COMPOSITIO MATHEMATICA

BRUCE HUNT

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Compositio Mathematica, tome 76, nº 1-2 (1990), p. 203-242

<http://www.numdam.org/item?id=CM_1990__76_1-2_203_0>

Compositio Mathematica **76**: 203–242, 1990.

A Siegel Modular 3-fold that is a Picard Modular 3-fold*

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Received 26 August 1988; accepted 4 October 1989

$-\infty$ Introduction

Let $\mathscr{D}$ be a bounded symmetric domain. $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ be a discrete, properly discontinuous group. If Γ is cocompact and acts freely, it has been known for several decades (Kodaira: [K], Hirzebruch: [Hi]) that $\Gamma\backslash\mathscr{D}$ is then an algebraic variety, and in fact of general type. The Hirzebruch proportionality theorem then tells us the (ratios of) Chern numbers of $X = \Gamma\backslash\mathscr{D}$, which allows us to recover $\mathscr{D}$ from the Chern numbers of X if we know only that X is of the form $\Gamma\backslash\mathscr{D}$ for *some* $\mathscr{D}$. The group Γ is then of course just the fundamental group $\pi_1(X)$. So it can't happen, for example, that $X = \Gamma\backslash\mathscr{D} = \Gamma\backslash\mathscr{D}'$ for 2 non-isomorphic bounded symmetric domains $\mathscr{D}$ and $\mathscr{D}'$.

Arithmetically the condition Γ cocompact means that Γ has $\mathbb{Q}$-rank zero. Although occasionally such groups occur in algebraic geometry (Shimura curves, for example), in most cases Γ will only be of finite covolume, and the space $\Gamma\backslash\mathscr{D}$ occur as moduli spaces of some sort, usually as moduli spaces of varieties with special properties (e.g. classes of abelian varieties, K3-surfaces, curves, etc.) More generally, the *period domains* according to Griffiths [GS] (classifying spaces of Hodge structures) are all of the form $\Gamma\backslash G/H$, where G is a real simple non-compact Lie group and H is a compact subgroup, Γ a discrete subgroup. If one has a Torelli theorem for a corresponding class of varieties, then these spaces, locally homogenous complex manifolds are the moduli spaces for that class of varieties. These spaces are all fibre bundles over some symmetric spaces, for example

$$\mathrm{SO}(a,b)/\mathrm{U}(h^0) \times \cdots \times \mathrm{U}(h^{m-1}) \times \mathrm{SO}(h^m) \to \mathrm{SO}(a,b)/\mathrm{SO}(a) \times \mathrm{SO}(b),$$
$$(a = h^0 + h^2 + \cdots + h^{2m}, b = h^1 + h^3 + \cdots + h^{2m-1})$$
$$\mathrm{Sp}(2a,\mathbb{R})/\mathrm{U}(h^0) \times \cdots \times \mathrm{U}(h^m) \to \mathrm{Sp}(2a,\mathbb{R})/\mathrm{U}(a),$$
$$(a = h^0 + h^1 + \cdots + h^m).$$

*Research supported in part by NSF Grant 8500994A2

Returning to the situation in which $\Gamma\backslash\mathscr{D}$ is a moduli space, in general $\Gamma\backslash\mathscr{D}$ will not be compact, and the compactification is instrumented by adding moduli of the *degenerations* at the boundary. This leads one to consider (smooth) algebraic varieties X which contain $\Gamma\backslash\mathscr{D}$ as a Zariski-open subset. We call such X locally symmetric.

Locally symmetric varieties, together with a divisor D (usually assumed to be normal crossings) such that $X - D = \Gamma\backslash\mathscr{D}$, still contain a lot of "symmetry", that is, structure determined by the group structure of $\mathrm{Aut}(\mathscr{D})$, and so we can think of a correspondence of pairs (X, D) and pairs $(\mathscr{D},\Gamma)$, where now $\Gamma = \pi_1(X-D)$. Thus it is now quite conceivable that given an algebraic variety X, there exist 2 different divisors D_1 and D_2 such that (X, D_1) corresponds to $(\mathscr{D}_1, \Gamma_1)$ and (X, D_2) corresponds to $(\mathscr{D}_2, \Gamma_2)$ in the above sense, but $\mathscr{D}_1$ and $\mathscr{D}_2$ are not isomorphic, nor even of the same $\mathbb{R}$-rank for that matter. This phenomena seemed worthy of study as soon as it occurred somewhere. For surfaces no such examples were, up to now, known.

Let X be the following modification (blow-up) of $\mathbb{P}^3$ (for notations and details see 2.5.1. below). Let $D_1, \ldots, D_4$ be the 4 coordinate planes (regular tetrahedron), $D_5, \ldots, D_{10}$ the 6 symmetry planes of this tetrahedron, and $D_{11}, \ldots, D_{15}$ be the 5 exceptional $\mathbb{P}^2$'s gotten by blowing up $\mathbb{P}^3$ at the corners and the center of the tetrahedron. Finally let $E_1, \ldots, E_{10}$ be the $\mathbb{P}^1 \times \mathbb{P}^1$'s gotten by blowing up along the 10 3-fold lines of the 10 planes (i.e. of the *arrangement* $D = D_1 \cup \cdots \cup D_{10}$, see 2.5). The proof and study of the following result is the object of this paper:

THEOREM. (i) (X, D) *corresponds to* $(\mathbb{S}_2, \Gamma(2))$ *in the above sense, where* $\mathbb{S}_2$ = *Siegel upper half space of degree* 2, $\Gamma(2)$ = *principal congruence subgroup of level* 2.

(ii) (X, E) *corresponds to* $(\mathbb{B}^3, \Gamma(1-\rho))$ *in the above sense,* $\mathbb{B}^3$ = *complex hyperbolic* 3-*ball,* $\Gamma(1-\rho)$ = *principal congruence subgroup of* $\mathrm{U}((3,1), \mathcal{O}_K)$, $K = \mathbb{Q}(\sqrt{-3})$, $\rho = e^{2\pi i/3}$. *Here* $D = \Sigma D_i, i = 1, \ldots, 15$, $E = \Sigma E_\lambda, \lambda = 1, \ldots, 10$.

There are several directions into which this result can be further investigated. On the group level, one can associate to the discrete group Γ a Tits building with scaffolding. Here we find that the corresponding Tits building with scaffoldings are *dual* to each other. It is not yet clear whether this is implied by the double structure as locally symmetric space or whether this is perhaps as additional coincidence of this particular example.

It is not difficult to determine the ring of modular forms of $\Gamma(2)$, using theta constants (Igusa [I1], [I2]). As for the ring corresponding to the group $\Gamma(1-\rho)$, the structure can be determined by utilizing results of Holzapfel [Ho1], together with results of Deligne-Mostow [DM]. Using in addition a result of Shimura [S] which characterises (X, E) as a moduli space of Picard curves, one can even describe the ring in terms of theta constants. Viewing things this way, the duality above turns into projective duality of varieties (the singular Baily-Borel embeddings), and this duality is in fact classical (cf. [B]).

It turns out that both (X, D) and (X, E) have moduli interpretations, and it is easthetically pleasing to see this duality in terms of degenerations. $X - D - E$ parametrises on the one hand hyperelliptic curves $y^2 = p_6(x)$, genus 2 curves, and on the other hand Picard curves $y^3 = p_6(x)$, genus 4 curves, the correspondence being given by the zeros of $p_6(x)$. But where zeroes of $p_6(x)$ coincide ($x \in D \cup E$), the types of degenerations seem to be dual to each other. For the hyperelliptic curves D corresponds to curves aquiring doublepoints, E to curves splitting into 2 components. For the Picard curves it is the other way around. This is summarised in the table 5.4.

For each such family of curves, one can consider the Picard-Fuchs differential equation corresponding to the dependency of the periods on the moduli. Both our families of curves are associated with the famous hypergeometric differential equation, but with different parameters. Here we use results of [DM] on the one hand, and of Sasaki and Yoshida [KSY] on the other.

The variety we are studying in this paper is probably one of the most thoroughly studied algebraic 3-folds, so we don't claim to be deriving new results on this variety. Rather, our object is to study in detail the 2 structures of locally symmetric spaces and their interrelations. The Siegel picture (§1) is very well known whence we only sketch the necessary statements and facts; we refer to [V] for general results and to [LW] for combinatorial and cohomological results. The Picard picture (§2) is, to the best of our knowledge, essentially new, so we give this side of the picture in much greater detail. However, this Picard picture is an almost straightforward 3-dimensional generalisation of the Picard picture of a surface for which extremely detailed results are available, namely Holzapfel's monograph [Hol]. Hence here there is also little which is original.

As this subject also has an interesting history, as well as being a model for our efforts in dimension 3, we now recall some of the background and results of the surface case. As long ago as 1769, L. Euler considered the following partial differential equation in connection with acoustics:

$$(E_a) \quad \frac{\partial^2 z}{\partial x \partial y} - \frac{a}{x-y}\frac{\partial z}{\partial x} + \frac{a}{x-y}\frac{\partial z}{\partial y} = 0.$$

About a century later Riemann constructed solutions of this equation by an inversion process, during which hypergeometric functions occurred. In 1880 Appell gave the generalisation to several variables of the hypergeometric function which also occurs in recent work of Mostow and Deligne. In 1881 E. Picard studied these and found the famous integral representation for Appell's hypergeometric series:

$$\int_0^1 u^\alpha(1-u)^\beta(1-xu)^\gamma(1-yu)^\delta\,\mathrm{d}u.$$

He in particular studied the integral

$$I_{g,h}(t_1,t_2) := \int_g^h \frac{\mathrm{d}x}{\sqrt[3]{x(x-1)(x-t_1)(x-t_2)}}, \quad g,h \in \{0,1,t_1,t_2,\infty\}$$

and discovered that this function (of t_1 and t_2) is a special solution of the Euler equation $(E_{1/3})$! In fact, 3 of these integrals form a fundamental system of solutions for a system of 3 partial differential equations.

One recognises immediately the integrals above as forming a base of the (1,0)-differentials on the trigonal curve

$$y^3 = x(x-1)(x-t_1)(x-t_2),$$

which is a genus 3 curve with Galois action by $\mathbb{Z}/3\mathbb{Z}$. These are the so-called Picard curves, of which we will be studying a suitable generalisation (we also call our curves, genus 4 curves, Picard curves).

We now recall for the reader's benefit some of Holzapfel's results in the study of the family of Picard curves. Consider the following subgroups of $\mathrm{Aut}(\mathbb{B}^2) = \mathrm{PSU}(2,1)$:

Eisenstein lattice	$\tilde{\Gamma} := \mathrm{PU}(2,1;\mathcal{O}_K), K = \mathbb{Q}(\sqrt{-3})$
Special Eisenstein lattice	$\Gamma := \mathrm{PSU}(2,1;\mathcal{O}_K)$,

$$\mathcal{O}_K = \text{ring of integers in } K.$$

Letting $\rho = e^{2\pi i/3}$ be a primitive cube root of unity, $(1-\rho)$ is an ideal in $\mathcal{O}_K$ (note that $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\rho)$ since $\rho = \frac{-1+\sqrt{-3}}{2}$. One considers also the congruence subgroups $\Gamma' = \Gamma(1-\rho)$, and $\tilde{\Gamma}' = \tilde{\Gamma}(1-\rho)$ (here we are using Holzapfel's notation). The following results are proved in [Ho1]:

I. The monodromy group of the system of partial differential equations alluded to above is $\tilde{\Gamma}'$.

II. The rings of automorphic forms for $\tilde{\Gamma}'$ and Γ are:

$$\bigoplus_{m=0}^{\infty} [\tilde{\Gamma}', m]_\chi = \mathbb{C}[\xi_1, \xi_2, \xi_3], \quad \bigoplus_{m=0}^{\infty} [\tilde{\Gamma}, m]_\chi = \mathbb{C}[G_2, G_3, G_4],$$

where the ξ_i have weight 1 and the G_j have weight j.

III. $\tilde{\Gamma}'\backslash\mathbb{B}^2 \cong \mathbb{P}^2$-{4 points}.

IV. If F_1, F_2 and F_3 are 3 fundamental solutions of the system alluded to above, then the automorphic forms of weight 1, $(\xi_1:\xi_2:\xi_3)$ give (up to coordinate transformations) the inverse to the many-valued function $(F_1:F_2:F_3)$, in other words, $(\xi_1:\xi_2:\xi_3)\circ(F_1:F_2:F_3)$ is $1-1$, and

$$\begin{tikzcd} & \mathbb{B}^2 \arrow[dr, "(\xi_1:\xi_2:\xi_3)"] & \\ \mathbb{P}^2\text{-}\{4 \text{ points}\} \arrow[ur, Rightarrow, "(F_1:F_2:F_3)"] \arrow[rr] & & \mathbb{P}^2\text{-}\{4 \text{ points}\} \end{tikzcd}$$

commutes.

V. There is also a commutative diagram

$$\begin{tikzcd} (\mathbb{B}^2)^* \arrow[r] \arrow[ddr] & \{\text{Picard curves}\}/\text{projective equivalence} \\ \ni P \arrow[r, mapsto] & C(P) := (y^3 = x^4 + G_2(P)x^2 + G_3(P)x + G_4(P)\} \arrow[dl, leftrightarrow] \\ & \tilde{\Gamma}\backslash(\mathbb{B}^2)^* \end{tikzcd}$$

where $(\mathbb{B}^2)^* = \mathbb{B}^2 \cup \{\tilde{\Gamma}\text{-rational cusps}\}$, $\tilde{\Gamma}\backslash(\mathbb{B}^2)^*$ is the Baily-Borel compactification.

VI. The (image of) $\tilde{\Gamma}$-fixed points of $\mathbb{B}^2$ are the set of Picard curves with automorphism group larger than $\mathbb{Z}/3\mathbb{Z}$.

In our work below we are able to give reasonable generalisations of I, II, III and IV to dimension 3. It would be challenging and extremely interesting to also get a nice equation in terms of modular forms (V) and to generalise VI also to dimension 3.

Finally we would like to remark that during the last year it has started to emerge that this example is the first in some finite list of examples with similar properties. An upcoming paper with Weintraub will give more details on the other known examples related to this one.

Acknowledgements

The author would like to thank the organisers of the conference at the Humboldt Universitat for the invitation and the opportunity to speak on this topic there. Also conversations with R.-P. Holzapfel and J.-M. Feustel were helpful. Finally

I am greatly indebted to S. Weintraub for many explanations and conversations on this and related topics, in particular for the technicalities of 2.1.

0. Notations and conventions

All varieties considered are over the complex number field $\mathbb{C}$. We use freely the standard notions of hemitian symmetric spaces. For a non-compact locally hermitian symmetric space X, X^* usually denotes the Baily-Borel compactification, $X^\wedge$ some desingularisation.

Throughout, we use the following notations:

$\mathbb{P}^n$	projective space (over $\mathbb{C}$)
$\mathbb{B}^n$	the complex n-ball
Σ_n	symmetric group on n letters
A_n	alternating group on n letters
Γ	some arithmetic group, particular cases of which are:
$\Gamma(2), \Gamma(4)$	principal congruence subgroups (Siegel case)
$\Gamma(1-\rho), S\Gamma(1-\rho), \Gamma(1-\rho)^2$	lattices in Picard groups
$\mathbb{F}_p$	field with p elements (p prime), which is sometimes also denoted $\mathbb{Z}/p\mathbb{Z}$
$\mathrm{Sp}(n,\mathbb{R}), \mathrm{U}(p,q), \mathrm{SU}(p,q)$	the usual classical groups
K	number field,
$\mathcal{O}_K$	ring of integers in K
$M_n(\mathbb{C})$	$n \times n$ matrices with $\mathbb{C}$-coefficients
$\pi_1(X)$	fundamental group.

1. A Siegel Modular 3-fold

1.1. Let $\mathbb{S}_2 = \mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(2) = \{Z \in M_2(\mathbb{C}) \mid Z = {}^tZ.\ \mathrm{Im}(Z) > 0\}$ be the Siegel upper half-space of degree 2, a 3-dimensional, $\mathbb{R}$-rank 2 bounded symmetric domain. $\mathrm{Sp}(2,\mathbb{Z})$ is a lattice in $\mathrm{Sp}(2,\mathbb{R})$ which has $\mathbb{Q}$-rank 2. The action of $\Gamma = \mathrm{Sp}(2,\mathbb{Z})$ on $\mathbb{S}_2$ is $Z \mapsto (AZ+B)(CZ+D)^{-1}$. We are particularly interested in the principal congruence subgroup of level 2, defined by the following exact sequence:

$$1 \to \Gamma(2) \to \mathrm{Sp}(2,\mathbb{Z}) \to \mathrm{Sp}(2,\mathbb{Z}/2\mathbb{Z}) \to 1. \tag{1.1.1}$$

$\Gamma(2)$ is thus a normal subgroup, of index equal to $|\mathrm{Sp}(2,\mathbb{Z}/2\mathbb{Z})| = 720$, since

$\mathrm{Sp}(2, \mathbb{Z}/2\mathbb{Z}) = \Sigma_6$, the symmetric group on 6 letters, as is well known. $\Gamma(2)$ does not act freely, but the quotient is smooth [I1], [I3], [C]. Let $X(2)$ denote the non-compact quotient $\Gamma(2)\backslash\mathbb{S}_2$.

1.2. A compactification of $X(2)$ is constructed in the standard way, i.e. Baily-Borel. Adjoin to $\mathbb{S}_2$ the rational (with respect to Γ) boundary components, which are copies of $\mathbb{S}_1$ in dimension 1 (rank 1), and points in dimension 0 (rank 0). The action of $\Gamma(2)$ extends to the rational boundary of $\mathbb{S}_2$, and the quotient $\Gamma(2)\backslash\mathbb{S}_2^*$ is a compact Hausdorff space. The action of $\Gamma(2)$ on one of the $\mathbb{S}_1$'s on the boundary is via the principal congruence subgroup of level 2 of $\mathrm{Sp}(1, \mathbb{Z}) = \mathrm{Sl}(2, \mathbb{Z})$, which has 3 inequivalent cusps. Since $\Gamma(2) \subset \mathrm{Sp}(2, \mathbb{Z})$ has 15 inequivalent 1-dimensional cusps as well as 15 inequivalent 0-dimensional ones, one gets the following configuration on $X(2)^* = \Gamma(2)\backslash\mathbb{S}_2^*$:

15 curves $C_i = \Gamma(2)\backslash\mathbb{S}_1^*$
15 points $P_{ij} = C_i \cap C_j =$ cusp of C_i and C_j.
3 different C_i meet at each P_{ij}.
Each C_i contains 3 cusps P_{ij}

1.3. In order to describe the boundary components precisely it is convenient to work in $(\mathbb{Z}/2\mathbb{Z})^4$. Since any two cusps are equivalent under $\mathrm{Sp}(2, \mathbb{Z})$ the exact sequence 1.1.1 implies the natural action of $\mathrm{Sp}(2.\mathbb{Z}/2\mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^4$ gives exactly the action of $\Gamma(2)$ on the boundary components (see also [LW]):

$$\text{1-dimensional cusps of } X(2)^* \leftrightarrow \begin{cases} l \in (\mathbb{Z}/2\mathbb{Z})^4, & 1 \neq 0 \\ l = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), & \varepsilon_j = 0 \text{ or } 1. \end{cases}$$

$$\text{0-dimensional cusps of } X(2)^* \leftrightarrow \begin{cases} h = l_1 \wedge l_2, \text{ 2-dimensional} \\ \text{isotropic subspaces.} \end{cases}$$

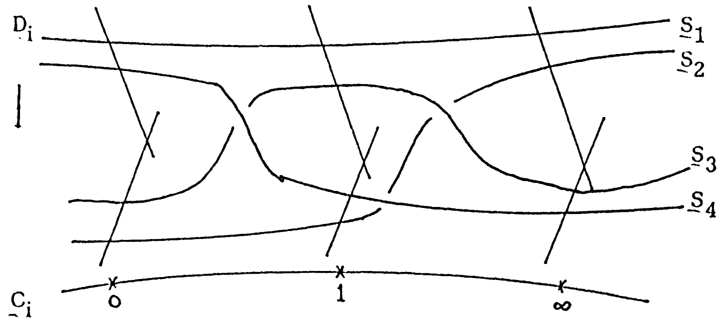
In this scheme the curves C_i are numbered by 4-tuples $(\varepsilon_1, \ldots, \varepsilon_4)$, $\varepsilon_j = 0, 1$. Then $C_i \cap C_j$ is the 2-plane spanned by

$$\begin{pmatrix} \varepsilon_1^i & \varepsilon_2^i & \varepsilon_3^i & \varepsilon_4^i \\ \varepsilon_1^j & \varepsilon_2^j & \varepsilon_3^j & \varepsilon_4^j \end{pmatrix},$$

and the third curve C_k meeting $C_i \cap C_j$ is (ε_μ^k), $\varepsilon_\mu^k = \varepsilon_\mu^i + \varepsilon_\mu^j$.

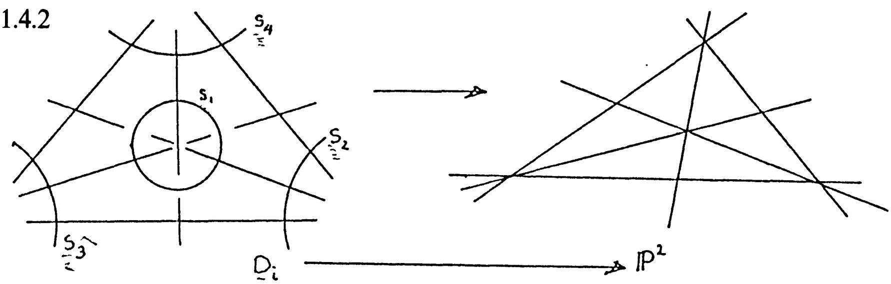
1.4. A desingularisation of $X(2)^*$ was constructed by Igusa in [I3] by blowing up along the sheaf of ideals defining the boundary. In [V] van der Geer explains how to obtain the desingularisation directly by means of toroidal embeddings of $X(2)$. The result is the same, and is as follows: there are 15 divisors $D_1, \ldots, D_{15}$, each itself an algebraic fibre space $D_i \to C_i$, whose generic fibre is a Kummer curve = $\mathbb{P}^1$ (= elliptic curve/involution) and whose 3 special fibres are degenerate conics in $\mathbb{P}^2$, consisting of 2 copies of $\mathbb{P}^1$ meeting at a point

(1.4.1)



The fibre space $D_i \to C_i$ has 4 sections $S_1, \ldots, S_4$. Another way to view the D_i is as $\mathbb{P}^2$ blown up in the 4 3-fold points of the line arrangement in $\mathbb{P}^2$:

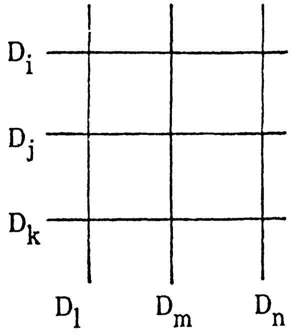
1.4.2



The fibering $D_i \to C_i$ is given by the pencil of conics passing through the 4 points, and the singular fibres are the 6 lines of the arrangement, 2 of them at a time forming a degenerate conic, and the sections S_i are the exceptional $\mathbb{P}^1$'s of the blow-up. Therefore on each D_i one can blow down the 4 sections, the result being $\mathbb{P}^2$. The D_i intersect 2 at a time along the singular fibres and 3 at a time at the double points of those fibres. This describes the structure of the normal crossings divisor $D = \Sigma D_i$. For more details on the intersection behavior see 2.5. We denote the Igusa desingularisation by $X(2)^\wedge$.

1.5. We now describe another important set of divisors, the Humbert surfaces of discriminant 1. For each natural number $\Delta \equiv 0$ or $1 \bmod (4)$ there is such a Humbert surface H_Δ ([V], §2), but we will only describe H_1 here. The diagonal $\mathbb{S}_1 \times \mathbb{S}_1 \subset \mathbb{S}_2$ has 10 inequivalent transforms under $\Gamma(2)$, and the action of $\Gamma(2)$ restricted to each copy is by $\Gamma_1(2) \times \Gamma_1(2) \subset \mathrm{Sl}(2, \mathbb{Z}) \times \mathrm{Sl}(2, \mathbb{Z})$. (Here $\Gamma_1(N)$ denotes the principle congruence subgroup of level N and degree 1, i.e. in $\mathrm{SL}(2, \mathbb{Z})$.) Let $E_1, \ldots, E_{10}$ be the images in $X(2)^*$ of these diagonals. Then each $E_j = \Gamma_1(2)\mathbb{S}_1^* \times \Gamma_1(2)\backslash\mathbb{S}_1^*$ is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. The E_j

are disjoint, but intersect the D_i in $X(2)^\wedge$ at the sections of each. Each E_λ intersects 6 D_i, 3 in each direction:



$$\begin{array}{cccc} \varepsilon_1^i & \varepsilon_2^i & \varepsilon_3^i & \varepsilon_4^i \\ \varepsilon_1^j & \varepsilon_2^j & \varepsilon_3^j & \varepsilon_4^j \\ \varepsilon_1^k & \varepsilon_2^k & \varepsilon_3^k & \varepsilon_4^k \end{array}$$

1.6. To describe the incidences $E_\lambda \cap D_i \neq \varnothing$, we follow [LW, §2]. Let $\Delta = \{\delta, \delta^\perp\}$, an unordered pair of δ, a non-singular plane and $\delta^\perp$, its orthogonal complement. Such Δ are in $1-1$ correspondence with the E_λ, and $E_\lambda \cap D_i \neq \varnothing$ iff $(\varepsilon_\alpha^i) \in \delta$ or $\in \delta^\perp$. Hence the E_λ can be numbered by $\{(\varepsilon_\alpha^i) \wedge (\varepsilon_\alpha^j), (\varepsilon_\alpha^k) \wedge (\varepsilon_\alpha^l)\}$, for example. We just give one example of this. Say E_1 will be numbered by $(1,0,0,0) \wedge (0,0,1,0)$ and $(0,1,0,0) \wedge (0,0,0,1)$, and the other D_i's meeting E_1 are $(1,0,1,0)$ and $(0,1,0,1)$ which, plugging into the above scheme, describes all intersections quite explicitly. The action of $\mathrm{Sp}(2,\mathbb{Z}/2\mathbb{Z})$ induces an action of the E_λ, i.e. $gE_\lambda = E_\mu$, E_λ associated to Δ and E_μ associated to Δg, for $g \in \mathrm{Sp}(2,\mathbb{Z}/2\mathbb{Z})$.

1.7. Finite covers

We just state a result here which follows from Theorem 3.3.2. and the proof of Theorem 2.7.1, but whose statement belongs here. Let $\Gamma(4)$ denote the principle congruence subgroup of level 4. This is a normal subgroup of $\Gamma(2)$ and $P\Gamma(2)/P\Gamma(4) = (\mathbb{Z}/2\mathbb{Z})^9$ (it is the projective groups that are acting effectively). This is the Galois group of the Fermat cover of degree 2 branched along the arrangement H (see 2.7. and [Hu] for Fermat covers), and in fact

THEOREM 1.7.1. *The (smooth) Fermat cover $Y(2,H)$ branched along H,*

$$Y(2,H) \to \hat{\mathbb{P}}^3$$

is the (Igusa compactification of the) Siegel modular 3-fold of level 4.

2. A Picard Modular 3-fold

2.1. Let $\mathbb{B}^3 := \mathrm{SU}(3,1)/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(1)) = \{z \in \mathbb{C}^3 \mid \sum |z_i|^2 < 1\}$ be the complex hyperbolic 3-ball, a 3-dimensional, $\mathbb{R}$-rank 1 bounded symmetric domain. The best-known lattices in SU(3, 1) are the Picard modular groups. For each square-free integer d let $K = \mathbb{Q}(\sqrt{-d})$ be the imaginary quadratic field associated with d and $\mathcal{O}_K$ = the ring of integers in K. Then $\mathcal{O}_K \subset \mathbb{C}$ is a lattice, and SU(3, 1; $\mathcal{O}_K$) is the *Picard Modular group of discriminant D* (D the discriminant of K) which is a lattice in SU(3,1) (note that whereas Sp(2, $\mathbb{R}$) is a group with real coefficients, so integer coefficients give a lattice, SU(3, 1) is a group of complex matrices, so we need coefficients in a lattice in $\mathbb{C}$). We shall be concerned in this paper with the field of Eisenstein numbers $K = \mathbb{Q}(\sqrt{-3})$, and the corresponding Picard modular group. Actually, the group more basic to our applications is the lattice U(3, 1; $\mathcal{O}_K$). These lattices are related as follows: SU $\lhd$ U, and U/(SU $= \mathbb{Z}/3\mathbb{Z}$, since an element $\in$ U(3, 1; $\mathcal{O}_K$) may have determinant $= \rho$ or ρ^2, as well as 1. We will refer to U as the Picard lattice, and to SU as the special Picard (or Eisenstein) lattice.

The action of Aut($\mathbb{B}_3$) = PU(3, 1) (PU since the center acts trivially) is by fractional linear transformations. For $\gamma \in \text{Aut}(\mathbb{B}^3)$, $z = (z_1, z_2, z_3) \in \mathbb{B}^3$, this action is described as follows:

$$\gamma z = \frac{1}{\sum a_{i4} z_i} \begin{pmatrix} \sum a_{i1} z_i \\ \sum a_{i2} z_i \\ \sum a_{i3} z_i \end{pmatrix}, \quad \gamma = (a_{ij}), \quad 1 \leqslant i,j \leqslant 4. \tag{2.1.1}$$

The jacobian of the action is $(\sum a_{i4} z_i)^{-1}$ at $z = (z_i)$. When considering lattices Γ in U(3, 1) or SU(3, 1) we will, without mentioning it, take their images in PU(3, 1). However, as this is a potential source of confusion, we describe this in some detail. Since the result is quite different in dimensions 2 and 3 we describe both. Let Γ, $\tilde{\Gamma}$, Γ', $\tilde{\Gamma}'$ be as in the introduction (Holzapfel's notation) and let a P in front of one of the groups denote the projectivised group. Let $Z(G)$ denote the center of a group G. Then:

$$Z(\tilde{\Gamma}) = \{\pm 1, \pm\rho, \pm\rho^2\}, \qquad Z(\Gamma) = \{1, \rho, \rho^2\}$$
$$Z(\tilde{\Gamma}') = \{1, \rho, \rho^2\}, \qquad Z(\Gamma') = \{1, \rho, \rho^2\}.$$

Thus we have the diagrams:

$$\begin{array}{ccc} \Gamma & \xrightarrow{1-6} & \tilde{\Gamma} \\ \big\downarrow{\scriptstyle 3-1} & & \big\downarrow{\scriptstyle 6-1} \\ P\Gamma & \xrightarrow{1-3} & P\tilde{\Gamma} \end{array} \qquad\qquad \begin{array}{ccc} \Gamma' & \xrightarrow{1-3} & \tilde{\Gamma}' \\ \big\downarrow{\scriptstyle 3-1} & & \big\downarrow{\scriptstyle 3-1} \\ P\Gamma' & \xrightarrow{1-3} & P\tilde{\Gamma}' \end{array}$$

and then a diagram relating the congruence subgroups:

$$\begin{tikzcd}
1 \arrow[r] & P\tilde{\Gamma}' \arrow[r] & P\tilde{\Gamma} \arrow[r] & PU(2,1;\mathbb{F}_3) = \Sigma_4 \arrow[r] & 1 \\
1 \arrow[r] & P\Gamma' \arrow[r] \arrow[u, hook] & P\Gamma \arrow[r] \arrow[u, hook] & PSU(2,1;\mathbb{F}_3) = \Sigma_4 \arrow[r] \arrow[u, equal, "\wr"] & 1
\end{tikzcd}$$

the bottom sequence of which Holzapfel proves is exact [Hol, 1.3.1]. Both inclusions $P\Gamma' \subset P\tilde{\Gamma}'$ and $P\Gamma \subset P\tilde{\Gamma}$ are of index 3.

We now give the corresponding diagrams in dimension 3, and fix, for the rest of this paper, the following notations:

$$\begin{aligned}
&\Gamma := U(3,1;\mathcal{O}_K), \quad \Gamma(1-\rho) \text{ congruence subgroup,} \\
&S\Gamma := SU(3,1;\mathcal{O}_K), \quad S\Gamma(1-\rho) \text{ congruence subgroup.}
\end{aligned}$$

Here we have

$$\begin{aligned}
&Z(\Gamma) = \{\pm 1, \pm\rho, \pm\rho^2\}, \quad Z(S\Gamma) = \{\pm 1\} \\
&Z(\Gamma(1-\rho)) = \{1, \rho, \rho^2\}, \quad Z(S\Gamma(1-\rho)) = \{1\},
\end{aligned}$$

giving the following diagrams:

$$\begin{tikzcd}
S\Gamma \arrow[r, "1-6"] \arrow[d, "2-1"] & \Gamma \arrow[d, "6-1"] \\
PS\Gamma \arrow[r, "1-2"] & P\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
S\Gamma(1-\rho) \arrow[r, "1-3"] \arrow[d, "1-1"] & \Gamma(1-\rho) \arrow[d, "3-1"] \\
PS\Gamma(1-\rho) \arrow[r, "1-1"] & P\Gamma(1-\rho).
\end{tikzcd}$$

Furthermore, from the Atlas of finite simple groups we have

$$\begin{aligned}
&U(3,1;\mathbb{F}_3) = \Sigma_6 \times 2 \\
&PU(3,1;\mathbb{F}_3) = \Sigma_6 \\
&SU(3,1;\mathbb{F}_3) = A_6 \times 2 \\
&PSU(3,1;\mathbb{F}_3) = A_6.
\end{aligned}$$

We have the following sequence

$$\begin{tikzcd}
1 \arrow[r] & P\Gamma(1-\rho) \arrow[r] & P\Gamma \arrow[r] & PU(3,1;\mathbb{F}_3) \cong \Sigma_6 \arrow[r] & 1 \\
1 \arrow[r] & PS\Gamma(1-\rho) \arrow[r] \arrow[u, equal, "\wr"] & PS\Gamma \arrow[r] \arrow[u, hook] & PSU(3,1;\mathbb{F}_3) \cong A_6 \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

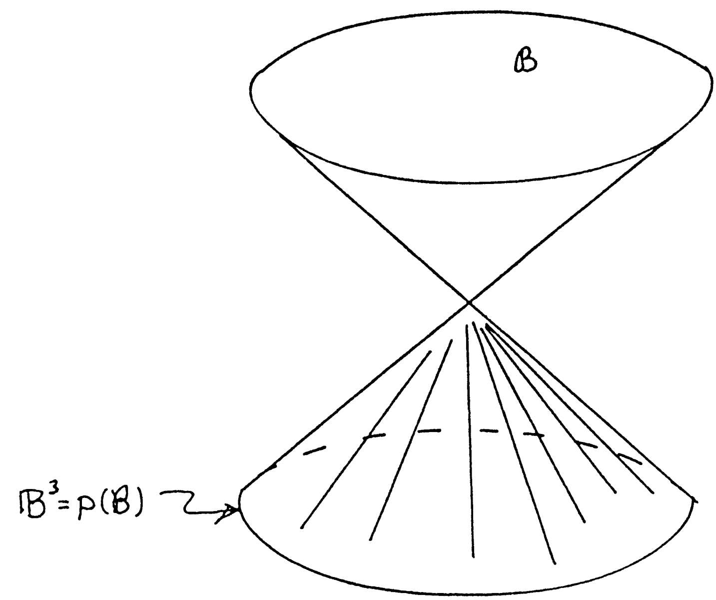
(where A_6 is the alternating group on 6 letters), and $PS\Gamma(1-\rho) \subset P\Gamma(1-\rho)$ is an isomorphism, whereas $PS\Gamma \subset P\Gamma$ is a subgroup of index 2.

Now just like $\Gamma(2)$ in Section 1, $\Gamma(1-\rho)$ does not act on $\mathbb{B}^3$ freely but it turns out that the quotient is nonetheless smooth. The subgroup $\Gamma(1-\rho)^2$ does act freely, and the singularities of $\Gamma(1-\rho)\backslash\mathbb{B}^3$ can be analysed by studying the cover corresponding to $\Gamma(1-\rho)^2 \subset \Gamma(1-\rho)$. We will describe this in 2.8 below. Let $Y(1-\rho)$ denote the non-compact quotient $\Gamma(1-\rho)\backslash\mathbb{B}^3$.

2.2. We now discuss the boundary components of $\mathbb{B}^3$ with respect to Γ. To do this, think of $\mathbb{B}^3 \subset V^4$, a 4-dimensional vector space over $\mathbb{C}$ with hermitian form

$$\Phi(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 - x_4\bar{y}_4.$$

Let $\mathscr{B} \subset V$ be the positive cone, i.e. $\mathscr{B} = \{v \subset V\,\Phi(v, v) > 0\}$. This looks as follows:



Then $\mathbb{B}^3 = p(\mathscr{B})$, $p\colon V \to \mathbb{P}(V)$ the natural projection. From this one sees that $\partial\mathbb{B}^3 = p(\partial\mathscr{B}) = p(\mathscr{E})$, $\mathscr{E} \subset V$ the set of isotropic vectors, $\mathscr{E} = \{v \in V \mid \Phi(v, v) = 0\}$. With this picture in mind it is obvious that the K-boundary components (or the boundary components with respect to Γ) are: (here we should fix an embedding $K \subset \mathbb{C}$),

$$\partial_K\mathbb{B}^3 = \{v \in K^4 \mid \Phi(v, v) = 0\}/K^* = p(\mathscr{E}_K), \mathscr{E}_K = \mathscr{E} \cap K^4;$$

these are the isotropic lines in $\mathbb{P}(K^4)$. Hence the number of Γ-cusps is the order of $\Gamma\backslash\partial_K\mathbb{B}^3$. This question is discussed in [Z, I] for any discriminant and corresponding Picard group. The answer is: Γ_d has $\mu(d)$ cusps, where $\mu(d) = \#\,|\text{equivalence classes of hermitian unimodular lattices in } K^4|$. This number has been calculated by Hashimoto [HK] and for $d = 3$ there is a unique equivalence class (1 cusp).

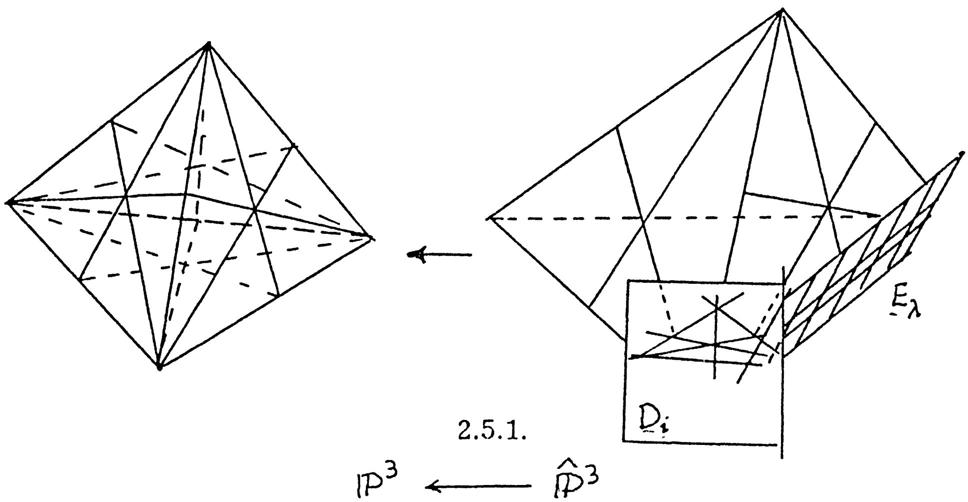
2.3. Hence, to determine the number of cusps of $\Gamma(1-\rho)$, just consider the exact sequence 2.1.2, and the ensuing action of $\Gamma/\Gamma(1-\rho) = \mathrm{PU}(3,1;\mathbb{Z}/3) = \Sigma_6$ on $(\mathbb{Z}/3)^4$. The number of cusps is just the number of totally isotropic lines $\{v \in \mathbb{Z}/3^4 \mid \Phi(v,v) = 0\}/\pm 1$ (note that $\mathbb{F}_3^* = \pm 1$), just reducing the form $\Phi \bmod 3$. This works as follows. The form on $\mathbb{C}^4$ is $z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - z_4\bar{z}_4$. The involution $z \to \bar{z}$ in $\mathcal{O}$ (where it is given by $\rho \to \rho^2$) descends to the trivial involution on $\mathbb{Z}/3$ (as it must, $\mathbb{Z}/3$ having no non-trivial automorphisms) which can be seen by the fact that $\rho \equiv 1 \bmod(1-\rho)$ so $\rho^2 \equiv 1^2 = 1 \equiv \rho \bmod(1-\rho)$. It is easily checked that there are, up to sign, 10 isotropic vectors: $(1,0,0,1)$, $(1,0,0,-1)$, $(0,1,0,1)$, $(0,1,0,-1)$, $(0,0,1,1)$, $(0,0,1,-1)$, $(1,1,1,0)$, $(1,1,-1,0)$, $(1,-1,1,0)$ and $(-1,1,1,0)$. I am indebted to Steve Weintraub for this nice exposition.

2.4. A desingularisation of $Y(1-\rho)^*$ is constructed by blowing up at the cusps. The resolving divisor is A/ι, $A = E \times E$ an abelian surface which is a product of 2 copies of the elliptic curve with complex multiplication by $\sqrt{-3}$ and ι is the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$, in other words the resolving divisors are $\mathbb{P}^1 \times \mathbb{P}^1$'s. This can be extracted from the standard construction ([He], [Hol]). This is of course the same as a desingularisation by means of toroidal embeddings which in the case of $\mathbb{Q}$-rank 1 yields isolated resolving divisors. Let $Y(1-\rho)^\wedge$ denote this resolution. We will see below that $Y(1-\rho)^\wedge$ is actually already smooth.

2.5. We now describe another important set of subvarieties, which we call modular subvarieties. Let $D = \mathbb{B}^2 \subset \mathbb{B}^3$ be a totally geodesic embedding such that Γ_D act properly discontinously, that is $\Gamma_D := \{\gamma \in \Gamma \mid \gamma D = D\} \subset \mathrm{Aut}(\mathbb{B}^2)$ is properly discontinous. That is to say the diagram

$$\begin{array}{ccc} D & \subset & \mathbb{B}^3 \\ \Big\downarrow{\scriptstyle \Gamma_D} & & \Big\downarrow{\scriptstyle \Gamma} \\ \Gamma_D\backslash\mathbb{B}^2 & \subset & \Gamma\backslash\mathbb{B}^3 \end{array}$$

commutes. This is what is usually called a modular embedding $(\Gamma_D, D) \subset (\Gamma, \mathbb{B}^3)$. We will be interested in the Γ_D such that $\Gamma_D = \mathrm{U}(2,1;\mathcal{O}_K(1-\rho))$ is the Picard (surface) group. We can then draw on the very detailed results of Holzapfel for these surfaces (i.e. the whole book [Hol]). As we will see below there are 15 modular subvarieties of this kind on $Y(1-\rho)$. These 15 meet in a union of special curves on each copy, the Γ-reflection discs of [Hol, 1.3.3]. To describe their intersections (which will be identified as such below) we use the figure alluded to in the introduction. Let $H_1, \ldots, H_{10}$ be the 10 planes consisting of the 4 facet planes and 6 symmetry planes of a regular tetrahedron in $\mathbb{P}^3$:



2.5.1.

This arrangement has the following singularities (i.e. is not a normal crossings divisor because of):

5	6-fold points	(4 corners and the center)
10	3-fold lines	(6 edges and 4 diagonals).

The divisor $H = \sum H_i$ is turned into a normal crossings divisor by blowing up $\mathbb{P}^3$ at the 5 points, then along the 10 lines just mentioned. Let $\hat{\mathbb{P}}^3$ denote this blow up, $[H_i] =: D_i$ the proper transforms, $D_{11}, \ldots, D_{15}$ the exceptional $\mathbb{P}^2$'s and $E_1, \ldots, E_{10}$ the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s. Then under the isomorphism $Y(1-\rho)^\wedge = \hat{\mathbb{P}}^3$ (see 2.7) $D_1, \ldots, D_{15}$ are the modular varieties just introduced. Notice that after the blow-up each of the D_i, $i = 1, \ldots, 15$, is identical. Each is the blow up of $\mathbb{P}^2$ at 4 points in general position, the 3-fold points of the linear arrangement 1.4.2, hence in each D_i there are 10 $\mathbb{P}^1$'s, all of which have self-intersection (-1) in each D_i. These are of course the same surfaces occurring in 1.4.

Σ_5 acts in a natural way permuting the 10 $\mathbb{P}^1$'s; in fact $D_i - P$, ($P = \Sigma P_i$, P_i the exceptional divisors under the modification $D_i \to \mathbb{P}^2$) is a GIT-quotient, arising as follows: Let (x_i, y_i), $i = 0, \ldots, 4$ be a set of homogenous coordinates on

$(\mathbb{P}^1)^5$. Let $X \subset (\mathbb{P}^1)^5$ be the Zariski open subset consisting of those (x_i, y_i) such that no 3 of the 5 are identical. PGL(2, $\mathbb{C}$) acts freely on X, and the quotient can be compactified to $\mathbb{P}^2$ blown up in 4 points ([Y], p. 140). The action of Σ_5 is then just permutation of the factors on X.

On the other hand we have the natural action of Σ_4 on the D_i as described in [Ho1, 1.3.ff]. This comes about as soon as you have chosen a subset of 4 (disjoint) out of the 10 $\mathbb{P}^1$'s to be blown down, i.e. identified a set of cusps.

2.6. We now give the combinatorial description of the D_i in $\mathbb{Z}/3\mathbb{Z}^4$. Obviously, in K^4 each such $D_i(K)$ (K-valued points) is given by the intersection of the cone of 2.1 with a hyperplane, fixed by Γ as in 2.5. However, in $\mathbb{Z}/3\mathbb{Z}^4$ there is no distinction between signature (3, 0) and (2, 1), so we must find the hyperplanes $H \subset K^4$ such that Φ restricted to $H \cap \mathbb{B}$ has signature (2, 1), *then* take their images in $\mathbb{Z}/3\mathbb{Z}^4$. Note that we can find representatives of the cusps (cf. 2.3) in $\mathbb{Z}^4 \subset K^4$:

$$(1,0,0,1),(0,1,0,1),(0,0,1,1),(-1,0,0,1),(0,-1,0,1),$$
$$(0,0,-1,1),(1,-2,-2,3),(1,2,2,3),(2,1,2,3),(2,2,1,3).$$

Letting the cusps (now in $\mathbb{Z}^4$) be denoted by v_i ($i = 1, \ldots, 10$), there are $\binom{10}{3} = 120$ sets of 3 of them. For each such triple, say (v_i, v_j, v_k), we can find an orthogonal base of the 3-space they span:

$$w_1 = v_i + v_j$$
$$w_2 = v_i - v_j$$
$$w_3 = -(v_j, v_k)v_i - (v_i, v_k)v_j + (v_i, v_j)v_k.$$

(Here (,) denotes the form Φ for notational simplicity). Using this base we can calculate the signature: (note $(v_i, v_j) < 0$ for any i, j)

$$(w_1, w_1) = (v_i + v_j, v_i + v_j) = 2(v_i, v_j) < 0$$
$$(w_2, w_2) = (v_i - v_j, v_i - v_j) = -2(v_i, v_j) > 0$$
$$(w_3, w_3) = -2(v_i, v_j)(v_j, v_k)(v_i, v_k) > 0,$$

so on *any* such 3-space, the form Φ has signature (2, 1). Of these 120, there are exactly 15 which contain a 4th cusp, and the images of these 15 subspaces of $\mathbb{Z}^4$ in $\mathbb{Z}/3\mathbb{Z}^4$ give the combinatorial description of the modular subvarieties. This amounts then, a postiori, to a linear combination, *in* $\mathbb{Z}/3\mathbb{Z}$, of the cusps given in 2.3. For example, the 3-plane spanned by (v_1, v_2, v_8) also contains v_9: $-v_1 + v_2 + v_8 \equiv v_9 (\mathrm{mod}\, 3)$.

2.7. We now come to the proof of

THEOREM 2.7.1. *$Y(1-\rho)^\wedge = \hat{\mathbb{P}}^3$, the D_i are the modular subvarieties of* 2.5.

$i = 1, \dots, 15$ and the E_λ are the compactification divisors of 2.4. $\lambda = 1, \dots, 10$.

Proof. Let $Y(3, H)$ be the Fermat cover of degree 3 associated to the arrangement H (see [Hu] for details on this construction), i.e. the variety whose function field is

$$\mathbb{C}[{}^{x_1}/x_0, {}^{x_2}/x_0, {}^{x_3}/x_0][({}^{l_2}/l_1)^{1/3}, \dots, ({}^{l_{10}}/l_1)^{1/3}]$$

where $\{l_i = 0\} = H_i$. In [Hu] I constructed a desingularisation and calculated the Chern numbers of $Y(3, H)$, as well as the logarithmic Chern classes of $(Y(3, H), \bar{E})$, where $\bar{E} = \pi^{-1}(\sum E_\lambda)$. It turned out that $\bar{c}_1{}^3(Y, \bar{E}) = 3\bar{c}_1 \bar{c}_2(Y, \bar{E})$ (logarithmic Chern numbers) so by Kobayashi's generalisation of Yau's theorem quoted in [Hu], (also proved by Yau), $Y - \bar{E}$ is a smooth, non-compact ball quotient, Y its compactification. The desingularisation described in [Hu] is affected by blowing up $\mathbb{P}^3$ in exactly the same manner as above, so the smooth cover $Y \to \hat{\mathbb{P}}^3$ is a branched cover of $\hat{\mathbb{P}}^3$, or put differently, $\hat{\mathbb{P}}^3$ is a ball quotient; we just have to identify the group. Let Γ_Y be the group such that $\Gamma_Y \backslash \mathbb{B}^3 = Y - \bar{E}$. Then $\Gamma \supset \Gamma_Y$, $\Gamma = \pi_1(\hat{\mathbb{P}}^3 - E - D)$, as $\Gamma_Y \backslash \mathbb{B}^3$ is a cover of $\hat{\mathbb{P}}^3$ which is unramified over $\hat{\mathbb{P}}^3 - E - D$. The quotient $\Gamma/\Gamma_Y = (\mathbb{Z}/3\mathbb{Z})^9$ is the Galois group of $Y \to \hat{\mathbb{P}}^3$.

Utilising known results on the hypergeometric differential equation, Γ is the monodromy group of Appell's equation, number 1 in the [DM] list in dimension 3. Later we will identify this differential equation with the Picard Fuchs equation of the periods of Picard curves, whose monodromy group is easily identified with $\Gamma(1 - \rho)^\wedge = \mathrm{U}(3, 1; \mathcal{O}_K(1 - \rho))$ (see §5–§6). The identification of this particular group is thus by means of the scheme:

$$(\text{Fermat cover}) \xrightarrow{1} ([\text{DM}]\text{-}\#) \xrightarrow{2} (\text{Picard curves}) \xrightarrow{3} (\text{monodromy group}).$$

Step 1 was done in [Hu]. Step 2 will be done in Sections 5–6, Step 3 in Section 6. The statements about D_i and E_λ follow from [Ho1] (identification of the D_i) and direct calculations (showing E_λ is an abelian variety as in 2.4).

Let us just mention that much of this is more or less well-known.

2.8. Finite covers

In this section we clarify a few questions which were left untouched up till now. We consider the following coverings:

$$\begin{tikzcd}
\Gamma(4)\backslash\mathbb{S}_2 \arrow[dr] & \subset & (\Gamma(4)\backslash\mathbb{S}_2)^\wedge \arrow[dr, "(\mathbb{Z}/2\mathbb{Z})^9"'] & & (\Gamma(1-\rho)^2\backslash\mathbb{B}^3)^\wedge \arrow[dl, "(\mathbb{Z}/3\mathbb{Z})^9"] & \supset & \Gamma(1-\rho)^2\backslash\mathbb{B}^3 \arrow[dl] \\
& \Gamma(2)\backslash\mathbb{S}_2 & \subset & \hat{\mathbb{P}}^3 & \supset & \Gamma(1-\rho)\backslash\mathbb{B}^3 &
\end{tikzcd}$$

We explain now the inclusion $Y(2) \subset \hat{\mathbb{P}}^3$, which will be given an easy proof in the next section. Delete $\cup D_i$ from $\hat{\mathbb{P}}^3$. Then the inclusion $Y(2) \subset \hat{\mathbb{P}}^3$ is such that the Humbert surfaces are the divisors $E_1, \ldots, E_{10}$. As mentioned above, each D_i is a Kummer modular surface (compactification divisor), and $\hat{\mathbb{P}}^3$ is the Igusa desingularisation of $Y(2)^*$.

Taking that for granted, one has two natural covers,

$$Y(4)^\wedge \to Y(2)^\wedge \quad \text{and}$$
$$Y(1-\rho)^{2\wedge} \to Y(1-\rho)^\wedge$$

(using obvious notations). We claim these are in fact both Fermat covers, of degrees 2 and 3, respectively. To see this, first note that both $P\Gamma(4)/P\Gamma(2)$ and $P\Gamma(1-\rho)/P\Gamma(1-\rho)^2$ are abelian. In fact, it is true that $\Gamma(\mathfrak{D})/\Gamma(\mathfrak{D}^2)$ is abelian for any ideal $\mathfrak{D}$. (Steve Weintraub pointed this out to me. Just calculate $(A + B)^2 \bmod(\mathfrak{D}^2)$.) The coefficients are in $\mathbb{Z}/4\mathbb{Z}/\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{O}_K(1-\rho)/\mathcal{O}_K(1-\rho)^2 = \mathbb{Z}/3\mathbb{Z}$, respectively, and a matrix in $\mathrm{PSp}(2, \mathbb{Z})$ and $\mathrm{PU}(3, 1; \mathcal{O}_K)$, respectively, will have 9 independent entries.

Once we know the groups are correct, we just have to note that the fixed point set under these Galois groups which consists of the union $\cup D_i \cup E_j$ of 2.5, are hermitian symmetric , and in fact identical in the Fermat covers as well as in $Y(4)^\wedge \to Y(2)^\wedge$ and $Y(1-\rho)^{2\wedge} \to Y(1-\rho)^\wedge$, respectively. This also allows us to count modular subvarieties and compactification divisors:

Siegel: $10 \cdot 2^4 = 160$ modular subvarieties, $15 \cdot 2^3 = 120$ compactification divisors,

Picard: $15 \cdot 3^3 = 135$ modular subvarieties, $10 \cdot 3^4 = 270$ compactification divisors.

We remark that this discussion finishes, modulo the proof of 3.3.3. below, the proof of 1.7.1. above.

3. Modular forms

In this paragraph we prove the theorem stated in the introduction, utilising for the proof modular forms. First we recall the structure of $R(\Gamma(2))$, a result due to Igusa [I1]. We then deduce the structure of $R(\Gamma(1-\rho))$. It turns out that these rings are dual to each other, i.e. the projective varieties $\mathrm{Proj}(R(\Gamma(2)))$ and $\mathrm{Proj}(R(\Gamma(1-\rho)))$ are dual. This implies they are birational, and our theorem follows.

3.1. Theta Constants

In this section we review the work of Igusa [I1]–[I2]. For later use we will need theta constants of genus 2 and 4, so in this section we give definitions and results for any g. Let $\tau \in \mathbb{S}_g$, $z \in \mathbb{C}^g$, and $m = (m', m'') \in \mathbb{Q}^{2g}$.

DEFINITION 3.1.1. The theta function of degree g and characteristic m is

$$\theta_m(\tau, z) = \sum_{n \in \mathbb{Z}^g} \exp(\tfrac{1}{2}{}^t(n + m')\tau(n + m') + {}^t(n + m')(z + m'')).$$

The corresponding theta constant is

$$\theta_m(\tau) := \theta_m(\tau, 0).$$

Igusa has studied these theta constants. Some of his results are:

LEMMA 3.1.2. $\theta_m(\tau) \equiv 0 \Leftrightarrow m \bmod(1)$ *satisfies* $\exp(4\pi i {}^t(m')(m'')) = -1$.

The Siegel modular group $\Gamma_g(1) := \mathrm{Sp}(g, \mathbb{Z})$ acts on the arguments (τ, z) by:

$$M(\tau, z) = ((A\tau + B)(C\tau + D)^{-1}, (C\tau + D)^{-1}z) \tag{3.1.2a}$$

and on the characteristic itself by

$$M\colon m = (m', m'') \mapsto \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} m + \frac{1}{2}\begin{pmatrix} \operatorname{diag}(C{}^tD) \\ \operatorname{diag}(A{}^tB) \end{pmatrix}. \tag{3.1.2b}$$

The behavior of the thetas under M is given by

LEMMA 3.1.3. (Igusa's transformation law), [I1], p. 226

$$\theta_{Mm}(M(\tau, z)) = \kappa(M)\exp(2\pi i\phi_m(M))\det(C\tau + D)^{1/2} \times \\ \times \exp(\pi i {}^tz(C\tau + D)^{-1}Cz)\theta_m(\tau, z),$$

where $\kappa(M)$ *is some 8th root of unity,*

$$\phi_m(M) = -\tfrac{1}{2}{}^tm'BDm' + {}^tm''{}^tACm'' - 2{}^tm''{}^tBCm'' - {}^t\operatorname{diag}(A{}^tB)(Dm' - Cm'').$$

In particular, for the theta constants the formula becomes

$$\theta_{Mm}(M\tau) = \kappa(M)\exp(2\pi i\phi_m(M))\det(C\tau + D)^{1/2}\theta_m(\tau).$$

3.2. The ring of modular forms for $\Gamma(2)$.

We review the results of Igusa describing $R(\Gamma(2))$, the ring of modular forms for $\Gamma(2)$. For the rest of this section we take the following particular case of 3.1: $g = 2$,

$m = (m', m'') \in \frac{1}{2}\mathbb{Z}^4$. There are 16 such characteristics, 6 of which yield odd theta functions ($\theta_m(\tau, z) = -\theta_m(\tau, -z)$) and hence zero theta constants (this is a special case of 3.1.2). There are 10 even characteristics, and among their fourth powers there are 5 linear relations (Riemann theta formula). In fact,

THEOREM 3.2. [I1, p. 397] *Let*

$$\begin{aligned}
&y_0 = \theta^4_{(0110)}(\tau), \quad y_1 = \theta^4_{(0100)}(\tau), \quad y_2 = \theta^4_{(0000)}(\tau),\\
&y_3 = \theta^4_{(1000)}(\tau) - \theta^4_{(0000)}(\tau), \quad y_4 = -\theta^4_{(1100)}(\tau) - \theta^4_{(0000)}(\tau).\\
&\chi_{10} = \Pi\theta^2_m
\end{aligned}$$

Then,

$R(\Gamma(2)) = \mathbb{C}[y_0, \ldots, y_4, \chi]/_{\mathscr{E}}$, $\mathscr{E}$ *the ideal generated by*

$$\mathscr{E} = \begin{cases} R_1 = (y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(\Sigma y_i) \\ R_2 = \chi^2_{10} - \frac{1}{4}f(y_0, \ldots, y_4), \quad f \text{ homogenous of degree } 5. \end{cases}$$

This theorem implies in particular, that the (singular) quartic defined by R_1 is the Baily-Borel embedding of $X(2)$. The y_i are modular forms of weight 2 with respect to $\Gamma(2)$, as follows from the transformation law 3.1.3. A more symmetric description of the same variety is given by van der Geer in [V, §5]. This is done by taking all thetas $\theta^4_m(\tau)$, and the map into $\mathbb{P}^9$,

$$\phi\colon (\Gamma(2)\backslash\mathbb{S}_2)^* \to \mathbb{P}^4 \subset \mathbb{P}^9, \tag{3.2.1}$$

The image being onto the $\mathbb{P}^4$ which is cut out by the 5 linear relations. The equation for $X(2)^*$ is then ([V], 5.2)

$$\left(\sum \theta^8_m\right)^2 - 4\left(\sum \theta^{16}_m\right) = 0. \tag{3.2.2}$$

In this description the action of $\mathrm{Sp}(2, \mathbb{Z}/2\mathbb{Z})$ on $X(2)^*$ is the action on the characteristics 3.1.2b. It is known that each theta vanishes along exactly one of the Humbert surfaces, so we can identify this action with the action of Σ_6 on H_1 described in 1.6. This replaces the rather artificial action 3.1.2b by the much more natural one of Σ_6 on $(\mathbb{Z}/2\mathbb{Z})^4$ described in 1.6. This observation is due to van der Geer [V] and Lee and Weintraub [LW].

3.3. The ring of modular forms on $Y(1 - \rho)^*$

We now proceed in the following manner: first we recall the (well-known) identification of $Y(1 - \rho)^*$ with the Segre Cubic. Then, since we know the coordinate ring of the Segre cubic, we know the coordinate ring of $Y(1-\rho)^*$. Finally it is not difficult to see that the natural coordinates used are indeed modular forms.

The Segre Cubic

There is a unique cubic 3-fold S in $\mathbb{P}^4$ with 10 ordinary doublepoints. S is given most symmetrically by the following 2 equations in the homogenous variables $x_0, \dots, x_5$ on $\mathbb{P}^5$:

$$\sum x_i = 0$$
$$\sum x_i^3 = 0.$$

The double points are $(1,1,1,-1,-1,-1)$ and its permutations under Σ_6, which acts naturally on S by permuting coordinates. There are 15 $\mathbb{P}^2$'s lying on S:

$$x_{\sigma(1)} + x_{\sigma(4)} = x_{\sigma(2)} + x_{\sigma(5)} = x_{\sigma(3)} + x_{\sigma(6)} = 0, \quad \sigma \in \Sigma_6.$$

The GIT-Quotient

It was already mentioned above that the following was proved in [LW], see also [KLW]:

LEMMA 3.3.1. $\tilde{S} = \hat{\mathbb{P}}^3 =$ GIT-*quotient* #1 *in* [DM]. ($\tilde{S}$ *denotes the* (*big*) *resolution of* S).

From this and 2.7.1 it follows first that $S = Y(1-\rho)^*$ and from this that the coordinate ring of $Y(1-\rho)^*$ coincides with that of S, i.e.

$$R(Y(1-\rho)^*) = \mathbb{C}[x_0, \dots, x_5]/\mathscr{E} \tag{3.3.2}$$
$$\mathscr{E} = \begin{cases} R_1 = \sum x_i \\ R_2 = \sum x_i^3. \end{cases}$$

An important additional bit of information we get from the argument of 2.7. is the following: The D_i, being in the branch locus of the cover $Y \to \hat{\mathbb{P}}^3$, are pointwise fixed under the Galois group, hence subball quotients. (The only submanifolds in the universal cover $\mathbb{B}^3$ fixed under automorphisms are subballs.) This implies on the one hand the fact (that we already know) that the D_i are modular subvarieties, and on the other implies that the divisors D_i are the zero-loci of modular forms (since roots of their quotients give the field extension of the finite cover $Y \to \hat{\mathbb{P}}^3$ described in 2.5). Hence there are 15 modular forms δ_i, $i = 1, \dots, 15$, with the property that each vanishes identically along precisely one of the D_i and does not vanish identically at any other. Taking all of these forms as coordinates we get a map

$$\begin{aligned} &\phi\colon Y(1-\rho) \to \mathbb{P}^5 \subset \mathbb{P}^{14} \\ &t \to \delta_i(t) \in S \end{aligned}$$

onto the Baily-Borel compactification (the x_i above will be linear combinations of the δ_i) much in the spirit of van der Geer's (3.2.1.–3.2.2. above).

THEOREM 3.3.3. $\hat{\mathbb{P}}^3$ *is the common resolution of singularities of* $X(2)^*$ *and* $Y(1-\rho)^*$, *i.e. we have a diagram*

$$\begin{tikzcd} & \hat{\mathbb{P}}^3 \quad \text{GIT} \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \text{Siegel } X(2)^* & & Y(1-\rho)^* \quad \text{Picard.} \end{tikzcd}$$

Using the notation in the introduction, we have

$(\hat{\mathbb{P}}^3, D)$ *corresponds to* $(\mathbb{S}_2, \Gamma(2))$
$(\hat{\mathbb{P}}^3, E)$ *corresponds to* $(\mathbb{B}^3, \Gamma(1-\rho))$.

This follows from the above results as follows: we know $X(2)^*$ is the quartic described in 3.2, $Y(1-\rho)^*$ is the Segre cubic, and classically (1880's !) it is known that these varieties are dual to each other. This implies they are birational, and σ_1 and σ_2 have been explicitly described above. σ_1 blows down the D_i to $\mathbb{P}^1$'s, σ_2 blows down the E_j to ordinary double points.

3.4. Theta constants of degree 4

Assume for the moment we have a modular embedding $(\mathbb{B}^3, \Gamma(1-\rho)) \subset (\mathbb{S}_4, \Gamma(?))$, where $\Gamma(?)$ is a not further specified level subgroup. We will see later in Section 5 that this exists via Jacobians of Picard curves. Consider the diagram

$$\begin{tikzcd} \mathbb{B}^3 \arrow[d] & \subset & \mathbb{S}_4 \arrow[d] \\ \Gamma(1-\rho)\backslash\mathbb{B}^3 & \subset & \Gamma(?)\backslash\mathbb{S}_4 =: X_4(?) \end{tikzcd}$$

It follows from Igusa's results ([I2, Cor., p. 235]) that theta constants can be used to embed $X_4(?)$ in projective space (ala Satake-Baily-Borel). Restricting these thetas to $\Gamma(1-\rho)\backslash\mathbb{B}^3$ yields Picard modular forms. This is the idea behind Feustel's proof (in dimension 2) of

THEOREM 3.4.1 [F, II]. *The modular forms ξ_i mentioned below in 6.1.8 for* $U(2,1;\mathcal{O}_K(1-\rho))$ *can be written as follows in terms of theta constants*:

$$y_i := \theta\begin{bmatrix} 0 & 1/6 & 0 \\ i/3 & 1/6 & i/3 \end{bmatrix}, \quad i=0,1,2.$$

$$\xi_1 = \sum y_i^3$$
$$\xi_2 = -3y_1^3 + y_2^3 + y_3^3$$
$$\xi_3 = y_1^3 - 3y_2^3 + y_3^3$$
$$\xi_4 = y_1^3 + y_2^3 - 3y_3^3.$$

This gives at least partial information on $\Gamma(?)$: it contains the group $\Gamma(6,36)$. Since we have D_i given by the equations, say

$$x_1 = -x_4, \quad x_2 = -x_5, \quad x_3 = -x_6,$$

we can use x_1, x_2 and x_3 as coordinates on $\Gamma(1-\rho)\backslash\mathbb{B}^2$, so by 3.4.1, the x_i can be written in terms of genus 3 thetas when restricted to the D_i. Hence in principle at least, we can take the modular forms in 3.3.2 to be genus 4 theta constants which restrict on the modular subvarieties D_i to the theta constants in 3.4.1. It would be very interesting to get some explicit results in this direction. This is, however, a highly non-trivial task. One would first have to find an explicit embedding $(\mathbb{B}^3,\Gamma)\subset(\mathbb{S}_4,\Gamma(?))$ together with 15 compatible sub-modular embeddings

$$\begin{array}{ccc} (\mathbb{B}^3,\Gamma) & \subset & (\mathbb{S}_4,\Gamma(?)) \\ \cup & & \cup \\ (\mathbb{B}^2,\Gamma_{2,1}) & \subset & (\mathbb{S}_3,\Gamma_3(?)). \end{array}$$

The 15 subballs will give the intersection of $(\mathbb{B}^3,\Gamma)$ with the zero locus of the sought for theta constants. Let us briefly describe what this looks like in local coordinates. Say

$$\mathbb{B}^2 = \{(z_1,z_2)\in\mathbb{C} \mid |z_i|^2 < 1\},$$
$$\mathbb{B}^2 = \{z_3 = 0\} \subset \mathbb{B}^3 = \{(z_1,z_2,z_3)\in\mathbb{C}^3 \mid \sum|z_i|^2 < 1\},$$

and

$$\mathbb{S}_3 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 \\ \tau_1 & \tau_3 & \tau_4 \\ \tau_2 & \tau_4 & \tau_5 \end{bmatrix} \subset \mathbb{S}_4 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 & x_1 \\ \tau_1 & \tau_3 & \tau_4 & x_2 \\ \tau_2 & \tau_4 & \tau_5 & x_3 \\ x_1 & x_2 & x_3 & \tau_6 \end{bmatrix}.$$

The modular embeddings are:

$$\mathbb{B}^2 = \{z_1 = \tau_2, z_2 = \tau_4\} \quad \text{or} \quad \{z_1 = x_1, z_2 = x_2\}, \quad \mathbb{B}^3 = \{z_i = x_i\}.$$

These 4 spaces parameterise:

$\mathbb{S}_4$ = principally polarised abelian 4-folds
$\cup$
Ξ = Shottky divisor = locus of jacobians = genus 4 curves
$\cup$
$\mathbb{B}^3$ = Jacobians with complex multiplication = Picard curves
$\mathbb{S}_4 \supset \mathbb{S}_3$ = genus 3 curves (= degenerating abelian 4-folds)

$\mathbb{B}^2$ = Picard curves (Jacobians with complex multiplication)
$\|$
$\mathbb{B}^3 \cap \mathbb{S}_3$ = Degenerate Jacobians with complex multiplication.

Next one would have to find the right level groups acting on $\mathbb{S}_3$ and $\mathbb{S}_4$ and show the equivariance of the diagram above with respect to the groups involved. Finally, one would have to describe the genus 3 thetas as special values of genus 4 theta constants on $\mathbb{S}_4$. As the embeddings involved are rather complicated (see for example those given by Resnikoff-Tai), this would seem to be a formidable task.

4. Tits buildings with scaffoldings

4.1. Let $\Gamma \subset G_{\mathbb{Q}}$ be an arithmetic subgroup, and $P_1, \ldots, P_k$ a complete set of Γ-inequivalent maximal parabolics. One can construct a simplicial complex associated to Γ, the *Tits Building*, as follows:

vertices: $v_i \longleftrightarrow P_i$
1-edges: $v_{ij} \longleftrightarrow P_i \subset P_j$ inclusions
2-faces: $v_{ijk} \longleftrightarrow P_i \subset P_j \subset P_k$ flags
$\ldots \quad \ldots$

First, let $\Gamma = \mathrm{Sp}(2, \mathbb{Z})$. Then there are 2 maximal parabolics P_1, P_2, corresponding to the 0-dimensional and 1-dimensional boundary components, respectively. There is one inclusion $P_1 \subset P_2$. Hence the Tits building is:

$\mathrm{Sp}(2, \mathbb{Z})$: $\quad v_1 \circ\!\!-\!\!\!-\!\!\circ v_2.$

The building for $\Gamma_{\sqrt{-3}} = \mathrm{SU}(3, 1; \mathcal{O}_K)$ is even more boring, being just one point:

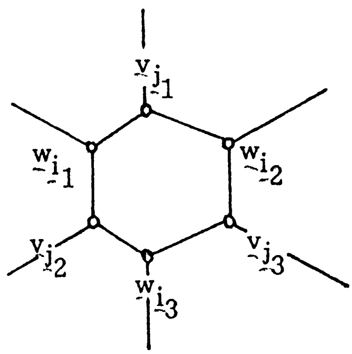
$\mathrm{SU}(3, 1; \mathcal{O}_K) \quad v_1 \circ$

Now we consider the congruence subgroups $\Gamma(2)$ and $\Gamma(1-\rho)$. The building of $\Gamma(1-\rho)$ is still not too interesting, consisting of 10 disjoint verticies:

$$\Gamma(1-\rho)\ v_1 \circ \cdots \circ v_{10}.$$

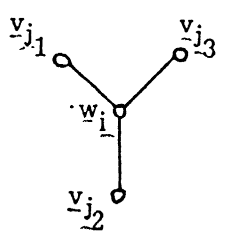
The interesting Tits building is that of $\Gamma(2)$. There are 15 verticies $v_1, \ldots, v_{15}$ corresponding to the 1-dimensional components, and 15 verticies $w_1, \ldots, w_{15}$ corresponding to the 0-dimensional components. Each w_i has 3 inclusions into v_j's, and v_j contains 3 w_i:

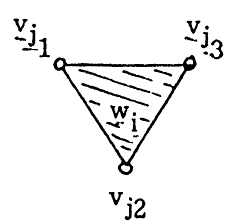
(4.1.1.)



There are therefore $3(15 + 15).\frac{1}{2} = 45$ edges, see Figure 4.1.2.

We now "blow-up" this Tits building, i.e. replace each w_i by a 2-simplex:





and we note that the blown-up Tits building is the dual complex of the normal crossings divisor D. This means we have 1 vertex for each component, 1 edge for each double intersection and 1 2-simplex for each triple point:

$$v_i \leftrightarrow D_i \text{ divisor} \qquad (15)$$

$$v_i v_j \leftrightarrow D_i \cap D_j \qquad (45)$$

$$\Delta v_i v_j v_k = w_\lambda \leftrightarrow D_i \cap D_j \cap D_k \qquad (15)$$

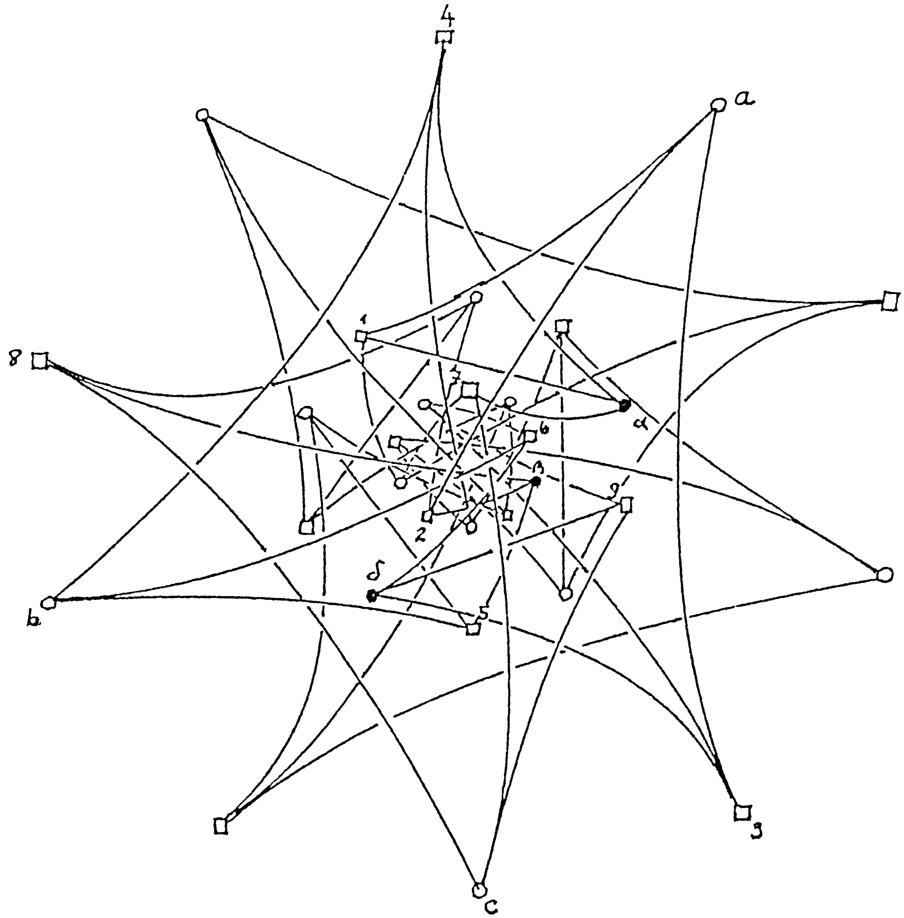


Figure 4.1.2.

4.2. Modular Scaffoldings

We consider a fixed locally symmetric space $(X, \Delta) = (\mathscr{D}, \Gamma)$ (we recall that this notational convenience explained in the introduction means $X - \Delta = \Gamma\backslash\mathscr{D}$), and modular subvarieties $(D_i, \Delta \cap D_i) = (\mathscr{D}_i, \Gamma)$ for $i \in$ Index set.

DEFINITION 4.2.1. A finite set $(\mathscr{D}_i, \Gamma_i) \subset (\mathscr{D}, \Gamma)$, $i = 1, \ldots, N$, is called a *Modular scaffolding of* $(\mathscr{D}, \Gamma)$, if:

(i) Each $(\mathscr{D}_i, \Gamma_i) \subset (\mathscr{D}, \Gamma)$ is a modular embedding.

(ii) Each intersection $(D_i \cap D_j, \Delta \cap (D_i \cap D_j)) = (\mathscr{D}_{ij}, \Gamma_{ij})$ is a modular embedding in both $(\mathscr{D}_i, \Gamma_i)$ and $(\mathscr{D}_j, \Gamma_j)$.

(iii) $\exists_{\lambda_i, \mu_j \in \mathbb{Q}}\, c_1(X) = \sum \lambda_i D_i + \sum \mu_j \Delta_j, \Delta = \sum \Delta_j$.

Now it is an easy matter to see that the modular subvarieties discussed in 1.5 and

2.5 form modular scaffoldings of $X(2)$ and $Y(1-\rho)$, respectively. The necessary Chern class calculations can be found in [V, §3], for example.

4.3. Scaffoldings on buildings

We now introduce a notion due to Lee-Weintraub ([LW1]), in a somewhat different fashion. According to 4.2 a modular scaffolding on (X, Δ) will be a normal crossings divisor.

DEFINITION 4.3.1. A *scaffolding* on the Tits building of (X, Δ) is the dual complex of a modular scaffolding.

This dual complex is a simplicial complex consisting of 1 vertex for each component, 1 edge for each simple intersection, and so on. Now just looking at the example above, we have:

PROPOSITION 4.3.2. *The scaffolding of $Y(1-\rho)$ is the blown-up Tits building of $X(2)$. The scaffolding of $X(2)$ is the Tits building of $Y(1-\rho)$.*

We now would like to combine the Tits building with the scaffolding of (X, Δ). The easy way to do this is to take the graph of the normal crossings divisor $\Delta + D$, Δ the compactification divisor and D the modular scaffolding as above. We call this complex the Tits building with scaffolding, Tbws.

PROPOSITION 4.3.3. *The Tits building with scaffolding of $\Gamma(2)$ and of $\Gamma(1-\rho)$ are the same, and the scaffolding of $\Gamma(2)$ is isomorphic to the Tits building of $\Gamma(1-\rho)$ and vice versa.*

This is what we mean by saying that $\Gamma(2)$ and $\Gamma(1-\rho)$ have dual Tits buildings with scaffolding. Now going back to the diagramm in 3.3.3 we recall the GIT-quotient had both Δ and D as compactification divisor. Consider its (blown-up) Tits building.

PROPOSITION 4.3.4. *The* GIT-*quotient's (blown-up) Tits building coincides with the* Tbws *mentioned in* 4.3.3. *The scaffolding of this* GIT-*quotient is trivial.*

Hence we see the Tits building of the GIT-quotient specialising to a Tbws in 2 different manners (dual to each other).

5. Modular interpretation

5.1. Families of marked lines

Let $S = \{1, 2, 3, 4, 5, 6\}$ and $\{x_i\}_{i \in S}$ a set of 6 points on $\mathbb{P}^1$, i.e. a map $\phi: S \to \mathbb{P}^1$. The set of all such maps is naturally isomorphic to $(\mathbb{P}^1)^6$. The "diagonals" Δ are the divisors $\{x_i = x_j,\ i \neq j = 1, \ldots, 6\}$ and their intersections. PGL(2) acts on $X := (\mathbb{P}^1)^6 - \Delta$ and the quotient, which is the GIT quotient discussed in 3.3, is isomorphic to $\mathbb{P}^3 - H$. H the arrangement consisting of the 10 planes of 2.5. In other words we have a fibre space

$$(\mathbb{P}^1)^6 - \Delta \to \mathbb{P}^3 - H$$

with fibre PGL(2). The explicit form of this map is given in [LW, §6], as mentioned above in Section 3. Here we are using the identification of $\mathbb{P}^3 - H$ with $S - D$, S the Segre cubic, derived in 3.3. We let $\mathscr{F}_0$ denote this family of marked $\mathbb{P}^1$'s over $\mathbb{P}^3 - H$.

5.2. Hyperelliptic curves

Let the $\{x_i\}_{i \in S}$ be as in 5.1, and consider the hyperelliptic curve

$$y^2 = \prod_1^6 (t - x_i) \tag{5.2.1}$$

which is a branched cover of $\mathbb{P}^1$, branched at each of the x_i. We let $\mathscr{M}_0 \to \mathscr{F}_0$ be the double cover of $\mathscr{F}_0$ branched at the $\{\phi_i\}$. To be precise, this object only exists in the category of algebraic stacks, i.e., the universal curve only exists locally, and cannot be globally constructed as $-1 \in \Gamma(2)$. However this is sufficient for our purposes. Hence, for each $\{x_i\}$ mod PGL(2) $\in \mathbb{P}^3 - H$, the fibre $(\mathscr{M}_0)_{\{x_i\}}$ is the double cover of $\mathbb{P}^1$ branched at the $\{x_i\}$ mod PGL(2). Thus we get a "fibre space"

$$\mathscr{M}_0 \to \mathbb{P}^3 - H = X(2) - E,$$

of genus 2 curves.

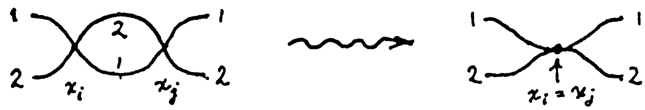
We now describe the degenerations of 5.2.1 corresponding to the divisors D (compactification divisor, 1.4) and E (Humbert surfaces, 1.5). We shall employ the following notation:

$D_i, i = 1, \ldots, 15$:	2 of the x_n coincide
$D_{ij} = D_i \cap D_j$:	2 pairs of the x_n coincide
$D_{ijk} = D_i \cap D_j \cap D_k$:	3 pair of the x_n coincide
$E_\lambda, \lambda = 1, \ldots, 10$:	3 of the x_n coincide
$D_i \cap E_\lambda$:	1 pair and 1 triple of the x_n coincide
$D_{ij} \cap E_\lambda$:	2 triples of the x_n coincide

D_i: If 2 of the x_n coincide, then we have a double cover with 5 branch points; this is a genus one curve (elliptic) with one double point.

D_{ij}: Reasoning as above we get here a rational curve with 2 double points.

D_{ijk}: Now we have only 3 branching points and at each branch point the degeneration is:



so that the double cover splits into 2 curves which are permuted by the Galois group, and each branch point is now a double point of the covering.

E_λ: when 3 of the x_n coincide, the curve is $y^2 = \Pi_1^3(t - x_n)$, an elliptic curve. To see precisely what is going on, let

$$y^2 = \prod_1^6 (t - x_n)$$

be the original equation, and write it as

$$y^2 = (t - x_1)(t - x_2)(t - x_3)(t - \lambda x_4)(t - \lambda x_5)(t - \lambda x_6)$$

and the degeneration is then given by letting $\lambda \to \infty$. The limit curve is

$$y^2 = \prod_1^3 (t - x_n), \tag{5.2.2}$$

with 4th branch point at infinity, and changing variable to $\tilde{t} = \lambda t$ we get

$$y^2 = (\tilde{t} - \lambda x_1)(\tilde{t} - \lambda x_2)(\tilde{t} - \lambda x_3)(\tilde{t} - x_4)(\tilde{t} - x_5)(\tilde{t} - x_6)$$

which for $\lambda \to \infty$ becomes

$$y^2 = \prod_4^6 (\tilde{t} - x_i), \tag{5.2.3.}$$

with 4th branch point at infinity. Therefore, over $p \in E_\lambda$ the corresponding degeneration consists of 2 elliptic curves 5.2.2 and 5.2.3, meeting at their common branch point.

REMARK. The degeneration just described of course depends on the birational model of $X(2)^\wedge$ used. That described here corresponds to the big resolution described in 2.4. There is also a small resolution of the ordinary double point, and in that case the degeneration is just an elliptic curve.

It is now obvious what the remaining degenerations are. A summary is given below in table 5.4.

5.3. Picard curves

Let $\{x_n\}_{n\in S}$ be again as in 5.1 and consider the trigonal curve

$$y^3 = \prod_1^6 (t - x_n). \tag{5.3.1}$$

This curve is called a Picard curve, being studied by Picard more than a century ago (actually it was the genus 3 curve he studied [P]). It is a genus 4 curve, whose Jacobian has complex multiplication by a cube root of unity, coming from the Galois action on the curve. As above we construct the triple cover $\mathscr{C}_0 \to \mathscr{F}_0$ whose fibre $(\mathscr{C}_0)_{\{x_n\}}$ is the curve 5.3.1. We get a fibering

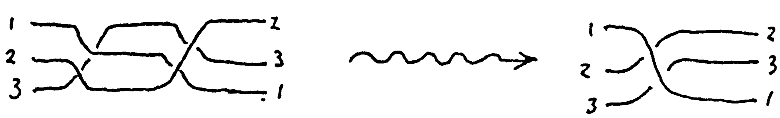
$$\mathscr{C}_0 \to \mathbb{P}^3 - H = Y(1-\rho) - D,$$

of genus 4 curves. This result was first proven by Shimura [S].

We now describe the degenerations of 5.3.1. corresponding to the divisors D (modular subvarieties, 2.5) and E (compactification divisors, 2.4). We use the same notations as in 5.2.

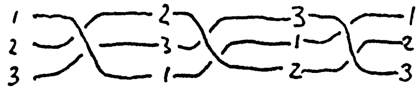
D_i: If one pair of the x_n coincide, we get a 3-fold cover, branched at 5 points, which is a smooth, genus 3 Picard curve (see the introduction and 2.5.)

D_{ij}: Now the equation becomes $y^3 = \prod_1^4(t - x_n)$, and each branch point induces:



so after 2 branch points we see that the cover splits into one elliptic and one rational component.

D_{ijk}: the picture now becomes:



so the cover splits into 3 rational curves.

E_λ: At a point where 3 of the x_n coincide, the action of the Galois group is $\begin{smallmatrix}1\\2\\3\end{smallmatrix}\!-\!-\!\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ at that branch point, so it is a double point of the curve, and checking euler numbers it has genus 2.

$E_\lambda \cap D_i$: As above, where the pair of $\{x_n\}$ coincide the cover is still smooth, with the double point of E_λ (from the triple of the $\{x_n\}$ which coincide), and an euler number calculation shows it is elliptic.

$E_\lambda \cap D_{ij}$: Here one gets a rational curve with 2 double points.

We can extend the family $\mathscr{C}_0 \to Y(1-\rho) - D$ to all of $Y(1-\rho)^\wedge$ by adding in the degenerations just described. We denote this by $\mathscr{C}^\wedge \to Y(1-\rho)^\wedge$. We note that this is not a semi-stable family of curves (the arithmetic genus changes).

5.4. Totally degenerate stable curves of genus 3.

There is (at least) yet another moduli interpretation of $\hat{\mathbb{P}}^3$, described in detail in [GHv]. I am indebted to F. Herrlich and L. Gerritzen for this.

DEFINITION 5.4.1. A connected projective curve C, of arithmetic genus g, is called a *totally degenerate stable curve of genus g* if:

(a) every irreducible component of C is a rational curve.

(b) every singular point of C is an ordinary doublepoint, and

(c) every non-singular component C_i of C meets $C - L$ in at least 3 points.

Naturally associated with such a degenerate curve is a more combinatorial object, a tree;

DEFINITION 5.4.2. A connected set of mutually intersecting $\mathbb{P}^1$'s is called an *n-pointed tree of projective lines* if;

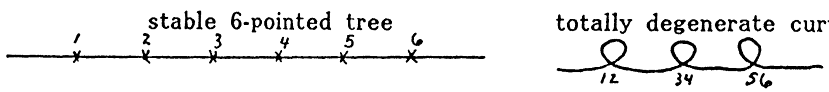
(i) the components intersect in ordinary double points

(ii) The intersection graph is a tree, and

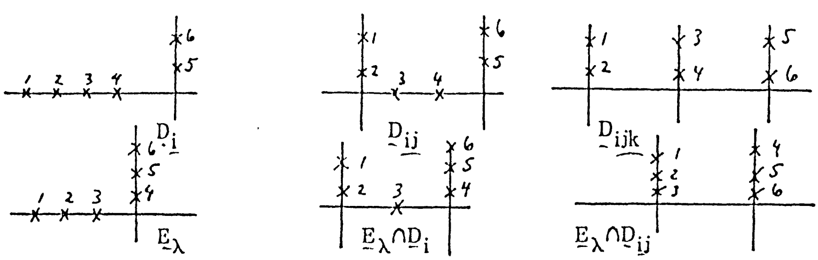
(iii) given is a set $\{p_1, \ldots, p_n\}$ of distinct ("marked") points on the components. It is called *stable* if, in addition,

(iv) on every component there are at least 3 points which are either singular or marked.

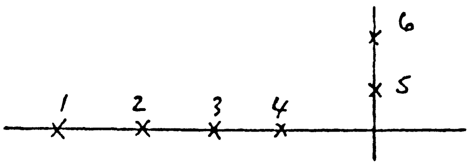
Totally degenerate curves are also parameterised by the $(x_i) \in \hat{\mathbb{P}}^3$. The generic curve is as follows:



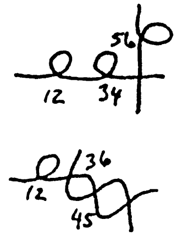
In the following we describe the 6-pointed trees corresponding to the loci D_i and F_λ as above:



The corresponding degenerate curves depend in addition on the identification of the double points, for example the tree:



gives both



We have included these in the following table.

6. Differential equations

6.1. Picard-Fuchs equations and monodromy

Let $\pi: \mathscr{C} \to S$ be a family of smooth curves over a parameter space S, i.e. $\mathscr{C}$, S are algebraic varieties, and π is a holomorphic map. We consider the sheaf $R^1\pi_*\mathbb{C}$, which is a locally constant sheaf whose stalk at $x \in S$ is just $H^1(F_x, \mathbb{C})$, where F_x is the fibre at x. Since F_x is a Riemann surface this stalk splits according to the Hodge decomposition:

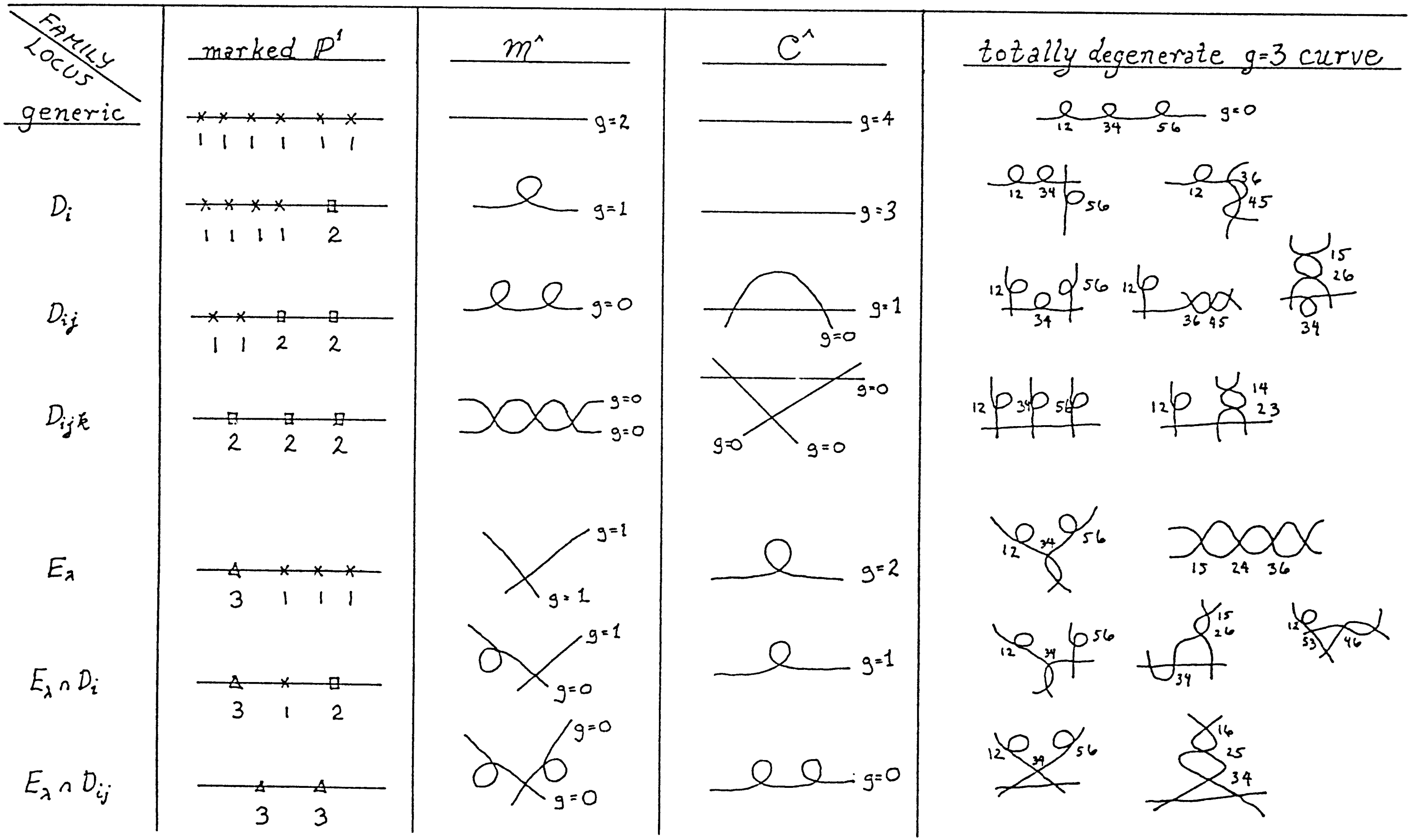
$$H^1(F_x, \mathbb{C}) = H^{1,0}(F_x) \oplus H^{0,1}(F_x).$$

$H^{1,0}(F_x)$ is the vector space of holomorphic 1-forms, and the decomposition is such that the position of $H^{1,0}(F_x)$ in $H^1(F_x, \mathbb{C}) = H^1(F, \mathbb{C})$ depends holomorphically on x (here F is a typical fibre). Fix a base $\omega_1(x), \ldots, \omega_g(x)$ of $H^{1,0}(F_x)$, and let $\delta_1, \ldots, \delta_{2g}$ be 1 cycles forming a basis for $H_1(F, \mathbb{Z})$. The period matrix of S is the $g \times 2g$ matrix

$$\Omega(x) = \begin{pmatrix} \int_{\delta_1} \omega_1(x) \cdots \int_{\delta_{2g}} \omega_1(x) \\ \cdots\cdots\cdots \\ \int_{\delta_1} \omega_g(x) \cdots \int_{\delta_{2g}} \omega_g(x) \end{pmatrix}$$

which can be put in a normalised form $\Omega(x) = (1_g, Z(x))$, where $Z(x) \in \mathbb{S}_g$.

Table 5.4.


FAMILY
LOCUS
generic
D_i
D_{ij}
D_{ijk}
E_λ
$E_\lambda \cap D_i$
$E_\lambda \cap D_{ij}$
marked $\mathbb{P}^1$
$\hat{M}$
$\hat{C}$
totally degenerate g=3 curve
g=4
g=3
g=2
g=1
g=0

Associated with the family $\mathscr{C} \to S$ one then gets a holomorphic *period map*:

$$\begin{aligned} \omega: S &\to \mathbb{S}_g \\ x &\mapsto Z(x). \end{aligned} \tag{6.1.1}$$

Furthermore the Torelli theorem tells us that from $Z \in \mathbb{S}_g$ (mod $\mathrm{Sp}(g, \mathbb{Z})$) we can recover the curve F_x such that $Z = Z(x)$. This is the general picture, and well-known.

Now fix a section of the $(1, 0)$ part of $R^1\pi_*\mathbb{C}$, in other words, fix a holomorphic 1-form $\omega(x)$ on each fibre, depending smoothly on x. Then the periods

$$\int_{\delta_1} \omega, \ldots, \int_{\delta_{2g}} \omega$$

are functions of the parameters, i.e. of x. It was first observed by Fuchs that, given ω, the $2g$ periods $\int_{\delta_i} \omega$ are all solutions of a linear differential equation of degree $2g$, the *Picard-Fuchs* equation. See Katz [Ka] for general remarks and higher-dimensional analogues.

Getting back to our family $\mathscr{C} \to S$ it is intuitively clear that if the fibres are "special" this leads to "special" Picard-Fuchs equations. For example, the special case of cycloelliptic curves

$$y^n = \Pi(t - x_i)^{\beta_i} \tag{6.1.2.}$$

is discussed in detail in part II of Holzapfel's book [Ho1], yielding equations he calls of Euler-Picard type. Both of our families are in fact special cases of these curves (§5.2: hyperelliptic, §5.3.:trigonal). Our special Picard-Fuchs equations will turn out to be hypergeometric differential equations.

We now just sketch how the monodromy of an algebraic differential equation relates to the geometry of the base space. First of all, the Picard-Fuchs equations have regular singular points (see e.g. [Gr] for precise definitions and statement of results). The singular locus $\Sigma \subset S$ is a divisor, which may be taken as normal crossings. For any $x \in S - \Sigma$, local solutions $\phi(x)$ will be single-valued. If the space of solutions of the differential equation has dimension d, let $\phi_1, \ldots, \phi_d$ be a local base at some fixed point $_* \in S - \Sigma$. For $\gamma \in \pi_1(S - \Sigma, _*)$ we can consider analytic continuation of the ϕ_i, and the continued solution, upon returning to $_*$, can be written as a linear combination of the ϕ_i's one started with, yielding a representation

$$\rho: \pi_1(S - \Sigma) \to \mathrm{GL}(d, \mathbb{C}) \tag{6.1.3}$$

called the *monodromy representation*. Of course, interesting things might happen if the image of ρ lies in some particular subgroup. For example, suppose $\mathscr{C} \to S$ is a family of elliptic curves over a curve, i.e. $\mathscr{C}$ is an elliptic surface. The Picard-Fuchs equation here is a second-order linear differential equation (see [St])

$$(D^2 - P(x)D + Q(x))F = 0 \tag{6.1.4}$$

with P and Q being singular at the discriminant of the family $\mathscr{C} \to S$. One then has the theorem [St, §3]:

The differential equation 6.1.4 coming from a family $\mathscr{C} \to S$ of elliptic curves is characterised as follows:

(1) The dimension of the solution space is 2.

(2) If ω_1 and ω_2 are 2 linearly independent solutions, then $\omega_1(x)/\omega_2(x) \in \mathbb{S}_1$ for $x \in S - \Sigma$.

(3) The monodromy lies in $SL(2, \mathbb{Z})$.

If $S = \mathbb{P}_1$, $\Sigma = \{0, 1, \infty\}$, the resulting equation 6.1.4 is the hypergeometric differential equation (in 1 variable). In that case the curve $\mathscr{C}_x$ is given by the equation

$$y^2 = (t-1)t(t-x), \tag{6.1.5}$$

i.e. a double cover of $\mathbb{P}^1$ branched at $\{0, 1, \infty$ and $x\}$. The differential in this case is

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{(t-1)t(t-x)}},$$

and the solutions to the HGDE are $\omega_i = \int_{\gamma_i} \omega$, γ_1, γ_2 a basis of $H_1(\mathscr{C}_x, \mathbb{Z})$. Taking their quotient we get a many-valued function

$$\omega(x) := \omega_1(x)/\omega_2(x) = \int_{\gamma_1} \omega \Big/ \int_{\gamma_2} \omega$$

which takes values in the upper half plane (for $x \in S - \Sigma$), yielding a diagram:

$$\begin{tikzcd} & \mathbb{S}_1 \arrow[dr, "J"] & \\ S - \Sigma \arrow[ur, shift left=2] \arrow[ur, "\omega(x)"] \arrow[ur, shift right=2] \arrow[rr] & & \mathbb{P}^1 \end{tikzcd} \tag{6.1.6}$$

The composition of these two maps is now a well defined (single-valued) function from $S - \Sigma$ to $\mathbb{P}^1$, since the many-valuedness of $\omega(x)$ is precisely offset by the

SL(2, $\mathbb{Z}$)-invariance of the modular function J. For $S = \mathbb{P}^1$, $\Sigma = \{0, 1, \infty\}$, this map is an isomorphism onto $\mathbb{P}^1$- $\{0, 1, \infty\}$. For a complete list of examples in this case, i.e. elliptic surfaces over $\mathbb{P}^1$ with 3 singular fibres see [SH].

In the upcoming sections we will find diagrams analogous to 6.1.6 for the families $\mathscr{M}$ and $\mathscr{C}$ considered in Section 5. Before we go into some of the details, let us begin by describing some of the very detailed results of Holzapfel for the analogous case where S is 2-dimensional (in our cases it will be 3-dimensional). One might start by considering the simplest 2-dimensional analogue of 6.1.5.

$$y^2 = (t-1)t(t-x_1)(t-x_2),$$

however this of course doesn't define a family of smooth curves. Instead, consider

$$y^3 = (t-1)t(t-x_1)(t-x_2), \tag{6.1.7}$$

branched also at ∞. This is a genus 3 curve. Let $S = \mathbb{P}^2$, $\Sigma = l_1 \cup \cdots \cup l_6$, the six lines forming the arrangement 1.4.2. Then 6.1.7. defines a smooth family over $S - \Sigma$, and its period matrix has the form ($\eta_1 = \mathrm{d}x/y, \eta_2 = \mathrm{d}x/y^2, \eta_3 = x\,\mathrm{d}x/y^2$)

$$\prod = \begin{pmatrix} A_1 & A_2 & -\rho^2 A_1 & A_3 & \rho^2 A_2 & \rho A_3 \\ B_1 & B_2 & -\rho B_1 & B_3 & \rho B_2 & \rho^2 B_3 \\ C_1 & C_2 & -\rho C_1 & C_3 & \rho C_2 & \rho^2 C_3 \end{pmatrix}$$

where

$$A_i = \oint_{\alpha_i} \eta_1, \qquad B_i = \oint_{\alpha_i} \eta_2, \qquad C_i = \oint_{\alpha_i} \eta_3, \quad i = 1, 2, 3.$$

from which it follows that $\prod$ is *already determined by the first row*: $x \mapsto (\int_{\alpha_1}\eta_1 : \int_{\alpha_2}\eta_1 : \int_{\alpha_3}\eta_1)$ is a many valued map from $S - \Sigma$ into $\mathbb{P}^2$, whose image lands in $\mathbb{B}^2 \subset \mathbb{P}^2$. Hence, in analogy to 6.1.6 above, we get a diagram

$$\begin{tikzcd} & \mathbb{B}^2 \arrow[dr, "(\xi_1:\xi_2:\xi_3)"] & \\ S - \Sigma \arrow[ur, "\omega(x)"] \arrow[rr] & & \mathbb{P}^2 \end{tikzcd} \tag{6.1.8.}$$

where $\omega(x) = (\int_{\alpha_1}\eta_1(x) : \int_{\alpha_2}\eta_1(x) : \int_{\alpha_3}\eta_1(x))$, and the $(\xi_1 : \xi_2 : \xi_3)$ are automorphic forms of weight 1 ([Hol, pp. 20–21], see also 3.4.1 above).

6.2. The Hypergeometric Differential Equation (HDE)

A general reference for this section is Terada [T]. Let $\mathbb{P}^n$ be given synthetic coordinates $\{(x_0, \ldots, x_{n+1}) \mid \Sigma x_i = 0\}$. The natural way to get these coordinates is to consider the $\mathbb{C}^{n+1} \subset \mathbb{C}^{n+2}$:

$$\mathbb{C}^{n+1} = \{(x_i) \in \mathbb{C}^{n+2} \mid \Sigma x_i = 0\},$$

and we take $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ with the induced homogenous coordinates. Since Σ_{n+2} acts naturally on $\mathbb{C}^{n+2}$ by permuting the coordinates and $\mathbb{C}^{n+1}$ is an invariant subspace, we get the so-called *projective symmetric representation* of Σ_{n+2} (cf. [Ho2]). In different language Σ_{n+2} is a unitary reflection group, and as such defines an arrangement H of planes in $\mathbb{P}^n$ (cf. [OS]). The first 3 such arrangements are:

$$n = 1. \quad H = \{x_0 x_1 (x_0 - x_1) = 0\},$$
$$n = 2. \quad H = \{x_0 x_1 x_2 (x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0\} = 1.4.2.$$
$$n = 3. \quad H = \{x_0 x_1 x_2 x_3 (x_0 - x_1) \ldots (x_3 - x_0) = 0\} = 2.5.1.$$

The hypergeometric differential equation is:

$$\begin{cases} (x_i - x_j)\partial_i \partial_j F + (\lambda_j - 1)\partial_i F - (\lambda_i - 1)\partial_j F = 0, \quad 1 \leqslant i < j \leqslant n, \\ x_i(x_i - 1)\partial_i^2 F + \Bigg[x_i(x_i - 1) \displaystyle\sum_{1 \leqslant \alpha \leqslant n, \alpha \neq i} (1 - \lambda_\alpha)/(x_i - x_\alpha) + \lambda_0 + \lambda_i - 2 + \\ \quad + (4 - 2\lambda_i - \lambda_0 - \lambda_{n+1})x_i \Bigg] \partial_i F + (\lambda_i - 1) \displaystyle\sum_{1 \leqslant \alpha \leqslant n, \alpha \neq n} x_\alpha (x_\alpha - 1)\partial_\alpha F / \\ \quad (x_i - x_\alpha) + \lambda_\infty (1 - \lambda_i) F = 0 \quad 1 \leqslant i \leqslant n. \end{cases} \tag{6.2.1}$$

where the λ_i are rational numbers with $\Sigma \lambda_i = n + 1$. This is an algebraic differential equation on $\mathbb{P}^n$ with regular singular points, non-singular off the arrangement $H \subset \mathbb{P}^n$ defined by Σ_{n+2}. A *solution* of 6.2.1 is the period of a holomorphic 1-form on the curve

$$y^\nu = t\mu_0 (t - 1)\mu_{n+1} (t - x_1)\mu_1 \ldots (t - x_n)\mu_n$$

for parameters $\mu_i / \nu = 1 - \lambda_i$. This is part of more general results of [DM]. The exact conditions on the parameters μ_i are known for the differentials $\omega_i = \int_{\gamma_i} y^{-1}\,dx$ to give a uniformisation into the ball $\mathbb{B}^n$. If this is the case, the monodromy group is a discrete subgroup in $PU(n, 1)$, and the parameter space therefore is locally symmetric. These conditions are:

$$(1 - \mu_i - \mu_j)^{-1} \subset \mathbb{Z} \cup \{\infty\}, \Sigma \mu_i = 2. \qquad \text{INT}$$

There are a finite number of parameter values $\{\mu_i\}$ satisfying INT, listed in [T] and [DM, §14].

6.3. Picard Curves

In this section we consider Picard curves as in Section 5,

$$y^3 = t(t-1)(t-x_1)(t-x_2)(t-x_3), \tag{6.3.1}$$

which are curves as considered in 6.2 with parameter values

$$(\mu_i) = (\tfrac{1}{3}, \ldots, \tfrac{1}{3})$$

yielding a particular HDE, the one listed as #1 in [DM]. Let $\rho: \pi_1(S - \Sigma) \to \mathrm{GL}(3, \mathbb{C})$ be the monodromy representation.

THEOREM 6.3.2. *The monodromy group of the* HDE *associated with the family* 6.3.1. *is the Picard lattice* $\Gamma(1-\rho)$.

Proof. There are several ways to prove this. The most straightforward is analogous to Holzapfels proof of the surface case. [Ho pp. 120–125]. We sketch this argument, in a series of lemmas.

LEMMA 6.3.3. *Let* Υ *be the monodromy group of the Picard family* 6.3.1., $\pi_1(S-\Sigma)$ *the fundamental group*, $\gamma_1, \ldots, \gamma_{10} \in \pi_1(S-\Sigma)$ *generators. Then* $\rho(\gamma_i) \in \mathrm{U}(3,1; \mathcal{O}_K(1-\rho))$.

COROLLARY 6.3.4. $\Upsilon \subset \mathrm{U}(3,1; \mathcal{O}_K(1-\rho))$ *has finite index.*

This follows from [DM]: Υ is an arithmetic lattice in PU(3, 1). From this one gets a diagram

$$\begin{tikzcd}
 & \mathbb{B}^3 \arrow[dl] \arrow[dr] & \\
(\Upsilon\backslash\mathbb{B}^3)^* \arrow[rr, "\chi"] & & \Gamma(1-\rho)\backslash\mathbb{B}^3)^*
\end{tikzcd}$$

and χ is a finite branched cover. 6.3.2 then follows from

LEMMA 6.3.5. χ *is unbranched.*

In fact, since $(\Gamma(1-\rho)\backslash\mathbb{B}^3)^*$ is simply connected, it has no unbranched covers, and 6.3.5 implies χ is an isomorphism. 6.3.5 could be proved in a manner similar to [Ho1], (his 6.3.11), which is a detailed analysis of χ near the cusps.

There is a somewhat easier proof of 6.3.2, which we now sketch. The idea is simple: we will identify fundamental domains of $\Gamma(1-\rho)$ and Υ in $\mathbb{B}^3$. First of all, both groups have 10 cusps. For $\Gamma(1-\rho)$ this was shown in 2.2, and for Σ it can be proved using the cover $Y \to \hat{\mathbb{P}}^3$ described in 2.7. Secondly, both have a scaffolding consisting of 15 modular subvarieties. Proof of this is the same. Now utilising the map

$$\chi\colon (\Sigma\backslash\mathbb{B}^3)^* \to (\Gamma(1-\rho)\backslash\mathbb{B}^3)^*$$

from above, we see both groups have the same fundamental domain in $\mathbb{B}^3$: the fundamental domain of Υ would consist of $\deg\chi$-copies of the fundamental domain of $\Gamma(1-\rho)$ if $\Gamma(1-\rho)$ acted freely. But since $\Gamma(1-\rho)$ and Υ have the same elliptic points (see the discussion in 2.8) this is sufficient.

Putting all this together, we get the following diagram:

$$\begin{tikzcd} & \mathbb{B}^3 \arrow[dr, "(\xi_i)"] & \\ S - \Sigma \arrow[ur, "\omega(x)"] \arrow[rr] & & \hat{\mathbb{P}}^3 \end{tikzcd} \qquad (6.3.6)$$

where $\omega(x)$ is the map given by the periods giving solutions of 6.2.1, and (ξ_i) are the modular forms discussed in 3.3.

6.4. Hyperelliptic curves

In this section we consider the family of Section 5.1.

$$y^2 = t(t-1)(t-x_1)(t-x_2)(t-x_3). \qquad (6.4.1)$$

Here, once again, we have exponents, this time

$$(\mu_i) = (\tfrac{1}{2}, \ldots, \tfrac{1}{2})$$

and a corresponding HDE. However this set of (μ_i) does not fulfill the condition INT, and so the situation here is different than that considered in 6.3 (as it should be). Sasaki and Yoshida [HSY] have succeeded in figuring this example out, yielding a diagram analogous to 6.3.6 in this case. This goes as follows:

THEOREM 6.4.2. *Let $E(\frac{1}{2})$ be the* HDE *in* 3 *variables with all parameters* $= \frac{1}{2}$. *Let $E \wedge E$ be the wedge product, a 6-dimensional system. Then for these parameter values, the solution space of $E \wedge E$ splits into an irreducible 5-dimensional space + complement. This 5-dimensional space is spanned by the* 2×2 *minors of* *the periods:*

$$\begin{pmatrix} \displaystyle\int_{\gamma_1} \frac{x\,dx}{y} \cdots \int_{\gamma_4} \frac{x\,dx}{y} \\ \displaystyle\int_{\gamma_1} \frac{dx}{y} \cdots \int_{\gamma_4} \frac{dx}{y} \end{pmatrix}$$

of the family 6.4.1.

Let us denote this solution by $\Omega = \{\sigma_1, \ldots, \sigma_5\}$. Then we get a diagram

$$\begin{tikzcd} & \mathbb{S}_2 \arrow[d, "(\theta_i)"] \\ S - \Sigma \arrow[ur, "\Omega"] \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r] & \hat{\mathbb{P}}^3. \end{tikzcd} \tag{6.4.3}$$

analogous to 6.3.6. Here we identify $\mathbb{S}_2$ with the non-compact dual of the hyperquadric $\mathbb{Q}_3$ in $\mathbb{P}^4$, and Ω maps onto projective coordinates.

There is the notion of dual differential equation, [HSY, §4], and one of the things proven in [HSY] is that $E(\frac{1}{2})$ is the unique HDE which is self-dual. This raises

PROBLEM. Is the self-duality of $E(\frac{1}{2})$ related to the double structure of $\hat{\mathbb{P}}^3$ as locally symmetric space?

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