. LEMMA. *Suppose $u_{(.)}(0) \mid W_{\lambda,\tau}$ is quasi-equivalent to $u_{(.)}(\lambda)$ for all $\lambda \in \Psi(T)$. Then there exists a family of increasing finite rank projections $\{e_n\}$ in $H(\tau)$ such that the following conditions hold:*

(a) $e_n u_w = u_w e_n$, *for all* $w \in W_\tau$.

(b) $e_n \xrightarrow{s} I$.

(c) *For any $\lambda \in \Psi(T)$, there exists N such that $u^n_{(.)}(0) \mid W_{\lambda,\tau}$ is quasi-equivalent to $u^n_{(.)}(\lambda)$ for all $n \geq N$, where $u^n_w = e_n . u_w$.*

Proof. Since $H(\tau)$ is a separable Hilbert space, there are only countably many ρ entering into the decomposition (3.5.1), say $\rho_1, \rho_2 \ldots$. Let e_n be the orthogonal projection from $H(\tau)$ onto $V_{\rho_1} \oplus \cdots \oplus V_{\rho_n}$. Then $e_n \xrightarrow{s} I$ and $\{e_n\}$ is an increasing sequence of projections of finite rank. By (3.5), we have

$$\mathfrak{a}(w, \lambda\tau)V_{\rho_i} = V_{\rho_i} \quad \text{for all } \rho_1, \rho_2, \ldots \quad \text{and} \quad w \in W_\tau,\ \lambda \in \Psi(T).$$

Then we have

$$e_n u_w(\lambda) = u_w(\lambda) e_n, \quad \text{for all } \lambda \in \Psi(T) \quad \text{and} \quad w \in W_\tau.$$

Note that $u_{(.)}(\lambda)$ is a representation of $W_{\lambda,\tau}$. We see that every irreducible subspace of $u_{(.)}(\lambda)$ is contained in some V_{ρ_i}. Since $W_{\lambda,\tau}$ is a finite group, there are only finitely many distinct irreducible components of $u_{(.)}(\lambda)$. Therefore, for each $\lambda \in \Psi(T)$, there exists N such that, up to isomorphism, $e_n(H(\tau))$ contains all irreducible subspaces of $u_{(.)}(\lambda)$ for all $n \geq N$, that is $u_{(.)}(\lambda) \approx u^n_{(.)}(\lambda)$ for all $n \geq N$. The notation $\approx$ denotes quasi-equivalence. Using the same arguments, we also have

$$u_{(.)}(0) \mid W_{\lambda,\tau} \approx u_{n(.)}(0) \mid W_{\lambda,\tau} \text{ for } n \text{ large enough.}$$

Since $u_{(.)}(0) \mid W_{\lambda,\tau} \approx u_{(.)}(\lambda)$, for all $\lambda \in \Psi(T)$, therefore, for each λ, we can choose N large enough such that

$$u^n_{(.)}(0) \mid W_{\lambda,\tau} \approx u^n_{(.)}(\lambda) \quad \text{for all } n \geq N.$$

3.8. Let σ be a unitary character of T and $C(\sigma)$ be the commuting algebra of

$\mathrm{Ind}_B^G\sigma$. Throughout this paper, we use the following notations:

$$\sigma_\alpha(t) = \sigma(h_\alpha(t)),\ \alpha\in\Delta,\ t\in\mathbb{Q}_p^\times.$$

$$\Delta'_\sigma = \{\alpha\in\Delta_+ : \sigma_\alpha(t) = 1\}$$

$$R(\sigma) = \{w\in W_\sigma : w(\Delta'_\sigma) = \Delta'_\sigma\}$$

$$W'_\sigma = \langle w_\alpha : \alpha\in\Delta'_\sigma\rangle.$$

Then $W_\sigma = W'_\sigma \rtimes R(\sigma)$ (semi-direct product) and $W'_\sigma = \{w\in W_\sigma : \mathfrak{a}(w,\sigma)\in\mathbb{C}I\}$ as in [7]. The following lemma is a particular case of Keys' result [7].

3.8.1. LEMMA. *Suppose $R(\sigma)$ is abelian. Then*

(a) *The number of inequivalent irreducible components of* $\mathrm{Ind}_B^G\sigma$ *is equal to the order of $R(\sigma)$.*
(b) *The multiplicity of each irreducible component of* $\mathrm{Ind}_B^G\sigma$ *is equal to* 1.
(c) $\{\mathfrak{a}(w,\sigma) : w\in R(\sigma)\}$ *forms a basis for $C(\sigma)$.*
(d) $C(\sigma)\cong\mathbb{C}[R(\sigma)]$ *the group algebra of $R(\sigma)$.*

Proof. Keys [7].

3.8.2. COROLLARY. *Suppose $R(\sigma)$ is abelian. Let r be the regular representation of $R(\sigma)$. Let $\pi_0(w) = \mathfrak{a}(w,\sigma)$, $w\in W_\sigma$. Then*

$$\pi_0\,|\,R(\sigma)\approx r.$$

Proof. By 3.8.1(a) and (b), we have the decomposition $\mathrm{Ind}_B^G\sigma\cong\pi_1\oplus\cdots\oplus\pi_k$, where π_i are irreducible components of $\mathrm{Ind}_B^G\sigma$ which are mutually inequivalent and $k=|R(\sigma)|$. Let H_i be the representation space of π_i. Then H_i is an invariant subspace of $\pi_0\,|\,R(\sigma)$. Then $\pi_0\,|\,R(\sigma) = \chi_1.I_H\oplus\cdots\oplus\chi_k I_H$ where the χ_i are irreducible representations of $R(\sigma)$. By (c) and (d), we see that $\chi_i\neq\chi_j$ for all $i\neq j$. Therefore the corollary follows.

3.9. The unitary characters τ and λ are defined as before.

3.9.1. LEMMA. *Suppose $R(\lambda\tau)\subseteq R(\tau)$ and both are abelian. Then*

(a) $W'_{\lambda\tau}\subseteq W'_\tau$.
(b) $\mathfrak{a}(.,\lambda\tau)\,|\,R(\lambda\tau)\approx\mathfrak{a}(.,\tau)\,|\,R(\lambda\tau)$.

Proof. (a) Since $T = {}^0T.A$ (direct product), we have

$$\Delta'_{\lambda\tau}\subseteq\Delta'_\tau.$$

Therefore we have $W'_{\lambda\tau}\subseteq W'_\tau$.
(b) By Corollary 3.8.2 and the Frobenius reciprocity theorem, the lemma follows.

3.9.2. LEMMA. *Suppose $\tau_\alpha\neq 1$ for all $\alpha>0$ and $R(\lambda\tau)$ is abelian for all $\lambda\in\Psi(T)$.*

Then

$$\mathfrak{a}(.,\lambda\tau) \approx \mathfrak{a}(.,\tau)\,|\,W_{\lambda,\tau} \quad \textit{for all } \lambda\in\Psi(T).$$

Proof. Since $\tau_\alpha \neq 1$ for all $\alpha > 0$, we have $\Delta'_\tau = \varnothing$. Therefore we have $W'_\tau = \{1\}$. So $R(\tau) = W_\tau$. Since $T = {}^0T.A$, we see that $(\lambda\tau)_\alpha \neq 1$ for all $\alpha > 0$ and $\lambda\in\Psi(T)$. For the same reason, we have $R(\lambda\tau) = W_{\lambda\tau}$, for all $\lambda\in\Psi(T)$. Then we have

$$R(\lambda\tau) \subseteq R(\tau) \quad \text{for all } \lambda\in\Psi(T).$$

By Lemma 3.9.1, the proposition follows.

REMARK. In Lemma 3.9.2, the condition $\tau_\alpha \not\equiv 1$ (for all $\alpha > 0$) cannot be omitted. Consider $G = SL(4, \mathbb{Q}_p)$. Then T is the diagonal subgroup. Let $\psi\in\widehat{\mathcal{O}}$ be the Legendre symbol. Let $\tau = [\psi, 1, \psi, 1]\in({}^0T)\hat{}$ and $\lambda = [i, -1, -i, 1]\in\Psi(T)$. Then $\mathfrak{a}(.,\lambda\tau)$ is *not* quasi-equivalent to $\mathfrak{a}(.,\tau)\,|\,W_{\lambda,\tau}$.

3.10. We now explain exactly how Sections 2 and 3 are related to each other. In order to apply Section 2 to Section 3, we make the following identifications:

$$S = \Psi(T)$$

y in S corresponds to λ in $\Psi(T)$

$$u_w(\lambda) = \mathfrak{a}(w^{-1}, \lambda\tau)^{-1}$$

$$\Gamma = W_\tau$$

$$\Gamma' = W'_\tau$$

$$\Gamma_1 = R(\tau)$$

$$\Gamma_y = W_{\tau,\lambda}.$$

Under these identifications, the fixed-point algebra A^β becomes

$$C^*(G,\tau) = \{f\in C(\Psi(T), k(H(\tau)) : f(w\lambda) = \mathfrak{a}(w,\lambda\tau)\,f(\lambda)\mathfrak{a}(w,\lambda\tau)^{-1},\ w \text{ in } W_\tau\}$$

and where $k(H(\tau))$ is the C^*-algebra of compact operators on $H(\tau)$. From [11], the C^*-algebra $C^*(G,\tau)$ is a C^*-component of the reduced C^*-algebra $C^*_r(G)$.

3.11. THEOREM. *Let G be a p-adic Chevalley group with maximal torus T. Let τ be a unitary character of ${}^0T$. Suppose $\tau_\alpha \neq 1$ for all positive roots α. Moreover assume W_τ is abelian. Then the C^*-component $C^*(G,\tau)$ is stably isomorphic to $C(\Psi(T)) \rtimes W_\tau$.*

Proof. Recall that W_τ is the isotropy subgroup $\{w\in W : w\tau = \tau\}$. Since $\tau_\alpha \neq 1$ for all positive roots α, we have $\Delta'_\tau = \varnothing$, $R(\tau) = W_\tau$ and $\Gamma' = \{1\}$ in the statement of Theorem 2.13. Let σ be any character of T such that $\sigma\,|\,{}^0T = \tau$. Now σ can be chosen so that $W_\sigma = W_\tau$. Such a σ will be denoted σ_0. Then

$$R(\sigma) = W_\sigma \subset W_\tau = W_{\sigma_0}$$

for every σ with $\sigma \mid {}^0T = \tau$. So $R(\sigma)$ is certainly abelian. The theorem now follows immediately from Lemma 3.9.2, Lemma 3.7, Corollary 3.8.2, and Theorem 2.13.

3.12. The condition $\tau_\alpha \neq 1$ for all positive roots α is the exact analogue for p-adic Chevalley groups of an *essential* representation in the discrete series of ${}^0T$, as in [17]. When $\tau_\alpha \neq 1$ for all positive roots α we have

$$R\text{-group } R(\tau) = \text{isotropy subgroup } W_\tau.$$

We shall then refer to W_τ as an *essential* isotropy subgroup. In the next section we compute all essential isotropy subgroups.

4. Essential isotropy

4.1. Throughout this section, assume $p \neq 2$ and let $\pi \in ({}^0T)\hat{}$ and $\lambda \in \Psi(T)$. Now τ is extended to T as in 3.3, and $\mathcal{O}, \mathcal{O}_1$ are defined as in 3.1. Recall that $\mathcal{O}^\times \cong \langle \varepsilon \rangle \times \mathcal{O}_1$, $\mathcal{O}_1 \cong \mathbb{Z}_p$ as in (3.1). In this section, we are going to classify the groups $R(\tau)$ when G is a simple p-adic Chevalley group. In [7] Keys classifies all R-groups $R(\sigma)$ with σ in $\hat{T}$. We are confining our attention to those σ which are trivial on A. We will see that the results are similar to those of the unramified case, Keys [8].

4.2. Type A_n.

Let $\check{\Delta}$ be the co-root system of Δ. Recall that $\Delta = \{e_i - e_j : 1 \leqslant i \neq j \leqslant n+1\}$ and $\Delta = \Delta'$. The Weyl group W acts as permutations of e_i.

4.2.1. LEMMA. *$R(\tau)$ is cyclic.*

Proof. According to Keys [7, p. 366], $R(\tau)$ embeds as a finite subgroup of $(\mathbb{Q}_p^\times)\hat{}$. By our choice of τ, $R(\tau)$ embeds as a finite subgroup of $(\mathcal{O}^\times)\hat{}$. But

$$(\mathcal{O}^\times)\hat{} \cong \mathbb{Z}/p-1 \times \mathbb{Z}/p^\infty$$

where $\mathbb{Z}/p^\infty$ is the direct limit of the cyclic groups $\mathbb{Z}/p^k$, $k = 1, 2, 3, \ldots$. Hence $R(\tau)$ is isomorphic to a subgroup of $\mathbb{Z}/p-1 \times \mathbb{Z}/p^k$ for some k. But $\mathbb{Z}/p-1 \times \mathbb{Z}/p^k$ is cyclic. Therefore $R(\tau)$ is cyclic.

Now suppose $d \mid n+1$ and $\mathcal{O}^\times$ admits a character of order d. Let τ_{α_1} be a character of $\mathcal{O}^\times$ of order d and χ a character of order p^m $(m \gg n)$. Let $\tau_{\alpha_i} = \tau_{\alpha_1}$ for $1 \leqslant i \leqslant n$, $i \neq kd$ and $\tau_{\alpha_{kd}} = \chi$. Then

$$R(\tau) = \langle (12 \cdots d)(d+1 \cdots 2d) \cdots (d-n+2 \cdots n+1) \rangle \cong \mathbb{Z}/d,$$

and $\tau_\alpha \neq 1$ for all positive roots α. Now we have the following proposition.

4.2.2. PROPOSITION. $R(\tau) \cong \{1\}$ *or* $\mathbb{Z}/d$ *with* $d \mid n+1$. *The group* $\mathbb{Z}/d$ *can occur if and only if* $\mathcal{O}^\times$ *admits a character of order* d.

4.3. Using similar arguments, we can classify $R(\tau)$ for the other types

Type $B_n: R(\tau) \cong \{1\}$ or $\mathbb{Z}/2$.

Type $C_n: R(\tau) \cong \{1\}$ or $\mathbb{Z}/2$.

Type D_n: n even: $R(\tau) = \{1\}, \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$.
n odd: $R(\tau) = \{1\}, \mathbb{Z}/2$ or $\mathbb{Z}/4$.

$\mathbb{Z}/4$ can occur if and only if $\mathcal{O}^\times$ admits a character of order 4 iff 4 divides $p-1$.

Type $E_6: R(\tau) = \{1\}$ or $\mathbb{Z}/3$.

$\mathbb{Z}/3$ can occur if and only if $\mathcal{O}^\times$ admits a character of order 3 iff $p = 3$.

Type $E_7: R(\tau) = \{1\}$ or $\mathbb{Z}/2$.

The $R(\tau)$ of types E_8, F_4 and G_2 are all trivial.

5. Arithmetic aspect in the case of SL(l)

5.1. We shall work with the p-adic Chevalley group $SL(l, \mathbb{Q}_p)$. We shall suppose that l is prime and that l divides $p-1$.

By (4.2.2), the R-group $R(\tau)$ must be trivial or $\mathbb{Z}/l$. If $R(\tau) = \mathbb{Z}/l$ then the conditions of Theorem (3.11) are satisfied and so we have the stable isomorphism

$$C^*(G, \tau) \cong C(\Psi(T)) \rtimes \mathbb{Z}/l.$$

The compact torus $\Psi(T)$ admits l fixed points under this $\mathbb{Z}/l$-action. For each fixed point σ, the R-group $R(\sigma)$ is $\mathbb{Z}/l$. We are interested in the arithmetic significance of these fixed points.

Let σ be such a fixed point. Let $x = \text{diag}(x_1, \ldots, x_n)$ be an element in the standard maximal torus T in $SL(l)$. Then

$$\sigma(x) = \psi(x_1)\psi^2(x_2) \cdots \psi^{l-1}(x_{l-1})$$

where ψ is a unitary character of the multiplicative group $F^\times$ of the p-adic field F. Let α_1 be the root $\varepsilon_1 - \varepsilon_2$ of $SL(l)$ and let $\alpha_1^\vee$ be the co-root. Then

$$\sigma_{\alpha_1}(x) = \sigma(\alpha_1^\vee(x)) = \psi(x)\psi^2(x^{-1}) = \psi^{-1}(x)$$

so that ψ is determined by σ. In fact we could have used any simple root instead of α_1.

5.2. The kernel of ψ is a subgroup of the multiplicative group of F of index l. By the local reciprocity law of local class field theory, a subgroup N of index l corresponds uniquely to an extension field E of degree l. The correspondence is given by

$$N = N_{E|F}(E^\times)$$

where $N_{E|F}(E^{\times})$ is the norm group of $E^{\times}$. Moreover we have a canonical isomorphism

$$G(E \mid F) \cong F^{\times}/N_{E|F}E^{\times}$$

where $G(E \mid F)$ is the Galois group of E with respect to F. Our reference for class field theory is Neukirch [9].

The fixed point σ determines uniquely an extension field E. The correspondence is given by

$$N_{E|F}E^{\times} = \ker(\sigma_{\alpha})$$

where α is a simple root of $SL(l)$.

We are going to prove that all such extension fields are totally ramified, and that every totally ramified extension field of degree l is accounted for in this way.

The ramification index e of E is defined by the equation

$$\text{val}(N_{E|F}E^{\times}) = (l/e)\mathbb{Z}$$

where val denotes valuation. The field E is totally ramified when $e = l$.

We now determine all subgroups of F of index l. Now $\mathbb{Z}$ has a unique subgroup of index l, namely $l\mathbb{Z}$. The group $\mathbb{Z}/(p-1)$ has a unique subgroup of index l, namely the image of $l\mathbb{Z}$ in the map $\mathbb{Z} \to \mathbb{Z}/(p-1)$. The subgroups of finite index of $\mathbb{Z}_p$ are $p^n\mathbb{Z}_p$ with $p = 1, 2, 3, \ldots$. These subgroups have index equal to a power of p, and therefore $\mathbb{Z}_p$ does not admit a subgroup of index l. Since the multiplicative group of the p-adic field F has the form

$$F^{\times} \cong \mathbb{Z} \times \mathbb{Z}/(p-1) \times \mathbb{Z}_p$$

it follows that we can concentrate on subgroups of index l in $\mathbb{Z} \times \mathbb{Z}/(p-1)$.

We now make use of a standard basis Lemma [4, p. 333] for finitely generated free abelian groups such as $\mathbb{Z} \times \mathbb{Z}$, and adapt this lemma to $\mathbb{Z} \times \mathbb{Z}/(p-1)$. There are $l + 1$ subgroups of index l of $\mathbb{Z} \times \mathbb{Z}/(p-1)$ and we list their generators:

$$\langle (1, c), (0, l) \rangle = N_c \quad \text{with} \quad 0 \leqslant c < l$$

$$\langle (l, 0), (0, 1) \rangle = M$$

Now

$$\text{val}(N_c) = \mathbb{Z}$$

$$\text{val}(M) = l\mathbb{Z}$$

and so $N_0, N_1, \ldots, N_{l-1}$ correspond, via the local reciprocity map, to the l totally ramified extensions of F of degree l. The subgroup M corresponds, by local reciprocity, to the unique unramified extension of F of degree l.

Now $F^{\times}$ contains a subgroup of order $p - 1$ which is cyclic and unique; let ε be a generator of this subgroup. If $x \in F^{\times}$ then x can be written uniquely $x = p^n \varepsilon^a v$ where $n \in \mathbb{Z}$, $a = 0, 1, 2, \ldots, p-2$, and v is a 1-unit; $n = \text{val}(x)$, a is determined by $p^{-n}x \equiv \varepsilon^a \bmod p$. Let $\omega = e^{2\pi i/l}$. There are l^2 characters ψ of $F^\times$ of order l, each one specified by two natural numbers r, s with $0 \leqslant r, s < l$. We have $\psi(x) = \omega^{rn}\omega^{sa}$. Then $\ker(\psi)$ comprises all pairs (n, a) such that

$$rn + sa \equiv 0 \bmod l.$$

This kernel equals N_c iff

$$r + sc \equiv 0 \bmod l.$$

If $s \neq 0$ then c is determined uniquely by r since $\mathbb{F}_l$ is a field. Therefore, as r runs through $0, 1, 2, \ldots, l-1$ the kernels of $x \mapsto \omega^{rn}\omega^{sa}$ are distinct and correspond to l distinct totally ramified field extensions. All totally ramified field extensions of degree l arise in this way.

5.3. The group of unramified quasi-characters of T can be identified with the complex torus ${}^LT^0$ in the L-group ${}^LG^0$, see, for example, Gelbart and Shahidi [6]. In that case ${}^LT^0$ is the complexification of S. The pairs (G, T) and $({}^LG^0, {}^LT^0)$ have a common Weyl group W. This Weyl group W acts on S. The action of $R(\tau)$ on $\Psi(T)$ is then the restriction from W to $R(\tau)$.

Let G be the p-adic Chevalley group $SL(l)$. Then

$${}^LG^0 = PGL(l, \mathbb{C})$$

$$W = \text{symmetric group } S_l$$

$$S = \Psi(T) = \mathbb{T}^l/\mathbb{T}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The unramified character λ sends $\operatorname{diag}(x_1, \ldots, x_l)$ in T to

$$z_1^{\operatorname{val}(x_1)} z_2^{\operatorname{val}(x_2)} \cdots z_l^{\operatorname{val}(x_l)}$$

with $z_1, z_2, \ldots, z_l \in \mathbb{T}$. Each unramified unitary character λ has projective coordinates $(z_1 : z_2 : \cdots z_l)$ with each $z_j \in \mathbb{T}$. The cyclic group $\mathbb{Z}/l$ acts by cyclic permutation of $z_1, z_2, \ldots, z_l$.

Each conjugacy class of Levi subgroups contributes to the reduced C^*-algebra of $SL(l)$, as in [11]. Now the standard maximal torus T of $SL(l)$ is a minimal Levi subgroup. So the conjugacy class of T makes its contribution to the reduced C^*-algebra. Part of this contribution is described in the following theorem, which results from Theorem 3.11 and Sections 5.1–5.3.

5.4. THEOREM. *The reduced C^*-algebra of the p-adic group $SL(l)$ admits $l-1$ direct summands stably isomorphic to*

$$C(\mathbb{T}^l/\mathbb{T}) \rtimes \mathbb{Z}/l.$$

Each $\mathbb{Z}/l$-fixed point σ in $\mathbb{T}^l/\mathbb{T}$ is associated with a unique totally ramified cyclic extension field E of degree l, and all such extension fields are accounted for in this way. Each σ determines an induced representation with l irreducible components.

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