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Mixed Hodge structures on the intersection cohomology of links

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Abstract. The theory of mixed Hodge modules is applied to obtain results about the mixed Hodge structure on the intersection cohomology of a link of a subvariety in a complex algebraic variety. The main result, whose proof uses the purity of the intersection complex in terms of mixed Hodge modules, is a generalization of the semipurity theorem obtained by Gabber in the l-adic case. An application is made to the local topology of complex varieties.

Introduction

Let X be a complex algebraic variety, assumed irreducible and of dimension n, and let Z be a closed subvariety. This paper studies the mixed Hodge structure on the intersection cohomology of the link of Z in X, derives a semipurity result, and deduces some topological consequences. The mixed Hodge structure is obtained using the theory of mixed Hodge modules developed by the second author.

Although the concept of 'link' of Z in X is intuitively obvious, its precise meaning is unclear. In this paper, we will define it as the nearby level set of a suitable nonnegative real valued distance function which vanishes exactly on Z. If Z is compact, a reasonable concept of link results if the distance function is assumed to be real analytic. Another stronger, concept results if N is a neighborhood of Z in X and if there is a proper continuous retraction map r from ∂N to Z such that the closure of N is the total space of the mapping cylinder of r; the distance function is then the projection of the mapping cylinder to $[0, 1]$. For most of this paper, however, we will use a third, weaker type of link and distance function which combines topological and homological properties. In fact, the homological notions of link alone are enough for most of our results. These homological notions are functors which can be canonically defined in the derived category and fit well with the theory of mixed Hodge modules which we

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will be using. The material on the various definitions of link is in the beginning of Section 2.

Whichever definition is used, a link L is an oriented topological pseudomanifold of (real) dimension $2n - 1$ with odd-dimensional strata. In particular, its Goresky-MacPherson middle-perversity intersection homology is defined. If L is a rational homology manifold, then intersection homology is ordinary homology. Furthermore, $N \backslash Z$ is a rational homology manifold if and only if L is.

This paper applies the theory of mixed Hodge modules to put a mixed Hodge structure on the intersection cohomology with rational coefficients of a link L. Various elementary properties of these groups are derived. The main result is as follows:

THEOREM 4.1. *If* $\dim X = n$ *and* $\dim Z \leqslant d$, *then* $IH_c^k(L)$ *has weights* $\leqslant k$ *for* $k < n - d$.

Duality then immediately shows that $IH^k(L)$ has weights $> k$ for $k \geqslant n + d$. The notation here is as follows: The intersection cohomology group $IH_c^k(L)$ is (topologically) the homology group of geometric $(2n - 1 - k)$-dimensional intersection chains with compact support, and $IH^k(L)$ is the similar group with closed support.

For varieties over finite fields and Z a point this result was proved by Gabber [Ga]. The result of Gabber is equivalent to the local purity of the intersection complex by definition and self duality; it implies the purity of intersection cohomology by Deligne's stability theorem for pure complexes under direct images for proper morphisms [De2]. This local purity also follows from the existence of the weight filtration on mixed perverse sheaves, since intersection complexes are simple [BDD 4.3.1].

For varieties over the complex numbers, Z a point and $X \backslash Z$ smooth, the above result for mixed Hodge structures on ordinary cohomology was deduced by several people [St1, El] using the characteristic 0 decomposition theorem. Later Steenbrink and Navarro found a more elementary proof using Hodge theory [St2, Na]. We show that Theorem 4.1 follows naturally from the second author's theory of mixed Hodge modules combined with the theory of gluing t-structures from [BBD]. The proof is in the spirit of the second proof of local purity in the l-adic case. (See the end of 1.4.)

This theorem is then used to show that certain products in the intersection homology of a link must vanish (Theorems 5.1 and 5.2). This result is a generalization of [DH], which treated the case where Z is a point and $X \backslash Z$ is smooth. For example, Theorem 5.2 implies that the five-torus $S^1 \times \cdots \times S^1$ is not a link of a compact curve Z in a three-fold X, and Theorem 5.1 implies that certain pseudomanifolds L cannot be links of points in a complex variety. The only previous result is this area is, we believe, one of Sullivan: If L is a link of a compact subvariety, then the Euler characteristic of L vanishes [Su]. Sullivan's proof is entirely topological and holds for any compact oriented pseudomanifold L with only odd-dimensional strata. The above examples are independent of this result, and hence provide new restrictions on the topology of complex algebraic varieties.

Although the results and proof of this paper are given in terms of mixed Hodge modules, they can actually be read in three settings:

The derived category of sheaves on X: On a first reading, this paper can be understood in the derived category $D(X)$ of sheaves of vector spaces over a field on X, together with the derived functors $f_*, f_!, \ldots$ (Following recent convention, we omit the R or L when referring to derived functors.) We use no more properties than those summarized in [GM2 Sect. 1]. Various properties of the intersection homology of links are stated and proved in this language. Of course no conclusions can be drawn about weights (Sect. 4) or the resulting corollaries on the topological structure of varieties (Sect. 5).

The category of mixed Hodge modules on X: The additional material we need about mixed Hodge modules is basically the same as the formalities of [BBD]. This material is summarized in the first section of this paper.

The category of mixed l-adic sheaves on a variety in characteristic p: Lastly, this paper can be read in the setting of [BBD], with the conclusions of Sect. 4 about weights for varieties in characteristic p. Of course, the concept of link as topological space makes no sense here, so the isomorphisms of 2.11 should be taken as the definition of cohomology, homology and intersection cohomology of 'link' in this case. However the applications of Sect. 5 to the topology of varieties can still be obtained by the methods of reduction modulo p as in [BBD Sect. 6].

Throughout the paper references are given for each of these three settings.

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1. Background material

General references for the following material on intersection homology and derived functors are [GM2, Bo, GM3 §1, Iv]. All groups will be assumed to have the rational numbers as coefficients, unless otherwise indicated. Following the convention of [BBD], we will use the same symbol for a functor and its right or left derived functor.

1.1. Suppose that W is a topological pseudomanifold of dimension m with strata of even codimension. For example, W can be a complex algebraic variety or a link of a subvariety. The following groups (all with rational coefficients) can be associated to W:

$H^k(W)$, resp. $H_c^k(W)$: The kth cohomology group, resp. cohomology group with compact supports.

$H_k(W)$, resp. $H_k^{BM}(W)$: The kth homology group, resp. Borel-Moore homology group (homology with closed supports).

$IH_k(W)$, resp. $IH_k^{BM}(W)$: The kth intersection homology group with middle perversity k-dimensional chains with compact support, resp. closed support [GM2, GM3 1.2; Bo I]. Since W has even codimensional strata, the middle perversity is well defined.

$IH^k(W)$, resp. $IH_c^k(W)$: The kth intersection cohomology group, defined (topologically) as $IH_{m-k}^{BM}(W)$, resp. cohomology with compact supports, defined as $IH_{m-k}(W)$.

1.2. We also have the following:

$a_W\colon W \to pt$

$\mathbf{Q}_W = (a_W)^*\mathbf{Q}$ = the constant sheaf on W

$\mathbf{D}_W = (a_W)^!\mathbf{Q}$ = the dualizing sheaf on W

$\mathbf{IC}_W^{\text{top}}$ = the intersection complex on W [GM2 2.1]

These complexes have the following properties:

$$H^k(W) = H^k(W, \mathbf{Q}_W) \text{ and } H_c^k(W) = H_c^k(W, \mathbf{Q}_W)$$

$$H_k(W) = H_c^{-k}(W, \mathbf{D}_W) \text{ and } H_k^{BM}(W) = H^{-k}(W, \mathbf{D}_W)$$

$$IH_k(W) = IH_c^{m-k}(W) = H_c^{-k}(W, \mathbf{IC}_W^{\text{top}}) \text{ and}$$

$$IH_k^{BM}(W) = IH^{m-k}(W) = H^{-k}(W, \mathbf{IC}_W^{\text{top}})$$

1.3. Let X be a complex algebraic variety (a reduced separated scheme of finite type over $\mathbf{C}$), assumed irreducible and of complex dimension n. A general reference for the following material is [BBD].

Let

$$\mathbf{IC}_X = \mathbf{IC}_X^{\text{top}}[-n]$$

as objects of $D_c^b(X)$ as in [BBD]. If $j\colon U \to X$ is the inclusion of a smooth dense Zariski-open set, then

$$\mathbf{IC}_X = j_{!*}\mathbf{Q}_U[n].$$

We also have

$D_c^b(X)$ = The derived category whose objects are bounded complexes of sheaves of **Q**-modules with constructible cohomology.

Perv(X) = The full subcategory of perverse sheaves over **Q**.

1.4. General references for the theory of mixed Hodge modules are [Sa1, Sa2]. Many of their formal properties are similar to those of mixed perverse sheaves [BBD 5.1].

We have

MHM(X) = the abelian category of mixed Hodge modules on X

rat: MHM(X)$\rightarrow$Perv(X), an additive, exact, faithful functor which assigns the underlying perverse sheaf over **Q**.

Functors f_*, $f_!$, f^*, $f^!$, **D**, $\otimes$, $\boxtimes$ on D^bMHM(X) compatible with the corresponding derived functors on $D_c^b(X)$ and with the corresponding perverse functors on Perv(X) via

$$\begin{array}{ccccc} D^b\mathrm{MHM}(X) & \xrightarrow{\mathrm{rat}} & D^b\mathrm{Perv}(X) & \xrightarrow{\mathrm{real}} & D_c^b(X) \\ \downarrow{\scriptstyle H^0} & & {\scriptstyle H^0}\downarrow & \swarrow {\scriptstyle {}^p\underline{\mathbf{H}}^0} & \\ \mathrm{MHM}(X) & \xrightarrow{\mathrm{rat}} & \mathrm{Perv}(X) & & \end{array}$$

where 'real' is an equivalence of categories [Be, BBD 3.1.10]. See also [BBD 1.3.6, 1.3.17(i), 3.1.14].

Adjoint relations for f^*, f_* and $f_!$, $f^!$, the natural morphism $f_! \rightarrow f_*$ and the usual relations $\mathbf{D}^2 = \mathrm{id}$, $\mathbf{D}f_* = f_!\mathbf{D}$ and $\mathbf{D}f^* = f^!\mathbf{D}$.

The fact that the category MHM(pt) is the category of polarizable mixed Hodge structures (over **Q**).

Let $\mathbf{Q}^H \in$ MHM(pt) be the mixed Hodge structure of weight 0 and rational structure rat($\mathbf{Q}^H$) = **Q**. In this language the cohomology of X has a mixed Hodge structure since we can write

$$H^\cdot(X) = H^\cdot((a_X)_*(a_X)^*\mathbf{Q}^H)$$
$$H_c^\cdot(X) = H^\cdot((a_X)_!(a_X)^*\mathbf{Q}^H)$$

and so forth. These mixed Hodge structures coincide with those of Deligne

[De1] at least if X is globally embeddable into a smooth variety (for example, if X is quasiprojective). We let

$$\mathbf{Q}_X^H = (a_X)^*\mathbf{Q}^H$$

$$\mathbf{D}_X^H = (a_X)^!\mathbf{Q}^H$$

$$\mathbf{IC}_X^H = j_{!*}\mathbf{Q}_U^H[n]$$

note that

$$\mathbf{DIC}_X^H = \mathbf{IC}_X^H(n)$$

where (n) denotes the Tate twist, which can be defined by $\boxtimes\mathbf{Q}^H(n)$.

As in the category of mixed complexes of l-adic sheaves, there is a weight filtration in MHM(X) and the notions of 'weight $\leqslant k$', etc., in D^bMHM(X) such that [BBD 5.1.8, 5.1.14, 5.3.2; Sa1 4.5]:

In MHM(pt) these are the usual notions of mixed Hodge theory.

$f_!$, f^* preserve weight $\leqslant k$.

f_*, $f^!$ preserve weight $\geqslant k$.

$j_{!*}$ preserves weight $= k$.

In particular, this implies the (local) purity of the intersection complex $\mathbf{IC}_X^H$, since $\mathbf{Q}_U^H[n]$ is pure and $\mathbf{IC}_X^H = j_{!*}(\mathbf{Q}_U^H[n])$ for j as in 1.3. This is the same argument as in the l-adic case, which uses stability by direct images and subquotients. Note that local purity can also be proved using only the existence of the weight filtration, since the weight filtration of $\mathbf{IC}_X^H$ must be trivial by the simplicity of $\mathbf{IC}_X$ and the faithfulness and exactness of the forgetful functor rat.

In the l-adic case, the existence of the weight filtration is proved [BBD 5.3.5] after showing the purity of intersection complexes with twisted coefficients [BBD 5.3.2]; in fact, this existence is not used in the definition of 'weight $\leqslant k$', etc., nor in the proof of its stability by the functors as above. However, in the mixed Hodge module case, the existence of the weight filtration is more or less assumed from the beginning, since the Hodge filtration and the rational structure are not together strong enough to determine the weight filtration uniquely. The latter fact is of course even true for mixed Hodge structures.

2. Links

We define a link L of a subvariety Z in a variety X to be the level set of a suitable distance function d whose zero locus is Z. We put three kinds of conditions on this distance function, and consequently get three different kinds of links: a 'weak topological link', which will be used for the rest of this paper, an 'analytic link', which can be defined if Z is compact, and a 'topological link', which is the strongest kind of link. These links are stratified topological pseudomanifolds [GM2 1.1]. We expect that an analytic link is a topological link. We also expect that an analytic link is unique in some sense as stratified topological pseudomanifold.

Since we cannot expect these links to be in general unique, we also introduce functors which describe the cohomology, homology and intersection cohomology of a link intrinsically. These are analogous with the vanishing cycle functors [DE3]. The 'local link cohomology functor' assigns a complex of sheaves on Z to a complex of sheaves on $X\backslash Z$. We will only apply this functor to the constant sheaf, the dualizing complex and the intersection complex on $X\backslash Z$. The 'global link cohomology functors' are obtained by composing the local link cohomology functor with the direct image (with and without compact supports) of the map of Z to a point. These link cohomology functors are intrinsically defined. Furthermore, they are naturally related to mixed theories (Hodge or l-adic). The connection between the topological types of link and the cohomological types of link is that a topological link determines a functor which is canonically isomorphic to the local link cohomology functor, and that a weak topological link correspondingly determines a functor canonically isomorphic to the global link cohomology functor.

2.1. Let X be an irreducible complex algebraic variety of dimension n, and Z a closed subvariety. Let

$$i: Z \hookrightarrow X,$$

$$j: U = X\backslash Z \hookrightarrow X.$$

In each of the following three definitions, there will be an open neighborhood N of Z in X and a *distance function*

$$d: N \to [0, 1)$$

with $Z = d^{-1}(0)$. Let N^* be defined as $N\backslash Z$ with stratification induced by a complex analytic Whitney stratification of X.

The link L will be of (real) dimension $2n - 1$, have odd-dimensional strata, and have an orientation induced from that of X. We will have

$$L = d^{-1}(\varepsilon)$$

for $0 < \varepsilon$ suitably small. Let

$$k\colon L \hookrightarrow U$$

be the inclusion whose image is $d^{-1}(\varepsilon)$. Also, N^* will be a rational homology manifold if and only if L is.

2.2. A stratified topological pseudomanifold L is a *weak topological link* if there exist an open neighborhood N of Z in X and a continuous distance function $d\colon N \to [0,1)$ such that $Z = d^{-1}(0)$ and such that the following conditions are satisfied:

(i) There is a stratified homeomorphism $\alpha\colon N^* \simeq L \times (0, 1)$ such that d is identified with the second projection.
(ii) For any F in $D_c^b(N^*)$ such that $\underline{\mathbf{H}}^i F$ is locally constant on each stratum of N^*, the natural morphisms $(\underline{\mathbf{H}}^{\cdot} d_* j_* F)_0 \to H^{\cdot}(Z, i^* j_* F)$ are isomorphisms.

Note that condition (ii) is always satisfied if d is proper.

2.3. Suppose Z is compact. A stratified topological pseudomanifold L is an *analytic link* of Z in X, if there exists a nonnegative real analytic distance function $d\colon N \to [0, 1)$ where N is an open neighborhood of Z in X, such that $Z = d^{-1}(0)$ and L is isomorphic to $d^{-1}(\varepsilon)$ as stratified space (up to a refinement of stratification) for $0 < \epsilon \ll 1$. (The stratification of $d^{-1}(\varepsilon)$ is induced by a complex Whitney stratification of X compatible with Z).

We may assume that d is proper over its image $[0, \delta)$ by taking two relatively compact open neighborhoods U_1, U_2 of Z in N such that $\bar{U}_1 \subset U_2$, and restricting d to $U_1 \backslash d^{-1} d(\overline{U_2 \backslash U_1})$.

When X is quasiprojective, analytic links can be shown to exist by arguments similar to [Du].

2.4. PROPOSITION. *An analytic link is a weak topological link.*

Proof. Condition (ii) is satisfied by the above remark. By the curve selection lemma we may assume that the restriction of d to the strata disjoint with Z is smooth. Thus, by the Thom isotopy theorem, the restriction of the Whitney stratification to $d^{-1}(\varepsilon)$ is well-defined and $N \backslash Z$ is isomorphic to $d^{-1}(\varepsilon) \times (0, \delta)$ as stratified space (See [Du]). Thus condition (i) is checked. □

2.5. A stratified topological pseudomanifold L is a *topological link* of Z in X, if there exist a continuous proper surjective map (the *retraction* of L onto Z)

$$r\colon L \to Z$$

and a homeomorphism

$$\alpha\colon N \to \tilde{N}$$

of an open neighborhood N of Z in X with an open neighborhood $\tilde{N}$ of Z in the open mapping cylinder $\mathrm{Cyl}^\circ(r)$ of r which is the identity on Z and induces a stratification preserving homeomorphism $N\backslash Z \simeq \tilde{N}\backslash Z$. Here $\mathrm{Cyl}^\circ(r)$ is defined as $Z \amalg L \times [0, 1)/r(x) \sim (x, 0)$, and the stratifications of $N\backslash Z$ and $\tilde{N}\backslash Z$ in $L \times (0, 1)$ are induced by a complex analytic Whitney stratification of X (defined on a neighborhood of Z) compatible with Z and the stratification of L, respectively.

This definition is inspired by [De3]. In this case we define the distance function d from Z to be the composition of α with the natural projection $\mathrm{Cyl}^\circ(r) \to [0, 1) \subset \mathbf{R}$. We define the retraction $\tilde{r}$ of N onto Z to be the composition of α with the natural morphism $\mathrm{Cyl}^\circ(r) \to Z$.

In the definition of topological link, we may assume that $\tilde{N} = \mathrm{Cyl}^\circ(r)$ and that ∂N is identified with L by replacing d with $(d/g) \circ r$ and N with the subset $\alpha^{-1}[Z \amalg \{(x, a) \in L \times (0, 1)\colon a < g(r(x))\}]$, where g is a continuous function on Z whose value is in $(0, 1]$ and goes to zero appropriately at infinity. Then $r = \tilde{r}k$ where k is as in 2.1.

If $X\backslash Z$ is smooth, the existence of topological links can be shown by reducing to the case where X is smooth and Z is a divisor with normal crossings. The existence of topological links in general will be treated elsewhere.

2.6. PROPOSITION. *A topological link is a weak topological link.*

Proof. We may assume $\tilde{N} = \mathrm{Cyl}^\circ(r)$ as above. Then the condition (i) is satisfied. For (ii) we have

$$(\underline{\mathbf{H}}^\cdot d_* j_* F)_0 = \varinjlim H^\cdot(d^{-1}(0, \delta), F)$$

and

$$H^\cdot(Z, i^* j_* F) = \varinjlim H^\cdot(V\backslash Z, F)$$

where V runs over open neighborhoods of Z in X. Since every V contains an open neighborhood V_g corresponding by α to $Z \amalg \{(x, a) \in L \times (0, 1)\colon a < g(r(x))\}$ for some continuous function g, and $H^\cdot(V_g\backslash Z, F)$ is independent of g by the hypothesis on F, we get the assertion. □

When Z is compact we expect that an analytic link is a topological link. The argument is similar to the proof of the Thom isotopy theorem, but also the limit of integral curves must be controlled using a good partition of unity. This problem will be treated elsewhere.

2.7. Assume that the stratifications of the objects in $D_c^b(U)$ and $D_c^b(Z)$ are algebraic. In the notation of 2.1, the *local link cohomology functor* Λ_Z of Z in X is defined for $F \in D_c^b(U)$ by

$$\Lambda_Z F = i^* j_* F$$

as an object of $D_c^b(Z)$. We also let

$$\Lambda'_Z F = i^! j_! F.$$

Note that the canonical isomorphism of 3.1 implies that

$$\Lambda'_Z F[+1] = \Lambda_Z F$$

and that there also is the duality isomorphism

$$\mathbf{D}\Lambda_Z = \Lambda'_Z \mathbf{D}$$

We will take F to be $\mathbf{Q}_U$, $\mathbf{D}_U$ and $\mathbf{IC}_U$ in the applications of this paper.

2.8. With the above notation, the *global link cohomology functors* of Z in X are defined by

$$(a_Z)_* \Lambda_Z F \quad \text{and} \quad (a_Z)_! \Lambda_Z F$$

for F as above, where $(a_Z)_*$: $Z \to pt$. Note that

$$\mathbf{D}(a_Z)_* \Lambda_Z F = (a_Z)_! \Lambda'_Z \mathbf{D} F.$$

2.9. PROPOSITION. *If L is a topological link of Z in X and if F is in $D_c^b(U)$ and $\underline{\mathbf{H}}^i F$ is locally constant on each stratum of the stratification of X in 2.5, then in $D^b(Z)$ there is a natural functorial isomorphism*

$$\Lambda_Z F = r_* k^* F.$$

Proof. Let $\tilde{r}'$ be the restriction of $\tilde{r}$ to N^*. We have natural morphisms of $\tilde{r}'_* F$ to $i^* j_* F$ and $r_* k^* F$, and since r is proper it is easy to check that they are quasi-isomorphisms. □

2.10. PROPOSITION. *If L is a weak topological link of Z in X, if F is in $D_c^b(U)$ and if $\underline{\mathbf{H}}^i F$ is locally constant on each stratum of the stratification of X in 2.2(i), then there are natural functorial isomorphisms in $D^b(pt)$*

$$(a_Z)_* \Lambda_Z F = (a_L)_* k^* F \quad \text{and} \quad (a_Z)_! \Lambda_Z F = (a_L)_! k^* F$$

Proof. The natural morphism $k^! \mathbf{Q}_U \otimes k^* F \to k^! F$ gives a natural isomorphism $k^* F[-1] = k^! F$. So by duality it is enough to show the first isomorphism. There is a commutative diagram

$$\begin{array}{ccccccc} Z & \xrightarrow{i} & N & \xleftarrow{j} & N^* & \xleftarrow{k} & L \\ \downarrow{\scriptstyle a_Z} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle a_L} \\ 0 & \xrightarrow{i'} & [0,1) & \xleftarrow{j'} & (0,1) & \xleftarrow{k'} & \varepsilon \end{array}$$

By 2.2(ii) and the commutativity of the diagram we have

$$(a_Z)_* \Lambda_Z F = i'^* d_* j_* F = i'^* j'_* d'_* F.$$

By 2.2(i), $\underline{\mathbf{H}}^{\bullet} d'_* F$ is constant on $(0,1)$, and the natural morphisms

$$i'^* j'_* d'_* F \leftarrow \mathbf{R}\Gamma([0,1), j'_* d'_* F) = \mathbf{R}\Gamma((0,1), d'_* F) \to (a_L)_* K^* F$$

are quasi-isomorphisms. □

If L is a topological link, the isomorphisms of 2.10 coincide with the direct images of the isomorphisms of 2.9.

2.11. PROPOSITION. *If L is a weak topological link, then*

(i) $H^k(L) = H^k((a_Z)_* \Lambda_Z \mathbf{Q}_U)$
(ii) $H_k(L) = H^{-k}((a_Z)_! \Lambda'_Z \mathbf{D}_U)$
(iii) $IH^{n+k}(L) = H^k((a_Z)_* \Lambda_Z \mathbf{IC}_U) = H^{k+1}((a_Z)_* \Lambda'_Z \mathbf{IC}_U)$

and the same for $H^k_c(L)$, $H^{BM}_k(L)$ and $IH^{n+k}_c(L)$ with $(a_Z)_*$ and $(a_Z)_!$ exchanged.

Proof. This follows immediately from 2.10. For the isomorphism (i), let $F = \mathbf{Q}_U$. For (ii), let $F = \mathbf{D}_U$ and use $k^! \mathbf{D}_U = \mathbf{D}_L$ [Bo V10.11]. For (iii), let $F = \mathbf{IC}^{\text{top}}_U$ and use $k^! \mathbf{IC}^{\text{top}}_U = \mathbf{IC}^{\text{top}}_L$ [GM2 5.4.1] and $\mathbf{IC}^{\text{top}}_U = \mathbf{IC}_U[n]$. The proofs in the other cases are similar. □

3. Mixed Hodge structures

In this section we derive various elementary properties for the mixed Hodge structure on the cohomology, homology and intersection homology of a link. Throughout the section, the notation is that of 2.1.

By the general theory of mixed Hodge modules and making the obvious modifications (replacing $\mathbf{Q}_U$ by $\mathbf{Q}^H_U$ and so forth), the right hand side of the six isomorphisms in 2.11 have mixed Hodge structures. We define the mixed Hodge structures on the cohomology, homology and intersection cohomology groups associated to a weak topological link by these isomorphisms. Various maps and operations on the right-hand groups are compatible with the corresponding operations on a weak topological link by [GM II]. Note that the results of this section and the following sections are actually proved for the groups on right-hand of 2.11. We define the mixed Hodge structures on the intersection homology groups by setting

$$IH_k(L) = IH_c^{2n-1-k}(L)(n),$$
$$IH_k^{BM}(L) = IH^{2n-1-k}(L)(n)$$

If U is smooth, these mixed Hodge structures have already been found by the methods of Deligne [El, St1, Na]. The mixed Hodge structures here agree with these, because $K := C(\Omega^{\cdot}_{\tilde{X}} \to \varepsilon_* \Omega^{\cdot}_{\tilde{D}_{\cdot}} \oplus \Omega_{\tilde{X}}(\log D))$ together with natural Hodge and weight filtrations is isomorphic to the filtered de Rham complex of the underlying filtered complex of $\mathscr{D}$-Modules of $C(\tilde{j}_! \mathbf{Q}_U^H \to \tilde{j}_* \mathbf{Q}_U^H)$ in a bifiltered derived category, where $\pi: (\tilde{X}, D) \to (X, Z)$ is an embedded resolution with $\tilde{j}: U = X \backslash Z \to \tilde{X}$, and $\varepsilon: \tilde{D}_{\cdot} \to \tilde{X}$ the semisimplicial space defined by the intersections of the irreducible components of D. In fact (K, F) is a filtered differential complex with weight filtration W as in [Sa3 2.5.8], and it can be checked that the filtered $\mathscr{D}$-Modules associated to $K_1 := C(\Omega^{\cdot}_{\tilde{X}} \to \varepsilon_* \Omega^{\cdot}_{\tilde{D}_{\cdot}})[-1]$ and $K_2 := \Omega_{\tilde{X}}(\log D)$ (together with the weight filtration) are canonically isomorphic to the underlying filtered $\mathscr{D}$-modules of $\tilde{j}_! \mathbf{Q}_U^H$ and $\tilde{j}_* \mathbf{Q}_U^H$ respectively by the calculations as in [Sa2 §3]. Moreover the morphism $K_1 \to K_2$ induced by the natural morphism $\Omega^{\cdot}_{\tilde{X}} \to \Omega_{\tilde{X}}(\log D)$ corresponds to the natural morphism $\tilde{j}_! \mathbf{Q}_U^H \to \tilde{j}_* \mathbf{Q}_U^H$ by the inverse of the de Rham functor [Sa3 §2], which is the identity on U.

3.1. LEMMA. *In the notation of* 2.1:

(i) *For* $F \in D^b\mathrm{MHM}(U)$ *there are isomorphisms in* $D^b\mathrm{MHM}(X)$

$$i_* i^* j_* F = \mathrm{cone}(j_! F \to j_* F) = i_! i^! j_! F[+1]$$

(ii) *For* $G \in D^b\mathrm{MHM}(X)$ *there are isomorphisms in* $D^b\mathrm{MHM}(Z)$

$$i^* j_* j^* G = \mathrm{cone}(i^! G \to i^* G) = i^! j_! j^! G[+1]$$

Proof. (i) For $G \in D^b\mathrm{MHM}(X)$ there is a distinguished triangle [Sa1 4.4.1, BBD 1.4.34, GM2 1.11]

$$j_! j^! G \to G \to i_* i^* G.$$

Letting $G = j_* F$ gives

$$j_! F \to j_* F \to i_* i^* j_* F$$

which gives the first isomorphism of (i). The dual of this triangle with $\mathbf{D}F$ replacing F is

$$i_! i^! j_! F \to j_! F \to j_* F$$

which shows the second isomorphism of (i).

(ii) The dual of the first triangle in the proof of (i) is

$$i_* i^! G \to G \to j_* j^* G.$$

Taking i^* gives

$$i^! G \to i^* G \to i^* j_* j^* G$$

which shows the first isomorphism of (ii).

The second follows by duality as above. □

3.2. PROPOSITION. *If L is a weak topological link of Z in X, the natural morphisms*

$$H^k(L) \to IH^k(L) = IH^{BM}_{2n-1-k}(L)(-n) \to H^{BM}_{2n-1-k}(L)(-n)$$

(and similarly with compact supports) are morphisms of mixed Hodge structures. These morphisms are isomorphisms if $X \backslash Z$ is a rational homology manifold in a neighborhood of Z.

Proof. There is a morphism in $D^b\mathrm{MHM}(U)$ [Sa 4.5.8; GM2 5.1]

$$\mathbf{Q}^H_U[n] \to \mathbf{IC}^H_U.$$

The dual of this is

$$\mathbf{IC}^H_U(n) \to \mathbf{D}^H_U[-n]$$

which combines with the first morphism to give

$$\mathbf{Q}^H_U[n] \to \mathbf{IC}^H_U \to \mathbf{D}^H_U(-n)[-n].$$

Taking $i^* j_*$ and $H^{k-n}(Z, -)$ gives

$$H^k(Z, i^* j_* \mathbf{Q}^H_U) \to H^{k-n}(Z, i^* j_* \mathbf{IC}^H_U) \to H^{k-2n}(Z, i^* j_* \mathbf{D}^H_U)(-n)$$

so we get the first assertion. The statement with compact supports is similar. The last assertion is well-known, and follows immediately from the definition of intersection complex [GM2]. □

3.3. PROPOSITION. *If L is a weak topological link of Z in X, the duality isomorphism* [Bo p. 149]

$$IH^k(L) \simeq [IH_c^{2n-1-k}(L)(n)]^*$$

is an isomorphism of mixed Hodge structures.

Proof. In $D^b\mathrm{MHM}(pt)$

$$\mathbf{D}(a_X)_* i_! i^! j_! \mathbf{IC}_U^H = (a_X)_! i_* i^* j_* \mathbf{IC}_U^H(n) = (a_X)_! i_! i^! j_! \mathbf{IC}_U^H(n)[+1]$$

where the second isomorphism is by 3.1(i). Taking cohomology, we get the proposition. □

3.4. PROPOSITION. *If L is a weak topological link of Z in X, the natural intersection pairings* [GM2 5.2]

$$H^p(L) \otimes H^q(L) \to H^{p+q}(L)$$

$$H^p(L) \otimes IH^q(L) \to IH^{p+q}(L)$$

$$H^p(L) \otimes H_q^{BM}(L) \to H_{q-p}^{BM}(L)$$

$$IH^p(L) \otimes IH^q(L) \to H_{2n-1-p-q}^{BM}(L)(-n)$$

(and similarly with compact supports) are morphisms of mixed Hodge structures.

Proof. Let Δ: $X \to X \times X$ be the diagonal embedding. We have the natural morphisms

$$\mathbf{Q}_X^H \otimes F := \Delta^*(\mathbf{Q}_X^H \boxtimes F) \simeq F$$

for $F = \mathbf{Q}_X^H$, $\mathbf{IC}_X^H$ and $\mathbf{D}_X^H$, and

$$\mathbf{IC}_X^H \otimes \mathbf{IC}_X^H := \Delta^*(\mathbf{IC}_X^H \boxtimes \mathbf{IC}_X^H) \to \mathbf{D}_X^H(-n) = a^!_X \mathbf{Q}^H(-n)$$

In fact the first is clear by $\mathbf{Q}_X^H \boxtimes = pr_2^*$ and the functoriality of pull-backs. The second is equivalent to

$$(a_X)_! \mathbf{IC}_X^H \otimes \mathbf{IC}_X^H \to \mathbf{Q}^H(-n)$$

by the adjoint relation. For a nonsingular Zariski-open dense subset V of X we have

$$H_c^i(V, \mathbf{IC}_V \otimes \mathbf{IC}_V) \xrightarrow{\sim} H_c^i(X, \mathbf{IC}_X \otimes \mathbf{IC}_X) \quad \text{for } i \geqslant 0$$

using a stratification and the support condition of $\mathbf{IC}_X$. Moreover they are equal to 0 for $i > 0$ and $\mathbf{Q}(-n)$ for $i = 0$, because $\mathbf{IC}_V \otimes \mathbf{IC}_V = \mathbf{Q}_V[2n]$ and V is connected since X is irreducible. So the morphism is obtained by

$$\mathrm{Hom}((a_X)_!\mathbf{IC}_X^H \otimes \mathbf{IC}_X^H, \mathbf{Q}^H(-n)) \xrightarrow{\sim} \mathrm{Hom}(H_c^0(X, \mathbf{IC}_X^H \otimes \mathbf{IC}_X^H), \mathbf{Q}^H(-n)) \xrightarrow{\sim} \mathbf{Q}.$$

By the adjoint relation for j^*j_*, we have the morphisms

$$j_*\mathbf{IC}_U^H \otimes j_*\mathbf{IC}_U^H \to j_*\mathbf{D}_U^H(-n)$$

and

$$i^*j_*\mathbf{IC}_U^H \otimes i^*j_*\mathbf{IC}_U^H \to i^*j_*\mathbf{D}_U^H(-n)$$

because $\Delta^* \circ (i \times i)^* = i^* \circ \Delta^*$ (same for the others). Then the assertion follows from

$$(a_{X \times X})_*(F \boxtimes G) = (a_X)_*F \otimes (a_X)_*G$$

and the same for $(a_X)_!$, and the canonical morphism

$$F \boxtimes G \to \Delta_*\Delta^*(F \boxtimes G) = \Delta_*(F \otimes G).$$

Note $\Delta_! = \Delta_*$ because Δ is a closed embedding. □

In the l-adic case this result follows by an argument similar to [GM 5.2]. In general we have a canonical isomorphism

$$\mathrm{Hom}(F \otimes G, \mathbf{D}_X^H) \simeq \mathrm{Hom}(F, \mathbf{D}G)$$

(i.e. $\mathrm{Hom}(\mathbf{Q}_X^H, \underline{\mathrm{Hom}}(F, G)) \simeq \mathrm{Hom}(F, G)$ by duality) for $F, G \in D^b\mathrm{MHM}(X)$, and we can use it to obtain the morphism $\mathbf{IC}_X^H \otimes \mathbf{IC}_X^H \to \mathbf{D}_X^H(-n)$.

If S is the family of supports in U defined by $S = \{C \subset U : C \text{ is closed in } X\}$, then $H_S^k(U, F) = H^k(X, j_!F)$; see [Go]. If X is compact, then $H_S^k(U, F) = H_c^k(U, F)$.

3.5. PROPOSITION. *If L is a weak topological link of Z in X, the following are exact sequences of mixed Hodge structures:*

(i) $\cdots \to H^k_S(U) \to H^k(U) \to H^k(L) \to \cdots$
(ii) $\cdots \to IH^k_S(U) \to IH^k(U) \to IH^k(L) \to \cdots$
(iii) $\cdots \to H^k_Z(X) \to H^k(Z) \to H^k(L) \to \cdots$

Proof. By 3.1(i) for $F \in D^b\mathrm{MHM}(U)$ there is a distinguished triangle

$$j_!F \to j_*F \to i_*i^*j_*F.$$

The long exact sequence of hypercohomology on X with $F = \mathbf{Q}^H_U$ gives (i), and with $F = \mathbf{IC}^H_U$ and 3.1(i) gives (ii).

By 3.1(ii) there is for $G \in D^b\mathrm{MHM}(X)$ a distinguished triangle

$$\to i^!G \to i^*G \to i^*j_*j^*G.$$

The long exact sequence of hypercohomology on Z with $G = \mathbf{Q}^H_X$ gives (iii). □

4. Semipurity

This section contains the main result of this paper.

4.1. THEOREM. *Let L be a weak topological link of Z in X. If* $\dim X = n$ *and* $\dim Z \leqslant d$, *then the intersection cohomology* $IH^k_c(L)$ *has weights* $\leqslant k$ *for* $k < n - d$.

For the case Z a point, see also [Sa2 1.18].

Proof. We use the notation of 3.4. By [BBD 1.4.23(ii)]

$$j_{!*}\mathbf{IC}_U = \tau^Z_{\leqslant -1} j_*\mathbf{IC}_U$$

where $\tau^Z_{\leqslant -1}$ is from [BBD 1.4.13]. Now $\mathbf{IC}_X = j_{!*}\mathbf{IC}_U$, so the above together with the distinguished triangle of [BBD 1.4.13] with $X = j_*\mathbf{IC}_U$ gives a triangle

$$\to \mathbf{IC}_X \to j_*\mathbf{IC}_U \to i_*\, {}^p\tau_{>-1} i^*j_*\mathbf{IC}_U \to$$

Taking i^* gives

$$\to i^*\mathbf{IC}_X \to i^*j_*\mathbf{IC}_U \to {}^p\tau_{>-1} i^*j_*\mathbf{IC}_U \to$$

so $i^*\mathbf{IC}_X \to i^*j_*\mathbf{IC}_U$ is identified with ${}^p\tau_{\leqslant -1} i^*j_*\mathbf{IC}_U \to i^*j_*\mathbf{IC}_U$ [BBD 1.3.3].

Since rat: $\mathrm{MHM}(X) \to \mathrm{Perv}(X)$ is faithful and $\mathrm{rat} \circ \tau = {}^p\tau \circ \mathrm{rat}$, we have

$$\to i^*\mathbf{IC}_X^H \to i^*j_*\mathbf{IC}_U^H \to \tau_{>-1}i^*j_*\mathbf{IC}_U^H \to$$

Thus there is a long exact sequence

$$\to H_c^{k-n}(Z, i^*\mathbf{IC}_X^H) \to H_c^{k-n}(Z, i^*j_*\mathbf{IC}_U^H) \to H_c^{k-n}(Z, \tau_{>-1}i^*j_*\mathbf{IC}_U^H) \to$$

The middle term above is $IH_c^k(L)$. The left term has weight $\leqslant k$, since $\mathbf{IC}_X^H$ has weight n, which implies $i^*\mathbf{IC}_X$ has weight $\leqslant n$. The right term is zero for $k - n < -d = -\dim Z$, since $\underline{\mathbf{H}}^i({}^p\tau_{>-1}i^*j_*\mathbf{IC}_U) = 0$ for $i < -d$ as a consequence of the support condition for a perverse sheaf. □

4.2. COROLLARY. *The group $IH^k(L)$ has weights $> k$ for $k \geqslant n + d$.*

This follows immediately from 4.1 and 3.3.

5. Applications

In this section the results of the previous sections are applied to the topology of a complex variety. We use again the notation of 2.1. Let L be a weak topological link of Z in X. Let

$$v\colon IH^k(L) = IH^{BM}_{2n-1-k}(L)(-n) \to H^{BM}_{2n-1-k}(L)(-n)$$

be the morphism from 3.2. The following theorem is concerned with the composite

$$IH_c^p(L) \times IH_c^q(L) \xrightarrow{\cup} H_{2n-1-p-q}(L)(-n) \xrightarrow{u} H^{BM}_{2n-1-p-q}(L)(-n)$$

where the first morphism is the cup product from 3.4, and the second morphism is the obvious one.

5.1. THEOREM. *Let L be a weak topological link of Z in X, and assume that* $\dim X = n$ *and* $\dim Z \leqslant d$. *If $\alpha \in IH_c^p(L)$, $\beta \in IH_c^q(L)$, and $u(\alpha \cup \beta)$ is the image of v, with $p, q < n - d$ and $p + q \geqslant n + d$, then $u(\alpha \cup \beta) = 0$.*

Proof. Weight $\alpha \leqslant p$ and weight $\beta \leqslant q$ by 4.1, so weight $u(\alpha \cup \beta) \leqslant p + q$ by 3.4. Since weight $(\operatorname{im} v) > p + q$ by 4.2 and 3.2, this implies $u(\alpha \cup \beta) = 0$. □

Let

$$w\colon H_c^k(L) \to H^k(L)$$

be the natural map.

5.2. THEOREM. *Let X, Z and L be as in 5.1. Suppose $X\backslash Z$ is a rational homology manifold in a neighborhood of Z. If $k_1, \dots, k_m < n - d$ and $k = k_1 + \cdots + k_m \geqslant n + d$, then the composition*

$$H_c^{k_1}(L) \times \cdots \times H_c^{k_m}(L) \overset{\cup}{\to} H_c^k(L) \overset{w}{\to} H^k(L)$$

is the zero morphism.

The proof is similar to the proof of 5.1.

Note that since $H_c^{2n-1}(L)$ has weight $2n$ by 3.3, the above theorem is true without the composition with w when $k = 2n - 1$. Also note that if Z is compact, both u and w are isomorphisms.

5.3. EXAMPLES. Let

$$T^k = \underbrace{S^1 \times \cdots \times S^1}_{k}$$

(i) If $n \geqslant 2$ and $d = 0$, then $L = (T^{2n-1}$ with two odd-dimensional submanifolds identified) cannot be a weak topological link of Z in X: In Theorem 5.1 take $p = q = n - 1$. The map $IH^{n-1}(L) \otimes IH^{n-1}(L) \to H_1(L)(-n)$ is nonzero and factors through $IH_1(L)(-n) \to H_1(L)(-n)$ as is seen by using $IH(L) \simeq H(T^{2n-1})$ [GM1 4.2] and the diagram

$$\begin{array}{ccccc} H^{n-1}(T^{2n-1}) \otimes H^{n-1}(T^{2n-1}) & \to & H_1(T^{2n-1})(-n) & \overset{\simeq}{\leftarrow} & IH_1(T^{2n-1})(-n) \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ IH^{n-1}(L) \otimes IH^{n-1}(L) & \to & H_1(L)(-n) & \leftarrow & IH_1(L)(-n) \end{array}$$

(ii) If Z is compact and $n - d > 1$, then T^{2n-1} cannot be a weak topological link of Z in X: In Theorem 5.2 take $k_i = 1$.

Note that the above examples L are compact oriented topological pseudomanifolds with odd-dimensional strata, so that they are not excluded from being links by [Su].

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