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Compositio Mathematica, tome 76, n° 1-2 (1990), p. 247-264

http://www.numdam.org/item?id=CM_1990__76_1-2_247_0

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Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d \rangle$

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Received 23 September 1988; accepted in revised form 20 September 1989

This work has been possible thanks to a Scientific Agreement between the Universidad Complutense and the Humboldt Universitaet. This cooperation agreement supported our stay in the Bereich Algebra and Departamento de Algebra respectively.

The aim of this paper is to classify map germs $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ and germs of curve singularities in \mathbb{C}^2 given by an equation of the type $f = (x^p + y^q)^2 + \sum_{i q + j p > 2 p q} a_{i j} x^i y^j = 0$ with a fixed Milnor number $\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$. Here we always suppose $p < q$ and $\gcd(p, q) = 1$.

The moduli space $M_{p, q, \mu}$ of the map germs described above is an affine Zariski-open subset of $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$ divided by a suitable action of μ_{2pq} (the group of $2pq$ -roots of unity) depending on $\mu(f)$.

The moduli space $T_{p, q, \mu}$ of all plane curve singularities described above (which is the moduli space of all plane curve singularities with the semigroup $\langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$ if μ is even) is $\mathbb{C}^{(p-2)(q-2) + [q/p] - 1}$ divided by a suitable action of μ_d , $d = \mu - (2p-1)(2q-1)$.

In both cases we also get an algebraic universal family. It turns out that the Tjurina-number $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (f, \partial f / \partial x, \partial f / \partial y) = \mu(f) - (p-1)(q-1)$ depends only on $\mu(f)$ and p and q .

Constructing the moduli spaces we use the graduation of $\mathbb{C}[[x, y]]$ defined by p, q : $\deg x^i y^j = i q + j p$.

We use the following idea to construct the moduli spaces: Let $\mu = (2p-1)(2q-1) + d$. We prove that for all f of the above type we can choose the same monomial base of $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$ (Lemma 2). We choose α, β and that $\alpha < p, \alpha q + \beta p = 2pq + d$. Hence $\mu(f_0) = \mu$ with $f_0 = (x^p + y^q)^2 + x^\alpha y^\beta$. Then we consider a universal μ -constant unfolding of f_0 as a "global" family (Lemma 3). The parameter space U of that unfolding is an affine open subset of $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$. The group μ_{2pq} acts on U and $M_{p, q, \mu} = U / \mu_{2pq}$.

To construct $T_{p,q,\mu}$ we consider the Kodaira-Spencer map of the universal μ -constant unfolding. The Kernel of the Kodaira-Spencer map is a Lie-algebra acting on U . The integral manifolds of that Lie-algebra are the analytically trivial subfamilies of the unfolding.

We choose a suitable section transversal to those integral manifolds, which turns out to be isomorphic to $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}$. The group μ_d acts on the corresponding family and we prove that $T_{p,q,\mu} = \mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$.

1. A normal form for map germs $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ with initial term $(x^p + y^q)^2$

LEMMA 1. *Let*

$$f = \left(x^p + y^q + \sum_{iq+jp > pq} h_{ij} x^i y^j \right)^2 + \sum_{iq+jp \geq 2pq+d} w_{ij} x^i y^j$$

then $\mu(f) \geq (2p - 1)(2q - 1) + d$, and $\mu(f) = (2p - 1)(2q - 1) + d$ iff

$$f_d := \sum_{iq+jp=2pq+d} (-1)^{[i/p]} w_{ij} \neq 0.$$

Proof. Either f is irreducible or the components of f have the same tangent direction. This implies that

$$\mu(\tilde{f}) = \mu(f) - 2p(2p - 1),$$

where \tilde{f} is the blowing up

$$\begin{aligned} \tilde{f} = \frac{f(xy, y)}{y^{2p}} &= \left(x^p + y^{q-p} + \sum h_{ij} x^i y^{i+j-p} \right)^2 + \sum w_{ij} x^i y^{i+j-2p} \\ &= \left(x^p + y^{q-p} + \sum_{i(q-p)+jp > (q-p)p} h_{i,j-i+p} x^i y^j \right)^2 + \\ &\quad + \sum_{i(q-p)+jp \geq 2(q-p)p+d} w_{i,j-i+2p} x^i y^j. \end{aligned}$$

Using induction we may assume that

$$\mu(\tilde{f}) \geq (2p - 1)(2(q - p) - 1) + d$$

and

$$\begin{aligned} \mu(\tilde{f}) &= (2p - 1)(2(q - p) - 1) + d \text{ iff } 0 \neq \sum_{i+jp=2pq+d} (-1)^{\lfloor i/p \rfloor} w_{ij} \\ &= \sum_{i(q-p)+jp=2(p-q)p+d} (-1)^{\lfloor i/p \rfloor} w_{i,j-i+2p}. \end{aligned}$$

This yields the if part of the result. Now if f is as above and $\mu(f) > (2p - 1)(2q - 1) + d$, the condition $f_d = 0$ says that $(x^p + y^q)$ divides $\sum_{i+jp=2pq+d} w_{ij} x^i y^j$, and adding $-\frac{1}{2} \sum_{i+jp=2pq+d} w_{ij} x^i y^j$ to the first part of f one gets $f = (x^p + y^q + \dots)^2 +$ terms of degree greater than $2pq + d$. Continuing this way, we get the result.

LEMMA 2. *Let $f = (x^p + y^q)^2 + \sum_{i+jp > 2pq} h_{ij} x^i y^j$ and $\mu(f) = (2p - 1)(2q - 1) + d$. Let γ, δ such that $\gamma q + \delta p = 3pq - q - p + d, \gamma < p$. Let $B = \{(i, j) \in N^2 / i < 2p - 1, j < q - 1\} \cup \{(i, j) \in N^2 / i < p, j < q\} \cup \{(i, j), (i, j) \in N^2 / i < \gamma, j < \delta + q\}$. Then $\{x^i y^j\}_{(i,j) \in B}$ is a base of $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$.*

Proof. We use the algorithm of Mora (cf. [3]) to compute a Groebner base of the ideal $(\partial f / \partial x, \partial f / \partial y)$. We consider $\mathbb{C}[[x, y]]$ as a graded ring with $\deg x = q, \deg y = p$. Let $f_1 = 1/2p(\partial f / \partial x)$ and $f_2 = 1/2q(\partial f / \partial y)$.

Consider $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$ and let f_3 be the reduction of $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$ with respect to the initial terms x^{2p-1} resp. $x^p y^{q-1}$ of f_1 resp. f_2 , i.e.

$$s(f_1, f_2) = f_3 + h_1 f_1 + h_2 f_2$$

$$f_3 = \sum_{\gamma_i < p} l_i x^{\gamma_i} y^{\delta_i}$$

$$q\gamma_i + p\delta_i = 3pq - q - p + i$$

and the initial terms of h_1 resp. h_2 have degree $> pq - p$ resp. $> pq - q$. $f_3 \neq 0$ because of $\mu(f) < \infty$. Let k be the minimal such that $l_k \neq 0$, i.e. $l_k x^{\gamma_k} y^{\delta_k}$ is the initial term of f_3 . Consider now

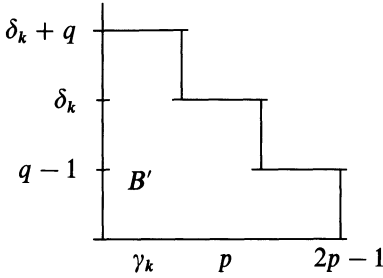
$$\begin{aligned} s(f_2, f_3) &= l_k y^{\delta_k - q + 1} f_2 - x^{p - \gamma_k} f_3 \\ &= l_k y^{\delta_k + q} + \text{terms of degree } > p\delta_k + pq \\ &=: f_4. \end{aligned}$$

It is not difficult to see that the reductions of $s(f_1, f_3)$ and $s(f_i, f_4) i = 1, 2, 3$ with respect to the initial terms of f_1, f_2, f_3, f_4 are zero, i.e. f_1, f_2, f_3, f_4 is a

Groebner base of $(\partial f/\partial x, \partial f/\partial y)$. This implies that

$$\{x^i y^j\}_{(i,j) \in B'}, B' = \{(i, j), i < 2p - 1, j < q - 1\} \cup \{(i, j), i < p, j < \delta_k\} \cup \{(i, j), i < \gamma_k, j < \delta_k + q\},$$

is a base of $\mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$.



This implies $\mu(f) = (p - 1)(q - 1) + q\gamma_k + p\delta_k$ and therefore $\gamma = \gamma_k$ and $\delta = \delta_k$ and $B = B'$. □

LEMMA 3. Let $f = (x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij} x^i y^j$ and $\mu(f) = (2p - 1)(2q - 1) + d$.

Let γ, δ be defined by

$$\gamma < p \text{ and } \gamma q + \delta p = 3pq - q - p + d$$

Let $B_0 = \{(i, j), iq + jp > pq, i \leq p - 2, j \leq q - 2\}$;

$$B_1 = \{(i, j), iq + jp \geq 2pq + d, i < p, j < \delta\} \cup \{(i, j), iq + jp \geq 2pq + d, i < \gamma, j < \delta + q\}.$$

There is an automorphism $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ such that

$$f(\varphi) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij} x^i y^j\right)^2 + \sum_{(i,j) \in B_1} w_{ij} x^i y^j$$

for suitable $h_{ij}, w_{ij} \in \mathbb{C}$.

Proof. Using Lemma 1 we may assume that

$$f = \left(x^p + y^q + \sum_{iq+jp > pq} b_{ij} x^i y^j\right)^2 + \sum_{iq+jp \geq 2pq+d} c_{ij} x^i y^j.$$

Assume that there is an automorphism $\varphi^{(k)}$ such that

$$f(\varphi^{(k)}) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq pq + k} b_{ij}^{(k)} x^i y^j \right)^2 + \\ + \sum_{(i,j) \in B_1} w_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq 2pq + k} c_{ij}^{(k)} x^i y^j,$$

$\varphi^{(1)} = \text{identity}$.

Now

$$\sum_{iq + jp = 2pq + k} c_{ij}^{(k)} x^i y^j = (x^p + y^q)H + \sum_{iq + jp = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} x^{i_0} y^{j_0}$$

for a suitable homogeneous H of degree $pq + k$ and $i_0 < p, i_0 q + j_0 p = 2pq + k$.

If $\sum_{iq + jp > 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} \neq 0$ and $(i_0, j_0) \notin B_1$ then $k \geq pq - q - p + d$ (Lemma 1) and $j_0 \geq \delta, i_0 \geq \gamma$ or $j_0 \geq \delta + q$.

Let α, β be defined by $q\alpha + p\beta = 2pq + d, \alpha < p$, then $w_{\alpha\beta} \neq 0$. Notice that $\alpha - 1 \equiv \gamma \pmod p$ and $\beta - 1 \equiv \delta \pmod q$.

Let

$$g := x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq pq + k} b_{ij}^{(k)} x^i y^j$$

and

$$\omega := e \cdot x^\xi y^\eta \left(\frac{\partial g}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right), \quad \xi q + \eta p = k - pq + p + q - d$$

$$e := \frac{1}{(\alpha q + \beta p) w_{\alpha\beta}} \cdot \sum_{ip + jq = 2pq + k} (-1)^{[i/p] - [\alpha - 1 + \xi/p] + 1} c_{ij}^{(k)}$$

$$\text{with } (\xi, \eta) = \begin{cases} (i_0 - \gamma, j_0 - \delta) & \text{if } j_0 \geq \delta, \quad i_0 \geq \gamma \\ (i_0 - \gamma + p, j_0 - \delta - q) & \text{if } j_0 \geq \delta + q. \end{cases}$$

Let $\psi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ the automorphisms corresponding to the vector field ω , then $g(\psi) = g$.

Hence,

$$f(\psi \circ \varphi^{(k)}) = g^2 + \sum_{(i,j) \in B_0} w_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq 2pq + k} \bar{c}_{ij}^{(k)} x^i y^j$$

and

$$\begin{aligned} & \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} \bar{c}_{ij}^{(k)} \\ &= \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} + (-1)^{[\alpha - 1 + \xi/p]} (\alpha q + \beta p) w_{\alpha\beta} \cdot e. \end{aligned}$$

If $(i_0, j_0) \notin B_1$ we may assume now that $\sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} = 0$.

Let $g_1 := g + \frac{1}{2}H$ and

$$\sum_{i_q + j_p = pq + k} b_{ij}^{(k)} x^i y^j + \frac{1}{2}H + m_k \frac{\partial g_1}{\partial x} + n_k \frac{\partial g_1}{\partial y} = \sum_{(i,j) \in B_0} d_{ij}^{(k)} x^i y^j.$$

The degree of the initial part of m_k resp. n_k is $q + k$ resp. $p + k$.

We define $\varphi^{(k+1)}$ by

$$\varphi^{(k+1)}(x) = \varphi^{(k)}(x) + m_k$$

$$\varphi^{(k+1)}(y) = \varphi^{(k)}(y) + n_k$$

and

$$h_{ij}^{(k+1)} = h_{ij}^{(k)} + d_{ij}^{(k)}$$

$$w_{ij}^{(k+1)} = w_{ij}^{(k)} \quad \text{if } (i, j) \neq (i_0, j_0)$$

$$w_{i_0, j_0}^{(k+1)} = w_{i_0, j_0}^{(k)} + \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} \quad \text{if } (i_0, j_0) \in B_1.$$

Then

$$\begin{aligned} f(\varphi^{(k+1)}) &= \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq pq + k + 1} b_{ij}^{(k+1)} x^i y^j \right)^2 + \\ &+ \sum_{(i,j) \in B_1} w_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq 2pq + k + 1} c_{ij}^{(k+1)} x^i y^j \end{aligned}$$

for suitable $b_{ij}^{(k+1)}, c_{ij}^{(k+1)}$. □

LEMMA 4. Let

$$f_t = (x^p + y^q)^2 + \sum_{i_q + j_p > 2pq} a_{ij}(t) x^i y^j, \quad a_{ij}(t) \in \mathbb{C}[t]$$

and $\mu(f_t) = (2p - 1)(2q - 1) + d$ for $t \in \mathbb{C}$.

Let γ, δ, B_0, B_1 be as in Lemma 3. There is a $\mathbb{C}[t]$ -automorphism $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ such that

$$f_t(\varphi_t) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}(t)x^i y^j \right)^2 + \sum_{(i,j) \in B_1} w_{ij}(t)x^i y^j$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$.

The proof is similar to that of Lemma 3. □

Let us consider the family

$$F(x, y, H, W) = \left(x^p + y^q + \sum_{(i,j) \in B_0} H_{ij}x^i y^j \right)^2 + \sum_{(i,j) \in B_1} W_{ij}x^i y^j$$

depending on the parameters $H = (H_{ij})_{(i,j) \in B_0}, W = (W_{ij})_{(i,j) \in B_1}$ and define $N = \# B_0 + \# B_1$, then $\mu(F) = (2p - 1)(2q - 1) + d$ on the open set U defined by $W_{\alpha\beta} \neq 0, \alpha q + \beta p = 2pq + d$, in $\mathbb{C}^N = \text{Spec } \mathbb{C}[H, W]$. Notice that $N = 2(p - 1)(q - 1) - p - q + 2 + [q/p]$ is not depending on d !

The group of $2pq$ -roots of unity acts on U :

$$\lambda \in \mu_{2pq}, \lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})). \quad \square$$

THEOREM 1. U/μ_{2pq} is the moduli space of all functions

$$f = (x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij}x^i y^j$$

with $\mu(f) = (2p - 1)(2q - 1) + d$ and F is the universal family.

Proof. Using Lemma 3 we have to prove the following

LEMMA 5. Let φ be an automorphism $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ such that

$$F(\varphi(x), \varphi(y), \bar{h}, \bar{w}) = F(x, y, h, w) \quad (*)$$

for $(\bar{h}, \bar{w}), (h, w) \in U \subseteq \mathbb{C}^N$ then $\lambda \cdot (\bar{h}, \bar{w}) = (h, w)$ for a suitable $\lambda \in \mu_{2pq}$.

Proof. Let $\bar{x} := \varphi(x), \bar{y} := \varphi(y)$ then grouping the squared part of (*) one gets:

$$\begin{aligned} & \left(x^p + y^q + \bar{x}^p + \bar{y}^q + \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \times \\ & \times \left(x^p + y^q - \bar{x}^p - \bar{y}^q - \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \\ & = \sum \bar{w}_{ij} \bar{x}^i \bar{y}^j - \sum w_{ij} x^i y^j. \end{aligned}$$

This equation implies obviously that the degree of the initial term of $\varphi(x)$ is $\geq q$ and

$$\bar{x} = \lambda^q \left(x + \sum_{iq+jp>q} a_{ij}^{(1)} x^i y^j \right) \quad \bar{y} = \lambda^p \left(y + \sum_{iq+jp>p} a_{ij}^{(2)} x^i y^j \right), \lambda \in \mu_{2pq}.$$

We may assume that $\lambda = 1$ and prove $(\bar{h}, \bar{w}) = (h, w)$.

Let the degree of the leading parts of both sides of the above equation be $2pq + m$ and let r be the degree of φ , i.e. $a_{ij}^{(1)} = 0$ if $iq + jp < q + r$, $a_{ij}^{(2)} = 0$ if $iq + jp < p + r$ and $a_{ij}^{(1)} \neq 0$ or $a_{ij}^{(2)} \neq 0$ for suitable i, j with $iq + jp = q + r$ resp. $iq + jp = p + r$.

1. Step. We prove that

(a) $r \geq pq - p - q$

$$\sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} \cdot k, \quad \sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} \cdot k$$

(b) $h = \bar{h}$ and $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 3pq - p - q + d$.

First of all $m \geq d + r$ because the leading part of the left side of the equation is divisible by $x^p + y^q$ and $m < d + r$ would imply that the leading part of the right side is a monomial. This implies $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 2pq + d + r$. Now $h_{ij} = \bar{h}_{ij}$ if $iq + jp < pq + r$. Otherwise the leading part of the left side of the equation would be $2(x^p + y^q)(h_{ij} - \bar{h}_{ij})x^i y^j$ for some i, j with $iq + jp < pq + r$ and therefore of degree $2pq + r < 2pq + m$.

Now suppose $r < pq - p - q$. Then there is at most one monomial of degree $p + r$ resp. $q + r$.

If $iq + jp = pq + r$ for some $(i, j) \in B_0$ and

$$qi_0 + pj_0 = q + r$$

$$qi_1 + pj_1 = p + r$$

then

$$(h_{ij} - \bar{h}_{ij})x^i y^j - pa_{ioj_0}^{(1)} x^{i_0+p-1} y^{j_0} - qa_{i_1j_1}^{(2)} x^{i_1} y^{j_1+q-1} = 0$$

otherwise the leading part of the left side of the equation would have degree $2pq + r < 2pq + m$.

But $(i, j) \in B_0$, i.e. $i < p - 1$ and $j < q - 1$. This implies $h_{ij} = \bar{h}_{ij}$, $a_{ioj_0}^{(1)} = a_{i_1j_1}^{(2)} = 0$ (because of $r < pq - p - q$ we have $i_1 < p - 1$). This is a contradiction since $a_{ioj_0}^{(1)} \neq 0$ or $a_{i_1j_1}^{(2)} \neq 0$ by the definition of r .

Similarly one gets a contradiction if there is no $(i, j) \in B_0$ with $qi + pj = p + r$, resp. no i_0, j_0 with $qi_0 + pj_0 = q + r$ resp. no i_1, j_1 with $qi_1 + pj_1 = p + r$.

This proves that $r \geq pq - p - q$. With the same method we obtain

$$\sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} k \quad \text{and} \quad \sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} k.$$

(b) is clear now by the choice of B_0 and the fact that $r \geq pq - p - q$.

2. Step. We prove that $r \geq 2pq - p - q$.

Assume that $r < 2pq - p - q$. Then $\deg k < pq$, i.e., k is a monomial.

The leading part of the left side of the above equation is divisible by $x^p + y^q$.

The leading part L of the right side is

$$(\bar{w}_{ij} - w_{ij})x^i y^j + \bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp = 2pq + m$ for some $(i, j) \in B_1$ or

$$\bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp \neq 2pq + m$ for $(i, j) \in B_1$.

Let $k = \kappa \cdot x^\xi y^\eta$. If $\alpha + \xi - 1 < p$, then $i = \alpha + \xi - 1$ and $j = \beta + \eta + q - 1$. If $\alpha + \xi - 1 \geq p$, then $i = \alpha + \xi - 1 - p$ and $j = \beta + \eta + 2q - 1$. But $(\alpha + \xi - 1, \beta + \eta + q - 1) \notin B_1$ and $(\alpha + \xi - 1 - p, \beta + \eta + 2q - 1) \notin B_1$. This implies $L = \bar{w}_{\alpha\beta} \cdot \kappa x^{\alpha-1} y^{\beta-1} ((\alpha/p)y^q - (\beta/p)x^p)$ which is not divisible by $x^p + y^q$. This is a contradiction and therefore $r \geq 2pq - p - q$.

Now $iq + jp \leq 4pq - 2p - 2q + d$ for $(i, j) \in B_1$ then $w_{ij} = \bar{w}_{ij}$ for all $(i, j) \in B_1$. □

2. The construction of the moduli space

We will construct the moduli space of all plane curve singularities given by an equation $(x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij} x^i y^j = 0$ with fixed Milnor number μ .

For μ being even we get especially the moduli space for all irreducible plane curve singularities with the semigroup $\Gamma = \langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$.

We use the family

$$V(F) \subseteq U \times \mathbb{C}^2 \rightarrow U$$

constructed in Theorem 1.

U admits a \mathbb{C}^* -action defined by

$$\lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})).$$

We get

$$F(\lambda^q x, \lambda^p y, h, w) = \lambda^{2pq} F(x, y, \lambda \circ (h, w)).$$

If $\mu = (2p - 1)(2q - 1) + d$ and $\alpha q + \beta p = 2pq + d, \alpha < p$, then $U \subseteq \mathbb{C}^N$ was defined by $W_{\alpha, \beta} \neq 0$.

For the construction of the moduli space it is enough to consider the restriction of our family to the transversal section to the orbits of the \mathbb{C}^* -action defined by $W_{\alpha, \beta} = 1$.

Let W' be defined by $W = (W_{\alpha, \beta}, W')$ and $G(x, y, H, W') = F(x, y, H, 1, W')$. The parameter space of G is $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[H, W']$.

The group μ_d of d th roots of unity acts on the family

$$V(G) \subseteq \mathbb{C}^2 \times \mathbb{C}^{N-1} \rightarrow \mathbb{C}^{N-1}$$

induced by the above \mathbb{C}^* -action

$$G(\lambda^q x, \lambda^p y, h, w') = \lambda^{2pq} G(x, y, \lambda \circ (h, w')) \quad \lambda \in \mu_d.$$

LEMMA 6. *Let $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ be an automorphism and $u \in \mathbb{C}[[x, y]]$ a unit such that*

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

then there is a $\lambda \in \mu_d$ such that (h, w') and $\lambda \circ (\bar{h}, \bar{w}')$ are contained in an analytically trivial subfamily of $V(G) \rightarrow \mathbb{C}^{N-1}$.

Proof. Let

$$\varphi(x) = \sum a_{ij}^{(1)} x^i y^j \quad \text{and} \quad \varphi(y) = \sum a_{ij}^{(2)} x^i y^j, \quad u = \sum u_{ij} x^i y^j.$$

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

implies

- (1) $a_{ij}^{(1)} = 0$ if $iq + jp < q$
- (2) $a_{1,0}^{(1)2p} = a_{0,1}^{(2)2q} = a_{1,0}^{(1)p} a_{0,1}^{(2)q} = u_{0,0}^{-1}$.

Let $a_{1,0}^{(1)} = \lambda^q$ and $a_{0,1}^{(2)} = \lambda^p$.

We will prove later that $\lambda^d = 1$.

Now we may assume that $\lambda = 1$ and prove that (h, w') and (\bar{h}, \bar{w}') are contained in an analytically trivial subfamily of $V(G) \rightarrow \mathbb{C}^{N-1}$.

We choose

- (1) $u(t) \in \mathbb{C}[t][[x, y]]$ with the following properties $u(0) = 1$, $u(1) = u$ and u is a unity for all $t \in \mathbb{C}$.
- (2) $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ with the following properties $\varphi_0 = \text{identity}$, $\varphi_1 = \varphi$ and φ_t is an automorphism of positive degree for all $t \in \mathbb{C}$.

Let $H(t) := u(t)G(\varphi_t(x), \varphi_t(y), h, w')$ and apply Lemma 4. There is an $\mathbb{C}[t]$ -automorphism $\Phi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ such that

$$H(\Phi_t) = F(x, y, h(t), w(t))$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$ with the property

$$h(0) = h$$

$$w(0) = (1, w').$$

$H(\Phi_t)$ has a constant Milnor number, i.e. $w_{\alpha, \beta}(t)$ has to be constant.

This implies

$$H(\Phi_t) = G(x, y, h(t), w'(t)).$$

But,

$$G(x, y, h(1), w'(1)) = H(\Phi_1) = G(\Phi_1(x), \Phi_1(y), \bar{h}, \bar{w}').$$

Using Lemma 5 and the fact that Φ_1 has positive degree we get

$$\bar{h} = h(1)$$

$$\bar{w}' = w'(1),$$

i.e. (h, w') and (\bar{h}, \bar{w}') are in the trivial family

$$G(x, y, h(t), w'(t)) = u(\Phi_t)G(\Phi_t \varphi_t, h, w').$$

To finish the proof of Lemma 6 we have to prove

LEMMA 7. *Let*

$$f_k = \left(x^p + y^q + \sum_{i+jp > pq} a_{ij}^{(k)} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{i+jp > 2pq+d} b_{ij}^{(k)} x^i y^j, \quad k = 1, 2$$

$$\alpha < p, \alpha q + \beta p = 2pq + d.$$

Let $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ be an automorphism with the property

$$\varphi(x) = \lambda^q x + \text{terms of degree } > q$$

$$\varphi(y) = \lambda^p y + \text{terms of degree } > p$$

and u a unit, such that

$$f_1(\varphi) = f_2 \cdot u$$

then $\lambda^d = 1$.

Proof. $u = \lambda^{2pq} + \text{terms of higher degree}$.

$$u \cdot f_2 = \lambda^{2pq} \left(x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(2)} x^i y^j \right)^2 + \lambda^{2pq} x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \bar{b}_{ij}^{(2)} x^i y^j$$

$$f_1(\varphi) = \lambda^{2pq} \left(x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(1)} x^i y^j \right)^2 + \lambda^{2pq+d} x^\alpha y^\beta + \sum_{iq+jp \geq 2pq+d} \bar{b}_{ij}^{(1)} x^i y^j$$

for suitable $\bar{a}_{ij}^{(k)}, \bar{b}_{ij}^{(k)}$.

This implies

$$\begin{aligned} & \left(2x^p + 2y^q + \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} + \bar{a}_{ij}^{(2)}) x^i y^j \right) \cdot \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} - \bar{a}_{ij}^{(2)}) x^i y^j \\ &= (1 - \lambda^d) x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \lambda^{-2pq} (\bar{b}_{ij}^{(2)} - \bar{b}_{ij}^{(1)}) x^i y^j. \end{aligned}$$

Because the leading term of the left side of the equation is divisible by $x^p + y^q$, we get $\lambda^d = 1$. \square

We consider now the Kodaira-Spencer map of the family

$$V(G) \rightarrow \mathbb{C}^{N-1}:$$

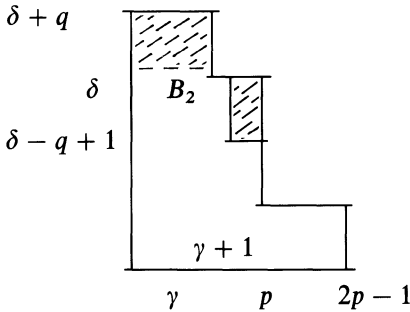
$$\rho: \text{Der}_{\mathbb{C}} \mathbb{C}[H, W'] \rightarrow \mathbb{C}[H, W'][[x, y]] / \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

defined by

$$\rho(\delta) = \text{class}(\delta G).$$

The kernel of the Kodaira-Spencer map is a Lie-algebra L and along the integral manifolds of L the family is analytically trivial. We will choose a transversal section to the integral manifolds of L and divide by the action of μ_d to get the moduli space. To describe this transversal section we choose a suitable subset of B_1 :

$$B_2 = \{(i, j) \in B_1, i \leq \gamma, j \leq \delta\} \cup \{(i, j) \in B_1, j \leq \delta - q\}.$$



Let $M := \#B_0 + \#B_2 = N - (p - 1)(q - 1) = (p - 2)(q - 2) + [q/p] - 1$. Let $W'' := (W_{ij})_{(i,j) \in B_2}$ and $\mathbb{C}^M = \text{Spec } \mathbb{C}[H, W'']$

$$G_u(x, y, H, W'') := \left(x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{(i,j) \in B_2} W_{ij} x^i y^j$$

As before μ_d acts on the family $V(G_u) \subseteq \mathbb{C}^2 \times \mathbb{C}^M \rightarrow \mathbb{C}^M$.

THEOREM 2. \mathbb{C}^M/μ_d is the moduli space of all plane curve singularities defined by an equation

$$(x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{ij} x^i y^j = 0$$

with Milnor numbers $\mu = (2p - 1)(2q - 1) + d$ and G_u is the corresponding universal family.

Especially the Tjurina number $\tau = \mu - (p - 1)(q - 1)$ only depends on μ for these singularities.

COROLLARY. Let $\Gamma = \langle 2p, 2q, 2pq + d \rangle$, d odd, a semigroup.

Then $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$ is the moduli space of all irreducible plane curve singularities with the semigroup Γ .

G_u is the corresponding universal family.

Proof. To prove the theorem we compute generators of the kernel of the

Kodaira-Spencer map.

Let

$$G^{(0)} = x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j$$

$$G^{(1)} = x^a y^b + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} W_{ij} x^i y^j, \text{ i.e.}$$

$$G = G^{(0)2} + G^{(1)}.$$

Let $\delta \in \text{Der}_{\mathbb{C}} \mathbb{C}[H, W']$ be a vector field which belongs to the kernel of the Kodaira-Spencer map, i.e.

$$\delta G \in \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Now

$$\delta G = 2G^{(0)} \sum_{(i,j) \in B_0} \delta H_{ij} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} \delta W_{ij} x^i y^j = S \cdot G \bmod \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$$

for a suitable $S \in \mathbb{C}[H, W'][[x, y]]$.

We will associate to any monomial $x^a y^b$, $(a, b) \neq (0, 0)$, a vector field $\delta_{a,b} \in \text{Der}_{\mathbb{C}}[H, W']$ such that

$$\delta_{a,b} G = x^a y^b G \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Obviously $\{\delta_{a,b}\}$ generate the kernel of the Kodaira-Spencer map as $\mathbb{C}[H, W']$ -module.

Now consider

$$x^a y^b G = x^a y^b G^{(0)2} + x^a y^b G^{(1)}.$$

$$\text{Let } x^a y^b G^{(0)} = \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + L_1 \frac{\partial G^{(0)}}{\partial x} + L_2 \frac{\partial G^{(0)}}{\partial y}$$

for suitable $E_{ij}^{ab} \in \mathbb{C}[H, W']$, $L_1, L_2 \in \mathbb{C}[H, W'][[x, y]]$,

$$L_1 = \frac{1}{p} x^{a+1} y^b + \text{terms of higher degree}$$

$$L_2 = \frac{1}{q} x^a y^{b+1} + \text{terms of higher degree,}$$

then

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

The leading term of

$$x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y}$$

is $-(d/2pq)x^{\alpha+a}y^{\beta+b}$.

Using Lemma 2 we get

$$\begin{aligned} & x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \\ &= \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right) \end{aligned}$$

for suitable $D_{ij}^{ab} \in \mathbb{C}[H, W']$.

This implies

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

We define for $(a, b) \neq (0, 0)$

$$\delta_{a,b}(H_{ij}) := \frac{1}{2} E_{ij}^{ab}$$

$$\delta_{a,b}(W_{ij}) := D_{ij}^{ab}, \text{ i.e.,}$$

$$\delta_{a,b} = \frac{1}{2} \sum E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \sum D_{ij}^{ab} \frac{\partial}{\partial W_{ij}}$$

The vector fields $\delta_{a,b}$ have the following properties:

- (1) $\delta_{a,b}$ is zero if $aq + bp > 2pq - 2p - 2q$
- (2) $\delta_{a,b}(W_{ij}) = 0$ if $iq + jp < 2pq + d + aq + bp$
- (3) $\delta_{a,b}(W_{ij}) = -d/2pq$ if $(i, j) = (\alpha + a, \beta + b)$ or $(i, j) = (\alpha + a - p, \beta + b + q)$ (in this case $iq + jp = 2pq + d + aq + bp$).

- (4) $\delta_{a,b}(H_{ij}) = 0$ for all $(i, j) \in B_0$ if $aq + bp \geq pq - 2p - 2q$
- (5) For $iq + jp \geq 3pq + d - q$ and $(i, j) \in B_1$ there is (a', b') such that

$$(i, j) = (\alpha + a', \beta + b') \text{ or } (i, j) = (\alpha + a' - p, \beta + b' + q),$$

i.e. $\delta_{a',b'}(W_{ij}) = -d/2pq$.

- (6) For any (a, b) , $aq + bp < pq - q$, always $(\alpha + a, \beta + b)$ or $(\alpha + a - p, \beta + b + q) \in B_1$, i.e. $\delta_{a,b}(W_{ij}) = -d/2pq$ for the corresponding $(i, j) \in B_1$.

(1) and (4) hold because of the fact that

$$iq + jp \leq 2pq - 2p - 2q \quad \text{if } (i, j) \in B_0$$

$$iq + jp \leq 4pq + d - 2p - 2q \quad \text{if } (i, j) \in B_1.$$

(2) and (3) hold because of the fact that the leading term of $G^{(1)}$ has degree $2pq + d$ and because of Lemma 2. To prove (5) we consider two cases

1. Case $i \geq \alpha - 1$

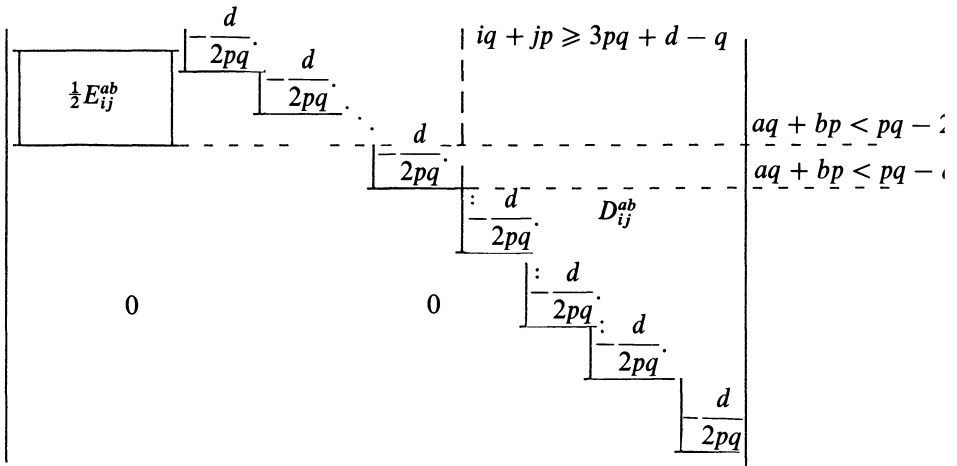
In this case $(i, j) \in B_1$ implies $i \leq p - 1$. But $iq + jp \geq 3pq + d - q$ implies $j \geq \beta$. Then $a' = i - \alpha, b' = j - \beta$ have the required properties. Notice that $i < \alpha - 1$ and $iq + jp \geq 3pq + d - q$ implies $(i, j) \notin B_1$

2 Case $i < \alpha - 1$

Now $iq + jp \geq 3pq + d - q$ implies $j \geq \beta + q$ then $a' = p + i - \alpha, b' = j - \beta - q$ have the required properties. (6) is similar to (5):

We may assume that $(\alpha + a, \beta + b) \notin B_1$. This implies $2p - 3 \geq \alpha + a \geq p$ and $\alpha \geq 2$ because $b \leq q - 2, a \leq p - 2$. Suppose $(\alpha + a - p, \beta + b + q) \notin B_1$ then $\beta + b + q \geq \beta + 2q - 1$, i.e. $b \geq q - 1$, or $\alpha + a - p \geq \alpha - 1$, i.e. $a \geq p - 1$, but this is not possible. □

For the coefficients to the vectorfields $\delta_{a,b}$ we get, because of (1)–(6), the following matrix:



This implies that the kernel of the Kodaira–Spencer map is generated (as $\mathbb{C}[H, W']$ -module) by the vector fields

$$\delta'_{ij} := \frac{\partial}{\partial W_{ij}} \quad (i, j) \in B_1, \quad iq + jp \geq 3pq + d - q$$

and

$$\begin{aligned} \delta'_{l,m} = & -\frac{pq}{d} \sum_{(i,j) \in B_0} E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \frac{\partial}{\partial W_{l,m}} + \\ & + \sum_{\substack{(i,j) \in B_1 \\ 2pq + d + aq + bp < iq + jp < 3pq + d - q}} \left(-\frac{2pq}{d} \right) D_{ij}^{ab} \frac{\partial}{\partial W_{ij}} \\ & \times (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}), \quad lq + mp < 3pq + d - q, \end{aligned}$$

with $(l, m) = \begin{cases} (\alpha + a, \beta + b) & \text{if } l \geq \alpha \\ (\alpha + a - p, \beta + b + q) & \text{else.} \end{cases}$

The vectorfields $\delta'_{l,m}$ act nilpotently on $\mathbb{C}[H, W']$. Namely, if we consider $\mathbb{C}[H, W']$ as a graded algebra defined by $\deg H_{ij} = pq - iq - jp < 0$, $\deg W_{ij} = 2pq - iq - jp < 0$ then the E_{ij}^{ab} resp. D_{ij}^{ab} are polynomials in $\mathbb{C}[H, W']$ of degree $\geq aq + bp + pq - iq - jp$ resp. $\geq aq + bp + 2pq - iq - jp$. Notice that their degree is always ≤ 0 . Let $A \in \mathbb{C}[H, W']$ be any polynomial of degree $0 \geq \deg A = s$ ($\deg A =$ minimum of the degrees of the monomials in A). Then the degree of $\delta'_{lm}(A) > s$. Therefore there is some n with $\delta'^n_{lm}(A) = 0$.

LEMMA 8. *Let A be a ring of finite type over a field k . $L \subseteq \text{Der}_k(A)$ a Lie-Algebra.*

Let $\delta_1, \dots, \delta_r$ vector fields with the following properties:

- (1) $\delta_1, \dots, \delta_r \in L$ and $L \subseteq \sum \delta_i A$
- (2) $[\delta_i, \delta_j] \in \sum_{k > \max\{i,j\}} \delta_k A$
- (3) *There are $x_1, \dots, x_r \in A$ such that*

$$\delta_i(x_i) = 1 \text{ and } \delta_j(x_i) = 0 \quad j > i$$

- (4) $\delta_1, \dots, \delta_r$ act nilpotently on A .

Then $A^L[x_1, \dots, x_r] = A$.

The Lemma is not difficult to prove. A similar lemma was used in the construction of the moduli space for curve singularities with the semi-group $\langle p, q \rangle$ (cf. [1], [2]).

Obviously A^L is the ring of all elements of A being invariant under $\delta_1, \dots, \delta_r$.

Now $A^{\delta_r}[x_r] = A$ and the conditions (2)–(4) of the lemma are satisfied for $\delta_1, \dots, \delta_{r-1}$ acting on A^{δ_r} .

Now we may apply the Lemma 8 to the kernel of the Kodaira-Spencer map and its generators $\{\delta'_{lm}\}$.

Because of the lemma the geometric quotient of $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[H, W']$ by the action of the kernel of the Kodaira-Spencer map exist and is isomorphic to the transversal section to the maximal integral manifolds (which intersect therefore each of these integral manifolds exactly in one point) defined by

$$W_{l,m} = 0, \quad (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}).$$

Now we use Lemma 6 and get Theorem 2. Notice that

$$\{G^{(0)}x^i y^j\}_{(i,j) \in B_0} \cup \{x^i y^j\}_{(i,j) \in B_2} \cup \{x^i y^j, iq + jp \leq 2pq, (i, j) \in B\}$$

is a base of the free $\mathbb{C}[H, W']$ -module $\mathbb{C}[H, W'][[x, y]]/(G, \partial G/\partial x, \partial G/\partial y)$. This implies $\mu - \tau = \#(B_1 \setminus B_2) = (p - 1)(q - 1)$. □

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