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# Fano bundles of rank 2 on surfaces

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0. We say that a vector bundle $\mathscr{E}$ on $X$ is Fano if $\mathbb{P}(\mathscr{E})$ is a Fano manifold. In [12], we classified rank-2 Fano vector bundles over the complex projective space $\mathbb{P}^3$ and over a smooth quadric $\mathbb{Q}_3 \subset \mathbb{P}^4$. This paper is thus complementary to [12].

Ruled Fano 3-folds were classified by Demin already in 1980. However, his paper [2] contains gaps and omissions, one of them corrected in the addendum. Here we correct the other. Moreover, we present a vector-bundle approach to the problem, i.e., we try to get our results by pure vector bundle methods. Even if the proofs that utilize Mori-Mukai-Iskovskich list of all Fano threefolds are sometimes shorter and even if we did not succeed, in some cases, to avoid using the classification list mentioned above (namely, to exclude the cases $c_1 = 0$, $c_2 = 6$ over $\mathbb{P}^2$ and $c_1 = (-1, -1)$, $c_2 = 5$ or 6 over $\mathbb{P}^1 \times \mathbb{P}^1$), we believe that our method is worthy of dealing with, also because it works for ruled Fano 4-folds, as well, see [12]. Besides, we are able to draw some conclusions on the geometry of stable vector bundles with $c_1 = 0$, $c_2 = 2$ or $c_2 = 3$, one of them being the 1-ampleness of such bundles.

THEOREM. *There are* 21 *types of ruled Fano* 3*-folds* $V = \mathbb{P}(\mathscr{E})$ *with a* 2*-bundle* $\mathscr{E}$ *on a surface. Up to a twist, they are the following:*

| | $\mathscr{E}$ | $-K^3$ | # in M-M-I list, [9] |
|---|---|---|---|
| over $\mathbb{P}^2$: | | | |
| 1 | $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ | 62 | 36, Table 2 |
| 2 | $\mathcal{O} \oplus \mathcal{O}(1)$ | 56 | 35 |
| 3 | $\mathcal{O} \oplus \mathcal{O}$ | 54 | 34 |
| 4 | $T_{\mathbb{P}^2}(-2)$ | 48 | 32 |
| 5 | $0 \to \mathcal{O} \to \mathscr{E} \to J_x \to 0, \quad x \in \mathbb{P}^2$ | 46 | 31 |
| 6 | $\mathscr{E}$ *stable with* $c_1 = 0$, $c_2 = 2$, *thus:* | | |
| | $0 \to \mathcal{O}(-1)^2 \to \mathcal{O}^4 \to \mathscr{E}(1) \to 0$ | 38 | 27 |
| 7 | $\mathscr{E}$ *stable*, $\mathscr{E}(1)$ *spanned*, $c_1 = 0$, $c_2 = 3$, | | |
| | *thus:* $0 \to \mathcal{O}(-2) \to \mathcal{O}^3 \to \mathscr{E}(1) \to 0$ | 30 | 24 |

over $\mathbb{P}^1 \times \mathbb{P}^1$:

| | | | |
|---|---|---|---|
| 8 | $\mathcal{O}(-1,-1) \oplus \mathcal{O}$ | 52 | 31, Table 3 |
| 9 | $\mathcal{O} \oplus \mathcal{O}$ | 48 | 27 |
| 10 | $\mathcal{O}(0,1) \oplus \mathcal{O}$ | 48 | 28 |
| 11 | $\mathcal{O}(0,-1) \oplus \mathcal{O}(-1,0)$ | 44 | 25 |
| 12 | $\mathscr{E}$ *stable*, $\mathscr{E}(1,1)$ *spanned*, $c_1 = (-1,-1), c_2 = 2$, thus: $0 \to \mathcal{O}(-1,-1) \to \mathcal{O}^3 \to \mathscr{E}(1,1) \to 0$ | 36 | 17 |

over the Hirzebruch surface $F_1$, with a blow-down map $\beta: F_1 \to \mathbb{P}^2$:

| | | | |
|---|---|---|---|
| 13 | $\beta^*(\mathcal{O} \oplus \mathcal{O}(-1))$ | 50 | 30, Table 3 |
| 10 | $\mathcal{O} \oplus \mathcal{O}$ | 48 | 28 |
| 14 | $\beta^*(T_{\mathbb{P}^2}(-2))$ | 42 | 24 |

over a Del Pezzo surface being a blow-up of $k \leqslant 8$ points on $\mathbb{P}^2$:

| | | | |
|---|---|---|---|
| 15–21 | $\mathcal{O} \oplus \mathcal{O}$ | $6(9-k)$ | 10/4. |

## 1. Preliminaries

Throughout the paper $\mathscr{E}$ is a rank-2 vector bundle on a smooth complex projective surface $S$ and $\xi_{\mathscr{E}}$ is the antitautological line bundle on $V = \mathbb{P}(\mathscr{E})$. By $p$ we denote the projection morphism $p: \mathbb{P}(\mathscr{E}) \to S$ and by $F$ – the fibre of $p$. We have the following exact sequence on $V = \mathbb{P}(\mathscr{E})$:

$$0 \to \mathcal{O}_V \to p^*(\mathscr{E})^\vee \otimes \xi_{\mathscr{E}} \to T_{V|S} \to 0. \tag{1.1}$$

with the relative tangent bundle $T_{V|S}$ fitting in

$$0 \to T_{V|S} \to T_V \to p^*(T_S) \to 0. \tag{1.2}$$

We call (1.1) the relative Euler sequence. We then obtain

(1.3) COROLLARY. $c_1(V) = p^*(c_1(S) - c_1(\mathscr{E})) + 2\xi_{\mathscr{E}}$.

The theorem of Leray and Hirsch yields that in the cohomology ring of $V$ the following holds

$$\xi_{\mathscr{E}}^2 - p^*(c_1(\mathscr{E}))\xi_{\mathscr{E}} + p^*(c_2(\mathscr{E})) = 0. \tag{1.4}$$

We infer that $S$ must be a Del Pezzo surface:

(1.5) PROPOSITION. *If $V = \mathbb{P}(\mathscr{E})$ is a ruled Fano threefold, then $S$ is a Del Pezzo surface.*

*Proof.* See [2]; and [12] for a more general version.

(1.6) PROPOSITION. *In the above notation, we have:*

$$b_2(V) = b_2(S) + 1, \quad \text{and} \quad b_3(V) = 0.$$

*Proof.* The first statement is obvious. To prove that $b_3$ vanishes, let us observe first that $h^{3,0}(V) = h^3(\mathscr{O}_V) = h^0(K_V) = 0$ by the Kodaira Vanishing Theorem. To show that $b_3 = 0$, it is then sufficient to prove that $b^{1,2} = b^{2,1} = 0$, i.e., that $h^1(\Omega^2) = h^2(\Omega^1) = 0$. Dualizing (1.2), we get

$$0 \to p^*(\Omega_S) \to \Omega_V \to (T_{V/S})^\vee \to 0.$$

Because $h^2(p^*(\Omega_S) = h^2(\Omega_S) = h^1(K_S) = h^1(\mathscr{O}_S) = 0$ by rationality, we have to show only that $h^2(T^\vee_{V|S}) = 0$. Recall that $T^\vee_{V|S}$ fits in a short exact sequence

$$0 \to (T_{V/S})^\vee \to p^*(\mathscr{E}) \otimes \xi^\vee \to \mathscr{O}_V \to 0.$$

Since $p^*(\mathscr{E})$ restricted to fibres of $p\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$ is a trivial 2-bundle and $\xi^\vee$ is $\mathscr{O}(-1)$ on fibres, we have $R^i p_*(p^*(\mathscr{E}) \otimes \xi^\vee) = 0$ for $i > 0$ and Leray's spectral sequence gives $h^2(p^*(\mathscr{E}) \otimes \xi^\vee) = 0$. Finally, vanishing of $H^1(\mathscr{O}_V)$ follows directly from the rationality of $V$.

## 2. Fano bundles over $\mathbb{P}^2$

This case is the most interesting one. We may assume $\mathscr{E}$ is normalized, i.e., $c_1(\mathscr{E}) = 0$ or $-1$.

(2.1) PROPOSITION. *If $\mathscr{E}$ is a normalized Fano bundle on $\mathbb{P}^2$, then $\mathscr{E}(2)$ is ample.*

*Proof.* Let $H = p^*(\mathscr{O}(1))$. By (1.3), $c_1(V) = -H + 2\xi_{\mathscr{E}(2)}$ if $c_1 = 0$ and $c_1(V) = 2\xi_{\mathscr{E}(2)}$ if $c_1 = -1$; but $H$ is nef and we are done.

(2.2) PROPOSITION. *Let $\mathscr{E}$ be a Fano bundle on $\mathbb{F}^2$ with $c_1(\mathscr{E}) = 0$. Then $c_2 \leqslant 3$ and in the cohomology ring of $\mathbb{P}(\mathscr{E})$ we have $H^3 = H\xi_{\mathscr{E}}^2 = 0$, $H^2\xi_{\mathscr{E}} = 1$, $\xi_{\mathscr{E}}^3 = -c_2(\mathscr{E})$.*

*Proof.* Vanishing of $H^3$ and the equality $H^2\xi_{\mathscr{E}} = 1$ is obvious. The relations between the generators of $\mathbb{P}(\mathscr{E})$ are then easy consequences of the Leray-Hirsch formulae. To prove that $c_2 \leqslant 3$, we calculate: $0 > K_V^3 = -c_1^3(V) = (3H + 2\xi_{\mathscr{E}})^3 = 8c_2 - 54$, hence $c_2 \leqslant 6$. However, as follows from the Mori-

Mukai-Iskovskich classification, [9], there are no Fano threefolds with $K_V^3 = 6$, 14, 22 and $b_2(V) = 0$, cf. (1.6); see also Remark in Section 3.

(2.3) PROPOSITION. *Let $\mathscr{E}$ be a normalized Fano bundle on $\mathbb{P}^2$. Then*
(a) *if $\mathscr{E}$ is not semistable, then either $\mathscr{E} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ or $\mathscr{E} = \mathcal{O} \oplus \mathcal{O}(-1)$;*
(b) *if $\mathscr{E}$ is semistable, but not stable, then either $\mathscr{E} = \mathcal{O} \oplus \mathcal{O}$ or $\mathscr{E}$ fits in $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_x \to 0$, where $\mathscr{I}_x$ is the sheaf of ideals of a point $x \in \mathbb{P}^2$;*
(c) *if $c_1(\mathscr{E}) = -1$, then either $\mathscr{E} = \mathcal{O} \oplus \mathcal{O}(-1)$* or *$\mathscr{E} = T_{\mathbb{P}^2}(-2)$,*

*Proof*. We start from (c). Let us take a line $L$ and let $\mathscr{E}(2)|L$ be $\mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2)$. We then have $a_1 + a_2 = c_1(\mathscr{E}(2)) = 3$ and $a_1, a_2 > 0$ by (2.1). Therefore $a_1 = 2$, $a_2 = 1$ up to permutation. By the Van de Ven theorem on uniform bundles, [13], $\mathscr{E}$ is as we claim. This proves (c). To show that (a) holds we may then assume that $c_1(\mathscr{E}) = 0$. If $\mathscr{E}$ is not semistable, there is a non-trivial section $s \in H^0(\mathscr{E}(-1))$. Assuming $s$ does not vanish anywhere we find a trivial subbundle $\mathcal{O} \subset \mathscr{E}(-1)$, hence the quotient of $\mathscr{E}(-1)$ by this trivial bundle is $\mathcal{O}(2)$ and we get (a). Assume now that $s$ vanishes at a point $x$. Let us take a line that contains a finite number of zeros of $s$. The bundle $\mathscr{E}(-1)|L$ then splits as $\mathcal{O}(k) \oplus \mathcal{O}(-2-k)$ with $k \geqslant 1$, so that $\mathscr{E}(2)|L = \mathcal{O}(k+3) \oplus \mathcal{O}(1-k)$ in contradiction with (2.1). This concludes the proof of (a). To show (b), assume $\mathscr{E}$ is not stable. Since for $c_1(\mathscr{E}) = -1$ a semistable rank-2 bundle is stable, we infer that $c_1(\mathscr{E}) = 0$. Pick a non-trivial section $s \in H^0(\mathscr{E})$. If the set $\{s = 0\}$ is empty, we get an embedding $\mathcal{O} \subset \mathscr{E}$ whose cokernel is also a trivial bundle and then $\mathscr{E} = \mathcal{O} \oplus \mathcal{O}$. If $s$ vanished at two points (not necessarily distinct), the line through these points would be a jumping one of type $(-2, 2)$, contradicting (2.1). Finally, if $s$ vanishes at a single point $x$, then $c_2(\mathscr{E}) = 1$ and (b) follows. It is known that any non-trivial extension as in (b) is a bundle, (see [10], ch.1, §1.5).

This proves 1 through 5 of our theorem. To conclude the case of $\mathbb{P}^2$ we must, in view of (2.2) and (2.3), study bundles with $c_1 = 0$, $c_2 = 2$ or 3 in more detail.

*Case* $c_1 = 0$, $c_2 = 2$. Let $\mathscr{E}$ be a stable bundle with $c_1(\mathscr{E}) = 0$, $c_2(\mathscr{E}) = 2$. The twisted bundle $\mathscr{E}(1)$ is then generated by global sections, though not ample, since there are lines $L$ such that $\mathscr{E}(1)|L = \mathcal{O} \oplus \mathcal{O}(2)$, [1]. Then $\xi_{\mathscr{E}(1)}$ is globally generated, hence nef. Recall that the cone of numerically effective divisors on $V$ is generated by two (classes of) divisors. Because $H$ and $\xi_{\mathscr{E}(1)}$ are not numerically equivalent, their sum must be then in the interior of the cone, i.e., $-K_V = H + 2\xi_{\mathscr{E}(1)}$ is ample. This gives 6 of our theorem.

REMARK. To see that any stable 2-bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 2$ fits in the exact sequence as in (6), observe first that $h^2(\mathscr{E}(1)) = h^2(\mathscr{E}) = h^0(\mathscr{E}) = 0$ by stability of $\mathscr{E}$ and of its dual. Then $h^1(\mathscr{E}) = -\chi(\mathscr{E}) = 0$ by Riemann-Roch, so $h^1(\mathscr{E}) = 0$ by the Castelnuovo criterion. By Horrock's criterion of decomposability the kernel of the evaluation $\mathcal{O}^4 \to \mathscr{E}(1) \to 0$ splits. Computing the Chern classes of the kernel then gives the sequence as in 6 of the Theorem.

APPLICATION 1. For $\mathscr{E}$ as above, the resulting Fano threefold arises from blowing up a twisted cubic in $\mathbb{P}^3$, [9]. The generic section $s$ of $\mathscr{E}(1)$ vanishes at three points whose associated lines in $(\mathbb{P}^2)^\vee$ form a triangle inscribed in the non-singular conic of jumping lines of $\mathscr{E}$, [1]. Let $Z = \text{zero}(s)$ and $0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(1) \to \mathscr{I}_Z(2) \to 0$ be the corresponding exact sequence. It gives rise to an embedding

$$S_3 := \mathbb{P}(\mathscr{I}_Z(2)) \subset \mathbb{P}(\mathscr{E}(1)) = \mathbb{P}(\mathscr{E})$$

(over $\mathbb{P}^2$) of the Del Pezzo surface $S_3$ (the blow-up of three points in $\mathbb{P}^2$). Since $\xi_{\mathscr{E}(1)}$ is spanned, $S_3$ is the inverse image of a plane in $\mathbb{P}^3 = \mathbb{P}(\Gamma(\mathscr{E}(1)))$. In other words:

(2.4) The Fano threefold $\mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ as above, admits two projections $p$: $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$, $q$: $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^3$ such that $q^{-1}$ is the blow-up of a twisted cubic and $p$ is a $\mathbb{P}^1$-bundle. On a generic plane $P \subset \mathbb{P}^3 = \mathbb{P}(\Gamma(\mathscr{E}(1)))$, the rational map $pq^{-1}: P \to \mathbb{P}^2$ is a quadratic map

$$(x_0, x_1, x_2) \to (x_0x_1, x_1x_2, x_2x_0)$$

that blows up the points where $P$ meets the twisted cubic and contracts the three exceptional curves that arise. The rational map $pq^{-1}$ is then a family of such "elementary" quadratic maps.

APPLICATION 2. Stable rank-2 vector bundles on $\mathbb{P}^2$ with $c_1 = 2$, $c_2 = 3$ are 1-ample, see [11] for the definition of 1-ampleness.

*Case* $c_1 = 0$, $c_2 = 3$.

(2.5) Let $\mathscr{E}$ be a stable rank-2 vector bundle on $\mathbb{P}^2$ with $c_1 = 0, c_2 = 3$. Then $h^0(\mathscr{E}(1)) = 3$.

*Proof.* As a rank-2 bundle with $c_1 = 0$, $\mathscr{E}$ is autodual, hence $h^0(\mathscr{E}) = h^2(\mathscr{E}) = h^2(\mathscr{E}(1)) = 0$. From the Riemann-Roch formula we obtain $h^1(\mathscr{E}) = -\chi(\mathscr{E}) = 1$. As a stable bundle, $\mathscr{E}$ has the generic splitting type $\mathcal{O} \oplus \mathcal{O}$ (the theorem of Grauert and Mülich) and then an easy lemma of LePotier, [3], prop. 2.17, gives $h^1(\mathscr{E}(k)) = 0$ for $k \geqslant 1$. Hence $h^0(\mathscr{E}(k)) = \chi(\mathscr{E}(k))$ if only $k \geqslant 1$; in particular $h^0(\mathscr{E}(1))$ equals 3.

(2.6) PROPOSITION. *Let $\mathscr{E}$ be a stable, rank-2 vector bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 3$. Then $V := \mathbb{P}(\mathscr{E})$ is a Fano threefold if and only if $\mathscr{E}(1)$ is spanned.*

REMARK. For a general bundle $\mathscr{E} \in \mathscr{M}(0, 3)$, $\mathscr{E}(1)$ is spanned and from Barth's description of $\mathscr{M}(0.3)$ it follows that there are stable rank-2 vector bundles $\mathscr{E}$ with $c_1 = 0$, $c_2 = 3$ and $\mathscr{E}(1)$ not spanned (namely, type "a" in Section 7 of [1]).

To prove (2.6), assume first that $\mathscr{E}(1)$ is spanned. Then by the same arguments as we used in the case of bundles with $c_1 = 0, c_2 = 2$ (namely, that the sum $H + \xi_{\mathscr{E}(1)}$ lies in the interior of the cone of numerically effective divisors and hence is ample) we conclude that $-K_V$ is ample. Assume then that $\mathscr{E}(1)$ is not spanned, so that the base point locus $Bs|H + \xi|$ is not empty.

(2.7) *Claim.* $|\xi + H|$ has no base components.

*Proof.* Assume then that $|H + \xi|$ has a fixed component $B_0$ and consider a divisor $D$ in the system $|H + \xi|$. Let $D = B_0 + U$, with $U$ in the variable part of $|H + \xi|$. Because the fibres of $p\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$ are curves, $p(B_0)$ and $p(U)$ are at least one-dimensional. If $p(B_0)$ and $p(U)$ were curves, they would give rise to sections of $\mathscr{E}(1) \otimes \mathscr{I}_{P(B_0)}$ and of $\mathscr{E}(1) \otimes \mathscr{I}_{P(U)}$, contradicting stability. Hence $P(B_0)$ and $P(U)$ are the whole $\mathbb{P}^2$, so that $B_0F \geqslant 1$, $UF \geqslant 1$, where $F$ is the fibre of $p$. Then $1 = 0 + 1 = HF + \xi F \geqslant B_0F + UF \geqslant 2$, a contradiction.

Since $(H + \xi)^3 = 0$, $Bs|\xi + H|$ is not zero-dimensional thus in view of (2.7) it contains one-dimensional components. Let $B$ the sum of them counted with multiplicities so that we can write $D_1 \cdot D_2 = B + C$ where 1-cycle $C$ does not contain one-dimensional components of the base point locus $Bs|\xi + H|$ and $D_1, D_2$ are general divisors from $|\xi + H|$.

(2.8) *Claim.* The cycle $C$ contains at least one fibre $F$ of the projection $p\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$.

*Proof.* Let $x \in p(B)$. We show that $h^0(\mathscr{E}(1) \otimes \mathscr{I}_x) = 2$. Indeed, if $h^0(\mathscr{E}(1) \otimes \mathscr{I}_x)$ were 1, then at $x$ there would exist two independent sections of $\mathscr{E}(1)$ (cf. (2.5)), hence it would be generated by global sections at $x$ which is not the case. Let us now take a line $L$ through $x$ such that $\mathscr{E}(1)|L = \mathscr{O}(1) \oplus \mathscr{O}(-1)$. The inclusion $H^0(\mathscr{E}(1)) \to H^0(\mathscr{E}(1)|L)$ induces the embedding $H^0(\mathscr{E}(1) \otimes \mathscr{I}_x) \to H^0(\mathscr{E}(1) \otimes \mathscr{I}_x|L)$ and from $h^0(\mathscr{E}(1) \otimes \mathscr{I}_x|L) = 2$ we infer that $h^0(\mathscr{E}(1) \otimes \mathscr{I}_x) \leqslant 2$ and therefore equals 2. To conclude the proof of (2.8). let us take two sections of $\mathscr{E}(1)$ that vanish at $x$. The corresponding divisors of $|\xi + H|$ then vanish along the fibres over $x$.

(2.9). In the above notation, $(H + \xi) \cdot C \geqslant 1$.

*Proof.* In general, if a curve $C$ is not contained in the base point set of a linear system $\Lambda$, then $\Lambda \cdot C \geqslant 0$. This shows that in our situation $(H + \xi) \cdot C \geqslant 0$. As $C$ contains the fibre, then $(H + \xi) \cdot C \geqslant 1$.

(2.10). $HB \leqslant 2$. Indeed, we know already that $(H + \xi)^2 = B + C$. Since $(H + \xi)^2 H = 2$ and $H$ is nef, we have $H \cdot C \geqslant 0$, so that $HB \leqslant 2$.

(2.11). $(H + \xi)B \leqslant -1$. Indeed, let us take two divisors as in (2.8). Since $(H + \xi)^3 = 0$, then by inequality (2.9) we get (2.11).

To conclude the proof of (2.6), let us notice that, by (2.11) and (2.10), $c_1(V)B = 2(H + \xi)B + HB \leqslant 0$, i.e., $c_1(V)$ cannot be ample. $\square$

To study the structure of $\mathbb{P}(\mathscr{E})$ more closely, we consider the evaluation morphism $\mathcal{O}^3 \to \mathscr{E}(1)$. Its kernel is a line bundle with $c_1 = -2$, i.e., we have an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^3 \to \mathscr{E}(1) \to 0. \tag{2.12}$$

Now, $\mathscr{E}(1)$ admits a section $s$ with four ordinary zeros at points $x_1, x_2, x_3, x_4$; see e.g. [5], proposition 1.4b. No three of these points are collinear, since otherwise $\mathscr{E}(1)$ would have a jumping line of type $(3, -1)$ or $(4, -2)$ which contradicts (2.1), Hence, in the terminology of [1], $\mathscr{E}$ is a Hulsbergen bundle. By standard arguments (see e.g. [1], §5.2), every such $\mathscr{E}$ is obtained as an extension $0 \to \mathcal{O} \to \mathscr{E}(1) \to \mathscr{J}(2) \to 0$ with $\mathscr{J} \subset \mathcal{O}$ the ideal sheaf of $Z = \{x_1, x_2, x_3, x_4\}$.

Recall that $\Upsilon := \mathbb{P}(\mathscr{J}_Z(2))$ is the blow-up of $\mathbb{P}^2$ at $x_1, x_2, x_3, x_4$ and is then a Del Pezzo surface. The above extension gives rise to a $\mathbb{P}^2$-embedding of $\Upsilon$ into $\mathbb{P}(\mathscr{E}(1))$. Let $C_1, C_2, C_3$ and $C_4$ be the blow-ups of $x_1, x_2, x_3, x_4$ and $H'$ be the pull-back of the divisor of a line in $\mathbb{P}^2$. We have:

(a) $-K_\Upsilon = 3H' - \sum C_i$, [7], Proposition 25.1(*i*);

(b) $\xi_{\mathscr{E}(1)} \,|\, \Upsilon = 2H' - \sum C_i$ and $(\xi_{\mathscr{E}(1)} \,|\, \Upsilon)^2 = \xi^3_{\mathscr{E}(1)} = (H + \xi_{\mathscr{E}})^3 = 0$.

(c) because $-K_\Upsilon$ is ample, by Mori's Cone Theorem the cone of curves is spanned by the extremal ones, [8], Theorem 1.2. Since $\Upsilon$ is neither $\mathbb{P}^2$, nor a $\mathbb{P}^1$-bundle over a curve, all extremal rational curves are exceptional, [8], Theorem 2.1., hence they are of the form $C_i$ or $H' - C_i - C_j$, as follows directly from [7], Proposition 26.2. From the above discussion, the following geometrical interpretation of (2.12) emerges:

(2.13). Let $s$ be a generic section of $\mathscr{E}(1)$ such that zero$(s) = \{x_1, x_2, x_3, x_4\}$. Let $L \subset \mathbb{P}^2 = \mathbb{P}(\Gamma(\mathscr{E}(1)))$ be a line corresponding to the section $s$ viewed at as an element of $H^0(\mathscr{E}(1))$. Then $\Upsilon \to L$ is a conic bundle whose fibres are (strict transforms of) conics through $x_1, x_2, x_3, x_4$. In particular, three fibres are reducible – they correspond to pairs of lines through $x_i, x_j$ and $x_k, x_l$, $(i, j, k, l)$ being a permutation of the indices $(1, 2, 3, 4)$. It follows that the map $p^{-1}$ composed with $\Upsilon \to L$ contracts all conics through our points.

COROLLARY. *$\mathscr{E}(1)$ is 1-ample.*

(2.14) REMARK. Globally generated bundles are dense in the moduli space $\mathscr{M}_{\mathbb{P}^2}(0, 3)$. Indeed, by standard cohomological algebra (cf. e.g. [10], ch. II, Lemma 4.1.3) the bundle $\mathscr{E}(1)$ is uniquely determined by choosing an embedding $\mathcal{O}(-2) \subset \mathcal{O}^3$. Giving such an embedding is, in turn, equivalent to picking a three-dimensional linear system in $|\mathcal{O}(2)|$ at each point $x \in \mathbb{P}^2$. Such systems are in an 1–1 correspondence with an open set of 2-planes in $\mathbb{P}^5$, i.e., with points of an open set in Grass $(3, 6)$. Since dim Grass $(3, 6) = 9 = \dim \mathscr{M}_{\mathbb{P}^2}(0, 3)$ and the latter space is irreducible, globally generated bundle form a dense subspace. From the (proof of) Lemma (5.4) in [1] we also infer that a general bundle in $\mathcal{M}_{\mathbb{P}^2}(0,3)$ has a non-singular cubic as its curve of jumping lines.

## 3. Fano bundles over $\mathbb{P}^1 \times \mathbb{P}^1$

Let $D_1$ and $D_2$ be the divisors corresponding to the two rulings of $\pi_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and let $D = D_1 + D_2$. It is easy to derive the Riemann-Roch formula for rank-2 bundles on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\chi(\mathscr{E}) = 2 + \tfrac{1}{2}c_1^2 - c_2 + c_1 \cdot D.$$

To study whether a bundle $\mathscr{E}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is Fano, we may assume $\mathscr{E}$ to be normalized, i.e., $c_1(\mathscr{E}) = (a_1, a_2)$ with $-1 \leqslant a_i \leqslant 0$, $i = 1,2$. Let us denote $H = p^*(D)$, $H_i = p^*(D_i)$, $i = 1,2$. As in Section 1, we obtain a formula for $c_1(V)$, where $V = \mathbb{P}(\mathscr{E})$: if $c_1(\mathscr{E}) = (0,0)$, then $c_1(V) = 2H + 2\xi$; if $c_1(\mathscr{E}) = -H_i$, then $c_1(V) = 2H + H_i + 2\xi$ and if $c_1(\mathscr{E}) = (-1,-1)$, then $c_1(V) = 3H + 2\xi$. Because $H$ is numerically effective, we easily derive an analogue of (2.1):

(3.1). If $\mathscr{E}$ is normalized 2-bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathbb{P}(\mathscr{E})$ is a Fano manifold, then $\mathscr{E}(2,2) := \mathscr{E} \otimes \mathcal{O}(2,2)$ is ample. If $c_1(\mathscr{E}) = 0$ then already $\mathscr{E}(1.1)$ is ample.

Because a bundle $\mathscr{F} = \oplus\, \mathcal{O}(a_i)$ on a line is ample iff all $a_i$'s are positive, we easily obtain the following corollaries:

(3.3). If $c_1(\mathscr{E}) = 0$, then $\mathscr{E} \mid D_i = \mathcal{O} \oplus \mathcal{O}$ for $i = 1,2$;

(3.4). If $c_1(\mathscr{E}) = (-1,0)$, then $\mathscr{E} \mid D_2 = \mathcal{O} \oplus \mathcal{O}(-1)$ and $\mathscr{E} \mid D_1 = \mathcal{O} \oplus \mathcal{O}$ with the obvious symmetry when $c_1(\mathscr{E}) = (0,1)$;

(3.5). If $c_1(\mathscr{E}) = (-1,-1)$, then $\mathscr{E} \mid D_1 = \mathcal{O} \oplus \mathcal{O}(-1)$ and $\mathscr{E} \mid D_2 = \mathcal{O} \oplus \mathcal{O}(-1)$.

Let us notice that in cases (3.3) and (3.4) the push-forward $\pi_{i*}(\mathscr{E})$ is a rank-2 vector bundle on $\mathbb{P}^1$ (for $i$ chosen such that $\mathscr{E} \mid D_i = \mathcal{O} \oplus \mathcal{O}$). Moreover, the natural morphism $\pi_i^* \pi_{i*}(\mathscr{E}) \to \mathscr{E}$ is an isomorphism and hence we have

(3.6) If $c_1(\mathscr{E}) = (0,0)$, then $\mathscr{E} = \mathcal{O} \oplus \mathcal{O}$;

(3.7) If $c_1(\mathscr{E}) = (-1,0)$ or $c_1(\mathscr{E}) = (0,-1)$, then either $\mathscr{E} = \mathcal{O}(-1,0) \oplus \mathcal{O}$ or $\mathscr{E} = \mathcal{O}(0,-1) \oplus \mathcal{O}$, respectively.

It remains then to study the cases $c_1(\mathscr{E}) = (-1,-1)$. If this is the case, the (1.3) reads as $c_1(V) = 3H + 2\zeta_{\mathscr{E}}$ and in the cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^1$ the following relations hold

$$H_1^3 = H_2^3 = 0,\ H_1 H_2 \xi = 1,\ H^2 \xi = 2,\ H_1 \xi^2 = -1,\ H\xi^2 = 2,\ \xi^3 = 2 - c_2(\mathscr{E}),$$

and thus $0 < (c_1(V))^3 = (3H + 2\xi)^3 = 56 - 8c_2$, i.e., $c_2 \leqslant 6$. Combining this with (1.7) and Table 3 in [9], we get $c_2 \leqslant 4$. Then we show that cases $c_2 = 3$ or 4 cannot occur:

(3.8). If $\mathscr{E}$ is a Fano 2-bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_1(\mathscr{E}) = (-1, -1)$, then $c_2(\mathscr{E}) \leqslant 2$.

*Proof.* First we show that $H^0(\mathscr{E}(-1, -1)) = 0$. Indeed, assume that $s$ is a non-zero section of $\mathscr{E}(-1, -1)$ and $\mathcal{O} \to \mathscr{E}(-1, -1)$ is the corresponding inclusion. On a line $L$ we have then the exact sequence

$$0 \to \mathcal{O}_L(3) \to \mathscr{E}(2, 2)\,|_L \to Q \to 0,$$

where rank-1 quotient sheaf $Q$ has degree 0 in contrary with the ampleness of $\mathscr{E}(2,2)$.

In general, for a 2-bundle $\mathscr{F}$ we have $\mathscr{F} = \mathscr{F}^\vee \otimes \det \mathscr{F}$, hence in our situation $H^2(\mathscr{E}(1, 1)) = H^0\mathscr{E}((-1, -1)) = 0$ by Serre duality and then the Riemann-Roch formula gives

$$\chi(\mathscr{E}(1, 1)) = 5 - c_2.$$

We see that if $c_2 \leqslant 4$, then $H^0(\mathscr{E}(1, 1)) \neq 0$. Let $s$ be a non-zero section.

*Claim.* If $c_2$ were 3 or 4, then it would exist a curve $C \in |\mathcal{O}(1, 1)|$ such that the multiplicity of the zero set $Z$ of $s$ on $C$ was at least three.

Indeed, let us consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ determined by the linear system $|\mathcal{O}(1, 1)|$. Let us pick a plane in $P^3$ meeting $Z$ at three points, counted with multiplicities. The intersection of the plane and the cubic surface $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ is the curve $C$, therefore $C$ is a conic. If $C$ were reducible, e.g. $C = C_1 \cup C_2$, the zero set of $s$ would meet one of $C_i$'s with multiplicity at least two. It would give risė to an embedding $\mathcal{O}_{\mathbb{P}^2}(2) \to \mathscr{E}(1, 1)$, in a contrary with $\mathscr{E}\,|\,D_i = \mathcal{O} \oplus \mathcal{O}(-1)$. Hence $C$ must be a smooth conic. Assume then that $\mathscr{E}(1, 1)\,|\,C = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, $a_1 \geqslant a_2$. As $c_1(\mathscr{E}(1, 1)) = (1, 1)$, we have $a_1 + a_2 = 2$, but $s\,|\,C$ gives an embedding $\mathcal{O}(3)\,|\,C \to \mathscr{E}(1, 1)\,|\,C$, so that $a_1 \geqslant 3$ and then $a_1 - a_2 \geqslant 4$, which contradicts Lemma 1.5 in [12] and hence proves (3.8).

REMARK. A similar method may be used to exclude (without using the M.-M.-I. classification) the case $c_1 = 0, c_2 = 4, 5$ oń $\mathbb{P}^2$.

(3.9) PROPOSITION *A Fano bundle $\mathscr{E}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_1(\mathscr{E}) = (-1, -1)$ fits into the exact sequence*

$$0 \to \mathcal{O}(0, -c_2) \to \mathscr{E} \to \mathcal{O}(-1, c_2 - 1) \to 0 \tag{3.10}$$

*and for $c_2 \leqslant 1$ the sequence splits.*

*Proof.* We know that $\mathscr{E}\,|\,D_i = \dot{\mathcal{O}} \oplus \mathcal{O}(-1)$, $i = 1,2$, hence the push-forward

$\pi_{i*}(\mathscr{E})$ is a line bundle on $\mathbb{P}^1$, say $\mathcal{O}(k)$. The natural morphism $\pi_i^*\pi_{i*}(\mathscr{E}) \to \mathscr{E}$ is an evaluation on every fibre and because the sections of $\mathcal{O} \oplus \mathcal{O}(-1)$ are constant (in particular, they do not vanish), we have an exact sequence

$$0 \to \pi_i^*\pi_{i*}(\mathscr{E}) \to \mathscr{E} \to Q \to 0$$

with a line bundle $Q$ as a cokernel. Calculating the Chern classes we obtain (3.10). Finally, the fact that for $c_2 \leqslant 1$ this sequence splits follows immediately from the vanishing of first cohomology groups of appropriate bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. This proves (3.9).

COROLLARY. *For $\mathscr{E}$ as above, $c_2(\mathscr{E}) \geqslant 0$.*

*Proof.* (3.10) gives the exact sequence

$$0 \to \mathcal{O}(2,.2 - c_2) \to \mathscr{E}(2,2) \to \mathcal{O}(1, c_2 + 1) \to 0$$

and $c_2 < 0$ would contradict the ampleness of $\mathscr{E}(2,2)$. This proves (8) and (11) of the Theorem.

(3.11) PROPOSITION. *If $\mathscr{E}$ is a Fano bundle and $c_1(\mathscr{E}) = (-1,-1)$, $c_2(\mathscr{E}) = 2$, then $\mathscr{E}(1,1)$ is globally generated and fits in an exact sequence*

$$0 \to \mathcal{O}(-1,-1) \to \mathcal{O}^{\oplus 3} \to \mathscr{E}(1,1) \to 0.$$

*Proof.* By (3.9) we have $h^i(\mathscr{E}(0,1)) = h^i(\mathscr{E}(1,0)) = \mathcal{O}$, all $i$, and $h^i(\mathscr{E}(1,1)) = 3$ if $i = 0$ and 0 otherwise. Restricting $\mathscr{E}$ to the ruling $D_1$ gives

$$0 \to \mathscr{E}(0,1) \to \mathscr{E}(1,1) \to \mathscr{E}(1,1)|_{D_1} \to 0.$$

The induced evaluation morphism $H^0(\mathscr{E}(1,1)) \to H^0(\mathscr{E}(1,1)|_{D_1})$ is then an isomorphism. But $\mathscr{E}(1,1)|_{D_i}$ is globally generated, so is $\mathscr{E}(1,1)$. Since $h^0(\mathscr{E}(1,1)) = 3$, computing the Chern classes of the kernel of the evaluation $\mathcal{O}^3 \to \mathscr{E}(1,1)$ gives (3.11). Conversely, if the inclusion $\mathcal{O} \to \mathcal{O}(1,1)^3$ corresponds to a non-vanishing section of $\mathcal{O}(1,1)^3$, the quotient is a 2-bundle. To complete our discussion of the case $c_2 = 2$, we must show that $\mathbb{P}(\mathscr{E})$ is Fano. Because $\mathscr{E}(1,1)$ is globally generated, it is nef and $H + 2\xi_{\mathscr{E}(1,1)}$ is nef, as well. Therefore, to prove that it is ample, it is sufficient (by the theorem of Moishezon and Nakai) to check that $H + 2\xi_{\mathscr{E}(1,1)}$ has positive intersections with curves in $\mathbb{P}(\mathscr{E})$. However, if $H \cdot C = 0$, then $C$ is contained in a fibre and then $\xi_{\mathscr{E}(1,1)} \cdot C > 0$.

## 4. Fano bundles over non-minimal Del Pezzo surfaces

Let us recall that any non-minimal Del Pezzo surface $S_k$ is a blow-up of $k$ points $x_i (1 \leqslant i \leqslant k \leqslant 8)$ on the plane, no three on one line and no six of them on a conic.

The canonical divisor of $S_k$ has the self-intersection number equal to $9-k$. $S_1$ is the same as the Hirzebruch surface $F_1$. Let $\beta\colon S_k \to \mathbb{P}^2$ be the blow-down morphism, $C_i$ be the exceptional divisors of $\beta$ and $H$ be the inverse image of the divisor of a line of $\mathbb{P}^2$. Let $\mathscr{E}$ be a Fano bundle on $S_k$. As in the preceeding sections, we may assume $\mathscr{E}$ to be normalized, i.e.,

$$-1 \leqslant c_1(\mathscr{E})\cdot C_i \leqslant 0 \quad \text{and} \quad -1 \leqslant c_1(\mathscr{E})\cdot H \leqslant 0.$$

Since $K\cdot C_i = 1$, we may apply the same methods as in Section 3 (using Lemma 1.5 from [12]) to conclude easily that $\mathscr{E}|_{C_i} = \mathcal{O} \oplus \mathcal{O}$ and consequently $\mathscr{E} = \beta^*(\mathscr{E}')$ with a 2-bundle $\mathscr{E}'$ on $\mathbb{P}^2$. Moreover, if $c_1(\mathscr{E}')\cdot H = 0$, then $\mathscr{E}$ is trivial. Indeed, let $\bar{L}$ be the strict transform of a line $L \subset \mathbb{P}^3$ that passes through one of the points $x_i$. Then $K_{S_k}\cdot\bar{L} \leqslant 2$ and in virtue of Lemma 1.5 in [12] we have $\mathscr{E}|\bar{L} = \mathcal{O}\oplus\mathcal{O}$, therefore $\mathscr{E}/L = \mathcal{O}\oplus\mathcal{O}$ and Van de Ven's theorem shows that $\mathscr{E}'$ is trivial, so is $\mathscr{E}$.

Let us notice that for $k \geqslant 2$ we can always choose a line $L$ that passes through two of the points $x_i$, so that $-K_{S_k}\cdot\bar{L} = -1$ and, as above, $c_1(\mathscr{E})\cdot\bar{L} = 0$, implying $c_1(\mathscr{E}')\cdot H = 0$. In other words, we have proved that for $k \geqslant 2$, the only ruled Fano 3-fold over a Del Pezzo surface $S_k$ is $\mathbb{P}^1 \times S_k$. Finally, on the Hirzebruch surface $F_1$ we have

(a) if $c_1(\mathscr{E}') = 0$, then, as above, $\mathscr{E} = \mathcal{O}\oplus\mathcal{O}$,

(b) if $c_1(\mathscr{E}') = -1$, then, as in (2.3), we infer that $\mathscr{E}' = \mathcal{O}\oplus\mathcal{O}(-1)$ or $\mathscr{E}' = T_{\mathbb{P}^2}(-2)$.

## References


1. Barth, W., Moduli of Vector Bundles on the Projective Plane. *Inv. Math.* 42, (1977), 63–91.
2. Demin, I.V., Three-dimensional Fano manifolds representable as line fiberings (Russian). *Izv. Acad. Nauk SSSR*, 44, no. 4 (1980). English translation in *Math. USSR Izv.* 17. Addendum to this paper in *Izv. Acad. Nauk SSSR*. 46, no. 3. English translation in *Math. USSR Izv.* 20.
3. Elencwajg, F. and Forster, O., Bounding Cohomology Groups of Vector Bundles on $\mathbb{P}^n$, *Math. Ann.* 246 (1980) 251–270.
4. Hartshorne, R., Ample Subvarieties of Algebraic Varieties. *Lecture Notes* 156 (1970).
5. Hartshorne, R., Stable Vector Bundles of Rank 2 on $\mathbb{P}^3$. *Math. Ann.* 238 (1978) 229–280.
6. Kawamata, Y., The cone of curves of algebraic varieties. *Ann. Math.* 119, 603–633 (1984).
7. Manin, Yu.I Cubic forms, Algebra, Geometry, Arithmetic. North Holland 1974.
8. Mori, Sh., Threefolds Whose Canonical Bundle is not Numerically Effective. *Ann. Math.* 116, 133–176 (1982).
9. Mori, Sh. and Mukai, Sh.: Classification of Fano 3-folds with $B_2 \geqslant 2$. *Manuscripta Math.* 36, 147–162 (1981).
10. Okonek, Ch., Schneider, M. and Spindler, H.: Vector Bundles on Complex Projective Spaces, Birkhäuser, 1981.
11. Schiffman, B, and Sommese, A. J., Vanishing theorems on complex manifolds. *Birkhäuser* 1985.
12. Szurek, M., Wiśniewski, J.A., Fano Bundles on $\mathbb{P}^3$ and $\mathbb{Q}_3$. *Pacific Journ. Math.* 140, no. 2. (1989).
13. Van de Ven, A., On uniform vector bundles. *Math. Ann.* 195 (1972) 245–248.