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On the Milnor fibrations of weighted homogeneous polynomials

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Let $\mathbf{w} = (w_0, \ldots, w_n)$ be a set of integer positive weights and denote by S the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ graded by the conditions $\deg(x_i) = w_i$. For any graded object M we denote by M_k the homogeneous component of M of degree k. Let $f \in S_N$ be a weighted homogeneous polynomial of degree N.

The Milnor fibration of f is the locally trivial fibration $f: \mathbb{C}^{n+1} \setminus f^{-1}(0) \to \mathbb{C} \setminus \{0\}$, with typical fiber $F = f^{-1}(1)$ and geometric monodromy $h: F \to F$, $h(x) = (t^{w_0}x_0, \ldots, t^{w_n}x_n)$ for $t = \exp(2\pi i/N)$. Since $h^N = 1$, it follows that the (complex) monodromy operator $h^*: H^\cdot(F) \to H^\cdot(F)$ is diagonalizable and has eigenvalues in the group $G = \{t^a; a = 0, \ldots, N-1\}$ of the N-roots of unity.

We denote by $H^\cdot(F)_a$ the eigenspace corresponding to the eigenvalue t^{-a}, for $a = 0, \ldots, N-1$.

When f has an isolated singularity at the origin, the only nontrivial cohomology group $H^k(F)$ is for $k = n$ and the dimensions $\dim H^n(F)_a$ are known by the work of Brieskorn [2]. But as soon as f has a nonisolated singularity, it seems that even the Betti numbers $b_k(F)$ are known only in some special cases, see for instance [9], [14], [17], [22], [25].

The first main result of our paper is an *explicit formula* for the cohomology groups $H^k(F)$ and for the eigenspaces $H^k(F)_a$. Let $\Omega^\cdot$ be the complex of global algebraic differential forms on $\mathbb{C}^{n+1}$, graded by the convention $\deg(u\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = p + w_{i_1} + \cdots + w_{i_k}$ for $u \in S_p$. We introduce a new differential on $\Omega^\cdot$, namely $D_f(\omega) = \mathrm{d}\omega - (|\omega|/N)\,\mathrm{d}f \wedge \omega$, for $\omega \in \Omega^k_p$ with $|\omega| = p$ the degree of ω and d the usual exterior differential, similar to Dolgachev [8], p. 61.

For $a = 0, \ldots, N-1$ we denote by $\Omega^\cdot_{(a)}$ the subcomplex in $\Omega^\cdot$ given by $\bigoplus_{s \geqslant 0} \Omega^\cdot_{-a+sN}$.

To a D_f-closed form $\omega \in \Omega^{k+1}$ we can associate the element $\delta(\omega) = [i^* \Delta(\omega)]$ in the de Rham cohomology group $H^k(F)$, where Δ is the contraction with the Euler vector field (as in [12], p. 467 in the homogeneous case and [8], p. 43 in the weighted homogeneous case) and $i: F \to \mathbb{C}^{n+1}$ denotes the inclusion.

THEOREM A. *The maps* $\delta: H^{k+1}(\Omega^\cdot, D_f) \to \tilde{H}^k(F)$ *and* $\delta: H^{k+1}(\Omega^\cdot_{(a)}, D_f) \to \tilde{H}^k(F)_a$ *are isomorphisms for any* $k \geqslant 0$, $a = 0, \ldots, N-1$, *with* $\tilde{H}$ *denoting reduced cohomology.*

The proof of this Theorem depends on a comparison between spectral sequences naturally associated to the two sides of these equalities see (1.8).

Our second main theme is that these spectral sequences can be used to perform *explicit computations* and to derive interesting *numerical formulas*, in spite of the fact that the E_1-term has infinitely many nonzero entries and that degeneration at the E_2-term happens only in special cases (see (3.10) and (3.11) below).

The eigenspaces $H^\cdot(F)_0$ are particularly interesting. If $\mathbf{P} = \mathbf{P}(\mathbf{w})$ denotes the weighted projective space Proj(S), V the hypersurface $f = 0$ in $\mathbf{P}$ and $U = \mathbf{P}\setminus V$ the complement, then there is a natural identification $H^\cdot(F)_0 = H^\cdot(U)$. We conjecture an inclusion between the filtration on $H^\cdot(F)_0$ induced by the spectral sequence mentioned above and the (mixed) *Hodge filtration* on $H^\cdot(U)$, having a substantial consequence for explicit computations and extending to the singular case an important result of Griffiths [12], see (2.7.ii) below.

To prove the analogous result for these filtrations on the whole $H^\cdot(F)$, we establish first some subtle properties of the *Poincaré residue operator* $R\colon H^\cdot(\mathbb{C}^{n+1}\setminus F) \to H^{\cdot-1}(F)$ (see (1.6), (1.20), (1.21) and (2.6)) which may be useful in their own.

Note that the Betti numbers $b_k(V)$ are completely determined by $b_k(U)$ and hence one can get by our method at least upper bounds for all $b_k(V)$ as well as the exact value of the top interesting one (i.e. $b_{n+m-1}(V)$ where $m = \dim f^{-1}(0)_{\text{sing}}$) in a *finite* number of steps see (2.8).

Then we specialize to the case when f has a one-dimensional singular locus, a situation already studied (without the weighted homogeneity assumption) by N. Yomdin and, more recently and more completely, by D. Siersma, R. Pellikaan, D. van Straten, T. de Jong. We relate the spectral sequence $(E_r(f)_0\, d_r)$ to some new spectral sequences associated to the *transversal singularities* of f, these being the intersections of $f^{-1}(0)$ with transversals to each irreducible component of $f^{-1}(0)_{\text{sing}}$. We hope that these intricate local spectral sequences will play a fundamental role in understanding better even the isolated hypersurface singularities (see for instance the nice *characterization* (3.10′) of weighted homogeneous singularities). Concerning the numerical invariants in this case, we get interesting and effective formulas for the Euler characteristics $\chi(V)$ and $\chi(F)$ extending in highly nontrivial way the known formulas for the homogeneous case (we conjecture them to hold in general and check them under certain assumptions on the transversal singularities of f, see (3.19.ii)).

The last section is devoted to explicit computations with our spectral sequence. The first two of them are just simple illustrations of our technique, while the third offers a more subtle example, for which we know no other method to get even the Betti numbers for V. It is interesting to remark that if one wants to compute the Euler characteristic $\chi(V)$ in this case using Theorem (3.1) in Szafraniec [26], then one is led to compute bases of Milnor algebras (and signatures of bilinear forms defined on them) of a huge dimension ($\simeq 6^7$) and this is an impossible task even for a computer!

A more theoretical application (improving a result of Scherk [20]) is given in the end, the key point in the proof being again an explicit computation with the spectral sequence.

A basic *open problem* is to decide whether the spectral sequence $(E_r(f), d_r)$ or its local analog $(E_r(g, 0), d_r)$ degenerates always in a finite number of steps and, in the affirmative case, to determine a bound for this number in terms of other invariants of f or g.

1. Some spectral sequences

In this section we shall use many notations and results from Dolgachev [8] without explicit reference.

Let $\Delta\colon \Omega^k \to \Omega^{k-1}$ denote the contraction with the Euler vector field $\sum w_i x_i \partial/\partial x_i$. For $k \geqslant 1$ we put $\bar{\Omega}^k = \ker(\Delta\colon \Omega^k \to \Omega^{k-1}) = \operatorname{im}(\Delta\colon \Omega^{k+1} \to \Omega^k)$ and let $\Omega^k_{\mathbf{P}}$ denote the associated sheaf on $\mathbf{P}$. One has also the twisted sheaves $\Omega^k_{\mathbf{P}}(s)$, for any $s \in \mathbb{Z}$.

Let $i\colon U \to \mathbf{P}$ denote the inclusion and put $\Omega^k_U(s) = i^*\Omega^k_{\mathbf{P}}(s)$.

The Milnor fiber F is an affine smooth variety and according to Grothendieck [13] one has $H^\cdot(F) = H^\cdot(\Gamma(F, \Omega^\cdot_F))$. Let $p\colon F \to U$ denote the canonical projection and note that

$$p_*\Omega^\cdot_F = \bigoplus_{a=0}^{N-1} \Omega^\cdot_U(-a). \tag{1.1}$$

If we let $A^\cdot_a = \Gamma(U, \Omega^\cdot_U(-a))$ and $A^\cdot = \bigoplus_{a=0}^{N-1} A^\cdot_a$, then we clearly have

$$H^\cdot(F) = H^\cdot(A), \quad H^\cdot(F)_a = H^\cdot(A^\cdot_a). \tag{1.2}$$

There is a natural increasing filtration F_s on $A^\cdot_a$, related to the order of the pole a form in $A^\cdot_a$ has along V, namely

$$F_s A^j_a = 0 \text{ for } s < 0 \text{ and } F_s A^j_a = \{\omega/f^s; \omega \in \Omega^j_{sN-a}\} \text{ for } s \geqslant 0 \text{ similar to [12].} \tag{1.3}$$

But for obvious technical reasons it is more convenient to consider the decreasing filtration.

$$F^s A^j_a = F_{j-s} A^j_a. \tag{1.4}$$

The filtration F^s is compatible with d, exhaustive (i.e. $A_a^\bullet = \bigcup F^s A_a^\bullet$) and bounded above ($F^{n+1}A^\bullet = 0$). Here d denotes the differential of the complex $A_a^\bullet$ which is induced by the exterior differential d in $\Omega_F^\bullet$ via (1.1) and which is given explicitly by the formula

$$\mathrm{d}(\omega/f^s) = \mathrm{d}_f(\omega)\cdot f^{-s-1} \quad \text{where} \quad \mathrm{d}_f(\omega) = f\mathrm{d}\omega - (|\omega|/N)\,\mathrm{d}f \wedge \omega. \tag{1.5}$$

By the general theory of spectral sequences e.g. [16], p. 44 we get the next *geometric* spectral sequence.

(1.6) PROPOSITION. *There is an E_1-spectral sequence $(E_r(f)_a, d_r)$ with*

$$E_1^{s,t} = H^{s+t}(F^s A_a^\bullet / F^{s+1} A_a^\bullet)$$

and converging to the cohomology eigenspace $H^\bullet(F)_a$.

Moreover one can sum these spectral sequences for $a = 0, \ldots, N-1$ and get a spectral sequence $(E_r(f), d_r)$ converging to $H^\bullet(F)$. And $(E_r(f)_0, d_r)$ and $(E_r(f), d_r)$ are in fact spectral sequences of algebras converging to their limits as algebras. Note that $H^\bullet(F)_0 \simeq H^\bullet(U)$, either using the fact that $U = F/G$, G acting on F via the geometric monodromy or the fact that $\Omega_U^\bullet$ is a resolution of $\mathbb{C}$ [24].

We pass now to the construction of some purely *algebraic* spectral sequences. Let (B_a, d', d'') be the double complex $B_a^{s,t} = \Omega_{tN-a}^{s+t+1}$, $d' = d$ and $d''(\omega) = -|\omega|/N\,\mathrm{d}f \wedge \omega$ for a homogeneous differential form ω. Note that the associated total complex $B_a^\bullet$, with $B_a^k = \bigoplus_{s+t=k} B_a^{s,t}$, $D = d' + d''$ is precisely the complex $(\Omega_a^{\bullet-1}, D_f)$.

Similarly $B^\bullet = \bigoplus B_a^\bullet = (\Omega^{\bullet-1}, D_f)$.

Consider the decreasing filtration F^p on $B_a^\bullet$ given by $F^p B_a^k = \bigoplus_{s \geqslant p} B_a^{s,k-s}$ and similarly on $B^\bullet$. Using the contraction operator Δ, we define the next complex morphisms, compatible with the filtrations:

$$\bar{\delta}\colon B_a^\bullet \to A_a^\bullet \quad \text{and} \quad \bar{\delta}\colon B^\bullet \to A^\bullet,$$
$$\bar{\delta}(\omega) = \Delta(\omega) f^{-t} \quad \text{for} \quad \omega \in B_a^{s,t}.$$

Note that $B^\bullet$ and $A^\bullet$ are in fact differential graded algebras, but $\bar{\delta}$ is not compatible with the products.

(1.7) PROPOSITION. *There is an E_1-spectral sequence $('E_r(f)_a, d_r)$ with*

$$E_1^{s,t} = H^{s+t}(F^s B_a^\bullet / F^{s+1} B_a^\bullet)$$

and converging to the cohomology $H^\bullet(B_a^\bullet)$. The operator $\bar{\delta}$ induces a morphism $\delta_r\colon ('E_r(f)_a, d_r) \to (E_r(f)_a . d_r)$ of spectral sequences.

Moreover one can sum these spectral sequences $'E_r(f)_a$ and get a spectral sequence $('E_r(f), d_r)$ converging to $H^\cdot(B^\cdot)$ and a morphism $('E_r(f), d_r) \to (E_r(f), d_r)$. The proof of these facts is standard e.g. [16], p. 49. Let $\tilde{E}_r(f)_0$ (resp. $\tilde{E}_r(f)$) denote the *reduced* spectral sequence associated to $E_r(f)_0$ (resp. $E_r(f)$) which is obtained by replacing the term at the origin $E_1^{0,0} = E_\infty^{0,0} = \mathbb{C}$ by zero. For $a \neq 0$, we put $\tilde{E}_r(f)_a = E_r(f)_a$.

We clearly have natural morphisms $\tilde{\delta}_r\colon 'E_r(f)_a \to \tilde{E}_r(f)_a, \tilde{\delta}_r\colon 'E_r(f) \to \tilde{E}_f(f)$ induced by δ_r. We can state now a basic result.

(1.8) THEOREM. *The morphisms $\tilde{\delta}_r$ are isomorphisms for $r \geqslant 1$ and they induce isomorphisms $H^\cdot(B_a) \cong \tilde{H}^\cdot(F)_a$ and $H^\cdot(B) \cong \tilde{H}^\cdot(F)$.*

Proof. Since $F^{n+1}B^\cdot = F^{n+1}A^\cdot = 0$, the filtrations F are strongly convergent [16], p. 50 and hence it is enough to show that $\tilde{\delta}_1$ is an isomorphism. The vertical columns in $'E_1(f)$ correspond to certain homogeneous components in the *Koszul complex* $K^\cdot$.

$$K^\cdot\colon 0 \to \Omega^0 \xrightarrow{\mathrm{d}f} \Omega^1 \xrightarrow{\mathrm{d}f} \cdots \xrightarrow{\mathrm{d}f} \Omega^{n+1} \to 0 \tag{1.9}$$

of the partial derivatives $f_i = (\partial f)/(\partial x_i)$, $i = 0, \dots, n$ in S. To describe the vertical columns in $\tilde{E}_1(f)$ is more subtle. Note that $fK^\cdot$ is a subcomplex in $K^\cdot$ and let $\bar{K}^\cdot$ denote the quotient complex $K^\cdot/fK^\cdot$. There is a map $\bar{\Delta}\colon \bar{K}^\cdot \to \bar{K}^{\cdot-1}$ induced by Δ which is a complex morphism and hence $\tilde{K}^\cdot = \ker \bar{\Delta}$ is a subcomplex in $\bar{K}^\cdot$.

Let $\tilde{\Delta}$ denote the composition $K^\cdot \to \bar{K}^\cdot \xrightarrow{\bar{\Delta}} \tilde{K}^{\cdot-1}$.

Then the vertical lines in $\tilde{E}_1(f)$ correspond to certain homogeneous components in the cohomology groups $H^\cdot(\tilde{K}^\cdot)$. The morphism $\tilde{\delta}_1$ corresponds to $\tilde{\Delta}^*\colon H^\cdot(K^\cdot) \to H^\cdot(\tilde{K}^{\cdot-1})$ and a well-defined inverse for $\tilde{\Delta}^*$ is given by the map

$$\nabla\colon H^\cdot(\tilde{K}^{\cdot-1}) \to H^\cdot(K^\cdot), \nabla[\Delta(\omega)] = [\mathrm{d}f \wedge \Delta(\omega)/(Nf)]. \tag{1.10}$$

To check this, use that $\mathrm{d}f \wedge \omega = 0$ implies $0 = \Delta(\mathrm{d}f \wedge \omega) = Nf\omega - \mathrm{d}f \wedge \Delta(\omega)$.

(1.11) EXAMPLE. Assume that f has an *isolated singularity* at the origin. Then $f_0, \dots, f_n$ form a regular sequence in S and we get $'E_1^{s,t}(f)_a = 0$ for $s + t \neq n$ and

$$'E_1^{n-t,t}(f)_a \simeq H^{n+1}(K^\cdot)_{tN-a} \simeq Q(f)_{tN-a-w}$$

where $Q(f) = S/(f_0, \dots, f_n), w = w_0 + \cdots + w_n$. Moreover, the Poincaré series for $Q(f)$ (see for instance [7], p. 109) implies that $Q(f)_k = 0$ for $k > (n+1)N - 2w$. Hence in this case all our spectral sequences are finite and degenerate at the E^1-term (the degeneracy of the component $a = 0$ being equivalent to Griffiths' Theorem 4.3 in [12]). Note that one can have $'E_1^{-1,n+1}(f)_a \neq 0$. In general, one has the next result about the size of the spectral sequence $'E_r(f)$.

(1.12) Proposition. *$'E_r^{s,t}(f) = 0$ for any $r \geqslant 1$ and $s + t < n - m$, where $m = \dim f^{-1}(0)_{\text{sing}}$.*

Proof. The result follows using the description of $'E_1^{s,t}(f)$ in terms of the Koszul complex and Greuel generalized version of the de Rham-Lemma, see [11], (1.7).

(1.13) Corollary. *$\tilde{H}^k(F) = 0$ for $k < n - m$.*

This result is implied also by [15], but (1.12) will be used below in (2.8) in a crucial way.

Now we show that our complexes can be used to describe very explicitly the *Poincaré residue isomorphism* $R\colon H^{k+1}(\mathbb{C}^{n+1} \backslash F) \to \tilde{H}^k(F)$ and the *Sebastiani-Thom isomorphism.*

When X is a smooth complex manifold and D is a smooth closed hypersurface in X there is a Gysin exact sequence

$$\cdots \to H^{k+1}(X) \xrightarrow{i^*} H^{k+1}(X \backslash D) \xrightarrow{R} \tilde{H}^k(D) \to \cdots \tag{1.14}$$

where i^* is induced by the inclusion $i\colon X \backslash D \to X$ and R is the Poincaré residue, see for instance [24], Section 8.

Let $C_f^{\bullet}$ denote the complex $\Omega^{\bullet}$ with the differential D_f introduced above (up to a shift $C_f^{\bullet} = B^{\bullet}$!) and note that

$$\begin{aligned} &\alpha\colon C_f^{\bullet} \to \Gamma(\mathbb{C}^{n+1} \backslash F, \Omega^{\bullet}_{\mathbb{C}^{n+1} \backslash F}) \\ &\alpha(\omega) = \omega - (\mathrm{d}f \wedge \Delta(\omega))/N(f-1) \end{aligned} \tag{1.15}$$

is a morphism of differential graded algebras (i.e. $\mathrm{d}\alpha(\omega) = \alpha D_f(\omega)$ and $\alpha(\omega_1 \wedge \omega_2) = \alpha(\omega_1) \wedge \alpha(\omega_2)$).

Using the definition of the Poincaré residue as in [12], p. 290 it follows that

$$R\alpha(\omega) = (-1)^k / N\delta(\omega). \tag{1.16}$$

Since R is an isomorphism by using (1.14) in the case $X = \mathbb{C}^{n+1}, D = F$ and δ is an isomorphism by Theorem A, it follows that $\alpha\colon H^{\bullet}(C_f) \to H^{\bullet}(\mathbb{C}^{n+1} \backslash F)$ is an isomorphism too.

To discuss the Sebastiani-Thom isomorphism (see for instance [17]), we introduce a new complex associated to f, namely $\bar{C}_f$ which is the complex $\Omega^{\bullet}$ with the differential $\bar{D}_f \omega = \mathrm{d}\omega - \mathrm{d}f \wedge \omega$.

Define $\theta\colon \bar{C}_f \to C_f$ to be the $\mathbb{C}$-linear map which on a homogeneous form ω with $k = |\omega|$ acts by the formula $\theta(\omega) = \lambda(k) \cdot \omega$, where $\lambda(k) = 1$ for $k \leqslant N$ and $\lambda(k) = (k - N) \cdots (k - tN) \cdot N^{-t}$ for $tN < k \leqslant (t+1)N$, $t \geqslant 1$.

Then it is obvious that θ induces a complex isomorphism between the corresponding reduced complexes. In particular we get isomorphisms $\theta\colon H^t(\bar{C}_f) \to H^t(C_f)$ for any $t \geqslant 1$.

Let $\mathbf{w}' = (w'_0, \ldots, w'_{n'})$ be a new set of weights and $f' \in \mathbb{C}[y_0, \ldots, y_{n'}]$ be a homogeneous polynomial of degree N with respect to these weights. Then it is easy to check that

$$\bar{C}_{f+f'} \simeq \bar{C}_f \otimes \bar{C}_{f'} \tag{1.17}$$

and that there is no such result for $C_{f+f'}$.

Using the isomorphisms θ and Theorem A we get the Sebastiani-Thom isomorphism

$$\tilde{H}^k(F'') = \bigoplus_{s+t=k-1} \tilde{H}^s(F) \otimes \tilde{H}^t(F') \tag{1.18}$$

where F', F'' denote the Milnor fibers of f' and $f+f'$ respectively.

Keeping trace of the homogeneous components in (1.17) we get

$$\tilde{H}^\cdot(F'')_0 = \bigoplus_c \tilde{H}^\cdot(F)_c \otimes \tilde{H}^\cdot(F')_{N-c}$$

with $c = 0, \ldots, N-1$ and $\tilde{H}^\cdot(F')_N = \tilde{H}^\cdot(F')_0$. When $f' = y_0^N$, Example (1.11) shows that $\tilde{H}^\cdot(F')_0 = 0$ and $\tilde{H}^\cdot(F')_c = \langle \delta(y_0^{N-c-1}\,\mathrm{d}y_0)\rangle$, a one-dimensional vector space for $c = 1, \ldots, N-1$. It follows that $\dim \tilde{H}^k(F'')_0 = \dim \tilde{H}^{k-1}(F)_{\neq 0}$ where $\tilde{H}^s(F)_{\neq 0} = \oplus_{c=1,N-1} H^s(F)_c$

This equality of dimensions is related to the next geometric setting. Let $H\colon y_0 = 0$ denote the hyperplane at infinity in the compactification $\mathbf{P}(\mathbf{w}, 1)$ of $\mathbb{C}^{n+1}$, let $V'' \subset \mathbf{P}(\mathbf{w}, 1)$ be the hypersurface given by $f(x) - y_0^N = 0$ and set $U'' = \mathbf{P}(\mathbf{w}, 1) \backslash V''$. Since $H \cap U'' = U, U'' \backslash H = \mathbb{C}^{n+1} \backslash F$, the Gysin sequence (1.14) applied to $X = U''$, $D = H \cap U''$ gives

$$\to H^k(U'') \xrightarrow{i^*} H^k(\mathbb{C}^{n+1} \backslash F) \xrightarrow{R} \tilde{H}^{k-1}(U) \to$$

(As a matter of fact U'' may be singular and then to apply (1.14) one has to do as follows. Let $q\colon \mathbf{P}^{n+1} \to \mathbf{P}(\mathbf{w}, 1)$ be the covering map induced by

$$(x_0 : \cdots : x_n : y) \mapsto (x_0^{w_0} : \cdots : x_n^{w_n} : y)$$

and let $\tilde{G}$ be the corresponding group of covering transformations.

If we set $\tilde{U} = q^{-1}(U''), \tilde{H} = q^{-1}(H)$, then there is a Gysin sequence associated to $X = \tilde{U}, D = \tilde{H} \cap \tilde{U}$. And the $\tilde{G}$-invariant part of this exact sequence is precisely the exact sequence which we have written above).

Note that

$$\begin{aligned}\dim H^k(\mathbb{C}^{n+1}\backslash F) &= \dim \tilde{H}^{k-1}(F) = \dim \tilde{H}^{k-1}(F)_{\neq 0} + \dim \tilde{H}^{k-1}(F)_0 \\ &= \dim H^k(U'') + \dim \tilde{H}^{k-}(U).\end{aligned}$$

It follows that the first and the last map in the above exact sequence are trivial. Note also that the geometric monodromy h acts on $\mathbb{C}^{n+1}\backslash F$ and hence it makes sense to define $H^s(\mathbb{C}^{n+1}\backslash F)_{\neq 0}$ as above.

It will be clear from what follows that the image of i^* is precisely $H^k(\mathbb{C}^{n+1}\backslash F)_{\neq 0}$ and hence we can write the next diagram of isomorphisms:

$$\begin{tikzcd}
H^k(F'')_0 \simeq H^k(U'') \arrow[r, "i^*"] & H^k(\mathbb{C}^{n+1}\backslash F)_{\neq 0} \arrow[r, "R"] & H^{k-1}(F)_{\neq 0} \\
H^{k+1}(C_{f+f'})_0 \arrow[u, "\delta"] & & H^k(C_f)_{\neq 0} \arrow[u, "\delta"'] \arrow[ul, "\alpha"] \\
H^{k+1}(\bar{C}_{f+f'})_0 \arrow[u, "\theta"] & & H^k(\bar{C}_f)_{\neq 0} \arrow[u, "\theta"'] \arrow[ll, "\psi"']
\end{tikzcd} \tag{1.19}$$

Here ψ is defined in a natural way: if $\omega \in H^k(\bar{C}_f)_c$ (i.e. ω is a sum $\omega_1 + \cdots + \omega_p$ of homogeneous forms such that $|\omega_i| \equiv -c$ modulo N) then $\psi(\omega) = \omega \wedge y_0^{c-1}\,\mathrm{d}y_0$

The formula (1.16) tells us that the triangle in the diagram (1.19) is commutative up to a constant. The big rectangle in the diagram is commutative in a similar way by the next result.

(1.20) LEMMA. *$Ri^*\delta\theta\psi = -1/N\delta\theta$.*

Proof. We have to show that both sides of this equality yield the same result when applied to an element $\omega = \omega_1 + \cdots + \omega_p \in H^k(\bar{C}_f)_c$ as above. Since these computations are rather tedious, we treat here only the case $p = 2$ and let the reader check that the general case is completely similar.

So let $\omega = \omega_1 + \omega_2$ with $q = tN - c = |\omega_1|$ and $q + N = |\omega_2|$ (when $|\omega_2| - |\omega_1| > N$ the forms ω_1 and ω_2 are themselves cycles in $H^k(\bar{C}_f)_c$ and the proof is easier!).

The condition $\bar{D}_f\omega = 0$ is equaivalent to

(i) $\mathrm{d}f \wedge \omega_2 = 0$

(ii) $\mathrm{d}f \wedge \omega_1 = \mathrm{d}\omega_2$

(iii) $0 = \mathrm{d}\omega_1$.

It is easy to see that

$$i^*\delta\theta\psi(\omega) = (-1)^k t![\omega_1/t(f-1)^t + \omega_2/(f-1)^{t+1}].$$

To compute the residue of this element we proceed as follows. First we apply Δ to the equality (i) and get

(iv) $\omega_2/(f-1) + \omega_2 = \mathrm{d}f \wedge \Delta(\omega_2)/(N(f-1))$.

Next we can divide this equality by $(f-1)^s$ and get

(v) $\omega_2/(f-1)^{s+1} + \omega_2/(f-1)^s = -\mathrm{d}(\Delta(\omega_2)/Ns(f-1)^s) + \mathrm{d}\Delta(\omega_2)/Ns(f-)^s$.

If we apply Δ to (ii), we get

$$\mathrm{d}\Delta(\omega_2) = (q+N)\omega_2 + \mathrm{d}f \wedge \Delta(\omega_1) - Nf\omega_1.$$

This should be put in (v), one should apply once more this trick getting a term containing $\mathrm{d}\Delta(\omega_1)$ and then replace this by $q\omega_1$ as follows by applying Δ to (iii).

Let $A_s = \omega_1/s(f-1)^s + \omega_2/(f-1)^{s+1}$ and note that A_s is a closed form on $\mathbb{C}^{n+1}\backslash F$ for any $s \geqslant 1$. The above computation implies that the associated cohomology classes satisfy $[A_s] = ((q - N(s-1))/Ns)[A_{s-1}]$ and hence

$$R[A_t] = (\lambda(q)/t!)R[A_1] = ((-1)^{k-1}\lambda(q)/Nt!)[q/N\delta(\omega_2) + \delta(\omega_1)].$$

This ends the proof of (1.20) in this case.

(1.21) REMARK. There is a nice geometric consequence of the existence of the diagram (1.19). One can think of the weighted projective space $\mathbf{P}(\mathbf{w}, 1)$ as a *compactification* of $\mathbb{C}^{n+1}\backslash F$ such that the complement $\mathbf{P}(\mathbf{w}, 1)\backslash(\mathbb{C}^{n+1}\backslash F)$ consists of two irreducible components, namely V'' and H. Using the isomorphism α, it follows that any cohomology class in $H^{\cdot}(\mathbb{C}^{n+1}\backslash F)$ can be represented by a closed differential form on $\mathbb{C}^{n+1}\backslash F$ having a pole of order 1 along V'' and a pole (possibly of a higher order) along H.

On the other hand, the isomorphism $i^*\delta\theta\psi$ shows that any class in $H^{\cdot}(\mathbb{C}^{n+1}\backslash F)_{\neq 0}$ can be represented by a closed differential form on $\mathbb{C}^{n+1}\backslash F$ having a pole on V'' and no poles at all along H. It can be shown similarly that any class in $H^{\cdot}(\mathbb{C}^{n+1}\backslash F)_0$ can be represented by a form having a pole of order 1 along H and a pole along V''. It would be nice to have a more geometric understanding of this phenomenon.

In conclusion, the natural isomorphism $H^k(F) = H^k(F)_0 \oplus H^k(F)_{\neq 0} = H^k(U) \oplus H^{k+1}(U'')$ shows that it is enough to concentrate on the cohomology groups $H^{\cdot}(U)$ and this is what we do in the next two sections.

2. The relation with the Hodge filtration

Let us consider the decreasing filtration F^s on $H^\bullet(U)$ defined by the filtration F^s on $A_0^\bullet$, namely

$$F^sH^\bullet(U) = \mathrm{im}\{H^\bullet(F^sA_0^\bullet) = H^\bullet(U)\}. \tag{2.1}$$

On the other hand there is on $H^\bullet(U)$ the decreasing *Hodge filtration* F_H^s introduced by Deligne [5].

(2.2) THEOREM. *One has* $F^sH^\bullet(U) \supset F_H^{s+1}H^\bullet(U)$ *for any* s *and* $F^0H^\bullet(U) = F_H^1H^\bullet(U)F_H^0H^\bullet(U) = H^\bullet(U)$.

Proof. Let $p\colon \mathbf{P}^n \to \mathbf{P}$ be the projection presenting $\mathbf{P}$ as the quotient of $\mathbf{P}^n$ under the group $G(\mathbf{w})$, the product of cyclic groups of orders w_i.

Then $\tilde{f} = p^*(f) = f(x_0^{w_0}, \ldots, x_n^{w_n})$ is a homogeneous polynomial of degree N and let $\tilde{U}$ be the complement of the hypersurface $\tilde{f} = 0$ in $\mathbf{P}^n$.

Since $H^\bullet(U)$ can be identified to the fixed part in $H^\bullet(\tilde{U})$ under the group $G(\mathbf{w})$ and since the monomorphism $p^*\colon H^\bullet(U) \to H^\bullet(\tilde{U})$ is clearly compatible with the filtrations F^s and F_H^s, it is enough to prove (2.2) for $\tilde{U}$.

To simplify the notation, we assume that $\mathbf{w} = (1, \ldots, 1)$ from the beginning. Then U is smooth and it is easier to describe the construction of the Hodge filtration [24].

Let $p\colon X \to \mathbf{P}^n$ be a proper modification with X smooth, $D = p^{-1}(V)$ a divisor with normal crossings in X and $\bar{U} = X \setminus D$ isomorphic to U via p.

From this point on it is more suitable to work with holomorphic differential forms on our algebraic varieties. If $\Omega_U^\bullet$ is this holomorphic sheaves complex, ${}^a\Omega_U^\bullet$ the algebraic version of it and $i\colon U \to \mathbf{P}^n$ is the inclusion, then one has inclusions $i_*({}^a\Omega_U^\bullet) \subset \Omega_{\mathbf{P}^n}^\bullet(*V) \subset i_*\Omega_U^\bullet$, where $\Omega_{\mathbf{P}^n}^\bullet(*V)$ denotes the sheaves of meromorphic differential forms on $\mathbf{P}^n$ with polar singularities along V. By Grothendieck [13], the inclusion $i_*({}^a\Omega_U^\bullet) \subset \Omega_{\mathbf{P}^n}^\bullet(*V)$ induces isomorphisms at the hypercohomology groups. And the same is true for the inclusions $\Omega_X^\bullet(\log D) \subset \Omega_X^\bullet(*D) \subset j_*\Omega_{\bar{U}}^\bullet$ where $j\colon \bar{U} \to X$ is the inclusion, $\Omega_X^\bullet(*D)$ is defined similarly to $\Omega_{\mathbf{P}^n}^\bullet(*V)$ and $\Omega_X^\bullet(\log D)$ is the complex of holomorphic differential forms with logarithmic poles along D[24].

Recall that there is a trivial filtration $\sigma_\geqslant$ on any complex $K^\bullet$, by defining $\sigma_{\geqslant s}K^\bullet$ to be the subcomplex of $K^\bullet$ obtained by replacing the first s terms in $K^\bullet$ by 0. The Hodge filtration is given by

$$F_H^sH^j(U) = \mathrm{im}\{\mathbf{H}^j(\sigma_{\geqslant s}\Omega_X^\bullet(\log D)) \to \mathbf{H}^j(\Omega_X^\bullet(\log D))\} \tag{2.3}$$

via the identifications

$$\mathbf{H}^\bullet(\Omega_X^\bullet(\log D)) = \mathbf{H}(j_*\Omega_{\bar{U}}^\bullet) = \mathbf{H}^\bullet(\Omega_{\bar{U}}^\bullet) = H^\bullet(\bar{U}) = H^\bullet(U).$$

The filtration F^s on the complex $A_0^{\cdot}$ is related to a filtration F^s on the complex $\Omega^{\cdot}_{\mathbf{P}^n}(*V)$ defined in the following way: $F^s\Omega^j_{\mathbf{P}^n}(*V)$ is the sheaf of meromorphic j-forms on $\mathbf{P}^n$ having poles of order at most $j-s$ along V for $j \geqslant s$ and $F^s\Omega^j_{\mathbf{P}^n}(*V) = 0$ for $j < s$.

Note that $F^s\Omega^j_{\mathbf{P}^n}(*V) \simeq \Omega^j_{\mathbf{P}^n}((j-s)N)$ for $j \geqslant s$. We get next a filtration on the complex $\Omega^{\cdot}_X(*D) \simeq p^*(\Omega^{\cdot}_{\mathbf{P}^n}(*V))$ by defining $F^s\Omega^{\cdot}_X(*D) = p^*(F^s\Omega^{\cdot}_{\mathbf{P}^n}(*V))$.

At stalks level, a germ $\omega \in \Omega^j_X(*D)_x$ belongs to $F^s\Omega^j_X(*D)_x$ if and only if $p^*(u)^{j-s}\cdot\omega \in \Omega^j_{X,x}$, where $u = 0$ is a local equation for V around the point y $p(x)$.If $v_1, \dots, v_n$ are local coordinates on X around x such that $v_1 \dots v_k = 0$ is a local equation for D. then $p^*(u)$ vanishes on D and hence $p^*(u) = v_1^{a_1} \dots v_k^{a_k} w$ for some germ $w \in \mathcal{O}_{X,x}$ and integers $a_i \geqslant 1$.

Using the definitiions, it follows that $\Omega^j_X(\log D) \subset F^s\Omega^j_X(*D)$ for $j > s$ and $j > 0$. And $\Omega^0_X(\log D) = \Omega^0_X \subset F^s\Omega^0_X(*D)$ for $s \leqslant 0$. We can state this as follows.

(2.4) LEMMA. (i) $\sigma_{\geqslant s+1}\Omega^{\cdot}_X(\log D \subset F^s\Omega^{\cdot}_X(*D)$ *for* $s > 0$;
(ii) $\Omega^{\cdot}_X(\log D) \subset F^0\Omega^{\cdot}_X(*D)$.

We can hence write the next commutative diagram

$$\begin{array}{ccccc}
\mathbf{H}^{\cdot}(\sigma_{\geqslant s+1}\Omega^{\cdot}_X(\log D)) & \to & \mathbf{H}^{\cdot}(F^s\Omega^{\cdot}_X(*D)) & \xleftarrow[\sim]{p^*} & \mathbf{H}^{\cdot}(F^s\Omega^{\cdot}_{\mathbf{P}^n}(*V)) \\
\downarrow & & & & \downarrow{\scriptstyle\beta} \\
\mathbf{H}^{\cdot}(\Omega^{\cdot}_X(\log D)) & \xrightarrow[\sim]{\qquad} & \mathbf{H}^{\cdot}(\Omega^{\cdot}_X(*D)) & \xleftarrow[\sim]{p^*} & \mathbf{H}^{\cdot}(\Omega^{\cdot}_{\mathbf{P}^n}(*V)).
\end{array}$$

Now $\mathbf{H}^{\cdot}(\Omega^{\cdot}_{\mathbf{P}^n}(*V)) = \mathbf{H}^{\cdot}({}^a\Omega^{\cdot}_U) = H^{\cdot}(A_0^{\cdot}) = H^{\cdot}(U)$. To compute $\mathbf{H}^{\cdot}(F^s\Omega^{\cdot}_{\mathbf{P}^n}(*V))$ we use the E_2-spectral sequence $E_2^{p,q} = H^p(H^q(\mathbf{P}^n, K^{\cdot}))$ converging to $\mathbf{H}^{\cdot}(K^{\cdot})$, where $K = F^s\Omega^{\cdot}_{\mathbf{P}^n}(*V)$ and Bott's vanishing theorem [8].

It follows that $E_2^{p,0} = H^p(F^sA_0^{\cdot}), E_2^{s,s} = H^s(\mathbf{P}^n, \Omega^s_{\mathbf{P}^n})$ and $E_2^{p,q} = 0$ in the other cases. The spectral sequence degenerates at E_2 since one can represent the generator of $E_2^{s,s}$ by a $\bar{\partial}$-harmonic form γ and hence $\bar{d}\gamma = 0$. On the other hand $\beta(\gamma) = 0$, since γ belongs to the kernel of the map $H^{2s}(\mathbf{P}^n) \xrightarrow{i^*} H^{2s}(U)$. In fact this map is zero for $s > 0$. To see this, it is enough to show that $i^*(c) = 0$, where $c = c_1(\mathcal{O}(1))$ is the first Chern class of the line bundle $\mathcal{O}(1)$ (in cohomology with complex coefficients!). But $Ni^*(c) = 0$, since it corresponds to the Chern class of $\mathcal{O}(N)|_U$ and this line bundle has a section (induced by f) without any zeros.

It follows that $\operatorname{im}(\beta) = F^sH^{\cdot}(\mathrm{U})$ and this gives the first part in (2.2).

The similar diagram associated to the inclusion (2.4. ii) gives $F^0H^{\cdot}(U) = F^0_HH^{\cdot}(U) = H^{\cdot}(\mathrm{U})$.

To see that $F^0_H = F^1_H$ we relate the mixed Hodge structure on $H^{\cdot}(U)$ to the mixed Hodge structure on $H^{\cdot}(V)$ Consider the exact sequence in cohomology with compact supports of the pair $(\mathbf{P}^n, V)$

$$\cdots \to H^k_c(U) \to H^k(\mathbf{P}^n) \to H^k(V) \to H^{k+1}_c(U) \to \cdots \tag{2.5}$$

This is an exact sequence of MHS (mixed Hodge structures) and it gives an isomorphism of MHS $H_c^{k+1}(U) \simeq H_0^k(V)$, the primitive cohomology of V [10]. Poincaré duality gives a natural identification (U is a $\mathbf{Q}$-homology manifold):

$$H^s(U) = \mathrm{Hom}(H_c^{2n-s}(U), H_c^{2n}(U)).$$

Since $H_c^{2n}(U) \simeq H^{2n}(\mathbf{P}^n) \simeq \mathbb{C}(-n)$, we get the following relations among mixed Hodge numbers

$$h^{p,q}(H^s(U)) = h^{n-p,n-q}(H_0^{2n-s-1}(V)).$$

This gives $h^{0,q}(H^s(U)) = 0$ for any q and s, which shows that $F_H^0 H^\cdot(U) = F_H^1 H^\cdot(U)$, ending the proof of (2.5).

(2.6) REMARK. In spite of the fact that $F^s H^\cdot(U) = F_H^{s+1} H^\cdot(\mathrm{U})$ for any s in many cases (e.g. when V is a quasi-smooth hypersurface or when V is a nodal curve in $\mathbf{P}^2$), this equality *does not hold* in general. A simple example is the next: take $V\colon x[xy(x+y)+z^3] = 0$ the union of a smooth cubic curve in $\mathbf{P}^2$ with an inflexional tangent. Then it is easy to show that in this case $\dim F^1H^2(U) = 2 > \dim F_H^2 H^2(U) = 1$.

There is a similar inclusion $F^s H^\cdot(F) \supset F_H^{s+1} H^\cdot(F)$ among the analogous filtrations on the cohomology of the Milnor fiber F. The proof of this fact can be reduced to (2.2) as follows. The geometric monodromy h is analgebraic map and hence h^* preserves both filtrations F^s and F_H^s on $H^\cdot(F)$. If we define $F^s H^\cdot(F)_a = F^s H^\cdot(F) \cap H^\cdot(F)_a$ it follows that $F^s H^\cdot(F) = \oplus_a F^s H^\cdot(F)_a$. And one has a similar result for the Hodge filtration F_H^s. In particular, it is enough to prove

(i) $F^s H^\cdot(F)_0 \supset F_H^{s+1} H^\cdot(F)_0$, and
(ii) $F^s H^\cdot(F)_{\neq 0} \supset F_H^{s+1} H^\cdot(F)_{\neq 0}$

where $F^s H^\cdot(F)_{\neq 0} = F^s H^\cdot(F) \cap H^\cdot(F)_{\neq 0} = \oplus_{a \neq 0} F^s H^\cdot(F)_a$ and similarly for F_H.

Now (i) is clearly implied by (2.2), since the isomorphism $H^\cdot(U) \xrightarrow[\sim]{P^*} H^\cdot(F)_0 \subset H^\cdot(F)$ is clearly compatible with both filtrations.

To get (ii) from (2.2) we use the diagram (1.19) and the next two facts.

The Poincaré residue map R is a morphism of MHS of type $(-1,-1)$ and hence

$$Ri^*(F_H^{s+1} H^\cdot(U'')) = F_H^s H^\cdot(F)_{\neq 0}.$$

Using the definition of the filtrations F^s and (1.20) it follows that

$$Ri^*(F^s H^\cdot(U'')) = F^{s-1} H^\cdot(F)_{\neq 0}.$$

Note also that the filtration F^s on $H^{\boldsymbol{\cdot}}(F)$ is very close to the filtrations considered by Scherk and Steenbrink in the isolated singularity case in [21].

(2.7) COROLLARY. (i) *$E_\infty^{s,t}(f)_0 =$ for $s < 0$ and $E_\infty^{s,t}(f)_a = 0$ for $s < -1$ and $a = 1 \ldots, N - 1$.*

(ii) *Any element in $H^k(U)$ can be represented by a differential k-form with a pole along V of order at most k.*

We note that (ii) can be regarded as an extension of Griffith's Theorem 4.2 in [12]. On the side of numerical computations of Betti numbers we get the following important consequence. Recall that $m = \dim f^{-1}(0)_{\text{sing}}$.

(2.8) THEOREM. *Let $b_j^0(V) = \dim H_0^j(V)$ denote the primitive Betti numbers of V. Then*

(i) *$b_j^0(V) = 0$ for $j < n - 1$ or $j > n - 1 + m$;*
(ii) *For $k \in [0, m]$ and $r > 1$ one has*

$$b_{n-1+k}^0(V) = b_{n-k}(U) \leqslant \sum_{s=0}^{n-k-1} \dim E_r^{s,n-k-s}(f)_0.$$

When $k = m$ and $r \geqslant n - m$ the above inequality is an equality.

Proof. Use (1.6), (1.7), (1.8), (1.12) and (2.7).

There is also an analog of (2.8) for $\dim H^j(F)_a$ but we leave the details for the reader.

3. The case of a one-dimensional singular locus

We assume in this section that f has a one-dimensional singular locus, namely

$$f^{-1}(0)_{\text{sing}} = \{z \in \mathbb{C}^{n+1}; \mathrm{d}f(x) = 0\} = \{0\} \cup \bigcup_{i=1,p} \mathbb{C}^* a_i$$

for some points $a_i \in \mathbb{C}^{n+1}$, one in each irreducible component of $f^{-1}(0)_{\text{sing}}$.

If H_i is a small transversal to the orbit $\mathbb{C}^* a_i$ at the point a_i, then the isolated hypersurface singularity $(Y_i, a_i) = (H_i \cap f^{-1}(0), a_i)$ is called the *transversal singularity* of f along the brach $\mathbb{C}^* a_i$ of the singular locus.

The weighted homogeneity of f easily implies that the isomorphism class ($\mathscr{K}$-equivalence) of the singularity (Y_i, a_i) does not depend on the choice of a_i (in the orbit $\mathbb{C}^* a_i$) or of H_i.

In this section we get a better understanding of the sequence $(E_r(f)_0, d_r)$ by relating it to some spectral sequences associated to the transversal singularities (Y_i, a_i) for $i = 1, \ldots, p$.

First we describe the construction of these new (local) spectral sequences.

Let $g: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ and let $(Y, 0) = (g^{-1}(0), 0)$ be the hypersurface singularity defined by g. Let $\Omega^{\bullet}_{g,0}$ denote the localization of the stalk at the origin of the holomorphic de Rham complex $\Omega^{\bullet}_{\mathbb{C}^n}$ with respect to the multiplicative system $\{g^s; s \geqslant 0\}$.

Choose $\varepsilon > 0$ small enough such that Y has a conic structure in the closed ball $B_\varepsilon = \{y \in \mathbb{C}^n; |y| \leqslant \varepsilon\}$ [4]. Let $S_\varepsilon = \partial B_\varepsilon$ and $K = S_\varepsilon \cap Y$ be the link of the singularity $(Y, 0)$. Then Thm. 2 in [13] implies the following.

(3.1) PROPOSITION. $H^{\bullet}(S_\varepsilon \backslash K) \simeq H^{\bullet}(\Omega^{\bullet}_{g,0})$.

One can construct a filtration F^s on $\Omega^{\bullet}_{g,0}$ in analogy to (1.4), namely

$$F^s\Omega^j_{g,0} = \{\omega/g^{j-s}; \omega \in \Omega^j_{\mathbb{C}^n,0}\} \text{ for } j \geqslant s \text{ and } F^s\Omega^j_{g,0} = 0 \text{ for } j < s.$$

(3.2) PROPOSITION. *There is an E_1-spectral sequence of algebras $(E_r(g,0), d_r)$ with*

$$E_1^{s,t} = H^{s+t}(Gr^s_F\Omega^{\bullet}_{g,0})$$

and converging to $H^{\bullet}(S_\varepsilon \backslash K)$ as an algebra.

Assume from now on that $(Y, 0)$ is an isolated singularity and let $L^{\bullet} = (\Omega^{\bullet}_{\mathbb{C}^n,0}\, \mathrm{d}g)$ denote the Koszul complex of the partial derivatives of g. In our case these derivatives form a regular sequence and hence $H^j(L^{\bullet}) = 0$ for $j < n$ and $H^n(L^{\bullet}) = M(g)$, the *Milnor algebra* of the singularity $(Y, 0)$, see for instance [7], p. 90. Let $I^{\bullet}$ denote the quotient complex $L^{\bullet}/gL^{\bullet}$ If g: $M(g) \to M(g)$ denotes the multiplication by g, it follows that $H^j(I^{\bullet}) = 0$ for $j < n-1, H^{n-1}(I^{\bullet}) = \ker(g)$ and $H^n(I^{\bullet}) = \operatorname{coker}(g) = T(g)$, the *Tjurina algebra* of $(Y, 0)$, see [7], p. 90.

There is the next analog of (1.8), computing $E_1(g, 0)$ in terms of $H^{\bullet}(\mathrm{I}^{\bullet})$.

(3.3) LEMMA. *The nonzero terms in $E_1(g, 0)$ are the following.*

(i) $E_1^{s,0}(g, 0) = \Omega^s_{\mathbb{C}^n,0}$ *for* $s \in [0, n]$
(ii) $E_1^{s,1}(g, 0) = \Omega^s_Y$ *for* $s \in [0, n-3]$, *there is an exact sequence* $0 \to \Omega^{n-2}_Y \xrightarrow{u} E_1^{n-2,1}(g, 0) \xrightarrow{v} \ker(g) \to 0$ *and* $E_1^{n-1,1}(g, 0) = \Omega^n_{\mathbb{C}^n,0}/g\Omega^n_{\mathbb{C}^n,0}$, *where*

$$\Omega^{\bullet}_Y = (\Omega^{\bullet}_{\mathbb{C}^n,0})/(g\Omega^{\bullet}_{\mathbb{C}^n,0} + \mathrm{d}g \wedge \Omega^{\bullet}_{\mathbb{C}^n,0}).$$

(iii) $E_1^{n-t-1,t}(g, 0) = \ker(g), E_1^{n-t,t}(g, 0) = T(g)$ *for* $t \geqslant 2$.

Proof. To get the more subtle point (ii), one uses the well-defined maps

$$u: \Omega^s_Y \to E_1^{s,1}(g, 0),\ u(\alpha) = [(\mathrm{d}g \wedge \alpha)/g]$$
$$v: E_1^{s,1}(g, 0) \to H^{s+2}(L^{\bullet}), v[\beta/g] = [(\mathrm{d}g \wedge \beta)/g]$$

and note that $\operatorname{im}(v) \subset \ker(g)$ for $s = n-2$.

(3.4) COROLLARY. *The only (possibly) nonzero terms in* $E_2(g,0)$ *are* $E_2^{0,0} = E_2^{0,1} = \mathbb{C}$ *and* $E_2^{n-1-t,t}, E_2^{n-t,t}$ *for* $t \geqslant 1$.

Proof. Use the exactness of the de Rham complexes [11]:

$$0 \to \mathbb{C} \to \Omega^0_{\mathbb{C}^n,0} \to \cdots \to \Omega^n_{\mathbb{C}^n,0} \to 0$$

$$0 \to \mathbb{C} \to \Omega^0_Y \to \cdots \to \Omega^{n-1}_Y.$$

We can also describe the differentials

$$d_1^t : \ker(g) = E_1^{n-1-t,t} \xrightarrow{d_1} E_1^{n-t,t} = T(g).$$

An $(n-1)$ form α induces an element in $\ker(g)$ if $dg \wedge \alpha = g\beta$ and then

$$d_1^t[\alpha] = [d\alpha - t\beta] \tag{3.5}$$

(3.6) EXAMPLE. Assume that $(Y,0)$ is a *weighted homogeneous singularity* of type $(w_1,\dots,w_n;N)$, i.e. $(Y,0)$ is defined in suitable coordinates by a weighted homogenous polynomial g of degree N with respect to the weights $\mathbf{w}$.

Then $M(g) = T(g) = \ker(g)$ and they are all graded $\mathbb{C}$-algebras.

Let $\alpha = \sum_{i=1,n}(-1)^{i+1} w_i x_i \, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_n$ and note that $dg \wedge \alpha = N \cdot g\omega_n$, with $\omega_n = dx_1 \wedge \cdots \wedge dx_n$. It follows that the class of α generates $\ker(g)$ For a monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ of degree $|x^a| = a_1 w_1 + \cdots + a_n w_n$ one has by (3.5)

$$d_1^t(x^a \alpha) = [(w + |x^a| - tN)x^a \omega_n]$$

with $w = w_1 + \cdots + w_n$.

It follows that $\ker d_1^t \simeq \operatorname{coker} d_1^t \simeq M(g)_{tN-w}$. Hence the E_2-term $E_2(g,0)$ has finitely many nonzero entries and the spectral sequence $E_r(g,0)$ degenerates at E_2 (compare to (1.11)).

The next result gives a large class of singularities having the E_3-term of the spectral sequence $E_r(g,0)$ with finitely many nonzero entries. The reader should have no difficulty in checking that this class contains in particular the next more familiar classes of singularities:

(i) all the non weighted homogeneous $\mathbb{R}$-*unimodal singularities*, see for instance [0], p. 184 for a complete list;

(ii) all the *semi weighted homogenous singularities* (see [7], p. 115 for a definition) of the form $g = g_0 + g'$ with g_0 weighted homogeneous of type $(w_1,\dots,w_n;N)$, g' weighted homogeneous of type $(w_1,\dots,w_n;N')$ and such that

$$N' \geqslant (n+1)N/2 - w_1 - \cdots - w_n.$$

To state the result, note that there is a linear map $d_1^0\colon \ker(g) \to T(g)$ defined by taking $t = 0$ in the formula (3.5).

(3.7) PROPOSITION. *Assume that the singularity $Y\colon g = 0$ satisfies the condition:*

(i) $g^2 = 0$ *in* $M(g)$ *and* $d_1^0 \mid (g) = 0$ (resp. (ii) $\mu(g) - \tau(g) = 1$).

Then $E_2^{n-1-t,t} = \ker d_1^t = (g)$ *for* $t \gg 0$ (*resp.* $\dim\ker \mathrm{d}_1^t = 1$ *for* $t \gg 0$ *and the lines* $\ker d_1^t$ *in* $\ker(g)$ *converge to the line* $\mathbb{C}\text{-}g$ *when* $t \to \infty$) *and the* E_3*-term* $E_3(g, 0)$ *has finitely many nonzero entries.*

Proof. (i) Let $K \subset M(g)$ be a vector subspace which is a complement of the ideal $\mathrm{Ker}\,(g) \subset M(g)$.

Then multiplication by g induces a vector space isomorphism $K \xrightarrow{\sim} gK = (g)$. For t large enough, it is clear using (3.5) that $\ker d_1^t = (g)$ and that the canonical projection $M(g) \to T(g)$ induces an isomorphism $K \simeq \operatorname{coker} d_1^t$.

Via these isomorphisms we may regard d_2^t as an endomorphism of K for $t \gg 0$.

Next $\mathrm{d}_1^t(ag) = 0$ implies that we may write $ag^2\omega_n = \mathrm{d}g \wedge \alpha$ and the $(n-1)$-form α satisfies $\mathrm{d}\alpha = \mathrm{d}g \wedge \beta + \lambda g \omega_n$ for some $(n-1)$-form β and function germ λ. But then we have

$$\mathrm{d}(\alpha/g^t) = (\lambda - t\cdot a)\omega_n/(g^{t-1}) + \mathrm{d}(\beta/(t-1)g^{t-1}) - (\mathrm{d}\beta)/((t-1)g^{t-1}))$$

This shows that the endomorphism d_2^t has a matrix of the form $-t\cdot Id + A + B(t-1)^{-1}$ for A, B some constant matrices. It follows that for $t \gg 0$ this matrix is invertible and this clearly ends the proof. The proof in case (ii) is similar.

Now we come back to our *global setting* and assume *first* that we are in the homogenous case, i.e $w_0 = \cdots = w_n = 1$. Let Z denote the singular locus of V.

Consider the restriction morphism

$$\rho\colon \Omega^{\cdot}_{\mathbf{P}^n}(*V) \to \Omega^{\cdot}_{\mathbf{P}^n}(*V)|_Z \tag{3.8}$$

and the associated morphisms

$$Gr_F^s\rho\colon Gr_F^s(\Omega^{\cdot}_{\mathbf{P}^n}(*V)) \to Gr_F^s(\Omega^{\cdot}_{\mathbf{P}^n}(*V)|_Z).$$

A moment thought shows that $Gr_F^s\rho$ is a quasi-isomorphism for $s < 0$. A computation using an E_2-spectral sequence shows that

$$\mathbf{H}^{\cdot}(Gr_F^s(\Omega^{\cdot}_{\mathbf{P}^n}(*V))) = H^{\cdot}(Gr_F^s A_0^{\cdot}).$$

Assume from now on that Z is a finite set, namely $Z = \{a_1, \dots, a_p\}$. Note that the singularity (V, a_i) is precisely the transversal singularity of f along the line $\mathbb{C}^* \cdot a_i$ as defined in the beginning of this section.

Choose the coordinates on $\mathbf{P}^n$ such that $H: x_0 = 0$ is transversal to V and $Z \subset \mathbf{P}^n \backslash H \simeq \mathbb{C}^n$. We denote again by a_i the corresponding points in $\mathbb{C}^n$ and let $g(y) = f(1, y)$.

Then $\Omega^{\cdot}_{\mathbf{P}^n}(* V)|_Z = \oplus_{j=1,p}\Omega^{\cdot}_{g,a_j}$, this identification being compatible with the F filtrations. Thus we get

$$\mathbf{H}^{\cdot}(Gr^s_F(\Omega^{\cdot}_{\mathbf{P}^n}(* V)|_Z)) = \bigoplus_{j=1,p} H^{\cdot}(Gr^s_F(\Omega^{\cdot}_{g,a_j})).$$

We can restate these considerations in the next form.

(3.9) THEOREM. *The restriction map ρ induces a morphism $\rho_r: E_r(f)_0 \to \oplus_{j=1,p}E_r(g, a_j)$ of spectral sequences such that at the E_1-level $\rho_1^{s,t}$ is an isomorphism for $s < 0$.*

(3.10) COROLLARY. *For a projective hypersurface $V: f = 0$ with isolated singularities the next statements are equivalent*

(i) *all the singularities of V are weighted homogeneous;*
(ii) $E_2^{s,t}(f)_0 = 0$ *for* $s < 0$;
(iii) $E_2^{s,t}(f)_0 \neq 0$ *for finitely many pairs* (s, t).

Proof. Using (3.6) and (3.9) we get (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is obvious. To prove (iii) $\Rightarrow$ (i) we compute the Euler Poincaré characteristic $\chi(U)$ in two ways. First we use the fact that $U = \mathbf{P}^n \backslash V$ and the well-known formula for $\chi(V)$ given in (3.12) below and get

$$\chi(U) = \chi(U_0) + (-1)^{n-1} \sum_{i=1,p} \mu(V, a_i)$$

where U_0 is the complement of a smooth hypersurface V_0 in $\mathbf{P}^n$.

Next using (1.8) and a standard property of spectral sequences we get

$$\chi(U) = 1 + \sum (-1)^{s+t} \dim E_2^{s,t}(f)_0$$

where the sum is finite by our assumption. Choose $m > n$ such that $E_2^{s,t}(f)_0 = 0$ for $t > m$. Then

$$\begin{aligned}\chi(U) - 1 &= (-1)^{n-1} \sum_{t=1,m} (\dim E_2^{n-1-t,t}(f)_0 - \dim E_2^{n-t,t}(f)_0)\\ &= (-1)^{n-1} \sum_{t=1,m} (\dim E_1^{n-1-t,t}(f)_0 - \dim E_1^{n-t,t}(f)_0)\\ &= (-1)^{n-1} [\dim E_1^{n-1-m,m}(f)_0 + \varphi]\end{aligned}$$

with

$$\varphi = \sum_{t=1,m} \dim E_1^{n-t,t-1}(f)_0 - \dim E_1^{n-t,t}(f)_0).$$

By (3.3.iii) and (3.9) it follows that

$$\dim E_1^{n-1-m,m}(f)_0 = \sum_{i=1,p} \tau(V, a_i)$$

where $\tau(V, a_i) = \dim T(g, a_i) = \dim \ker(g, a_i)$ are the corresponding Tjurina numbers. On the other hand, using the connection of $E_1(f)$ with the Koszul complex, it is easy to see that the sum φ does not depend on f. Since one can compute $\chi(U_0)$ in the same way, it follows that

$$\chi(U) = \chi(U_0) + (-1)^{n-1} \sum_{i=1,p} \tau(V, a_i).$$

Comparing the two formulas for $\chi(U)$ we get $\mu(V, a_i) = \tau(V, a_i)$ for any $i = 1, \dots, p$ and hence by *K. Saito's Theorem* (see for instance [7], p. 119 for a discussion) all the singularities (V, a_i) are weighted homogeneous.

Since for any isolated hypersurface singularity $(Y, 0)$ there is a projective hypersurface V having just one singular point a_1 and such that $(V, a_1) \simeq (Y, 0)$, see for instance [2], we get the next result using (3.6), (3.9) and (3.10).

(3.10′) COROLLARY. *For an isolated hypersurface singularity* $(Y, 0)$ *defined by* $g = 0$ *in* $(\mathbb{C}^n, 0)$, *the next statements are equivalent:*

(i′) $(Y, 0)$ *is a weighted homogeneous singularity;*
(ii′) *the spectral sequence* $E_r(g, 0)$ *degenerates at* E_2;
(iii′) $E_2^{s,t}(g, 0) \neq 0$ *for finitely many pairs* (s, t).

We *conjecture* in analogy with (3.10′) that the statements in (3.10) are equivalent to the next stronger version of (ii):

(iv) *the spectral sequence* $E_r(f)_0$ *degenerates at* E_2.

(3.11) REMARK. Let f be a homogenous polynomial such that V has an isolated singularity of the type considered in (3.7). Then $E_r(f)_0$ surely does not degenerate at E_2. Note that $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is *concentrated* in the terminology of [25], p. 206 and our spectral sequence $E_r(f)_0$ is a subobject in the *huge* spectral sequence considered in [25], p. 209. Hence in this case that spectral sequence does not degenerate at E_2 and this gives a negative answer to the question at the top of p. 209 in [25].

By Theorem (2.8) the interesting Betti numbers for V in the isolated singularities case are just $b_{n-1}(V), b_n(V)$ and we can get $b_n(V)$ from $E_{n-1}(f)_0$.

But one has a simple formula for the *Euler-Poincaré characteristic* in this case [6]:

$$\chi(V) = \chi(V_0) + (-1)^n \sum_{i=1,p} \mu(V, a_i) \tag{3.12}$$

where V_0 denotes a smooth hypersurface in $\mathbf{P}^n$ of degree N and $\mu(V, a_i) = \dim M(g, a_i)$ are the corresponding Milnor numbers.

In this way we get $b_{n-1}(V)$ knowing $b_n(V)$. We remark that there is a formula for $\chi(F)$ similarto (3.12) and which appears in the special case $n = 2$ as Theorem 6. A in [9].

(3.13) PROPOSITION. $\chi(F) = 1 + (-1)^n[(N-1)^{n+1} - N \sum_{i=1,p} \mu(V, a_i)]$.

Proof. If $\bar{F}$ denotes the closure of F in $\mathbf{P}^{n+1}$, one has $\chi(F) = \chi(\bar{F})\backslash\chi(V)$. One then use (3.12) and the remark that the singularities of $\bar{F}$ are just the N-fold suspensions of the singularities of V and hence

$$\mu(\bar{F}, (a_i:0)) = (N-1)\mu(V, a_i).$$

(3.14) REMARK. An important invariant of the singularity f is the *zeta function* $Z(h)$ of the monodromy operator h. Explicitly one has

$$Z(h) = \prod_{i \geqslant 0} \det(1 - th^* \mid H^i(F))^{(-1)^{i+1}} = \exp \sum_{k \geqslant 1} \wedge (h^k) t^k/k$$

where $\wedge(h^k)$ denotes the Lefschetz number of the map h^k. Using the second expression above for $Z(h)$ it follows that for any homogeneous polynomial f one has

$$Z(h) = (1 - t^N)^{-\chi(F)/N}.$$

When V has only isolated singularities, this formula may be used to compute $\dim H^n(F)_a$ for $a = 1, \ldots, N-1$ assuming that we know $\dim H^{n-1}(F)_a$ via computations with the spectral sequence $E_r(f)$ as in the remark after (2.8).

Next we describe briefly the *additional facts* necessary in order to treat the case when f has *arbitrary weights* $\mathbf{w} = (w_0, \ldots, w_n)$.

First we have to include a group action in the local setting. Let $G \subset U(n)$ be a finite group and consider the induced action on $\mathbb{C}^n$. Then the ball B_ε and the sphere S_ε are G-invariant subsets. Assume that $Y{:}g = 0$ is a reduced hypesurface singularity which is also G-invariant (i.e. $y \in Y, \gamma \in G \Rightarrow \gamma(y) \in Y$ for a representative Y of $(Y, 0)$ in B_ε).

There is an associated action of G on $\Omega^{\cdot}_{\mathbb{C}^n,0}$ given by $\gamma \cdot \omega = (\gamma^{-1})^* \omega$ And there is character $\chi_Y: G \to \mathbb{C}^*$ such that $\gamma \cdot g = \chi_Y(\gamma) g$ for any $\gamma \in G$. In this situation we call $(Y, 0)$ a G-singularity. Note that this setting is larger than in Wall [27] where one takes $\chi_Y = 1$, but coincides (in the case of G cyclic) to the hyperquotient singularity notion of M. Reid [19].

Let $(\Omega^{\cdot G}_{g,0}, d)$ be the subcomplex in $(\Omega^{\cdot}_{g,0}, d)$ consisting of the fixed elements under the obvious action of G. If $K^{\cdot}$ is any complex of $\mathbb{C}$-vector spaces with G-actions compatible with the differentials, then there is a natural isomorphism $H^{\cdot}(K^{\cdot G}) = H^{\cdot}(K^{\cdot})^G$ which says that taking cohomology commutes with taking the fixed parts under G. Moreover in Proposition (3.1) both cohomology groups have natural G-actions and the isomorphism considered there is compatible with these actions. It follows that

$$H^{\cdot}(\Omega^{\cdot G}_{g,0}) = H^{\cdot}(\Omega^{\cdot}_{g,0})^G = H^{\cdot}(S_\varepsilon \backslash K)^G = H^{\cdot}((S_\varepsilon \backslash K)/G). \tag{3.15}$$

Next, using again the above commutativity, we get an E_1-spectral sequence $(E_r(g, 0)^G, d_r)$ consisting of the fixed parts of the spectral sequence described in (3.2) and converging to $H^{\cdot}((S_\varepsilon \backslash K)/G)$.

Assume now that $(Y, 0)$ is an isolated singularity and note that G acts on the complex $L^{\cdot}$ considered above. Since the G-action commutes with the differentials in $L^{\cdot}$ up-to multiplicative constants, it follows that there is an induced action on the cohomology $H^{\cdot}(L^{\cdot})$. And one has exactly as in Wall [27] an isomorphism of G-vector spaces

$$H^n(L^{\cdot}) = M(g) \otimes \mathbb{C}\omega_{n+1}$$

with $\omega_{n+1} = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$. Let χ_0 be the character of the action of G on $\mathbb{C}\omega_{n+1}$. If W is any G-vector space and $\chi: G \to \mathbb{C}^*$ is a character we set

$$W^\chi = \{w \in W; \gamma \cdot w = \chi(\gamma) w \text{ for all } \gamma \in G\}.$$

With this notation, note that

$$\omega / g^t \in \Omega^{\cdot G}_{g,0} \text{ if and only if } \omega \in \Omega^{\cdot \chi_Y^t}_{\mathbb{C}^n,0}.$$

Combining these remarks we get the next analog of (3.3.iii):

$$E_1^{n-t-1,t}(g, 0)^G \simeq \ker(g)^{\chi_Y^t \chi_0^{-1}}$$

$$E_1^{n-t,t}(g, 0)^G \simeq T(g)^{\chi_Y^t \chi_0^{-1}} \tag{3.16}$$

for all $t \geqslant 2$, where $\ker(g)$ and $T(g)$ have the obviously induced G-actions.

We consider now the global setting. Let $a \in \mathbb{C}^{n+1} \setminus \{0\}$ be a point in the singular locus $f^{-1}(0)_{\text{sing}}$. Let G_a be the *isotropy subgroup* of a with respect to the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by

$$t \cdot x = (t^{-w_0} x_0, \ldots, t^{-w_n} x_n).$$

Then G_a is the finite cyclic group of the unity roots of order

$$k_a = \text{g.c.d.}\{w_j;\ \text{the component } a_j \text{ of a is nonzero}\}.$$

Take H to be a transversal to the orbit $\mathbb{C}^* . a$ at the point a which is G_a-invariant. For instance, we may assume that $a_0 \neq 0$ and then take $H\colon x_0 - a_0 = 0$. We identify the germs $(\mathbb{C}^n, 0)$ and (H, a) via the isomorphism φ given by $(y_1, \ldots, y_n) \mapsto (a_0, y_1, \ldots, y_n)$. Then the transversal singularity $(Y, a) = (H \cap f^{-1}(0), a)$ is in an obvious way a G_a*-singularity* and moreover

$$\chi_Y = N, \quad \chi_0 = w_0 + \cdots + w_n = w$$

under the identification of the (multiplicative) group of the characters of G_a with the (additive) group $\mathbb{Z}/k_a\mathbb{Z}$ (the character $t \mapsto t^m$ corresponds to the class of m modulo $k_a\mathbb{Z}$, denoted again by m!).

Note that the germ $(\mathbf{P}, a)$ (resp. (V, a)) can be identified to $(H/G_a, a)$ (resp. $(Y/G_a, a)$) and hence the latter is a *hyperquotient singularity* in the sense of Reid [19]. It follows that $\Omega^{\bullet}_{\mathbf{P},a} \simeq \Omega^{\bullet G_a}_{\mathbb{C}^n,0}$ and $\Omega^{\bullet}_{\mathbf{P}}(*V)_a \simeq \Omega^{\bullet G_a}_{g_a,0}$ where $g_a(y) = f(a_0, y_1, \ldots, y_n)$ is a local equation for (Y, a), compare with [24], Section 5.

Let $Z \subset V$ be the finite set corresponding to the singular locus $f^{-1}(0)_{\text{sing}}$. Then we have, (with exactly the same proof) the next analog of Theorem (3.9):

(3.17) THEOREM. *The restriction map* $\rho\colon \Omega^{\bullet}_{\mathbf{P}}(*V) \to \Omega^{\bullet}_{\mathbf{P}}(*V)|_Z$ *induces a morphism* $\rho_r\colon E_r(f)_0 \to \oplus_{a \in Z} E_r(g_a, a)^{G_a}$ *of spectral sequences such that at the* E_1*-level* $\rho_1^{s,t}$ *is an isomorphism for* $s < 0$.

As an application we derive now *new formulas* for the Euler characteristics $\chi(V)$ and $\chi(F)$ similar to (3.12), (3.13). Our result should be compared to the more explicit formulas of Siersma [22] (obtained in the very special case when $f^{-1}(0)_{\text{sing}}$ is a complete intersection and all the transversal singularities are of type A_1) and, on the other hand, to the very general formulas of Yomdin [28] (which involve some numerical invariants defined topologically and hence difficult to compute in general concrete cases).

Consider the Poincaré series

$$P(t) = ((1 - t^{N - w_0}) \cdots (1 - t^{N - w_n}))/((1 - t^{w_0}) \cdots (1 - t^{w_n})) = \sum_{k \geqslant 0} c_k(\mathbf{w}, N) t^k$$

associated to the weighted homogenity type $(\mathbf{w}, N)$. Define next the *virtual Euler characteristics of order* m of V and F by the formulas:

$$\chi_m(V(\mathbf{w}, N)) = n + (-1)^{n-1} \sum_{s=1,m} c_{sN-w}(\mathbf{w}, N)$$

$$\chi_m(F(\mathbf{w}, N)) = 1 + (-1)^{n} \sum_{s=1,mN-w} c_s(\mathbf{w}, N) \tag{3.18}$$

where $w = w_0 + \cdots + w_n$.

Note that if there is a weighted homogeneous polynomial of type $(\mathbf{w}, N)$ having an isolated singularity at the origin and if V' (resp. F') denotes the corresponding hypesurface in $\mathbf{P}$ (resp. Milnor fiber) then

$$\chi_m(V(\mathbf{w}, N)) = \chi(V') \quad \text{for} \quad m \geqslant n$$

(resp. $\chi_m(F(\mathbf{w}, N)) = \chi(F')$ for $m \geqslant n+1$). To see this you may find useful to read first the proof of (3.19. ii) below!

(3.19) PROPOSITION. (i) *Assume that a polynomial f' as above exists. Then*

$$\chi(V) = \chi(V') + (-1)^n \sum_{a \in Z} \dim M(g_a)^{-w}$$

$$\chi(F) = \chi(F') + (-1)^{n+1} \sum_{a \in Z} \sum_{j=1,N} \dim M(g_a)^{-N-w+j}.$$

(ii) *Assume that any transversal singularity $g_a = 0$ for $a \in Z$ is either weighted homogeneous or satisfies the assumptions in* (3.7). *Then*

$$\chi(V) = \chi_m(V(\mathbf{w}, N)) + (-1)^n \sum_{a \in Z} \dim M(g_a)^{mN-w}$$

$$\chi(F) = \chi_m(F(\mathbf{w}, N)) + (-1)^{n+1} \sum_{a \in Z} \sum_{j=1,N} \dim M(g_a)^{(m-1)N-w+j}$$

for all m large enough. When all the singularities g_a are weighted homogeneous, it is enough to take $m \geqslant n+1$.

Proof. On a *formal level*, note that the formulas in (i) are a special case of the formulas in (ii), obtained by taking m divisible by all $k_a = |G_a|$, $a \in Z$. The proof of (i) is purely topological and independent of our previous results. Let $a, H, \ldots,$ be as above. We may take f' close enough to f such that for all $a \in Z$ the intersection $F_a = B_\varepsilon \cap (f' \circ \varphi)^{-1}(0)$ can be identified to the Milnor fiber of the singularity (Y, a). Note also that F_a is G_a-invariant. Let $B'(a)$ be the image of the small ball B_ε

under the natural projection $(\mathbb{C}^n, 0) \to (\mathbf{P}, a)$. Then there is a homeomorphism $V \backslash \mathring{B}' \simeq V' \backslash \mathring{B}'$ where $B' = \bigcup_{a \in Z} B'(a)$. Moreover $B'(a) \cap V$ is contractible, while $B'(a) \cap V'$ can be identified to F_a/G_a and hence has middle Betti number

$$b_{n-1}(F_a/G_a) = \dim H^{n-1}(F_a)^G = \dim M(g_a)^{-w}$$

as in [27]. Then a Mayer-Vietories argument gives the result for $\chi(V)$. The result for $\chi(F)$ then follows from that for $\chi(V)$ as in the proof of (3.13).

To prove (ii) we use basically the same argument as in the proof of (3.10) (if necessary starting the computations with the E_3-term of the spectral sequence $E_r(f)_0$) together with (3.17) and (3.16). First we express the sum φ from the proof of (3.10) in terms of the Poincaré series $P(t)$. Following Siersma [22], we define a *new grading* on $\Omega^{\cdot}$ by setting for a homogeneous p-form $\omega \in \Omega^p$:

$$\deg \omega = (n - p + 1)N - w + |\omega|$$

where $|\omega|$ denote the degree of ω as defined in our introduction. Then multiplication by $\mathrm{d}f$ becomes a map of degree 0 and one has

$$P(t) = \sum_{k=0,n+1} (-1)^{n+1-k} P(\Omega^k)(t)$$

where $P(\Omega^k)$ is the Poincaré series of Ω^k with respect to this new grading [22]. Then it is obvious that

$$\varphi = - \sum_{s=1,m} c_{sN-w}(\mathbf{w}, N).$$

To treat the case of transversal singularities covered by (3.7) one has to use the next isomorphisms of vector spaces, which are clear by the proof of (3.7):

$$E_1^{n-1-m,m}(g,0)^G + E_2^{n-2-m,m+1}(g,0)^G = \ker(g)^{mN-w} + (g)^{(m+1)N-w}$$
$$= \ker(g)^{mN-w} + K^{mN-w} = (\ker(g) + K)^{mN-w} = M(g)^{mN-w}.$$

The case of singularities in (3.7.ii) can be treated similarly.

(3.20) EXAMPLE. The polynomial $f = x_0^{265} + x_0 x_1^{11} + x_0 x_2^8 + x_2 x_3^4$ has degree $N = 265$ with respect to the weights $\mathbf{w} = (1, 24, 33, 58)$. The singular set Z consists of one point, namely $a = (0, -1, 1, 0)$ with transversal singularity g_a of type A_3. The corresponding isotropy group G_a is $\mathbb{Z}/3\mathbb{Z}$ and acts on $M(g_a)$ such that $\dim M(g_a)^i = 1$ for any i. It is konwn that the Poincaré series $P(t)$ is a polynomial in this case, in spite of the fact that there is no isolated singularity f'

of this homogeneity type $(\mathbf{w}, N)$, see [0], p. 201. It follows that

$$\chi_m(F(\mathbf{w}, N)) = 1 - P(1) = -66515 \text{ for } m \gg 0 \text{ and}$$

$$\chi(F) = \chi_m(F(\mathbf{w}, N)) + 265 = -66250.$$

Using a computer to determine the coefficients of $P(t)$, one gets $\chi(V) = 254$.

(3.21) REMARKS. (i) We conjecture that the formulas (3.19.ii) hold for *any* transversal singularities.

(ii) If all the transversal singularities (Y, a) for $a \in Z$ have links which are $\mathbb{Q}$-homology spheres, then the hypersurface V is a $\mathbb{Q}$-homology manifold and hence satisfies the Poincaré duality over $\mathbb{Q}$. In this case $b_n(V) = b_n(\mathbf{P}^{n-1})$ and the remaining interesting Betti number $b_{n-1}(V)$ can be determined from $\chi(V)$ once this Euler characteristic is known.

For *concrete computations* it is useful to use the following general remark. Assume that $f_1, \ldots, f_n$ is a regular sequence in S(this can be always achieved by a linear change of coordinates in the homogeneous case!). Then the Koszul complex $K^\cdot$ (1.9) is quasi-isomorphic to the complex

$$0 \to Q_1(f) \xrightarrow{f_0} Q_1(f) \to 0 \tag{3.22}$$

where $Q_1(f) = S/(f_1, \ldots, f_n)$ and f_0 denotes multiplication by f_0. An indication of the dimensions of $H^{n+1}(K^\cdot)_k \simeq Q(f)_{k-n-1}$ and $H^n(K^\cdot)_k \simeq \ker(f_0)_{k-n}$ can be obtained from the exact sequence

$$0 \to \ker(f_0)_k \to Q_1(f)_k \xrightarrow{f_0} Q_1(f)_{k+N-1} \to Q(f)_{k+N-1} \to 0 \tag{3.23}$$

since the Poincaré series of $Q_1(f)$ is known.

4. Explicit computations

(4.1) EXAMPLE (Computation of $H^1(U)$).

Let $f = f_1^{a_1} \ldots f_k^{a_k}$ be the decomposition of f in distinct irreducible factors. Then it is known that $b_1(U) = b^0_{2n-2}(V) = k - 1$ and it is easy to check that the closed forms

$$\omega_i = (\mathrm{d}f_i)/(f_i) - (N_i/N(\mathrm{d}f)/(f)$$

where $N_i = \deg(f_i), i = 1, \ldots, k$ generate $H^1(U)$ with only one relation: $\sum a_i \omega_i = 0$ Compare to (2.7.ii).

(4.2) EXAMPLE (with isolated singularities for V).

Let $f = xyz(x + y + z)$, $n = 2$. Then V consists of 4 lines in general position in $\mathbf{P}^2$ and its topology is simple to describe. However, the dimensions of the eigenspaces $H^{\bullet}(F)_a$ are more subtle invariants.

First we compute explicit bases for the homogeneous components of $Q(f)$:

$$Q(f)_0 = \langle 1 \rangle, Q(f)_1 = \langle x, y, z \rangle, Q(f)_2 = \langle x^2, y^2, z^2, xy, yz, zx \rangle$$

$$Q(f)_3 = \langle x^3, y^3, z^3, x^2y, y^2z, z^2x, xyz \rangle \quad \text{and}$$

$$Q(f)_k = \langle x^k,\ y^k,\ z^k,\ x^{k-1}y,\ y^{k-1}z,\ z^{k-1}x \rangle \quad \text{for } k \geqslant 4.$$

Then we look for the elements in $H^2(K^{\bullet})$ and define:

$$\omega_{xy} = x(x + 2y + z)\,\mathrm{d}y \wedge \mathrm{d}z + y(2x + y + z)\,\mathrm{d}x \wedge \mathrm{d}z$$

and ω_{yz}, ω_{zx} by cyclic symmetry.

Then $\mathrm{d}f \wedge \omega_{xy} = \mathrm{d}f \wedge \omega_{yz} = \mathrm{d}f \wedge \omega_{zx} = 0$ and these three forms give a basis for $H^2(K^{\bullet})_4$.

The six forms $x\omega_{xy}, y\omega_{xy}, y\omega_{yz}, z\omega_{yz}, z\omega_{zx}, x\omega_{zx}$ generate $H^2(K^{\bullet})_5$ with one relation among them (their sum is trivial).

And the six forms $x^k\omega_{xy}, y^k\omega_{xy}, \ldots$, form a basis for $H^2(K^{\bullet})_{k+4}$ for any $k \geqslant 2$.

It is now easy to compute $d_k^1 : H^2(K^{\bullet})_k \to H^3(K^{\bullet})_k$ and the nontrivial kernels and cokernels are listed below together with $E_2^{0,0}(f)_0$:

$$E_2^{0,0}(f)_0 = E_2^{0,2}(f)_2 = E_2^{0,2}(f)_3 = E_2^{1,1}(f)_1 = \mathbb{C},$$

$$E_2^{0,1}(f)_0 = E_2^{1,1}(f)_0 = \mathbb{C}^3.$$

The computations also show that the spectral sequence degenerates at E_2 and hence we get the complete results. One can restate them by saying that the monodromy operator h^* acts trivially on $H^0(F) = \mathbb{C}, H^1(F) = \mathbb{C}^3$ and its action on $H^2(F) = \mathbb{C}^6$ has characteristic polynomial $(t - 1)^3(t + 1)(t^2 + 1)$.

(4.3) EXAMPLE (with nonisolated singularities for V).

An irreducible cubic surface in $\mathbf{P}^3$ with nonisolated singularities is projectively equivalent to one of the next normal forms [3]

(i) a cone on the nodal cubic curve;
(ii) a cone on the cuspidal cubic curve;
(iii) $S' \colon x^2z + y^2t = 0$;
(iv) $S \colon x^2z + y^3 + xyt = 0$.

The topology of the surfaces (i)–(iii) can be described easier e.g. using [18], so that we concentrate on the last case: $f = x^2z + y^3 + xyt$. The homogeneous components of $Q(f)$ are given by

$$Q(f)_0 = \langle 1 \rangle \quad \text{and} \quad Q(f)_k = \langle z, t \rangle_k + \langle z^{k-1}x, z^{k-1}y, z^{k-2}y^2 \rangle \quad \text{for } k \geqslant 1,$$

where $\langle z, t \rangle_k$ denotes the vector space of all homogeneous polynomials in z, t of degree k. Hence dim $Q(f)_k = k + 4$ for $k \geqslant 2$. Consider now the differential forms:

$$\begin{aligned}
\omega_1 &= x\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,\\
\omega_2 &= x\,\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t + t\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t - 3y\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,\\
\omega_3 &= t\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - 2z\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + x\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t,
\end{aligned}$$

Then some tedious computations show that:

$$\begin{aligned}
H^3(K^{\cdot})_4 &= \langle \omega_1, \omega_2, \omega_3 \rangle,\\
H^3(K^{\cdot})_5 &= \langle z, t \rangle_1 \omega_2 + \langle z, t \rangle_1 \omega_3 + \langle x\omega_2, y\omega_2, y\omega_3 \rangle,\\
H^3(K^{\cdot})_{k+4} &= \langle z, t \rangle_k \omega_2 + \langle z, t \rangle_k \omega_3 + \langle z^{k-1}x, z^{k-1}y, z^{k-2}y^2 \rangle \omega_3,
\end{aligned}$$

for $k \geqslant 2$. This last vector space has dimension $2k + 5$. And similarly one gets

$$\begin{aligned}
&H^2(K^{\cdot})_{k+4} = \langle z, t \rangle_k \omega \quad \text{with}\\
&\omega = (6yz - t^2)\,\mathrm{d}x \wedge \mathrm{d}y - xt\,\mathrm{d}x \wedge \mathrm{d}z - yt\,\mathrm{d}x \wedge \mathrm{d}t - 3xy\,\mathrm{d}y \wedge \mathrm{d}z - 3y^2\,\mathrm{d}y \wedge \mathrm{d}t.
\end{aligned}$$

After these complicated formulas it comes as a surprise that the spectral sequence $E_r(f)$ degenerates at E_2 and the only nonzero terms are $E_\infty^{0,0}(f)_0 = E_\infty^{0,2}(f)_1 = E_\infty^{0,2}(f)_2 = \mathbb{C}$.

It follows that $H^{\cdot}(S) \simeq H^{\cdot}(\mathbf{P}^2)$ and hence S has the same *rational homotopy type* as $\mathbf{P}^2$, according to Berceanu [1], who has proved that a projective complete intersection (with arbitrary singularities) is an intrinsically formal space.

Concerning the Milnor fiber one has $H^0(F) = \mathbb{C}$ with trivial action of $h^*, H^2(F) = \mathbb{C}^2$ with the characteristic polynomial of h^* equal to $t^2 + t + 1$ and $H^1(F) = H^3(F) = 0$.

Our next result is an improvement of Corollary (3.11) in Scherk [20] (to see the connexion between these two results have a look at the exact sequences (1.3) in [20]!).

Let $(Y, 0)$ be an isolated hypersurface singularity given by $g = 0$ in $\mathbb{C}^n$. We define the *μ-constant determinacy order* of $(Y, 0)$ (denote by $\mu-\det(Y, 0)$) to be the smallest integer $s > 0$ such that the family $g^t = g + th$ ($t \in [0, 1]$) is μ-constant for any $h \in (y_1, \ldots, y_n)^s$ with small enough coefficients. Note that μ-det$(Y, 0)$ can be easily computed for large classes of singularities (e.g. weighted homogeneous or Newton nondegenerate singularities) and is always less or equal to the strongly $\mathscr{K}$-determinancy order $O(g)$ see [7], p. 75. In Scherk's notation, one has.

$$S = \min\{j; (y_1, \dots, y_n)^{j+1} \subset (g, g_1, \dots, g_n)\} \leqslant O(g) - 2.$$

(4.4) PROPOSITION. *Let $V \subset \mathbf{P}^n$ be a hypersurface having just one singular point a and such that $N = \deg(V) > \mu\text{-}\det(V, a)$.*

Then $b_{n-1}(V) = b_{n-1}(V_0) - \mu(V, a)$ and $b_n(V) = b_n(V_0)$, where V_0 is a smooth hypersurface in $\mathbf{P}^n$ with $\deg(V_0) = N$.

Proof. Choose the coordinates on $\mathbf{P}^n$ such that $a = (1:0:\cdots:0)$ and $H: x_0 = 0$ is transversal to V. If $f = 0$ is an equation for V, then we set $g(y) = f(1, y_1, \dots, y_n) = g_2(y) + \cdots + g_N(y)$, with g_k a homogeneous polynomial of degree k. Using the assumptions, we can find a continuous family

$$g^t(y) = g_2^t(y) + \cdots + g_N^t(y) \quad \text{for } t \in [0, 1]$$

with the properties:

(i) $g^0 = g, g_k^t = g_k$ for $k < N - 1$;

(ii) For any $t > 0$, the hypersurfaces in $\mathbf{P}^{n-1}$

$$W_i^t \colon g_i^t = 0 \quad \text{for } i = N - 1, N$$

are smooth and intersect transversally;

(iii) g^t is a μ-constant family;

(iv) The projective hypersurfaces V^t with the affine equations $g^t = 0$ have no singularities except a.

According to [6], the cohomology of V^t is determined by a *lattice morphism*

$$\varphi^t \colon L_1^t \hookrightarrow L^t \to \bar{L} = L^t/\operatorname{Rad} L^t$$

where L_1^t (resp. L^t) is the Milnor lattice of the singularity $g^t = 0$ (resp. $g_N^t = 0$). When t varies, these Milnor lattices are constant and hence the morphism φ^t has to be constant too.

Hence $H^\bullet(V) = H^\bullet(V^1)$ and so we can assume from the beginning that g_{N-1}, g_N satisfy the condition (ii).

Let $\varphi\colon L_1 \overset{i}{\hookrightarrow} L \to \bar{L}$ be the lattice morphism in this case. We have to show that $i(L^1) \cap \operatorname{Rad} L = 0$, where i is the embedding of Milnor lattices arising from the small deformation $g^r(y) = g(r \cdot y) \cdot r^{-N}$ $(r \gg 0)$ of the singularity $g_N = 0$, see [6], proof of (1.2).

But we may think of g^r as being a even smaller deformation (of order r^{-2}) of the germ $g' = g_{N-1} \cdot r^{-1} + g_N$, which is a small deformation of g_N. If L' denotes the Milnor lattice of the singularity $g' = 0$, then the inclusion i above factorizes as $L_1 \hookrightarrow L' \overset{j}{\hookrightarrow} L$ and hence it is enough to show that

(v) $j(L') \cap \mathrm{Rad}\, L = 0$.

Now j is related to the cohomology of the hypersurface $V' \subset \mathbf{P}^n$ with the affine equation $g_{N-1} + g_N = 0$. Note that V' has just one singular point too, namely a. By a μ-constant argument as above, we can assume that

$$g_k(y) = y_1^k \cdots + y_n^k \quad \text{for } k = N-1, N.$$

Next (v) is equivalent to $H_0^n(V') = 0$ and we show this using the spectral sequence $E_r(f')$ for $f' = x_0 g_{N-1}(x_1, \ldots, x_n) + g_N(x_1, \ldots, x_n)$. It is enough to show that d_1 is injective. And this follows easily using the fact that a base for $H^n(K)$ is given by the forms $x_n^{a_0} \ldots x_n^{a_n} \cdot \omega$ with $a_i < N-2$ for $i = 1, \ldots, n$ and $\omega = \omega_1 \wedge \cdots \wedge \omega_n$, where the 1-forms

$$\omega_k = x_k \, dx_0 + [(N-1)x_0 + Nx_k] \, dx_k$$

are the obvious solution of the equation

$$df = \sum_{k=1,n} x_k^{N-2} \omega_k.$$

Compare to [22], [25], but note that here the transversal type is not A_1 for $N > 3$.

Note added in proof

The proof of Theorem (2.2) above contains an error on p. 11 lines 7 and 8. It is possible to repair this in some special cases, e.g. when all the singularities of V are isolated and weighted homogeneous. A more general result implying Theorem (2.2) has been proved by the author and P. Deligne (to whom I am very grateful for pointing out the above mentioned error!). For details, see our preprint "Hodge and order of the pole filtrations for singular hypersurfaces".

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