COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 76, n° 3 (1990), p. 367-398

<http://www.numdam.org/item?id=CM_1990__76_3_367_0>

Compositio Mathematica **76**: 367–398, 1990.

Deformations of theta divisors and the rank 4 quadrics problem

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Received 24 January 1989; accepted in revised form 14 February 1990

Introduction

Here is the precise statement that we prove in this paper (cf. Prop. 1 and Thm. 4).

THEOREM. *Let C be a non-hyperelliptic curve (i.e. complete, connected, non-singular, 1-dimensional scheme of finite type) of genus $g \geqslant 5$ over an algebraically closed field k of characteristic $\neq 2$. Let $\Theta = \Theta(C)$ be a theta divisor in the Jacobian $J = J(C)$ of C and let $I_2(C)$ denote the vector space of quadratic polynomials in the homogeneous ideal $I(C)$ of polynomials vanishing on the canonical model $\Phi_K(C)$ of C in $\mathbb{P}^{g-1} = \mathbb{P}T_0J$. Then the equations of tangent cones to Θ at rank 4 double points span $I_2(C)$ as a k-vector space.*

Both this problem and our approach to it arose from the classic paper of Andreotti-Mayer [A-M] whose results over $\mathbb{C}$ may be summarized, from our point of view, as follows: (Below, 'sg.Θ' denotes the locus of singular points on Θ and 'sg.$_2\Theta$' denotes the sublocus of double points on Θ. We use the notation Q_p both for a quadratic tangent cone and for its defining polynomial and we use the word 'trigonal' to refer to a curve which has a g^1_3 but no g^1_2.)

LEMMA A [A-M, Cor. p. 212]. *For any trigonal (and therefore also for any generic) curve C of genus $g \geqslant 4$, the subset $\varphi(\text{sg.}_2\Theta) \subset I_2(C)$ generates $I_2(C)$ as a vector space, where $\varphi(p) = Q_p$, the quadratic term of Riemann's theta function expanded about p.*

COROLLARY A (generic constructive Torelli). *For a generic (but* not *trigonal) curve C of genus $g \geqslant 5$ it follows, since $I_2(C)$ generates $I(C)$,* cf. [S-D], *that Θ determines the canonical model of C 'constructively' as an intersection of quadrics:*

$$\Phi_K(C) = \cap\{Q_p\colon p \in \text{sg.}_2\Theta\} \subset \mathbb{P}^{g-1}.$$

(This result can be found in [Mayer, Introd. pp. 1–4, Thm. 2 p. 26]; [cf. M4, p. 89]. It seems to be missing from [A-M], for which [Mayer] was apparently a preliminary draft.)

Now if $t: \tilde{\mathscr{M}}_g \to \mathscr{H}_g$ is the Torelli map $t(C) = J(C)$, from 'Torelli space' $\tilde{\mathscr{M}}_g$ [A-M, 10a, pp. 217–218] to Siegel upper half space $\mathscr{H}_g$, and if we denote its image by $\mathscr{J}_g = t(\tilde{\mathscr{M}}_g) \subset \mathscr{H}_g$, and the image of its differential by $T_C\mathscr{J}_g = t_*(T_C\tilde{\mathscr{M}}_g)$, then for every non-hyperelliptic curve C of genus $g \geqslant 4$, the infinitesimal theory of the Torelli map cf. [O-S] implies that $T_C\tilde{\mathscr{M}}_g \cong T_C\mathscr{J}_g$, and $I_2(C) = (T_C\mathscr{J}_g)^\perp \subset T_J^*\mathscr{H}_g$.

LEMMA B. *For any non-hyperelliptic curve C of genus $g \geqslant 4$, if a double point $p \in \mathrm{sg}._2\Theta$ persists (as a singularity of Θ) under first order deformation of (J, Θ) in a given direction $v \in T_J\mathscr{H}_g$, then $\varphi(p)(v) = 0$, where $\varphi(p) \in I_2(C)$ is interpreted as a linear functional orthogonal to $T_C\mathscr{J}_g$.*

(This variation of [A-M, Lemma 8, p. 214], resembles the statement in [M4, p. 87], and follows from [S-V1, Prop. 2.7, p. 653].)

COROLLARY B. *For trigonal (and also for generic) curves C of genus $g \geqslant 4$, the only directions $v \in T_J\mathscr{H}_g$ in which every double point of Θ persists, are those directions $v \in T_C\mathscr{J}_g$.*

([A-M, Cor., p. 212, Thm. 1, p. 213, part (ε) of pf. of Thm. 1, pp. 216–7, Lemma 8 p. 214, 10b, p. 218], imply this at least for generic C.)

LEMMA C [A-M, Cor. p. 212]. *For C any trigonal curve of genus $g \geqslant 5$, or a generic curve of genus $g = 4$, $\mathrm{sg}.\Theta$ has precisely two irreducible components, both of which are of dimension $g - 4$ and conjugate to each other under the involution of J taking x to $-x$.*

Note that since Θ is symmetric, a singular point on Θ smooths in a given direction if and only if the conjugate singular point does so.

COROLLARY C. *For C any trigonal curve of genus $g \geqslant 5$, or a generic curve of genus $g = 4$,*

(i) *whenever all the singularities of a $(g - 4)$ dimensional component of* $\mathrm{sg}.\Theta$ *persist in a given direction $v \in T_J\mathscr{H}_g$, then in fact all the singularities of Θ persist along v.*

(ii) *If $\mathscr{N}_{g-4}$ = the locus in $\mathscr{H}_g$ of period matrices defining principally polarized abelian varieties (p.p.a.v.'s) with* $\dim.\mathrm{sg}.\Theta \geqslant g - 4$, *and if $\bar{\mathscr{J}}_g$ is the closure of $\mathscr{J}_g$, then* $\dim_J(\bar{\mathscr{J}}_g) = \dim_J(\mathscr{N}_{g-4})$.

Proof of (ii). Denote the set theoretic tangent cone of X at x by $TC_x(X)$. Since it can be proved that a whole component of singular points on Θ persist along every direction v in $TC_J\mathscr{N}_{g-4}$, Cor. B and (i) imply $TC_J\mathscr{N}_{g-4} \subset T_C\mathscr{J}_g \subset TC_J\bar{\mathscr{J}}_g$. Finally, the previous inclusions are actually equalities since $\bar{\mathscr{J}}_g \subset \mathscr{N}_{g-4}$. (A-M prove instead, in [A-M, parts (γ), (δ), (ε) of pf. of Thm. 1, pp. 215–17], that if M is a component of $\mathscr{N}_{g-4}$ containing J, then the tangent space to M at a generic point of M has dim. $= 3g - 3$, and in fact the limits at J of the tangent vectors at smooth points ('Nash tangent cone') of some local analytic component of M, lie in $T_C\mathscr{J}_g$.)

Consequently, [A-M, Thm. 1, p. 213]:

Theorem of Andreotti-Mayer (generic 'geometric Schottky'). $\bar{\mathscr{J}}_g$ *is an irreducible component of* $\mathscr{N}_{g-4}$, (for $g \geqslant 4$), *both in* $\mathscr{H}_g$ *and in the coarse moduli space* $\mathscr{A}_g$ *of all p.p.a.v.'s of* dim. g.

Since the publication of [A-M], 'the rank 4 quadrics conjecture' has referred to the assertion (cf. Lemma A) that $\varphi(\mathrm{sg}._2\Theta)$ generates $I_2(C)$, for all non-hyperelliptic curves C of genus $g \geqslant 4$. (The terminology comes from the fact that all the quadrics Q_p, for $p \in \mathrm{sg}._2\Theta$, have rank $\leqslant 4$, and the fact [A-H] that these quadrics span linearly all rank 4 quadrics containing C.) The conjecture would imply that Corollary B also holds for all such curves C. The approach in this paper is to show that for any non-hyperelliptic curve C of genus $g \geqslant 5$ the assertion in Corollary B is essentially equivalent to that in Lemma A, and then to use deformation theory to establish Corollary B for such curves independently. Hence one obtains, over any algebraically closed field of characteristic $\neq 2$, both Lemma A (the rk.$4-Q$ conjecture) for all non-hyperelliptic curves of genus $g \geqslant 5$ and Corollary A (constructive Torelli) for all non-hyperelliptic, non-trigonal curves of genus $g \geqslant 5$ except (in genus 6) plane quintics; [cf. B-V, K5, for results in some of the exceptional cases].

The conjecture is relatively easy for $g \leqslant 5$. It was proved over $\mathbb{C}$, by Arbarello-Harris [A-H] for $g = 6$, and by Mark Green [Green] for all $g \geqslant 4$. Here is an intuitive explanation of the present approach:

(0) The 'rank 4 quadrics conjecture' is the assertion that $\{Q_p \mid p \in \mathrm{sg}._2\Theta\}$ spans $I_2(C)$ for any non-hyperelliptic curve C of genus $\geqslant 5$.

(1) By the infinitesimal Torelli theory, $T_C\mathscr{J}_g = (I_2(C))^\perp$ in $T_J\mathscr{H}_g$, so the rank 4 quadrics conjecture becomes the assertion that the following inclusion (of subspaces of $T_J\mathscr{H}_g$) is an equality:

$$T_C\mathscr{J}_g \subset \cap\{(Q_p)^\perp\colon p \in \mathrm{sg}._2\Theta\}.$$

(2) We will interpret the space on the right side of the previous inclusion by introducing the subset $\mathscr{E} \subset T_J\mathscr{H}_g$ of tangent directions to $\mathscr{H}_g$ in which all singular points of Θ persist. The density of $\mathrm{sg}._2\Theta$ in $\mathrm{sg}.\Theta$ and the Andreotti-Mayer criterion in Lemma B for persistence of double points (cf. [S-V1, Thm. (4.1), p. 658; Prop. (7.2), p. 671]) suggest that

$$\cap\{(Q_p)^\perp\colon p \in \mathrm{sg}._2\Theta\} = \mathscr{E}.$$

With this identification, the rank 4 quadrics conjecture would become the assertion that the inclusion (of subspaces of $T_J\mathcal{H}_g$):

$$T_C\mathcal{J}_g \subset \mathcal{E} = \text{`equisingular directions in moduli'}$$

is an equality. Thus a deformation-theoretic formulation of the rank 4 quadrics problem is the following assertion: *Every deformation of* $\Theta(C)$ *for which all the double points persist as singular points comes from a deformation of the curve* C.

(3) Since the Abel map $\sigma: C_{g-1} \to \Theta$ is a resolution of singularities, one is led to try to compute $\mathcal{E}$ by comparing the 'equisingular deformations' of the singular space Θ with deformations of its resolution C_{g-1} (=the $g-1$ fold symmetric product of the curve.) We may summarize the results in this paper roughly as follows: There exist linear injections:

$$T^1(C) \hookrightarrow \mathcal{E} \hookrightarrow T^1(C_{g-1}).$$

where T^1 denotes the functor of (isomorphism classes of) abstract first order deformations.

(4) Then Kempf's Theorem that $T^1(C_{g-1}) \cong T^1(C)$ forces the inclusion

$$(T^1(C) \cong) T_C\mathcal{J}_g \subset \cap(Q_p)^\perp (= \mathcal{E})$$

to be an equality, which would finish the argument.

The actual argument in the paper is organized as follows. Since we have not yet provided the foundations for equating the spaces $\cap(Q_p)^\perp$ and $\mathcal{E}$, we consider instead of $\mathcal{E}$ the kernel of a reduced local Kodaira-Spencer homomorphism called $\bar{s}$. This map $\bar{s}$ measures the infinitesimal effect that deforming Θ in a given tangent direction to $\mathcal{H}_g$ at J, has on the double points of Θ; $\bar{s}$ is defined in the proof of Prop. 1, and part (2) of the proof shows that $\cap(Q_p)^\perp = \ker(\bar{s})$. Thus the actual argument in this paper is to prove the existence of linear injections

$$T^1(C) \hookrightarrow \ker(\bar{s}) \hookrightarrow T^1(C_{g-1}).$$

and then to invoke Kempf's Theorem.

The first injection is provided by the inclusion $T_C\mathcal{J}_g \subset \ker(\bar{s})$ in Prop. 1, part (1). Then, the rank 4 quadrics conjecture is equivalent to the assertion that the first injection is an isomorphism. This equivalence is proved in Prop. 1, part (2). The second injection is obtained as the composition of an injection $\ker(\bar{s}) \hookrightarrow H^1(\mathcal{T}(U))$ (Lemma 32) followed by an injection $H^1(\mathcal{T}(U)) \hookrightarrow T^1(C_{g-1})$ (Cor. 28 and Prop. 29). The intermediate space $H^1(\mathcal{T}(U))$ is the space of locally trivial deformations of the open subset U of Θ, where $U = \Theta - \{\text{rank} \leqslant 3$ double points and points of multiplicity $\geqslant 3\}$ and the map from $\ker(\bar{s})$ is defined by establishing that, for $v \in T_J\mathcal{H}_g$, the vanishing of $\bar{s}(v)$ around a point $p \in U$ is equivalent to the local triviality near p of the deformation of Θ in the direction v. The appearance of $H^1(\mathscr{T}(U))$ as another incarnation of $\mathscr{E}$ is motivated by the fact that rank 4 double points are dense in sg.Θ (Lemma 6), and that, on a $(g-4)$-dimensional singular locus, a rank 4 double point cannot deform nontrivially as a singularity.

Since Kempf's theorem is valid in all characteristics, in this paper we replace the Siegel space $\mathscr{H}_g$ by a fine moduli space $\tilde{\mathscr{A}}_g$ for p.p.a.v.'s, replace Andreotti-Mayer's use of the transcendental heat equations by the algebraic version due to Welters, and prove our deformation-theoretic results also in a characteristic-free way. The arguments and conclusions in this paper are valid at least in all characteristics $\neq 2$. Mark Green was the first to prove [Green] that Kempf's theorem implies the rank 4 quadrics conjecture for all non-hyperelliptic curves over the complex numbers. The innovation in this paper is the use of deformation theory of singularities. Note that if one were to show independently that the inclusion $\ker(\bar{s}) \hookrightarrow T^1(C_{g-1})$ is an isomorphism, it would follow that Kempf's theorem is actually equivalent to the rank 4 quadrics conjecture. We hope that, because of their generality, the methods of this paper may lead to the solution of analogous problems for other abelian varieties, such as Prym varieties.

We have benefited principally from the works of Andreotti-Mayer, Green, Grothendieck, Kempf, Kodaira-Spencer, Mumford, Oort, Oort-Steenbrink, Schlessinger, and Welters, and we are grateful especially to Kempf and Schlessinger for encouragement as well as technical insight. In particular it was Kempf who drew our attention to the fact that the characteristic p case of the rank four quadrics conjecture remained unsolved, and pointed out that his theorem on deformations of symmetric products of curves, a key step in Green's proof over $\mathbb{C}$, held in all characteristics. Since the second author had already proved [V] Prop. 1 below over $\mathbb{C}$, it only remained to substitute Welters' abstract heat equations into his argument, and to prove a comparison theorem for deformations of Θ and C_{g-1}. This solution of the rank 4 quadrics problem was announced in [S-V1] and the methods and proofs were outlined in [S-V2]. Both authors are grateful to the N.S.F. for partial financial support, under grants DMS-8603281 and DMS-8803487, during work on this problem. The second author is grateful also to the Italian C.N.R. and to the Università di Firenze for support and hospitality during the spring of 1986. The first author is grateful for the opportunity to speak about this work and for support from the Universities of Pavia and of Erlangen-Nurenberg in the fall of 1987.

1. Deformation-Theoretic Formulation of the Rank 4 Quadrics Problem

(1.0) Throughout this paper we work over a fixed algebraically closed ground field k of characteristic $\neq 2$. For the convenience of the reader we begin by summarizing some fundamental definitions and terminology concerning abelian varieties, polarizations, and deformation theory. Our primary references for this material are Mumford's *Abelian Varieties* [M2], and Schlessinger's (unpublished but widely circulated) 1964 Harvard thesis [S1], cf. [R]. Since the lemmas and definitions in this subsection are not referred to in the rest of the paper, they are not numbered.

If A is an abelian variety, $\hat{A} = \mathrm{Pic}^0(A)$ is the dual abelian variety, and if D is an ample divisor on A, then the map $\lambda_D: A \to \hat{A}$ is called a *polarization* of A, where $\lambda_D(x) = [\text{divisor class of } D_x - D]$ and $D_x = D + x$. If λ_D is an *isomorphism* then λ_D is called a *principal* polarization of A, and in this case the pair (A, λ) is called a *principally polarized abelian variety* (p.p.a.v.). If (A, λ) is a p.p.a.v. and Θ is an *effective* ample divisor such that $\lambda_\Theta = \lambda$, then Θ is called a *theta-divisor* for (A, λ).

LEMMA. *If A is an abelian variety of dimension g, and if $\Theta \subset A$ is an effective divisor whose g-fold self intersection number is $\Theta^g = g!$, then λ_Θ defines a principal polarization on A, (for which Θ is a theta divisor).*

Proof. By [M2, p. 150] if $L = \mathcal{O}(\Theta)$, then the Euler characteristic $\chi(L) = (\Theta^g/g!) = 1$. Moreover $\lambda_\Theta = \lambda_L$ only depends on L, and if we denote $\mathrm{Ker}(\lambda_L)$ by $K(L)$, then the argument at the bottom of p. 152 in [M2] shows that $K(L)$ is finite. Then since Θ is effective, by [M2, p. 60, Appl. 1, (ii), (iv)], L (and hence also Θ) is ample on A. Then by [M2, pp. 124–5] there is an isomorphism $\hat{A} \cong (A/K(L))$, of the dual variety $\hat{A}$ with the quotient of A by the finite group scheme $K(L)$, which carries λ_L into the quotient map $\lambda: A \to (A/K(L))$. To show that this is an isomorphism and hence that λ_L is a principal polarization on A, it therefore suffices to check that $K(L) \cong \mathrm{spec}(k)$, where k is the ground field. Using [M2, p. 150], we have at least that $\mathrm{degree}(\lambda) = \mathrm{degree}(\lambda_L) = \chi(L)^2 = 1$. To finish the argument we want to appeal to Thm. 1 of [M2, pp. 111–112], so we check the relevant hypotheses. Since L is ample, A is projective, and hence every orbit in A under the action by the finite group $K(L)$ does lie in an affine open subset of A; (just embed A in projective space and choose a hyperplane that misses the finite set). We show next that the action of $K(L)$ on A is free, i.e. that the map $K(L) \times A \to A \times A$, defined by $(x, a) \to (x + a, a)$ is a closed immersion. This follows from decomposing the map into the *inclusion* map $K(L) \times A \to A \times A$, followed by the isomorphism $A \times A \to A \times A$ defined by $(x, a) \mapsto (x + a, a)$, and from the fact that $K(L)$ is a closed subscheme of A [M2, p. 123; Prop., p. 89] which implies that the previous inclusion is a closed immersion. Now it follows from [M2, p. 112] that $K(L) \cong \mathrm{spec}(R)$ where $\dim_k(R) = \mathrm{degree}(\lambda) = 1$. Hence $K(L) \cong \mathrm{spec}(k)$. □

REMARK. As a corollary of the 'theorem of the square' [M2, p. 60, b] any translate of the divisor Θ in the lemma determines the same polarization on A as does Θ.

Assume now that C is a complete, connected, non-singular curve of genus $g \geqslant 1$, let $J = \mathrm{Pic}^0(C)$, and for $d \geqslant 0$, let $W_d(C) = W_d = \{$effective line bundles of degree d on C, i.e. those with $h^0 \geqslant 1\} \subset \mathrm{Pic}^d(C)$.

LEMMA. *If* $\Theta \subset J$ *is any translate in* $\mathrm{Pic}^0(C)$, *of* $W_{g-1} \subset \mathrm{Pic}^{g-1}(C)$, *then the pair* (J, Θ) *determines a unique p.p.a.v. called the* Jacobian variety of C.

Proof. Using the previous lemma and the remark following it, we only need to know that $\Theta^g = g!$, which is proved in [Mat, Appendix]. □

LEMMA. (1) *If* (A, λ_L) *is any p.p.a.v., then* $h^0(L) = 1$, *and* $h^j(L) = 0$, *for* $j \geqslant 1$.

(2) *The data of a p.p.a.v.* (A, λ) *is actually* equivalent *to that of the pair* (A, Θ) *where* Θ *is an effective ample divisor given up to translation, such that* $\lambda_\Theta = \lambda$; *i.e. a theta divisor for a p.p.a.v. is* determined up to translation *by the principal polarization.*

Proof. (1) By hypothesis, L is ample and $K(L)$ is one reduced point. Consequently by Mumford's 'vanishing theorem' [M2, p. 150], L has non-vanishing cohomology precisely in one degree which is denoted $i(L) = \mathrm{index}(L)$. Now for large n, L^n is both ample and effective and so, by [M2, Appl. 1(ii), p. 60], $K(L^n)$ is finite, the vanishing theorem applies also to L^n, and $i(L^n) = 0$ since L^n is effective. Using the corollary on p. 159 of [M2], $i(L) = i(L^n) = 0$, so that L is also effective and $\chi(L) > 0$. By the 'Riemann-Roch' theorem [M2, p. 150], $\chi^2(L) = \deg(\lambda_L) = \#(K(L)) = 1$, and thus $\chi(L) = h^0(L) = 1$. □

(2) We will show that if D, D', are effective ample divisors on A with $h^0(D) = h^0(D') = 1$, and if $\lambda_D = \lambda_{D'}$, then D' is a translate of D; i.e. $D' = D_x = D + x$ for some (unique) x in A. By [M2, (a), p. 60], since $\lambda_D = \lambda_{D'}$, then $\lambda_{D-D'} = 0$, and therefore $D - D'$ represents an element of $\mathrm{Pic}^0(A)$, [M2, p. 74]. Hence by [M2, p. 77] there is an element x in A such that $D_x - D$ and $D' - D$ are linearly equivalent, hence also D' and D_x. Since $h^0(D') = 1$, we conclude that $D' = D_x$, as claimed. □

REMARK. Arguments like the ones above, using the formula $g! \cdot \chi(\mathcal{O}(D)) = D^g$ from [M2, p. 150], prove that the following properties are equivalent for a line bundle L on a g-dimensional abelian variety A:

(i) L defines a principal polarization, i.e. L is ample and λ_L is an isomorphism from A to $\hat{A}$.
(ii) $h^0(L) = 1$ and $h^j(L) = 0$ for all $j \geqslant 1$.
(iii) $h^0(L) \neq 0$ and $\chi(L) = 1$.
(iv) $L = \mathcal{O}(D)$, where D is effective and $D^g = g!$.

We give next a sketch of some fundamental constructions from deformation theory, specifically of T^1, $\mathcal{T}^1$, and $\mathcal{T}$ $(=\mathcal{T}^0)$, and of the relations between them. The basic concept is that of $T^1(X)$, the vector space of isomorphism classes of first order deformations of a scheme X. A first order deformation of X is a *flat* map $\pi\colon \mathbf{X} \to \mathbb{D}$, where $\mathbb{D} = \operatorname{spec}(k[\varepsilon])$, $k[\varepsilon] = k[t]/(t^2)$, together with an isomorphism $\varphi\colon X \to \pi^{-1}(0)$, where $0 \in \mathbb{D}$ denotes the inclusion of the closed subscheme $0 = \operatorname{spec}(k[\varepsilon]/(\varepsilon)) \hookrightarrow \mathbb{D}$. Two such deformations $(\mathbf{X}, \pi, \varphi)$ and $(\mathbf{X}', \pi', \varphi')$ are isomorphic if there exists an isomorphism $F\colon \mathbf{X} \to \mathbf{X}'$, over $\mathbb{D}$, (i.e. such that $\pi' \circ F = \pi$), and which respects the identifications of X with the fibres over 0, (i.e. such that $F \circ \varphi = \varphi'$). $T^1(X)$ is naturally a k-vector space, and in fact also a $\Gamma(X, \mathcal{O}_X)$-module [see S1, p. 31, Thm 1, (i), where in our case the trivial deformation (see definition below) provides a canonical origin for the principal homogeneous space defined by Schlessinger; (the module structure is on p. 18)]. We define $\mathscr{T}^1(X)$, the 'sheaf of first order deformations of X', to be the sheaf associated to the presheaf $(U \subset X) \mapsto T^1(U)$, [S1, (2.2.3) pp. 27–28]. This is a presheaf on X, i.e. restriction of first order deformations is well-defined and functorial since if $\pi\colon \mathbf{U} \to \mathbb{D}$ is a first order deformation of U, and $V \subset U$ is an open subset, we get a restriction $\mathbf{V}$ of the first order deformation to V, just by restricting the sheaf $\mathcal{O}_{\mathbf{U}}$ on U, to the open subset V. The point here is that U and $\mathbf{U}$ have the *same* underlying topological space and that $\mathbf{U}$ differs from U, and $\mathbf{V}$ from V, only by carrying a different structure sheaf. The $\mathcal{O}_X$-module structure on $\mathscr{T}^1(X)$ follows from the module structure on $T^1(X)$. [Although $\mathscr{T}^1(X)$ is called the sheaf of first order deformations, it is necessary to pass to isomorphism classes of deformations to obtain a reasonable set with algebraic structure. We may sometimes omit to mention 'isomorphism classes'.]

The canonical map from global sections of a presheaf to global sections of the corresponding sheaf, gives a fundamental map $T^1(X) \to H^0(X, \mathscr{T}^1(X))$, which is compatible with the $\Gamma(X, \mathcal{O}_X)$- structure. By definition, the kernel of this map consists of the (isomorphism classes of) 'locally trivial first order deformations'. (A first order deformation of Y is trivial if it is isomorphic to the product deformation $Y \times \mathbb{D}$, the fiber product of schemes over $\operatorname{spec}(k)$; hence locally trivial first order deformations of X are those which restrict on some open cover of X to trivial ones.) The computation of these locally trivial deformations follows the lines of classical Kodaira-Spencer theory in the algebraic setting, [S1, (9) pp. 28–29, Th. 1 pp. 31–32]; i.e. if $\mathscr{T}$ $(=\mathscr{T}^0)$ defined by $\mathscr{T}(X) = \mathscr{H}\mathcal{O}\mathscr{M}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X) = \mathscr{D}\mathscr{E}\mathscr{R}_k(\mathcal{O}_X, \mathcal{O}_X)$ is the sheaf of derivations, dual to $\Omega^1_{X/k}$, then there is an isomorphism of $H^1(X, \mathscr{T}(X))$ with the isomorphism classes of locally trivial first order deformations of X, and hence an exact sequence of $\Gamma(X, \mathcal{O}_X)$-modules, and in particular of k-vector spaces:

$$0 \to H^1(X, \mathscr{T}(X)) \to T^1(X) \to H^0(X, \mathscr{T}^1(X))$$

where the right-hand arrow is generally *not* surjective. [S1, pp. 28, 8, 1, 3, 6], [R, Lem. 4.4, p. 68 (p. 99 of vol.)]. One immediate corollary of this sequence is that whenever $\mathscr{T}^1(X) = 0$, then $T^1(X)$ is precisely equal to the cohomology space $H^1(X, \mathscr{T}(X))$. The proof in Lemma 3 that $T^1(U) = 0$ when U is smooth and affine, will imply in particular that when X is a smooth scheme $\mathscr{T}^1(X) = 0$ and hence also $T^1(X) \cong H^1(X, \mathscr{T}(X))$ [Gr1, Cor. 2, p. 13].

Now let $D \subset X$ be a hypersurface in a smooth scheme X. A first order embedded deformation of D in X is a closed subscheme $\mathbf{D} \subset X \times \mathbb{D}$ such that the map p: $\mathbf{D} \to \mathbb{D}$ induced by the projection $X \times \mathbb{D} \to \mathbb{D}$ is a deformation of D, i.e. p is flat and $\mathbf{D} \cap (X \times 0) = D \times 0 = D$.

$$\begin{array}{ccc} \mathbf{D} & \subset & X \times \mathbb{D} \\ {\scriptstyle p}\searrow & & \swarrow \\ & \mathbb{D} & \end{array}$$

Now let $X_\alpha \subset X$ be an open affine subscheme, and assume that $D_\alpha = (D \cap X_\alpha) \subset X_\alpha$ is the corresponding hypersurface in the smooth affine scheme X_α, with ideal $I(D_\alpha) = I$, structure sheaf $\mathcal{O}_{D_\alpha}$, and affine coordinate ring $\mathcal{O}(D_\alpha)$. If $D_\alpha \subset X_\alpha$ is principal, $I(D_\alpha) = (f_\alpha) \subset \mathcal{O}(X_\alpha)$, then a first order embedded deformation of D_α in X_α, $\mathbf{D}_\alpha \subset X_\alpha \times \mathbb{D}$, is also principal with ideal $(f_\alpha + \varepsilon\tilde{g}_\alpha) \subset \mathcal{O}(X_\alpha \times \mathbb{D}) = \mathcal{O}(X_\alpha)[\varepsilon]$, where $\tilde{g}_\alpha \in \mathcal{O}(X_\alpha)$. Then the set of first order embedded deformations of D_α in X_α is an $\mathcal{O}(D_\alpha)$-module. The set of all first order embedded deformations of $D \subset X$ is similarly a $\Gamma(D, \mathcal{O}_D)$-module, and in fact is isomorphic to the $\Gamma(D, \mathcal{O}_D)$-module $H^0(D, \mathcal{N}(D, X))$, the global sections of the normal sheaf (which here is a line bundle) to D in X [see Gr2, Prop. 5.1, p.21, for a general statement; and A, Th. 6.1, p. 27, for a proof in the case where X is $\mathbb{A}^n$]. Consider now the $\mathcal{O}(D_\alpha)$-module of sections $N(D_\alpha/X_\alpha) = \Gamma(D_\alpha, \mathcal{N}(D_\alpha/X_\alpha)) = \mathrm{Hom}_{D_\alpha}(I/I^2, \mathcal{O}(D_\alpha)) \cong \mathrm{Hom}_{X_\alpha}(I, \mathcal{O}(D_\alpha))$ of the normal sheaf $\mathcal{N}(D_\alpha/X_\alpha)$ of D_α in X_α. Then there is a natural sequence of $\mathcal{O}(D_\alpha)$-maps as follows: $N(D_\alpha/X_\alpha) \xrightarrow{\sim} \{$embedded 1st order defs. of D_α in $X_\alpha\} \to T^1(D_\alpha)$, defined by $(f_\alpha \mapsto g_\alpha) \mapsto (f_\alpha + \varepsilon\tilde{g}_\alpha) \mapsto$ (the isomorphism class of the 1st order def. of D_α defined by the embedded deformation with equation $f_\alpha + \varepsilon\tilde{g}_\alpha$), where $\tilde{g}_\alpha$ is a lift of g_α from D_α to X_α. The composite map, from $N(D_\alpha/X_\alpha)$ to $T^1(D_\alpha)$, fits into the following fundamental exact sequence of $\mathcal{O}(D_\alpha)$-module maps [A, p. 32]:

$$0 \to T(D_\alpha) \to T(X_\alpha)_{|D_\alpha} \to N(D_\alpha/X_\alpha) \to T^1(D_\alpha) \to 0$$

whose terms are explicitly given as follows:

$$0 \to \mathrm{Der}_k(\mathcal{O}(D_\alpha), \mathcal{O}(D_\alpha)) \to \mathrm{Der}_k(\mathcal{O}(X_\alpha),$$
$$\mathcal{O}(D_\alpha)) \to \mathrm{Hom}_{X_\alpha}(I, \mathcal{O}(D_\alpha)) \to T^1(D_\alpha) \to 0$$

where the first (non-trivial) map on the left is composition with the canonical surjection of affine coordinate rings $\mathcal{O}(X_\alpha) \to \mathcal{O}(D_\alpha)$, and the second map is restriction of a derivation from $\mathcal{O}(X_\alpha)$ to I. These exact sequences of $\mathcal{O}(D_\alpha)$-

modules are compatible with restrictions and hence yield the following corresponding exact sheaf sequence of $\mathcal{O}_D$-modules:

$$0 \to \mathcal{T}(D) \to \mathcal{T}(X)_{|D} \to \mathcal{N}(D/X) \to \mathcal{T}^1(D) \to 0,$$

which is given explicitly as follows:

$$0 \to \mathcal{DER}_k(\mathcal{O}_D, \mathcal{O}_D) \to \mathcal{DER}_k(\mathcal{O}_X, \mathcal{O}_D) \to \mathcal{HOM}_X(\mathcal{I}_D, \mathcal{O}_D) \to \mathcal{T}^1(D) \to 0.$$

From the theory of Jacobian varieties, we will use the Riemann-Kempf Singularities Theorem that if $\Theta = \Theta(C) \cong W_{g-1}$, then the set of points having multiplicity $\geqslant r+1$ on Θ corresponds precisely to the set W^r_{g-1}, where in general $W^r_d = \{\text{line bundles on } C \text{ of degree } d \text{ and with } h^0 \geqslant r+1\}$. In particular the sets sg.Θ and W^1_{g-1} are isomorphic via translation, and we give them the scheme structure of sg.Θ as the singular scheme of the hypersurface $\Theta \subset J$. [This seems appropriate since we study the behavior of singularities of Θ under deformation off the Jacobi locus, and the alternative 'Brill-Noether' scheme structure on W^r_d makes sense only for subsets of Jacobians. An important part of our argument is the fact that along the set of rank 4 double points of Θ, sg.Θ is a reduced scheme (see Lemma 7 below).] We take for granted moreover that for some suitable integer n, ($n \geqslant 3$, n even and, if the characteristic is positive, n relatively prime to the odd prime characteristic p), there exists a space $\tilde{\mathcal{A}}_g$, parametrizing isomorphism classes of triples $\{(A, \lambda, \alpha)$, $A = g$ dimensional abelian variety, $\lambda =$ a principal polarization, $\alpha =$ a level n structure$\}$, and equipped with a corresponding universal family $\mathfrak{X}$ of principally polarized abelian varieties (with level n structure) over it, and also a family of (symmetric) theta divisors $\boldsymbol{\Theta}$ over it defining the polarizations. Moreover we may choose $\tilde{\mathcal{A}}_g$ and $\mathfrak{X}$ smooth, and may choose the map between them defining the family to be a smooth map. [O, Th. (2.4.1), p. 244; M1, Th. 7.9, p. 139; W, p. 190; O-S, Th. (1.9), p. 163].

We also assume the existence of a smooth space $\tilde{\mathcal{M}}_g$ parametrizing isomorphism classes of smooth connected curves of genus g with level n structure, and a morphism $t: \tilde{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$, the Torelli map, assigning to the isomorphism class (C, α) of a curve with level n structure, the isomorphism class of the associated principally polarized Jacobian variety $(J(C), \lambda, \alpha)$ with level n structure [O-S, Th. 1.8, pp. 162–163]. Moreover $T_{(C,\alpha)}\tilde{\mathcal{M}}_g \cong T^1(C)$, i.e. the level n structure does not appear in the first order geometry of $\tilde{\mathcal{M}}_g$, [O-S, Prop. 2.5, p. 166–7, (n prime to p)]. (We will sometimes omit the level structure and the polarization in the notation for elements of $\tilde{\mathcal{M}}_g$ and $\tilde{\mathcal{A}}_g$.) Let $\mathcal{J}_g$, the 'locus of Jacobians', denote the image of this map, as a constructible subset of $\tilde{\mathcal{A}}_g$, and let $\bar{\mathcal{J}}_g$ denote its Zariski closure in $\tilde{\mathcal{A}}_g$. Since one of our main goals is to obtain a constructive version of Torelli's theorem, we prefer not to assume any form of the global Torelli result. We will however use the infinitesimal result [O-S, S-D] that for every non-hyperelliptic curve C, the differential of t is injective, and there is a natural k-linear identification of $I_2(C)$ with $(T_C\mathcal{J}_g)^{\perp}$ where we use the notation $T_C\mathcal{J}_g = t_*(T_C\tilde{\mathcal{M}}_g)$, the image of the differential of t. (In particular, for every non-hyperelliptic curve C of genus $g \geqslant 3$, $\dim(T_C\mathcal{J}_g) = \dim_C(\tilde{\mathcal{M}}_g) = 3g - 3$.) These identifications are explained more fully below in the proof of Prop. 1. We will also use some further properties of the sheaf $\mathcal{T}^1(\Theta)$ of isomorphism classes of first order deformations of Θ, which will be summarized in the proof of the following proposition.

(1.1) PROPOSITION 1. *If* $\mathcal{J}_g \subset \tilde{\mathcal{A}}_g$ is the locus of Jacobians and $J \in \mathcal{J}_g$ is the Jacobian of a non-hyperelliptic curve C of genus $g \geqslant 4$, then the (local) Kodaira-Spencer homomorphism s: $T_J\tilde{\mathcal{A}}_g \to H^0(\mathcal{T}^1(\Theta))$ defined in (1.2) below induces a map $\bar{s}$: $T_J\tilde{\mathcal{A}}_g \to H^0(\bar{\mathcal{T}}^1(\Theta))$, where $\bar{\mathcal{T}}^1(\Theta)$ denotes the restriction of $\mathcal{T}^1(\Theta)$ to the closed subscheme $\mathrm{sg}.\Theta_{\mathrm{red.}} = (\mathrm{sg}.\Theta)_{\mathrm{red}}$, the reduced singular locus of the theta divisor. Moreover,

(1) $T_C\mathcal{J}_g \subset \ker(\bar{s})$.
(2) The rank 4 quadrics conjecture is true for $C \Leftrightarrow [T_C\mathcal{J}_g = \ker(\bar{s})]$.

[REMARK. If $N_C(\mathcal{J}_g/\tilde{\mathcal{A}}_g) = (T_J\tilde{\mathcal{A}}_g)/(T_C\mathcal{J}_g) =$ the 'normal space in $\tilde{\mathcal{A}}_g$ to the Jacobi locus at C', then it follows from (1) that $\bar{s}$ induces a map

$$\bar{s}\colon I_2(C)^* = N_C(\mathcal{J}_g/\tilde{\mathcal{A}}_g) \to H^0(\bar{\mathcal{T}}^1(\Theta))$$

and the proof shows $(\bar{s}(\lambda))(\xi)(\theta|_{\xi}) = \lambda(Q_{\xi})$, where ξ is in sg.Θ, and where $Q_{\xi} \in (m_{\xi}^2/m_{\xi}^3) \cong S^2T_0^*J$ is the quadratic term at ξ (possibly zero) of a local equation θ for Θ, in J. (This uses the interpretation given below of $\mathcal{T}^1$ as the restricted normal bundle of Θ, hence the dual of the restricted conormal bundle, so that the fiber of $\mathcal{T}^1$ (and of $\bar{\mathcal{T}}^1$) at ξ is $\bar{\mathcal{T}}^1(\Theta)_{|\xi} = (\mathcal{I}_{\Theta}/m_{\xi}\mathcal{I}_{\Theta})^*$. Since $\theta_{|\xi}$ provides a basis of the space $(\mathcal{I}_{\Theta}/m_{\xi}\mathcal{I}_{\Theta})$, it follows that $(\bar{s}(\lambda))(\xi)$ should assign a number to $\theta_{|\xi}$. Moreover since $\bar{\mathcal{T}}^1$ is a line bundle on $\mathrm{sg}.\Theta_{\mathrm{red.}}$, $\bar{s}(\lambda)$ is determined by the map it induces $\mathrm{sg}.\Theta_{\mathrm{red.}} \to \cup\{\bar{\mathcal{T}}^1{}_{|\xi}\}$.) Then the rank 4 quadrics conjecture for C becomes the statement that the map induced by $\bar{s}$ on the normal space to the Jacobi locus at C is injective, i.e. that for $0 \neq v \in N_C(\mathcal{J}_g/\tilde{\mathcal{A}}_g)$ the open set $\{\bar{s}(v)(p) \neq 0\}$ is non-empty. Using Lemma B, Introduction, this suggests that the rk.4 $-$ Q conjecture is the assertion that in every direction normal to Jacobians, some double points of Θ smooth.]

Proof. Recall $\mathcal{T}^1(\Theta) =$ the sheaf of isomorphism classes of first order deformations of Θ. Next we show, since $\Theta \subset J$ is a hypersurface, $\mathcal{T}^1(\Theta)$ is a line bundle supported on sg.Θ, the scheme defined by $\{\theta_{\alpha}, \partial\theta_{\alpha}/\partial z_i, i = 1, \ldots, g\}$, where θ_{α} is a local equation for Θ in an affine open subset J_{α} of J, $z_1, \ldots, z_g$ are uniformizing parameters in J_{α}, and $\partial\theta_{\alpha}/\partial z_i$ are defined by $d\theta_{\alpha} = \Sigma(\partial\theta_{\alpha}/\partial z_i)\,dz_i$. We keep this notation throughout the proof.

LEMMA 2. *If* Θ *is any hypersurface in a smooth scheme* J, *then* $\mathscr{T}^1(\Theta) \cong \mathcal{O}_\Theta(\Theta)_{|\mathrm{sg}.\Theta}$, where $\mathcal{O}_\Theta(\Theta) = \mathcal{O}_J(\Theta)_{|\Theta}$.

Proof. We use the global presentation of $\mathscr{T}^1$ from section (1.0):

$$0 \to \mathscr{T}(\Theta) \to \mathscr{T}(J)_{|\Theta} \xrightarrow{\mu} \mathcal{N}(\Theta/J) \to \mathscr{T}^1(\Theta) \to 0,$$

and we let $\mathscr{I} = \mathscr{I}_\Theta$. The restricted tangent bundle $\mathscr{T}(J)_{|\Theta} = \mathscr{DER}_k(\mathcal{O}_J, \mathcal{O}_\Theta)$ is trivial with local $\mathcal{O}_\Theta$ basis $\{(\partial/\partial z_i)_{|\Theta}\}$, (which we denote by $(\partial/\partial z_i)$), and $\mathcal{N}(\Theta/J) = \mathscr{HOM}_\Theta(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_\Theta) \cong \mathscr{HOM}_\Theta(\mathcal{O}_\Theta(-\Theta), \mathcal{O}_\Theta) \cong \mathcal{O}_\Theta(\Theta)$. Since the map μ is induced by the restriction of a derivation to its action on $\mathscr{I}$, which takes $\partial/\partial z_i$ to $(\theta \mapsto \partial\theta/\partial z_i) = (\partial\theta/\partial z_i)\cdot(\theta \mapsto 1)$, $\mathscr{T}^1(\Theta) \cong \mathrm{coker}(\mu) \cong (\mathcal{O}_\Theta(\Theta))/\{\partial\theta/\partial z_i\}\cdot(\mathcal{O}_\Theta(\Theta)) = \mathcal{O}_\Theta(\Theta)_{|\mathrm{sg}.\Theta}$. □

(1.2) As mentioned above, there is a 'universal' family of theta divisors

$$\begin{array}{ccc} \Theta & \subset & \boldsymbol{\Theta} \\ | & & \downarrow \\ J & \in & \tilde{\mathscr{A}}_g \end{array}$$

parametrized by the fine moduli space $\tilde{\mathscr{A}}_g$. Here $\boldsymbol{\Theta}$ is an effective locally principal divisor in $\mathfrak{X}$ (= the universal p.p.a.v. over $\tilde{\mathscr{A}}_g$), whose scheme-theoretic fibre over (A, λ, α) in $\tilde{\mathscr{A}}_g$ is a (symmetric) theta divisor $\Theta \subset A$ representing the polarization λ. Thus at each A, and in particular at a Jacobian J, we have a Kodaira-Spencer map:

$$\begin{array}{rccc} s\colon & T_J\tilde{\mathscr{A}}_g & \to & H^0(\mathscr{T}^1(\Theta)) \\ & v & \mapsto & \{\boldsymbol{\Theta}|_v\} \end{array}$$

i.e. s factors into: $T_J\tilde{\mathscr{A}}_g \to T^1(\Theta) \to H^0(\mathscr{T}^1(\Theta))$, the restriction $[v \mapsto \{\boldsymbol{\Theta}|_v\}]$ followed by the map from T^1 to sections of $\mathscr{T}^1$. Since $\mathscr{T}^1(\Theta)$ is a line bundle on the scheme sg.Θ, s defines a linear system, and hence a rational map from sg.Θ to $\mathbb{P}T_J^*\tilde{\mathscr{A}}_g$. We want to give a formula equating this map with the 'Gauss map' of the family $\boldsymbol{\Theta}$. Choose a cover of $\Theta = \Theta(J)$ by (smooth) affine open sets $\mathfrak{X}_\alpha$ of $\mathfrak{X}$, and local equations $\boldsymbol{\theta}_\alpha$ defining $\boldsymbol{\Theta}_\alpha = \boldsymbol{\Theta} \cap \mathfrak{X}_\alpha$ in $\mathfrak{X}_\alpha$. Furthermore, put $J \cap \mathfrak{X}_\alpha = J_\alpha$ and $\Theta_\alpha = \Theta \cap \mathfrak{X}_\alpha$ so that $\Theta_\alpha \subset J_\alpha$ is a hypersurface in a smooth affine variety and $\boldsymbol{\Theta}_\alpha$ is a deformation of the affine open set $\Theta_\alpha \subset \Theta$. Then we have:

LEMMA 3. *For any* $v \in T_J\tilde{\mathscr{A}}_g$, $s(v) \in H^0(\Theta, \mathscr{T}^1(\Theta))$ *is given locally by the directional derivative of an equation for* $\boldsymbol{\Theta}$; *i.e. in the presentation*

$$0 \to T(\Theta_\alpha) \to T(J_\alpha)_{|\Theta_\alpha} \to N(\Theta_\alpha/J_\alpha) \xrightarrow{\beta} T^1(\Theta_\alpha) \to 0$$

of $T^1(\Theta_\alpha) = \Gamma(\Theta_\alpha, \mathscr{T}^1(\Theta_\alpha))$, we have $s(v)_{|\Theta_\alpha} = \beta(\theta_\alpha \mapsto D_v(\boldsymbol{\theta}_\alpha)_{|\Theta_\alpha})$.

Proof. We may assume $v \neq 0$. Embed $\mathbb{D}$ in $\tilde{\mathscr{A}}_g$ using v, and restrict $\boldsymbol{\Theta}_\alpha$ and $\mathfrak{X}_\alpha$ to $\mathbb{D}$, yielding a hypersurface $\boldsymbol{\Theta}'_\alpha = \boldsymbol{\Theta}_\alpha|_{\mathbb{D}}$ $(= s(v)_{|\Theta_\alpha})$ of $\mathfrak{X}'_\alpha = \mathfrak{X}_\alpha|_{\mathbb{D}}$.

Claim: $\mathfrak{X}'_\alpha \cong J_\alpha \times \mathbb{D}$ (as deformations of J_α over $\mathbb{D}$). Since $\mathfrak{X}'_\alpha$ is smooth and affine over $\mathbb{D}$, with J_α as central fiber, it suffices to show that $T^1(U) = 0$ for any smooth affine variety $U \subset \mathbb{A}^n$. Consider the sequence

$$0 \to I/I^2 \to (\Omega^1_{\mathbb{A}^n})_{|U} \to \Omega^1_U \to 0$$

of $\mathscr{O}(U)$-modules, where I is the ideal of U in $\mathbb{A}^n$, and Ω^1_U is the locally free module of Kähler differentials. This sequence is exact by [H, Th. 8.17, p. 178] and, since Ω^1_U is a projective $\mathscr{O}(U)$-module, also split-exact. Then $\mathrm{Hom}_{\mathscr{O}(U)}(\cdot, \mathscr{O}(U))$ preserves exactness, and yields the claim as follows:

$$0 \to T(U) \to T(\mathbb{A}^n)_{|U} \to N(U/\mathbb{A}^n) \to 0 = T^1(U).$$

Thus $\boldsymbol{\Theta}'_\alpha \subset J_\alpha \times \mathbb{D}$ is defined by $\theta_\alpha + \varepsilon \tilde{f}_\alpha$ for some $\tilde{f}_\alpha$ on J_α, whence $D_v(\boldsymbol{\theta}_\alpha) = (\mathrm{d}/\mathrm{d}\varepsilon)(\theta_\alpha + \varepsilon\tilde{f}_\alpha) = \tilde{f}_\alpha$. Thus $s(v)_{|\Theta_\alpha} = \beta(\tilde{f}_{\alpha|\Theta_\alpha}) = \beta(D_v(\boldsymbol{\theta}_\alpha)|_{\Theta_\alpha})$. □

(1.3) *Proof of Prop.* 1(1). Let $\bar{s}$: $T_J\tilde{\mathscr{A}}_g \to H^0(\bar{\mathscr{T}}^1)$ be the composition of s with the restriction map $H^0(\mathscr{T}^1(\Theta)) \to H^0(\bar{\mathscr{T}}^1)$. By Lemmas 2 and 3, the rational map γ: $\mathrm{sg}.\Theta_{\mathrm{red.}} \to \mathbb{P}T^*_J\tilde{\mathscr{A}}_g$ associated to $\bar{s}$ is $\xi \mapsto \gamma(\xi)$ where $\ker \gamma(\xi) = \{v : \bar{s}(v)(\xi) = 0\} = \{v \in T_J\tilde{\mathscr{A}}_g : D_v(\boldsymbol{\theta}_\alpha)(\xi) = 0\}$, the Gauss map γ of the family $\boldsymbol{\Theta}$; i.e. $\bar{s}(v)(\xi) = 0 \Leftrightarrow \gamma(\xi)(v) = 0$. Hence $T_C\mathscr{J}_g \subset \ker(\bar{s})$ if and only if $\gamma(\mathrm{sg}.\Theta_{\mathrm{red.}}) \subset \mathbb{P}N^*_C(\mathscr{J}_g/\tilde{\mathscr{A}}_g) = \mathbb{P}(T_C\mathscr{J}_g)^\perp$, the (projectivized) conormal space to Jacobians at C. We prove $\gamma(\mathrm{sg}.\Theta_{\mathrm{red.}}) \subset \mathbb{P}(T_C\mathscr{J}_g)^\perp$ as follows: the natural isomorphism

$$\begin{array}{l} \mathbb{P}T^*_J\tilde{\mathscr{A}}_g = \text{projectivized cotangent space to } \tilde{\mathscr{A}}_g \text{ at } J \\ \quad \wr\| \\ \mathbb{P}S^2T^*_0J = \{\text{quadrics in } \mathbb{P}^{g-1}\} \end{array}$$

induces an isomorphism from $\mathbb{P}(T_C\mathscr{J}_g)^\perp$ to $\mathbb{P}I_2(C)$ [cf. O-S]:

$$\begin{array}{ccc} & \mathbb{P}T^*_J\tilde{\mathscr{A}}_g & \supset \mathbb{P}(T_C\mathscr{J}_g)^\perp \\ & \wr\| & \wr\| \\ \{\text{quadrics in } \mathbb{P}^{g-1}\} = & \mathbb{P}S^2T^*_0J & \supset \mathbb{P}I_2(C) = \{\text{quadrics containing } \Phi_K(C)\} \end{array}$$

Then there is the following diagram relating two morphisms of $\mathrm{sg}._2\Theta$ to these projectivized dual spaces,

$$\begin{tikzcd}[row sep=small] & \mathbb{P}T_J^*\tilde{\mathscr{A}}_g \arrow[d, phantom, "\wr\|"] & \supset & \mathbb{P}(T_C\mathscr{J}_g)^\perp \arrow[d, phantom, "\wr\|"] \\ \mathrm{sg}._2\Theta \arrow[ur, "\gamma"] \arrow[dr, "\varphi"'] & & & \\ & \mathbb{P}S^2T_0^*J & \supset & \mathbb{P}I_2(C) \end{tikzcd}$$

where $\varphi(\xi) = (\partial^2\theta_\alpha/\partial z_i\,\partial z_j(\xi)) = Q_\xi$. Welters' heat equations [W, 3.6, p. 190] imply that γ, like φ, is defined on $\mathrm{sg}._2\Theta$ and that this diagram commutes. Hence to show that $\gamma(\mathrm{sg}._2\Theta) \subset \mathbb{P}(T_C\mathscr{J}_g)^\perp$ it suffices to show that $\varphi(\mathrm{sg}._2\Theta) \subset \mathbb{P}(I_2(C))$, i.e. that

$$\xi \in \mathrm{sg}._2\Theta \Rightarrow \Phi_K(C) \subset Q_\xi \text{ in } \mathbb{P}^{g-1}.$$

This is an immediate corollary of the Riemann-Kempf representation [K1], $Q_\xi = \cup\bar{D}$, union over all divisors $D \in |\mathscr{L}_\xi|$, where $\mathscr{L}_\xi$ is the line bundle corresponding to ξ under the (unique) translation-isomorphism $\Theta \cong W_{g-1} \subset \mathrm{Pic}^{g-1}(C)$. Q.E.D. for Prop. 1(1).

Proof of Prop. 1(2). The rank 4 quadrics conjecture is true for $C \Leftrightarrow \varphi(\mathrm{sg}._2\Theta)$ generates $\mathbb{P}I_2(C) \Leftrightarrow$ (by commutativity of the diagram) $\gamma(\mathrm{sg}._2\Theta)$ generates $\mathbb{P}(T_C\mathscr{J}_g)^\perp \Leftrightarrow \cap\{\ker\gamma(\xi)\colon\ \xi \in \mathrm{sg}._2\Theta\} = T_C\mathscr{J}_g$. Since for $v \in T_J\tilde{\mathscr{A}}_g$, $\gamma(\xi)(v) = 0 \Leftrightarrow \bar{s}(v)(\xi) = 0$, it follows that if $\gamma(\xi)(v) = 0$ for all $\xi \in \mathrm{sg}._2\Theta$, then $\bar{s}(v)(\xi) = 0$ also for all $\xi \in \mathrm{sg}._2\Theta$. Moreover, Welters' heat equations [W, 3.6, p. 190] imply as well $\bar{s}(v)(\xi) = 0$ for all $\xi \in (\mathrm{sg}.\Theta - \mathrm{sg}._2\Theta)$. Thus $\cap\{\ker\gamma(\xi)\colon \xi \in \mathrm{sg}._2\Theta\} = \ker(\bar{s})$. Q.E.D. for Prop. 1(2).

(1.4) By Proposition 1 the next result is the main goal of this paper:

THEOREM 4. *For C any non-hyperelliptic curve of genus $g \geqslant 5$, if $\bar{s}\colon T_J\tilde{\mathscr{A}}_g \to H^0(\bar{\mathscr{T}}^1(\Theta))$ is the reduced Kodaira-Spencer map defined in sections* (1.2) *and* (1.3) *above, then the following equality holds*:

$$T_C\mathscr{J}_g = \ker(\bar{s}).$$

Proof. The proof of Theorem 4, which is concluded just before Corollary 33, will occupy the remainder of this paper. We first show it suffices to consider double points of rank exactly 4 on Θ. Thus we let $U = \Theta - B$, where $B = \{$all points of multiplicity $\geqslant 3$ and all rank 3 double points on $\Theta\}$. We use this notation frequently in the remainder of the paper.

LEMMA 5. *Under the hypotheses of Theorem* 4,

(i) *the set* $U = \Theta - B$ *is open and dense in* Θ;
(ii) *the set* $\text{sg.}_2\Theta = \{$*double points of* $\Theta\}$ *is open and dense in* $\text{sg.}\Theta$.

Proof of (i). Since $\Theta = W_{g-1}$ is the image of C_{g-1} under the Abel map, Θ is irreducible, and it is reduced by definition. Thus $\bar{B}$, a closed subset of sg.Θ, is a proper subset of Θ [H, II.8.16], hence has open dense complement, so we need to show $B = \bar{B}$ is closed. Consider the map $\varphi: \xi \mapsto Q_\xi$ on $\text{sg.}\Theta_{\text{red}}$, which is seen (in char. $\neq 2$), to be regular by representing it locally by the second partials of an affine equation for Θ. Then $B \subset \text{sg.}\Theta$ is precisely the closed subset of sg.Θ defined as the inverse image of the set of quadrics of rank $\leqslant 3$, (including the 'zero quadric'). Thus U is open and dense in Θ. Q.E.D.(i)

Proof of (ii). Pulling back the open set $\{Q: Q \neq 0\}$ by the map $\varphi: \xi \mapsto Q_\xi$ we see that double points are open in sg.Θ. To show density we use the geometric RRT as in [A-M, Prop 8b, p. 209]; i.e. if $L = g^r_{g-1} \in W^1_{g-1} \cong \text{sg.}\Theta$ is any point, $(r \geqslant 1)$, then $D \in |L|$ is a divisor of $g - 1$ points spanning a linear space in $\mathbb{P}^{g-1}$ of codimension $r + 1$. By the Riemann-Kempf Singularities Theorem it suffices to show that D is a limit of effective divisors E with dim. $|E|$ equal to one. Consider any point p that appears multiply in D and replace p by a nearby point $q \in C$ not in $\overline{D - p}$ = the linear span of $D - p$ in canonical space $\mathbb{P}^{g-1}$. Since m_p^{k-1}/m_p^k has dimension one for $p \in C$, removing p lowers dim. $\bar{D}$ by at most one, hence dim. $\overline{D - p + q} \geqslant \text{dim.}\,\bar{D}$, and therefore dim. $|D - p + q| \leqslant \text{dim.}\,|D|$. Continuing, we either reach a nearby divisor of degree $g - 1$ and dimension 1, or we get a divisor E of degree $g - 1$ and projective dimension > 1, and made up of $g - 1$ distinct points. If the latter occurs, choose a divisor S, $0 \leqslant S \leqslant E$, consisting of at most $g - 3$ independent points, and with $\bar{S} = \bar{E}$. Consider the complementary divisor $E - S$ and discard the first point of $E - S$, replacing it by any nearby point $v \in C$ with $v \notin \bar{S}$. Then replace the second point of $E - S$ by a nearby point not in the span of $S \cup \{v\}$. Continuing this with all but one of the points of $E - S$ finishes the argument, obtaining a g^1_{g-1} with distinct points in this case. Thus double points are dense in sg.Θ. Q.E.D. for Lemma 5.

LEMMA 6. *With the hypotheses of Theorem* 4, *then in fact* sg. $U = \{$rank 4 *double points of* $\Theta\}$ *is open and dense in* sg. Θ.

Proof. [Note that the Jacobian of a non-hyperelliptic genus 4 curve with an effective even theta characteristic has a theta-divisor with exactly one double point, which is of rank 3, so this lemma fails in genus 4.] Since U is open and dense in Θ, sg. $U = U \cap \text{sg.}\,\Theta$ is open in sg. Θ. Since sg. $U = \{$rank 4 double points of $\Theta\}$ and $\text{sg.}_2\Theta$ is open dense in sg.Θ, to show density of sg. U in sg. Θ, it suffices to show that the set of rank 3 double points of Θ is nowhere dense in sg. Θ. We will use a dimension count. First we recall how to see that for C non-hyperelliptic of genus $g \geqslant 4$, sg. Θ is nonempty of pure dimension $g - 4$. We will use the Abel surjection $\sigma: C^1_{g-1} \twoheadrightarrow W^1_{g-1}$ and [K2, first theorem, p. 9] applied to the bundle map defined by the derivative $\sigma_*\colon \mathscr{T}(C_{g-1}) \to \sigma^*\mathscr{T}(J)$. In Kempf's notation we have $l = 1, g = g, f = g - 1$, and want to conclude that $Z_1(n) = \{$set where corank $\sigma_* \geqslant 1\}$ (by defn.) $= C^1_{g-1}$ (by non-singularity of the fibers of σ), is Cohen-Macaulay of pure codimension 2 in C_{g-1}. We must check his hypothesis that codim. $C^1_{g-1} \geqslant 2$ in C_{g-1}. Let Y be any component of C^1_{g-1}. By the proof of (our) Lemma 5, divisors D with $\dim.|D| = 1$ are dense in Y, whence $(\dim. Y) - 1 = \dim. \sigma(Y) \leqslant g - 4$, by [S-D, Thm. (2.4), p. 162]. Thus $\dim. Y \leqslant g - 3$, i.e. codim. $Y \geqslant 2$ in C_{g-1}. Now [K2, p. 9] does apply and says that for every component Y of C^1_{g-1}, $\dim. Y = g - 3$ and hence $\dim. \sigma(Y) = g - 4$. Thus every component of sg.Θ has dim. $= g - 4$. Finally, sg.Θ is nonempty by [K2, Cor. of Thm. 3, p. 15].

Now to finish the proof that rânk 4 double points are dense in sg.Θ for $g \geqslant 5$, we will check that rank 3 double points are contained in a subset of dimension $\leqslant g - 5$. We use the arguments in [M3, pp. 346–347] and [A-M, Lemma 4, p. 192] as follows: we will show as in [A-M] that every line bundle $L \in W^1_{g-1}$ at which the tangent cone is a rank 3 quadric has the form $L = M + N$ where M is another line bundle with $h^0 \geqslant 2$, and both N and $K - 2M - N$ are effective. Then Mumford's argument, using Clifford's Theorem, will bound the dimension of the set of such L. So let $L \in W^1_{g-1}$ be a rank 3 double point of W_{g-1}. Then the tangent cone

$$Q_L = \bigcup_{t \in |L|} \bar{E}_t.$$

to W^1_{g-1} at L is, by Riemann-Kempf, a union of the linear $g - 3$ dimensional spans of the divisors of $|L|$, and has equation $ab - c^2 = 0$. Since this quadric has only one family of such linear spaces, the $\bar{E}_t$ may be represented as follows: there is a unique linear space, the vertex of the quadric, of codimension 3 in $\mathbb{P}^{g-1}$, and a plane conic Z in $\mathbb{P}^{g-1}$, such that the spaces $\bar{E}_t$ are those spanned by the vertex plus a single point t of Z. Hence for a generic point t of Z the moving part of the divisor E_t is that part of E_t supported outside the vertex of Q, and the fixed divisor is the part supported in the vertex. If we denote by x the fixed part, then $E_t = D_t + x$, and so if $M = \mathcal{O}(D_t)$, and $N = \mathcal{O}(x)$, then $L = M + N$, and $D_t = \varphi^*(p)$ under the map $\varphi\colon C \to Z$ defined by M, and extending the projection from the vertex. Now if we choose a line tangent to Z at the point t and pull it back by φ, we are pulling back the Cartier divisor $2p$, while the line itself pulls back under projection to a hyperplane H through the vertex of Q. Thus this hyperplane cuts a canonical divisor on C that dominates $\varphi^*(2p) = 2D_t$. Since H contains $\bar{E}_t$ this canonical divisor dominates E_t and hence also x. Thus $(K - 2D_t - x)$ is effective.

Next we use the previous representation to bound the dimension of the set of rank 3 double points. Fix an integer d with $3 \leqslant d \leqslant g - 1$ and consider the subset: $\Sigma_d \subset (C_{g-1-d} \times C_{g-1-d} \times W_d^1)$ where $\Sigma_d = \{(x, y, M): h^0(M) \geqslant 2, x + y \in |K - 2M|\}$. Now let $\Sigma = \cup_d \Sigma_d$ and define the map $\Sigma \to W_{g-1}^1$ by $(x, y, M) \mapsto M(x)$. We have just shown that the image of this map contains the rank 3 double point locus, and we will show next that for every d, $\dim. \Sigma_d \leqslant g - 5$. If $d = g - 1$ we define C_0 to consist of one point, the divisor zero. Then for all (x, y, M) in Σ_{g-1}, $M^2 = K$ so there is only a finite set of such M, and a unique choice of x and y, (both zero), so Σ_{g-1} has dimension $= 0 \leqslant g - 5$ since we have assumed that $g \geqslant 5$. Now assume that $d < g - 1$ and project Σ_d to the factor W_d^1, $(x, y, M) \mapsto M$. By [S-D, Thm. (2.4)] we know $\dim. W_d^1 \leqslant d - 3$, since C is non-hyperelliptic. To bound the dimension of the fiber of the projection, it suffices to estimate the dimension of $|K - 2M|$. By Clifford, $\dim. |K - 2M| \leqslant (1/2)\deg.(K - 2M) - 1 = g - 2 - d$, again since C is non-hyperelliptic. Thus, $\dim.(\Sigma_d) \leqslant \dim. W_d^1 + \dim. |K - 2M| \leqslant d - 3 + g - 2 - d = g - 5$. □

LEMMA 7. *If* $U = \{$*smooth points and rank* 4 *double points of* $\Theta\}$, *as above, then* sg. U *is reduced and in fact smooth, in its natural scheme structure defined locally by the partials of an equation for* Θ.

Proof. For $p \in \text{sg.}\, U$, $T_p(\text{sg.}\, U) = T_p(\text{zero scheme of } \{\theta, \partial\theta/\partial z_i\}) =$ the linear subspace of $T_p J$ defined by $\{d(\partial\theta/\partial z_i)_p\} =$ kernel of the matrix $(\partial^2\theta/\partial z_i\, \partial z_j(p))$ $=$ vertex of Q_p. By definition of U, $\operatorname{rank} Q_p = 4 = \text{codim. sg.}\, U$ in J, so the Jacobian criterion applies. □

REMARK (Kempf). Lemma 7 follows (also in characteristic 2) from the existence of a 2×2 determinantal equation for Θ near a double point [K3].

2. Deformation theory

(2.1) First order deformations of hypersurfaces.

We have discussed deformations of a single scheme. To study the first order geometry of parameter spaces of principally polarized abelian varieties we must extend the treatment to cover schemes with added structure. We give next the definition of $T^1(X, D)$, the vector space of isomorphism classes of first order deformations of a pair of schemes (X, D), where $D \subset X$ is a closed subscheme of a scheme X.

A first order deformation of the pair (X, D) is a commutative diagram

$$\begin{tikzcd} \mathbf{D} \arrow[dr, "\mathrm{p}"'] & \subset & \mathbf{X} \arrow[dl, "\pi"] \\ & \mathbb{D} & \end{tikzcd}$$

where $\mathbf{D} \subset \mathbf{X}$ is a closed subscheme and $\pi: \mathbf{X} \to \mathbb{D}$ and $p = \pi|_{\mathbf{D}}: \mathbf{D} \to \mathbb{D}$ are flat maps, together with an isomorphism of pairs $\varphi: (X, D) \to (\pi^{-1}(0), p^{-1}(0))$, i.e. an isomorphism of schemes $\varphi: X \to \pi^{-1}(0)$ which carries the closed subscheme D isomorphically onto $p^{-1}(0)$. In analogy with the earlier notation for a first order deformation of X as $(\mathbf{X}, \pi, \varphi)$, we denote a first order deformation of a pair as $((\mathbf{X}, \mathbf{D}), \pi, \varphi)$. An isomorphism $((\mathbf{X}, \mathbf{D}), \pi, \varphi) \xrightarrow{\sim} ((\mathbf{X}', \mathbf{D}'), \pi', \varphi')$ of first order deformations of (X, D) is an isomorphism $F: \mathbf{X} \to \mathbf{X}'$ such that $\pi' \circ F = \pi$, F takes $\mathbf{D}$ isomorphically onto $\mathbf{D}'$, and $F \circ \varphi = \varphi'$.

Now let (X, D) denote a pair consisting of a smooth scheme X and a hypersurface $D \subset X$, let $T^1(X, D) = \{$isomorphism classes of first order deformations of the pair $(X, D)\}$, and let $T^1(D) = \{$isomorphism classes of first order deformations of the abstract hypersurface $D\}$. We will investigate when we can forget about X without changing the first order deformations, i.e. we will give a criterion for obtaining a natural isomorphism $T^1(X, D) \cong T^1(D)$. To do this we equate these first order deformation sets with the hypercohomology groups of certain 'tangent complexes' and then compare those cohomology groups. This approach to $T^1(X, D)$ occurs in various places in the literature, for instance [*C*] and [*W*], as well as [Ka], and we are grateful to Mike Schlessinger for sharing his insights into this point of view for $T^1(D)$.

To compare $T^1(X, D)$ with $T^1(D)$, we appeal to the following result.

PROPOSITION 8. *For any smooth scheme X and any hypersurface $D \subset X$, there exists a commutative diagram:*

$$\begin{array}{ll} \alpha: T^1(X, D) \to & \mathbb{H}^1(\mathscr{T}(X) \to \mathscr{N}(D/X)) \\ \qquad \downarrow & \qquad\qquad \downarrow \\ \beta: T^1(D) \quad \to & \mathbb{H}^1(\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X)) \end{array}$$

in which the left vertical map is the map forgetting X, the right vertical map is induced by the natural map (res, id) = (restriction, identity) of complexes, and the horizontal maps α, β are natural isomorphisms.

COROLLARY 9. *If $H^1(\mathscr{I}_D \cdot \mathscr{T}(X))$ and $H^2(\mathscr{I}_D \cdot \mathscr{T}(X))$ are both* 0, *then the map forgetting X: $T^1(X, D) \to T^1(D)$ is an isomorphism.*

Proof. By Proposition 8 it suffices to show that the corresponding map $\mathbb{H}^1(\mathscr{T}(X) \to \mathscr{N}(D/X)) \to \mathbb{H}^1(\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X))$ is an isomorphism. Recall that this hypercohomology map is induced by the natural map (res, id) of two-step complexes:

$$(\mathscr{T}(X) \to \mathscr{N}(D/X)) \to (\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X))$$

which embeds in the following short exact sequence of complexes:

$$\begin{aligned}0 &\to (\mathscr{I}_D \cdot \mathscr{T}(X) \to 0) \to (\mathscr{T}(X) \to \mathscr{N}(D/X))\\ &\to (\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X)) \to 0,\end{aligned}$$

whose left hand map is given by the obvious pair of inclusions. This yields the long exact sequence of hypercohomology:

$$\begin{aligned}\cdots\cdots\ &\mathbb{H}^1(\mathscr{I}_D \cdot \mathscr{T}(X) \to 0) \to \mathbb{H}^1(\mathscr{T}(X) \to \mathscr{N}(D/X)) \to\\ &\to \mathbb{H}^1(\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X)) \to \mathbb{H}^2(\mathscr{I}_D \cdot \mathscr{T}(X) \to 0) \to \cdots\end{aligned}$$

(The existence of this long exact sequence is implied by [C-E, Prop. 2.3, p. 80], since $\mathbb{H}$ can be computed as the total cohomology of the double complex given by Γ of an injective resolution of the two-step complex of sheaves.) Since $\mathbb{H}^i(\mathscr{I}_D \cdot \mathscr{T}(X) \to 0) \cong H^i(\mathscr{I}_D \cdot \mathscr{T}(X))$, our hypothesis implies that the two extreme cohomology groups in the portion of the long exact sequence displayed above are zero, hence the map of the middle two groups is an isomorphism. □

Sketch of Proof of Prop. 8. Since the proof is a straightforward but lengthy sequence of verifications with Čech representatives for elements of $\mathbb{H}^1$, we will limit ourselves to giving the definitions of the maps α, β. We define β first:

$$\beta\colon T^1(D) \to \mathbb{H}^1(\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X))$$

The complex $\mathscr{T}(X)_{|D} \to \mathscr{N}(D/X)$ is given by the natural map $\mathscr{DER}_k(\mathscr{O}_X, \mathscr{O}_D) \to \mathscr{HOM}_D(\mathscr{I}_D/\mathscr{I}_D^2, \mathscr{O}_D)$, which restricts a derivation from $\mathscr{O}_X$ to $\mathscr{I}_D$. (By the Leibniz rule it kills $\mathscr{I}_D^2$). To represent cohomology, we use the Čech complex. We only need enough to get a grip on $\mathbb{H}^1$:

$$\begin{array}{ccc} C^1(\{D_{\alpha\beta}\}, \mathscr{T}(X)_{|D}) & \to & C^1(\{D_{\alpha\beta}\}, \mathscr{N}(D/X)) \\ \uparrow & & \uparrow \\ C^0(\{D_\alpha\}, \mathscr{T}(X)_{|D}) & \to & C^0(\{D_\alpha\}, \mathscr{N}(D/X)) \end{array}$$

Here $\{D_\alpha\} = \{D \cap X_\alpha\}$ is an affine cover of D which is the restriction of an affine cover $\{X_\alpha\}$ of X such that the ideal $I_\alpha = (\theta_\alpha)$ of D_α is principal. The vertical maps are the Čech differentials, and the horizontal maps are obtained by applying $\mathscr{O}_D$-valued derivations of $\mathscr{O}_X$ to a local generator θ_α of $\mathscr{I}_D$.

Given the divisor $D \subset X$ and a (first order) deformation of D as abstract variety:

$$\begin{array}{ccc} D & \subset & \mathbf{D} \\ \downarrow & & \downarrow \\ 0 & \in & \mathbb{D} \end{array}$$

where $\mathbb{D} = \operatorname{spec}(k[\varepsilon])$, we want to produce an element in $\mathbb{H}^1$ of the complex.

(2.1.1) We need a pair $(\chi_{\alpha\beta}, v_\alpha)$, where $v \in C^0(\{D_\alpha\}, \mathcal{N}(D/X))$, and $\chi \in Z^1(\{D_{\alpha\beta}\}, \mathcal{T}(X)_{|D})$, and such that these two elements have the same image in $C^1(\{D_{\alpha\beta}\}, \mathcal{N}(D/X))$. In particular we need for all α, β:

(i) $\chi_{\alpha\beta} \in \operatorname{Der}_k(R_{\alpha\beta}, S_{\alpha\beta})$ (where $R_{\alpha\beta}$ = coord. ring of $X_{\alpha\beta}$, $S_{\alpha\beta}$ = coord. ring of $D_{\alpha\beta}$), and $\chi_{\alpha\beta} + \chi_{\beta\gamma} = \chi_{\alpha\gamma}$.
(ii) $v_\alpha \in \operatorname{Hom}_{S_\alpha}(I_\alpha/I_\alpha^2, S_\alpha)$, (where I_α = ideal of D_α in X_α). such that:
(iii) $\chi_{\alpha\beta}$ and $v_\beta - v_\alpha$ have the same value on a generator in $I_{\alpha\beta}$.

To get v, we use Grothendieck's theory of the Hilbert scheme and Schlessinger's theory of deformations of singular schemes [Gr2, Prop. 5.1, p. 21; S1, Th. 1(ii), p. 32, (8) p. 28] asserting that for X_α smooth and affine, and $D_\alpha = D \cap X_\alpha$, the following diagram commutes, with natural vertical isomorphisms, and the forgetful map across the bottom.

$$\begin{array}{ccc} H^0(D_\alpha, \mathcal{N}(D/X)) & \twoheadrightarrow & H^0(D_\alpha, \mathcal{T}^1(D)) \\ \uparrow & & \uparrow \\ \{\text{emb. defs.}/\mathbb{D} \text{ of } D_\alpha \subset X_\alpha\} & \twoheadrightarrow & \{\text{abstr. defs.}/\mathbb{D} \text{ of } D_\alpha, \text{ up to iso.}\} \end{array}$$

The bottom surjection gives us embeddings over $\mathbb{D}$, $\tilde{f}_\alpha\colon \mathbf{D}_\alpha \hookrightarrow X_\alpha \times \mathbb{D}$, and their images by the left vertical arrow yield candidates for the family $\{v_\alpha\}$.

(2.1.2) To see the $\{v_\alpha\}$ explicitly, we pass to the ring level. Let R_α = coord. ring of X_α, S_α = coord. ring of D_α, $\mathcal{S}_\alpha$ = coord. ring of $\mathbf{D}_\alpha$, and $R_\alpha[\varepsilon]$ = coord. ring of $X_\alpha \times \mathbb{D}$. Then the inclusion $D_\alpha \subset X_\alpha$ is given by a k-algebra surjection $0 \to (\theta_\alpha) \to R_\alpha \to S_\alpha \to 0$, (where $(\theta_\alpha) = I_\alpha$ = ideal of D_α), and the embedding $\tilde{f}_\alpha$ corresponds to a $k[\varepsilon]$-algebra surjection $0 \to (\theta_\alpha + b_\alpha\varepsilon) \to R_\alpha[\varepsilon] \to \mathcal{S}_\alpha \to 0$, which reduces mod ε to the previous map, (and where $(\theta_\alpha + b_\alpha\varepsilon) = \mathcal{I}_\alpha$ is the ideal of $\tilde{f}_\alpha(\mathbf{D}_\alpha)$). Now $H^0(D_\alpha, \mathcal{N}(D/X)) = \operatorname{Hom}_{S_\alpha}(I_\alpha/I_\alpha^2, S_\alpha) \cong \operatorname{Hom}_{R_\alpha}(I_\alpha, S_\alpha)$. Since $I_\alpha = (\theta_\alpha)$, v_α is defined to be the map sending θ_α to $b_\alpha (\operatorname{mod} I_\alpha) \in S_\alpha$.
(2.1.3) To get $\chi_{\alpha\beta}$, consider the pair of embeddings $\mathbf{D}_\alpha \hookrightarrow X_\alpha \times \mathbb{D}, \mathbf{D}_\beta \hookrightarrow X_\beta \times \mathbb{D}$,

and the two resulting restrictions to $\mathbf{D}_{\alpha\beta}$ which give a partially non-commutative diagram as follows:

$$(*)\qquad \begin{array}{ccccccc} D_{\alpha\beta} & \subset & \mathbf{D}_{\alpha\beta} & \hookrightarrow & X_{\alpha\beta} \times \mathbb{D} & \to & X_{\alpha\beta} \\ \| & & \| & & \| & & \| \\ D_{\beta\alpha} & \subset & \mathbf{D}_{\beta\alpha} & \hookrightarrow & X_{\beta\alpha} \times \mathbb{D} & \to & X_{\beta\alpha} \end{array}$$

since on the overlaps $\mathbf{D}_{\alpha\beta}$, the embeddings need not agree, the center square of this diagram may not commute. The rightmost square does commute where the horizontal maps are projections, and any two maps beginning at the upper left corner agree since both embeddings restrict over 0 to the inclusion of $D_{\alpha\beta} \subset X_{\alpha\beta}$. Thus the largest square, formed by the top, the bottom, and the extreme left and right verticals does commute, as does the leftmost square. To measure the lack of commutativity, we pass again to the ring level, keeping the same notation for the coord. rings as above. The two possibly different embeddings $\tilde{f}_\alpha$, $\tilde{f}_\beta$ of the same deformation $\mathbf{D}_{\alpha\beta}$ given by the top and bottom rows of the center square of diagram (*) give two surjections of the same rings, $f_{\alpha\beta}$: $R_{\alpha\beta}[\varepsilon] \to \mathscr{S}_{\alpha\beta} \to 0$, and $f_{\beta\alpha}$: $R_{\beta\alpha}[\varepsilon] \to \mathscr{S}_{\beta\alpha} \to 0$. We subtract them and restrict to $R_{\alpha\beta}$, i.e. form $(f_{\alpha\beta} - f_{\beta\alpha})$: $R_{\alpha\beta} \to \mathscr{S}_{\alpha\beta}$. The commutativity of the outermost rectangle in (*) says that this difference, when carried further into $S_{\alpha\beta}$ by modding out ε, becomes zero. Thus its image is in $\varepsilon \cdot \mathscr{S}_{\alpha\beta}$, an $R_{\alpha\beta}$-module. The flatness hypothesis on a deformation implies that the natural map $S_{\alpha\beta} \to \varepsilon \cdot \mathscr{S}_{\alpha\beta}$ sending 1 to ε is an isomorphism as $R_{\alpha\beta}$-modules, [S2, Lemma 3.3, p. 216; comment, p. 217; A, p. 28], hence the prescription $x \mapsto (\varepsilon^{-1} \cdot x)(\mathrm{mod}\ \varepsilon)$ gives a well defined $R_{\alpha\beta}$-module map $\varepsilon \cdot \mathscr{S}_{\alpha\beta} \to S_{\alpha\beta}$. Composing this map with the difference $(f_{\alpha\beta} - f_{\beta\alpha})$ gives a function $\chi_{\alpha\beta} = (\varepsilon^{-1}(f_{\alpha\beta} - f_{\beta\alpha}))(\mathrm{mod}\ \varepsilon)$: $R_{\alpha\beta} \to S_{\alpha\beta}$, which is our desired derivation. We summarize the definition in a picture:

$$\begin{array}{rccc} (f_{\alpha\beta} - f_{\beta\alpha}): & R_{\alpha\beta}[\varepsilon] & \to & \varepsilon \cdot \mathscr{S}_{\alpha\beta} \\ & \bigcup & & \uparrow \cong \\ \chi_{\alpha\beta}: & R_{\alpha\beta} & \to & S_{\alpha\beta} \end{array}$$

(2.1.4) Next we define the map α:

$$\alpha\colon T^1(X, D) \to \mathbb{H}^1(\mathscr{T}(X) \to \mathscr{N}(D/X))$$

This time the abstract deformation of D is embedded in an abstract deformation of X:

$$\begin{array}{ccc} \mathbf{D} & \subset & \mathbf{X} \\ & \searrow \ \swarrow & \\ & \mathbb{D} & \end{array}$$

and when we pass to an affine cover the deformation of X again trivializes over $\mathbb{D}$, $\mathbf{X}_\alpha \cong X_\alpha \times \mathbb{D}$, and by composing we again have embeddings $\tilde{f}_\alpha: (\mathbf{D}_\alpha \hookrightarrow X_\alpha \times \mathbb{D})$, but this time on overlaps the two deformations of X are abstractly isomorphic. So we get a diagram:

$$\begin{array}{ccccc} \mathbf{D}_{\alpha\beta} & \subset & \mathbf{X}_{\alpha\beta} & \to & X_{\alpha\beta} \times \mathbb{D} \\ \| & & \| & & \cong \uparrow \tilde{\varphi}_{\beta\alpha} \\ \mathbf{D}_{\beta\alpha} & \subset & \mathbf{X}_{\beta\alpha} & \to & X_{\beta\alpha} \times \mathbb{D} \end{array}$$

in which $\tilde{\varphi}_{\beta\alpha}$ is defined so as to make the diagram commute. If we now omit the middle equality and add projection maps on the right, we get the following diagram in which only the left square commutes:

$$\begin{array}{ccccc} \tilde{f}_{\alpha\beta}: \mathbf{D}_{\alpha\beta} & \hookrightarrow & X_{\alpha\beta} \times \mathbb{D} & \to & X_{\alpha\beta} \\ \| & & \cong \uparrow \tilde{\varphi}_{\beta\alpha} & & \| \\ \tilde{f}_{\beta\alpha}: \mathbf{D}_{\beta\alpha} & \hookrightarrow & X_{\beta\alpha} \times \mathbb{D} & \to & X_{\beta\alpha} \end{array}$$

Finally look at the ring version of this diagram, flipping it right to left:

$$(**) \quad \begin{array}{ccccc} R_{\alpha\beta} & \to & R_{\alpha\beta}[\varepsilon] & \to & \mathscr{S}_{\alpha\beta} \\ \| & & \cong \uparrow \varphi_{\beta\alpha} & & \| \\ R_{\beta\alpha} & \to & R_{\beta\alpha}[\varepsilon] & \to & \mathscr{S}_{\beta\alpha} \end{array}$$

so that the right square of (**) commutes but not the left one.

(2.1.5) To define $(\tilde{\chi}, \tilde{v}) \in \mathbb{H}^1(\mathscr{T}(X) \to \mathscr{N}(D/X))$, the Čech diagram is this:

$$\begin{array}{ccc} C^1(\{D_{\alpha\beta}\}, \mathscr{T}(X)) & \to & C^1(\{D_{\alpha\beta}\}, \mathscr{N}(D/X)) \\ \uparrow & & \uparrow \\ C^0(\{D_\alpha\}, \mathscr{T}(X)) & \to & C^0(\{D_\alpha\}, \mathscr{N}(D/X)) \end{array}$$

Define $\tilde{v}_\alpha$ exactly as v_α in (2.1.1) and (2.1.2). Define $\tilde{\chi}_{\alpha\beta}$ nearly the same as $\chi_{\alpha\beta}$, but since $\tilde{\chi}$ will be an element of $\mathrm{Der}(R_{\alpha\beta}, R_{\alpha\beta})$, in diagram (**) subtract only the two compositions from $R_{\alpha\beta}$ to $R_{\alpha\beta}[\varepsilon]$. Thus $\tilde{\chi}_{\alpha\beta} = [\varepsilon^{-1}(1 - \varphi_{\beta\alpha})](\mathrm{mod}\, \varepsilon)$.

This ends our description of the definitions of α, β.

(2.2) Restriction of deformations to an open subset.

Using M. Schlessinger's homological method (depth), we will prove the following comparison result:

THEOREM 10. *If X is an algebraic scheme (i.e. of finite type over k), $Z \subset X$ a closed subset, and $\mathrm{depth}_{(Z \cup \mathrm{sg}.X)}(X) \geq 3$, then $T^1(X) \to T^1(X - Z)$ is an isomorphism.*

[Recall that for a point $\xi \in X$, and an $\mathcal{O}_X$-module $\mathcal{F}$, that $\operatorname{depth}_\xi(\mathcal{F})$ is the maximum length of a regular sequence for $\mathcal{F}_\xi$ consisting of elements of m_ξ, and that for a closed set $Z \subset X$, $\operatorname{depth}_Z(\mathcal{F})$ is the minimum of $\operatorname{depth}_\xi(\mathcal{F})$ over all (not nec. closed) points $\xi \in Z$. We write depth(X) for depth($\mathcal{O}_X$).]

REMARK 11. In case X is Cohen-Macaulay and $(Z \cup \mathrm{sg}.X)$ is non-empty, the depth hypothesis holds if and only if every irreducible component of $(Z \cup \mathrm{sg}.X)$ has codimension $\geqslant 3$.

Proof of Theorem 10. We will break this proof into 2 parts, Proposition 12 and Proposition 15. These two Propositions imply Theorem 10 as follows. Consider the following commutative diagram of restrictions:

$$\begin{array}{ccc} T^1(X) & \to & T^1(X - Z) \\ \downarrow & & \downarrow \\ T^1(X - \mathrm{sg}.X) & \to & T^1(X - (\mathrm{sg}.X \cup Z)) \end{array}$$

The vertical arrows are isomorphisms by Proposition 12 and the bottom arrow is an isomorphism by Proposition 15. Hence the map across the top is also an isomorphism. Q.E.D. for Theorem 10, (assuming Prop. 12 and Prop. 15).

PROPOSITION 12. *If X is an algebraic scheme, and if $Z \subset \mathrm{sg}.X$ is a closed subset of the singular locus, and if $\operatorname{depth}_\xi(X) \geqslant 3$ for all (not necessarily closed) points $\xi \in \mathrm{sg}.X$, then the restriction map is an isomorphism $T^1(X) \to T^1(X - Z)$.*

Proof. We will employ two lemmas.

LEMMA 13. *Assume X is a normal scheme of dimension at least two, that $V = X - \mathrm{sg}.X$ is the subscheme of smooth points, and $\mathbf{X}$, $\mathbf{X}'$ are two deformations of X over $\mathbb{D} = \operatorname{spec}(k[\varepsilon])$. If $\varphi\colon \mathbf{V} \to \mathbf{V}'$ is an isomorphism of the restricted deformations of V, then φ extends uniquely to an isomorphism $\bar{\varphi}\colon \mathbf{X} \to \mathbf{X}'$. In particular, $T^1(X) \to T^1(X - \mathrm{sg}.X)$ is injective.*

Proof. In case X is affine, this is [A, Lemma (9.1), p. 47]. So take an affine cover of $X = \cup X_\alpha$, let $V_\alpha = V \cap X_\alpha$ and let $\varphi_\alpha\colon \mathbf{V}_\alpha \to \mathbf{V}'_\alpha$ be the restricted isomorphism. Then the lemma cited implies there exist unique isomorphisms $\bar{\varphi}_\alpha\colon \mathbf{X}_\alpha \to \mathbf{X}'_\alpha$ extending the φ_α. Moreover, the two restrictions $\bar{\varphi}_{\alpha\beta}$ and $\bar{\varphi}_{\beta\alpha}$ (of $\bar{\varphi}_\alpha$ and $\bar{\varphi}_\beta$) to $X_{\alpha\beta}$ both extend the map $\varphi_{\alpha\beta} = \varphi_{\beta\alpha} =$ restriction of φ to $V_{\alpha\beta}$. Since $X_{\alpha\beta}$ is affine, and $V_{\alpha\beta} = X_{\alpha\beta} - \mathrm{sg}.X_{\alpha\beta}$, the uniqueness part of [A, (9.1)] implies then that $\bar{\varphi}_{\alpha\beta} = \bar{\varphi}_{\beta\alpha}$. Consequently, the $\bar{\varphi}_\alpha$ patch together into a unique isomorphism $\bar{\varphi}\colon \mathbf{X} \to \mathbf{X}'$. □

With the hypotheses of Proposition 12, note that if $\mathrm{sg}.X \neq \varnothing$, then the depth hypothesis implies that X satisfies Serre's R_1 and S_2 [H, p. 185] hence is normal of dimension $\geqslant 3$. Lemma 13 thus applies to X. Since the restriction $T^1(X) \to T^1(X - \mathrm{sg}.X)$ is thus injective and equals the composite of the two restrictions $T^1(X) \to T^1(X - Z) \to T^1(X - \mathrm{sg}.X)$, the first of these is injective also, which gives injectivity in Prop. 12.

LEMMA 14. *If X is an algebraic scheme of depth $\geqslant 3$ at all (not nec. closed) points of sg.X, then the restriction $T^1(X) \to T^1(X - \mathrm{sg}.X)$ is bijective.*

Proof. Again Lemma 13 implies injectivity, so we must show surjectivity. When X is affine this is [A, (9.2)]. One can eliminate the affineness hypothesis here too, by using [A, (9.1)] and the general principle that "local existence and uniqueness implies global existence and uniqueness'. That is, given a deformation of $V = X - \mathrm{sg}.X$, after taking an affine cover X_α of X, each of the resulting deformations of the $V_\alpha = (X_\alpha - \mathrm{sg}.X_\alpha)$ extends to a deformation of X_α, by [A, (9.2)]. On overlaps $V_\alpha \cap V_\beta$ moreover, these deformations are isomorphic, and hence are isomorphic on $X_\alpha \cap X_\beta$ as well by [A, (9.1)]. Using uniqueness again on $X_\alpha \cap X_\beta \cap X_\gamma$ these isomorphisms are compatible and allow the deformations of the X_α to patch to a deformation of X. $\square$

Now we can prove bijectivity in Prop. 12., as follows. Look at the same composition as before: $T^1(X) \to T^1(X - Z) \to T^1(X - \mathrm{sg}.X)$. Since $X - \mathrm{sg}.X = (X - Z) - \mathrm{sg}.(X - Z)$, both the right hand map and the composition are bijective by Lemma 14, hence so is the left hand map. Q.E.D. for Prop. 12.

PROPOSITION 15. *Assume X is non-singular, and $Z \subset X$ is a closed subset of codimension $\geqslant 3$. Then the natural restriction map $T^1(X) \to T^1(X - Z)$ is an isomorphism.*

Proof. We will reduce this to a calculation with cohomology:

LEMMA 16. *For any closed subset Z of a smooth scheme X, there is a commutative diagram as follows, where the vertical maps are isomorphisms, the top map is restriction of deformations and the bottom map is induced by restriction of tangent vector fields:*

$$\begin{array}{ccc} T^1(X) & \to & T^1(X - Z) \\ \downarrow & & \downarrow \\ H^1(X, \mathscr{T}(X)) & \to & H^1(X - Z, \mathscr{T}(X - Z)) \end{array}$$

Proof. [Gr1, Cor. 2, p. 13]. $\square$

Hence Prop. 15 will follow from:

LEMMA 17. *Under the hypotheses of Proposition 15 the natural restriction map is an isomorphism of cohomology groups:*

$$H^1(X, \mathscr{T}(X)) \to H^1(X - Z, \mathscr{T}(X - Z)).$$

Proof. As the first step we recall the properties of depth which we wish to use repeatedly, in particular its relation to the vanishing of local cohomology groups and sheaves, and to the extension problem for sections:

PROPOSITION 18. *If $Z \subset X$ is a closed set, $n \geqslant 2$ an integer, and $\mathscr{F}$ is a coherent sheaf on X, then the following assertions are equivalent*:

(i) $\mathrm{Depth}_Z(\mathscr{F}) \geqslant n$.
(ii) $\mathscr{H}^i_Z(\mathscr{F}) = 0$, *for* $0 \leqslant i \leqslant n-1$.
(iii) $H^i_Z(V, \mathscr{F}_{|V}) = 0$, for $0 \leqslant i \leqslant n-1$, *all* $V \subset X$ *open*.
(iv) res: $H^j(V, \mathscr{F}_{|V}) \to H^j(V-(Z \cap V), \mathscr{F}_{|V})$ *is an isomorphism for* $0 \leqslant j \leqslant n-2$, (*where res is induced by restriction*), *and for all* V *open in* X.

Proof. That (i) and (ii) are equivalent is [Gr3, Th. 3.8, p. 44].

For (ii) $\Rightarrow$ (iii) we use the spectral sequence in [Gr3, Prop. 1.4, p. 5]. Fix V. We have $E_2^{p,q} = H^p(V, \mathscr{H}^q_Z(\mathscr{F})) = 0$, for $0 \leqslant q \leqslant n-1$. Hence $E_\infty^{p,q} = 0$, for $0 \leqslant p+q \leqslant n-1$. Thus also the limit $H_Z^{p+q}(V, \mathscr{F}_{|V})$, for which $E_\infty^{p,q}$ are the graded quotients, is zero, for $0 \leqslant p+q \leqslant n-1$.

For (iii) $\Rightarrow$ (ii), use [Gr3, Prop. 1.2, p. 4]. That (ii) is equivalent to (iv) is [Gr3, Prop. (1.11), p. 11–12]. □

REMARK. (i), (ii), and (iii) are also equivalent for $n = 1$.

LEMMA 19. *If X is smooth, then* $\mathrm{depth}_Z(\mathscr{T}(X)) \geqslant \mathrm{codim}.(Z \subset X)$.

Proof. X smooth implies $\mathscr{T}(X)$ locally free, so any regular sequence for $\mathscr{O}_X$ is also regular for $\mathscr{T}(X)$. Thus the length of a maximal such sequence for $\mathscr{T}(X)$ is at least as great as that for $\mathscr{O}_X$, so $\mathrm{depth}_Z(\mathscr{T}(X)) \geqslant \mathrm{depth}_Z X$. Also, on a smooth scheme, $\mathrm{depth}_Z X = \mathrm{codim}.(Z \subset X)$. □

Now we can finish the proof of Lemma 17 and hence of Proposition 15. Under the hypotheses (that X is non-singular and $Z \subset X$ is a closed subset of codimension $\geqslant 3$), we have by Lemma 19 that $\mathrm{depth}_Z(\mathscr{T}(X)) \geqslant 3$, so we can apply Prop. 18(iv) to conclude that $H^1(X, \mathscr{T}(X)) \to H^1(X-Z, \mathscr{T}(X-Z))$ is an isomorphism. Q.E.D. for Prop. 15.

(2.3) Lifting first order deformations to a resolution.

Recall that the space of isomorphism classes of locally trivial first order deformations of X, is precisely $\mathrm{Ker}\{T^1(X) \to H^0(\mathscr{T}^1(X))\}$, and is computed by $H^1(X, \mathscr{T}(X))$, where $\mathscr{T}(X)$ is the tangent sheaf of X. We want to compare these cohomology groups for a normal space X and for a small resolution Y of it, [cf. S3, §2, for an analogous result in the case of a quotient singularity]. Now for any map $f\colon Y \to X$, the low degree exact sequence coming from the Leray spectral sequence [Go, Th. 4.5.1., p. 82, Th. 4.17.1, pp. 201–202] gives an injection of the groups $H^1(f_*\mathscr{T}(Y)) \hookrightarrow H^1(\mathscr{T}(Y))$, so it is natural to try to relate $f_*\mathscr{T}(Y)$ to $\mathscr{T}(X)$. If these are isomorphic then we will have an injection from the locally trivial first order deformations of X into the locally trivial first order deformations of Y.

Note that this is a souped up version of relating $f_*(\mathcal{O}_Y)$ to $\mathcal{O}_X$, which are isomorphic whenever f is a birational projective map and X is normal.

The general result which we shall prove is this:

PROPOSITION 20. *There is a natural injection from locally trivial (first order) deformations of a normal space to (first order) deformations of any 'small' resolution, i.e. a resolution in which the exceptional set over the singular locus has codimension* $\geqslant 2$.

Proof. As remarked before the statement of the Proposition, it suffices to establish the following:

LEMMA 21. *Let* $f\colon Y \to X$ *be a small resolution of singularities of a normal algebraic scheme* X. *Then* $f_*(\mathcal{T}(Y)) \cong \mathcal{T}(X)$.

Proof. Since f is an isomorphism except over the singular locus of X, these sheaves have the same sections over sets which miss the singular locus. We will show that those sections determine all the sections by using Prop. 18 to extend sections of these sheaves across the singular set. As usual we must consider the codimension of the singular locus and of its inverse image as well as the depth of these sheaves along those sets. We start with a version of 'Schlessinger's Lemma'.

LEMMA 22. *Let* X *be an algebraic scheme and let* $Z \subset X$ *be a closed subset such that* $\mathrm{depth}_Z(X) \geqslant 2$. *If* $\mathcal{G}$ *is a coherent sheaf on* X, *and if* $\mathcal{G}^*$ *denotes the dual sheaf* $\mathcal{HOM}(\mathcal{G}, \mathcal{O}_X)$, *then the local cohomology sheaves* $\mathcal{H}^i_Z(\mathcal{G}^*)$ *vanish for* $i = 0, 1$. [*Thus if* $\mathcal{O}_X$ *has depth* $\geqslant 2$ *along* Z, *then so does any 'reflexive' sheaf.*]

REMARK 23. If X has dimension at least 2, and $Z = \mathrm{sg}.X$, then by Serre's $(R_1 + S_2)$ criterion, X has depth $\geqslant 2$ along Z if and only if X is normal.

Proof of Lemma 22 (similar to [S3, Lemma 1, p. 21]). By [Gr3, p. 44], the vanishing of local cohomology sheaves for $i \leqslant 1$ is equivalent to the depth of $\mathcal{G}^*$ being $\geqslant 2$ at every (not nec. closed) point of Z. Hence this is a local property and we can assume X is affine. Since $\mathcal{G}$ is coherent there is an exact sheaf sequence $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{G} \to 0$ in which $\mathcal{F}$ is finite free and $\mathcal{R}$ is coherent. Dualizing gives $0 \to \mathcal{G}^* \to \mathcal{F}^* \to \mathcal{Q} \to 0$, where $\mathcal{Q} \subset \mathcal{R}^*$. Since $\mathcal{R}$ is also a quotient of a finite free sheaf, dualizing gives $\mathcal{R}^* \subset$ (finite free) and hence also $\mathcal{Q} \subset$ (finite free). Now the long exact sequence of cohomology sheaves from [Gr3, Prop. (1.1)b, p. 4] gives:

$$0 \to \mathcal{H}^0_Z(\mathcal{G}^*) \to \mathcal{H}^0_Z(\mathcal{F}^*) \to \mathcal{H}^0_Z(\mathcal{Q}) \to \mathcal{H}^1_Z(\mathcal{G}^*) \to \mathcal{H}^1_Z(\mathcal{F}^*) \to \cdots$$

Since $\mathcal{F}^*$ is locally free, the depth hypothesis which implies vanishing of $\mathcal{H}^i_Z(\mathcal{O})$, for $i = 0, 1$ implies also $\mathcal{H}^i_Z(\mathcal{F}^*) = 0$ for $i = 0, 1$, in this sequence. Therefore $\mathcal{H}^0_Z(\mathcal{G}^*) = 0$ since in the sequence above, this group injects into the zero group. Next we claim $\mathcal{H}^0_Z(\mathcal{Q}) = 0$. Indeed, we noted $\mathcal{Q} \subset$ (finite free), so we have $0 \to \mathcal{Q} \to \oplus\, \mathcal{O} \to \mathcal{W} \to 0$, and thus $0 \to \mathcal{H}^0_Z(\mathcal{Q}) \to \mathcal{H}^0_Z(\oplus\, \mathcal{O}) = 0$. Therefore $\mathcal{H}^1_Z(\mathcal{G}^*) = 0$, from the sequence above. □

COROLLARY 24. *For a normal algebraic scheme X of dimension at least 2, $\mathscr{T}(X) = \mathscr{HOM}(\Omega^1_X, \mathcal{O}_X)$ has depth $\geqslant 2$ along the singular locus of X.*

Proof. By Remark 23, $Z = \mathrm{sg}.X$ satisfies the hypotheses of Lemma 22. Applying the Lemma with $\mathscr{G} = \Omega^1_X$, we get that $\mathscr{H}^i_Z(\mathscr{T}(X)) = 0$, for $i = 0, 1$; hence by Prop. 18, $\mathrm{depth}_Z \mathscr{T}(X) \geqslant 2$. □

Proof of Lemma 21 concluded. Since, by assumption, Y is smooth and $f^{-1}(\mathrm{sg}.X)$ has codimension $\geqslant 2$ in Y, it follows that $\mathcal{O}_Y$ has depth $\geqslant 2$ along $f^{-1}(\mathrm{sg}.X)$. Thus the locally free sheaf $\mathscr{T}(Y)$ also has depth $\geqslant 2$ there, and so sections of $\mathscr{T}(Y)$ extend across $f^{-1}(\mathrm{sg}.X)$. By definition of f_* then sections of $f_*(\mathscr{T}(Y))$ extend across $\mathrm{sg}.X$. Recall that $\mathscr{T}(X)$ has depth $\geqslant 2$ along $\mathrm{sg}.(X)$, by Corollary 24, and thus its sections too extend across $\mathrm{sg}.X$. Since f is an isomorphism over the smooth points of X, it follows that $f_*(\mathscr{F}(Y))$ and $\mathscr{T}(X)$ have the same sections on smooth points and thus, by the extension properties, they have the same sections everywhere. To be precise, if $Z = \mathrm{sg}.X$, we have natural isomorphisms for all open subsets $V \subset X$:

$$H^0(V, f_*(\mathscr{T}(Y))) \cong H^0(V - (Z \cap V), f_*(\mathscr{T}(Y)))$$
$$\cong H^0(V - (Z \cap V), \mathscr{T}(X)) \cong H^0(V, \mathscr{T}(X)).$$ Q.E.D. for Lemma 21, and thus Q.E.D. for Prop. 20.

3. Applications to abelian varieties and theta divisors

Now we apply the deformation theory from the previous section.

PROPOSITION 25. *If (A, Θ) is a p.p.a.v. with* $\dim(A) \geqslant 3$, *then the map 'forgetting A': $T^1(A, \Theta) \to T^1(\Theta)$, is bijective.*

Proof. This follows immediately from Corollary 9 provided we check that $H^1(\mathscr{I}_\Theta \cdot \mathscr{T}_A)$ and $H^2(\mathscr{I}_\Theta \cdot \mathscr{T}_A)$ are 0. Since the tangent bundle $\mathscr{T}_A$ of A is trivial (i.e. $\mathscr{T}_A \cong \mathcal{O}_A \otimes T_0A$), we have $H^i(\mathscr{I}_\Theta \cdot \mathscr{T}_A) \cong H^i(\mathscr{I}_\Theta \otimes T_0A) \cong H^i(\mathscr{I}_\Theta) \otimes T_0A$. Now note that $H^i(\mathscr{I}_\Theta) \cong H^i(\mathcal{O}(-\Theta)) = 0$, for $i < g$, by Mumford's vanishing theorem [M2, §16, p. 150]. (Since $L = \mathcal{O}(\Theta)$ defines a principal polarization, in his notation $K(L) = 0$, and $H^0(\mathcal{O}(\Theta)) = 1 \neq 0$ implies $H^g(\mathcal{O}(-\Theta)) \neq 0$, so that $H^i(\mathcal{O}(-\Theta)) = 0$, for $i < g$). □

PROPOSITION 26. *If C is a non-hyperelliptic curve of genus $\geqslant 4$ with Jacobian J, and $J \supset \Theta \supset U = \{$smooth points and rank four double points on $\Theta\}$, then the restriction $T^1(\Theta) \to T^1(U)$ is bijective.*

Proof. This follows from Proposition 12 with $X = \Theta$, $Z = B = \Theta - U$, once we check the depth condition there. Since Θ is a hypersurface in a smooth variety all its localizations are Cohen-Macaulay and hence the depth at a point is the codimension of the (closure of the) point [H, p. 184]. Thus, since $\mathrm{sg}.\Theta$ has (pure) codimension $= 3$ in Θ, the depth of Θ at all points of $\mathrm{sg}.\Theta$ is indeed $\geqslant 3$.

□

Next we will compare the deformations of $U \subset \Theta$ with those of its inverse image under the Abel resolution.

LEMMA 27. *Let $\sigma: C_{g-1} \to \Theta$ be the Abel map parametrizing the theta divisor of the Jacobian of a non-hyperelliptic curve of genus $\geqslant 4$. Then Θ is normal and σ is a small resolution of Θ.*

Proof. Θ is a hypersurface, hence Cohen Macaulay, so $\mathcal{O}_\Theta$ has depth along sg.Θ equal to the codimension of sg.$\Theta \subset \Theta$, which is 3. In particular Θ is normal; see Remark 23. Since σ is a resolution, it remains only to prove that $\sigma^{-1}(\text{sg.}\Theta)$ has codimension $\geqslant 2$ in C_{g-1}. By the Riemann-Kempf Singularities Theorem the fibers of σ over double points of Θ are copies of $\mathbb{P}^1$ so the set $\sigma^{-1}(\text{sg.}_2\Theta)$ has pure dimension $g-3$. Note that the proof of Lemma 5 shows that $\sigma^{-1}(\text{sg.}_2\Theta)$ is dense in $\sigma^{-1}(\text{sg.}\Theta)$. Therefore $\sigma^{-1}(\text{sg.}\Theta)$ also has pure dimension $g-3$, hence codimension $= 2$ in C_{g-1}. □

COROLLARY 28. *With the same hypotheses as in Lemma 27, if $\Theta \supset B = \{$rank 3 double points and points of mult. $\geqslant 3$ on $\Theta\}$, if $U = \Theta - B$ and $\mathscr{U} = \sigma^{-1}(U)$, then there is a natural injection $H^1(\mathscr{T}(U)) \hookrightarrow H^1(\mathscr{T}(\mathscr{U}))(\cong T^1(\mathscr{U})$ since $\mathscr{U}$ is smooth).*

Proof. By Lemma 27, $\sigma: C_{g-1} \to \Theta$ is a small resolution of a normal space, hence so is the restriction $\sigma: \mathscr{U} \to U$. Now apply Proposition 20. □

PROPOSITION 29. *If $\sigma: C_{g-1} \to \Theta$ is the Abel map parametrizing the theta divisor of a non-hyperelliptic curve of genus $g \geqslant 5$, and if $\mathscr{U} = \sigma^{-1}(U)$ is the inverse image of the set $U = \{$smooth points and rank 4 double points of $\Theta\}$, then the restriction map $T^1(C_{g-1}) \to T^1(\mathscr{U})$ is an isomorphism.*

Proof. Since C_{g-1} is nonsingular it suffices by Prop. 15 to compute the codimension of $(C_{g-1}) - \mathscr{U} = \sigma^{-1}(B)$, where B is the set defined in Cor. 28. We refer back to the proof of Lemma 27. We saw there that $\sigma^{-1}(\text{sg.}_2\Theta)$ has pure dimension $g-3$ and is dense in $\sigma^{-1}(\text{sg.}\Theta)$. It follows that no component of $\sigma^{-1}(\text{sg.}\Theta)$ lies entirely over points of mult. $\geqslant 3$ on Θ. Thus σ^{-1} (points of mult. $\geqslant 3$) has dimension $\leqslant g-4$, and thus has codimension $\geqslant 3$ in C_{g-1}. Since we also checked in the proof of Lemma 6 that the set of rank 3 double points has dimension $\leqslant g-5$, and since by Riemann-Kempf the fibers of σ over double points are copies of $\mathbb{P}^1$, we get dim. σ^{-1}(rank 3 double points) $\leqslant g-4$. Hence codim. $\sigma^{-1}(B) \geqslant 3$, as desired. □

PROPOSITION 30. *If C is a non-hyperelliptic curve of genus $g \geqslant 3$ there is a natural isomorphism $T^1(C_{g-1}) \cong T^1(C)$.*

Proof. This is a special case of a theorem of Kempf, proved in [K4].

COROLLARY 31. *If U is the open subset of Θ defined in Prop. 29 above, then the dimension of $H^1(\mathscr{T}(U))$ is at most $3g-3 =$ dim. $T^1(C) =$ dim. $T_C\tilde{\mathscr{M}}_g$.*

Proof. Combining all our results, $H^1(\mathscr{T}(U))$ injects into $T^1(C) \cong H^1(\mathscr{T}(C)) \cong k^{3g-3}$ by Serre duality and Riemann-Roch. □

Now we are ready to prove the central result, Theorem 4, of section (1.4).

Proof of Theorem 4. Let $\bar{s}$: $T_J\tilde{\mathscr{A}}_g \to H^0(\bar{\mathscr{T}}^1(\Theta))$ denote the map induced by the Kodaira-Spencer map of the family $\boldsymbol{\Theta}$ over $\tilde{\mathscr{A}}_g$. Then to prove Theorem 4 we must show, when $J = J(C)$, that $\ker(\bar{s}) = T_C\tilde{\mathscr{J}}_g$ $(=t_*(T_C\tilde{\mathscr{M}}_g)$, the image of the differential of the Torelli map t: $\tilde{\mathscr{M}}_g \to \tilde{\mathscr{A}}_g$). We will prove this by first proving there are (linear) injections $T_C\tilde{\mathscr{J}}_g \subset \ker(\bar{s}) \hookrightarrow H^1(\mathscr{T}(U))$, where U is again the open subset of Θ defined in Prop. 29 above, and then invoking Cor. 31. The first inclusion has already been proved as statement (1) in Proposition 1, in section (1.1), so we prove the second injection:

LEMMA 32. *There exists a linear injection* $\ker(\bar{s}) \hookrightarrow H^1(\mathscr{T}(U))$.

Proof. If we denote by $\tilde{\mathscr{A}}$ the fine moduli space $\tilde{\mathscr{A}}_g$ for triples $(A, \lambda, \alpha) = (g -$ dimensional abelian variety, principal polarization, level n structure (for suitable n, i.e. n prime to p and even, and $n \geqslant 3$)), it follows that $T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}} \cong \mathrm{MOR}(\{\mathbb{D}, 0\}, \{\tilde{\mathscr{A}}, (A, \lambda, \alpha)\}) \cong$ {isomorphism classes of families of triples over $\mathbb{D}$, plus identification of (A, λ, α) with the fiber over 0} $= T^1(A, \lambda, \alpha)$, where the isomorphism from $T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}}$ to $T^1(A, \lambda, \alpha)$ is the Kodaira-Spencer map (i.e. pull-back of the universal family from $\tilde{\mathscr{A}}$ to $\mathbb{D}$). Then one checks that a deformation of (A, λ, α) which is trivial on (A, λ) must be trivial on the level n structure α also [cf. O-S, p. 166], so that the natural forgetful map is injective $T^1(A, \lambda, \alpha) \hookrightarrow T^1(A, \lambda)$. Then use the restrictions to $\mathbb{D}$ of the universal theta divisor over $\tilde{\mathscr{A}}$ to define $T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}} \to T^1(A, \Theta)$. We claim this last map is injective by virtue of the following commutative diagram:

$$\begin{array}{ccc} T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}} & \to & T^1(A, \lambda, \alpha) \\ \downarrow & & \downarrow \\ T^1(A, \Theta) & \to & T^1(A, \lambda). \end{array}$$

That is, the top arrow is isomorphic and the right vertical arrow is injective so the left vertical arrow is also injective. Combining this with our result (Proposition. 25) that the map $T^1(A, \Theta) \to T^1(\Theta)$ is isomorphic we get an injection $T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}} \hookrightarrow T^1(\Theta)$ (which is pull-back of the family $\boldsymbol{\Theta}$ over $\mathbb{D}$, hence a linear Kodaira-Spencer homomorphism) such that the following composition is $\bar{s}$:

$$\bar{s}\colon \{T_{(A,\lambda,\alpha)}\tilde{\mathscr{A}} \hookrightarrow T^1(\Theta) \to H^0(\mathscr{T}^1(\Theta)) \to H^0(\bar{\mathscr{T}}^1(\Theta))\}.$$

If we now restrict to the case where $(A, \lambda, \alpha) = J$ is the Jacobian of a non-hyperelliptic curve C of genus $g \geqslant 5$, then restricting the two right hand maps from Θ to U yields the following commutative diagram of linear maps:

$$\begin{tikzcd}
T_J\tilde{\mathscr{A}} \arrow[r, hook] & T^1(\Theta) \arrow[r] \arrow[d] & H^0(\mathscr{T}^1(\Theta)) \arrow[r] \arrow[d] & H^0(\bar{\mathscr{T}}^1(\Theta)) \arrow[d] \\
& T^1(U) \arrow[r] & H^0(\mathscr{T}^1(U)) \arrow[r, phantom, "="] & H^0(\bar{\mathscr{T}}^1(U))
\end{tikzcd}$$

The equals sign in the bottom row follows from Lemma 7 in section (1.4). Since the left vertical arrow is bijective by Prop. 26, and the composition of the whole top row is $\bar{s}$, it follows that $\ker(\bar{s})$ injects into $\ker\{T^1(U) \to H^0(\mathscr{T}^1(U))\} = H^1(\mathscr{T}(U))$. □

Since by Corollary 31 this implies that $\ker(\bar{s})$ has dimension $\leqslant 3g - 3$, the inclusion $T_C\mathscr{J}_g \subset \ker(\bar{s})$ must be an equality. Q.E.D. for Theorem 4.

COROLLARY 33. For any non-hyperelliptic curve C of genus $g \geqslant 4$, the vector space $I_2(C)$ of quadrics containing the canonical model of C is generated by those quadrics arising as tangent cones to double points of the theta divisor of $J(C)$. Moreover, for $g \geqslant 5$, (and for $g = 4$ if C has no even effective theta characteristic) those quadric tangent cones to theta having rank exactly 4 generate $I_2(C)$.

Proof. For $g \geqslant 5$ this is what our proof has shown, and for $g = 4$ it is elementary that the only quadric containing $\Phi_K(C)$ is equal to the quadric tangent cone at each double point of Θ. □

Summary

The following diagram displays the whole argument:

$$\begin{tikzcd}
T^1(C) \arrow[r, "\approx"] \arrow[ddddd, leftrightarrow, "\approx"'] & T_C\mathscr{J}_g \arrow[r, phantom, "\subset"] & \ker(\bar{s}) \arrow[r, phantom, "\subset"] \arrow[ddd, hook] & T_J\tilde{\mathscr{A}}_g \arrow[d, hook, "(\approx)"] \arrow[ddrr, "\bar{s}"] & & \\
& & & T^1(J, \Theta) \arrow[d, hook, "(\approx)"] & & \\
& & & T^1(\Theta) \arrow[r] \arrow[d, hook, "(\approx)"] & H^0(\mathscr{T}^1(\Theta)) \arrow[r] \arrow[d] & H^0(\bar{\mathscr{T}}^1(\Theta)) \arrow[d] \\
& 0 \arrow[r] & H^1(\mathscr{T}(U)) \arrow[r] \arrow[d, hook] & T^1(U) \arrow[r] & H^0(\mathscr{T}^1(U)) \arrow[r, "="] & H^0(\bar{\mathscr{T}}^1(U)) \\
& & H^1(\mathscr{T}(\mathscr{U})) & & & \\
H^1(\mathscr{T}(C)) \arrow[rr, "\approx"] & & H^1(\mathscr{T}(C_{g-1})) \arrow[u, "(\approx)"] & & &
\end{tikzcd}$$

Briefly, $T^1(C) \cong T_C\mathscr{J}_g \subset \ker(\bar{s}) \subset \{\text{log. triv. defs. of } U\} \subset T^1(\mathscr{U}) \cong T^1(C_{g-1}) \cong T^1(C)$. Hence $T_C\mathscr{J}_g = \ker(\bar{s})$.

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