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On the classification of primitive ideals for complex classical Lie algebras, I

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Introduction

Let $\mathfrak{g}$ be a complex classical simple Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. If M is a simple $U(\mathfrak{g})$-module and $I = \mathrm{Ann}(M)$ is the annihilator of M in $U(\mathfrak{g})$, then I is called a primitive ideal. In the case where $\mathfrak{g}$ is of type A_n, the classification of primitive ideals is due to Joseph, [5, 6]. To each element $w \in W$ of the Weyl group there can be attached a primitive ideal I_w (of fixed infinitesimal character, see below for more detail). The classification has been reduced by Duflo [2] to the problem of determining when $I_w = I_{w'}$ for w, $w' \in W$. In this case we have $W = S_{n+1}$ the symmetric group on $n + 1$ letters. Joseph answered this question using the Robinson-Schensted algorithm and obtained a complete invariant of I_w, the Young tableau produced by the Robinson-Schensted algorithm, which we may call $T(w)$, i.e. $I_w \neq I_{w'}$ if and only if $T(w) \neq T(w')$. In [10], Vogan introduced the notion of the generalized τ-invariant of a primitive ideal, and showed that in case A_n it was a complete invariant (see also Jantzen, [4]). The aim of this paper, of which these six sections constitute Part I, is to prove analogous results to those of [5, 6] and [10] about the classification of primitive ideals for $\mathfrak{g}$ of types B_n, C_n, and D_n, and about the generalized τ-invariant of a primitive ideal.

The two main results of Part I are the following. We will prove the existence of an algorithm A for Weyl groups of types B_n, C_n, and D_n, with properties that make it the appropriate generalization of the Robinson-Schensted algorithm used by Joseph in [5, 6]. To an element $w \in W$, A associates a pair of standard domino tableaux (cf. 1.1.9i) $A(w) = (L(w), R(w))$. We will define another algorithm S which, given any standard domino tableau T, produces one, $S(T)$, of special shape (i.e. the corresponding representation of the Weyl group is special in the sense of Lusztig, [8]).

Although this will not be discussed in Part I, these algorithms yield the same

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parameters as used by Barbasch-Vogan in [1] to classify the primitive ideals for types B_n, C_n, and D_n, the domino tableaux of special shape. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Known results imply that it suffices to classify the primitive ideals with infinitesimal character $\lambda \in \mathfrak{h}^*$ for λ anti-dominant, regular and integral. Let ρ be half the sum of the positive roots for some choice of positive system of the roots of $\mathfrak{h}$ in $\mathfrak{g}$. Let I_w be the annihilator of the irreducible highest weight module of highest weight $w\lambda - \rho$. Duflo [2] has shown that every primitive ideal with infinitesimal character λ is of the form I_w for some $w \in W$. In [1], Barbasch-Vogan construct $(L(w), R(w))$ by a quite different method from that of this paper. They embed W in the symmetric group S_{2n}, apply the (ordinary) Robinson-Schensted algorithm which produces two Young tableaux $(T(w), T(w^{-1}))$ and then use a shuffling procedure to produce the domino tableaux $(L(w), R(w))$ from these. Now $L(w)$ does not depend only on I_w. Barbasch-Vogan showed that for every $w \in W$ there exists a unique standard domino tableau U with special shape such that there exists $v \in W$ with $U = L(v)$ and $I_v = I_w$.

Using the results of Part I of this paper, in Part II we will prove Vogan's conjecture [10] in cases B_n and C_n, that the generalized τ-invariant (together with the infinitesimal character) is a complete invariant. We will show that for $w, w' \in W, I_w = I_{w'}$ if and only if $S(L(w)) = S(L(w'))$, thus giving a new proof of the classification in cases B_n and C_n, the first part of which is essentially the same as in [1], and the second part of which replaces their use of asymptotic support and induction from and restriction to Weyl subgroups of smaller rank with the use of the generalized τ-invariant. It will follow from this that, given, $w, v \in W$ as above, we have $U = L(v) = S(L(w))$. We show in Part I that S and A can be run in reverse, so there is a determinate procedure, given w, to find $\{w' \in W: I_{w'} = I_w\}$. Vogan's conjecture is false when $\mathfrak{g}$ is of type D_n, $n \geqslant 6$. In a projected Part III of this paper we intend to give the definition of a generalization of the generalized τ-invariant, the generalized generalized τ-invariant, prove a modification of Vogan's conjecture, and deal with the classification of primitive ideals in that case.

This paper arose out of the project of using the classification of [1] for the problem of determining the annihilators of irreducible admissible Harish-Chandra modules of real Lie groups of classical type, generalizing the results for $U(p, q)$ and $\mathrm{GL}(n, \mathbb{R})$ of [3], which used the results of Joseph [5, 6] and Vogan [10] in the case of type A_n. The properties we prove of the generalized τ-invariant, as well as the better formulation of the algorithm A, and even more, the supplying of the algorithm S are crucial to the determination of the annihilators of irreducible admissible Harish-Chandra modules. This has already been accomplished in certain cases, and we intend to discuss this in a separate paper.

Furthermore, Part I of this paper is written in such a way that it will apply, as will be shown in Part II, to the theory of cells in classical Weyl groups [7], [8]. Shi [9] has generalized the Robinson-Schensted algorithm to the affine Weyl groups of type $\tilde{A}_n$, and one can expect that the results of Part I of this paper can be used in the same way for work on the affine Weyl groups of types $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$.

In more detail, this paper (that is, Part I) is organized as follows: there are six sections. In Section 1 we define various parameter sets which will be in constant use. We define domino tableaux in 1.1.8. Then we introduce some preliminaries for the definition of the domino analogue of the Robinson-Schensted algorithm (and its inverse). In Section 2 we define this algorithm as a map A defined on the parameter set $\mathscr{S}(M_1, M_2)$ (which is isomorphic to the Weyl group when $M_1 = M_2 = \{1, \ldots, n\}$). We give two definitions of A, (1.2.1) and (1.2.7). The first definition is useful in showing the relation between L and R: $L(w) = R(w^{-1})$ (1.2.3). The second definition is defined by means of an important map α (1.2.5). This definition will be used in all later proofs, and is most convenient for describing A as an algorithm, as will be illustrated in Sections 3 and 4. In (1.2.8) we prove the two definitions are equivalent. We show that A has an inverse B, and so is a bijection between W and a set of pairs of domino tableaux. In the third section we prove certain properties of the maps α and β which are useful for computing A and B. In Section 4 we illustrate these properties with examples which show in practice how to calculate A and B. The second definition of the algorithm A can be thought of as building up domino tableaux from the parameter of w starting from scratch and adjoining dominoes one by one. As in the (ordinary) Robinson-Schensted algorithm, the adjoining of each new domino is accompanied by an alteration in the positions of some of the preceding dominoes. This combined step is called α. It is repeated until standard domino tableaux of the right order are obtained.

In Section 5 we recall (a reformulation of) Lusztig's notion of special: we define the notion of a domino tableau's having special shape. We then give the algorithm S, which transforms a domino tableau T to a domino tableau with special shape. In order to do this, we define the concept of a cycle of a domino tableau. Then T is transformed into $S(T)$ by means of operations which we call "moving T through a cycle." Note that this is a quite different sort than the operations which define A (or B). The last, sixth, section illustrates S by examples.

Please note that the symbol $\setminus$ denotes set-theoretic difference.

Section 1

In this section we introduce the definitions of the parameter sets we will be using, preliminaries for the definition of the algorithm A (defined as a map on W) and its inverse, B.

(1.1.1) NOTATION. Let W_n be the Weyl group of a complex simple Lie algebra $\mathfrak{g}$ of type C_n. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{h}^*$

such that if $\alpha_1 = 2e_1, \alpha_i = e_i - e_{i-1}$ for $2 \leqslant i \leqslant n$ then $\pi = \{\alpha_1, \ldots, \alpha_n\}$ are the simple roots for a choice of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$. With respect to this basis an element $w \in W_n$ satisfies $w(e_i) = \varepsilon_i e_{\sigma(i)}$ for some $\sigma \in S_n, \varepsilon_i \in \{\pm 1\}$.

Let $W'_n \subseteq W_n$ be defined as follows: if $w \in W_n$, $w(e_i) = \varepsilon_i e_{\sigma(i)}$ then $w \in W'_n \Leftrightarrow |\{i \mid \varepsilon_i = -1\}|$ is even. W'_n is the Weyl group of a complex simple Lie algebra of type D_n.

(1.1.2) DEFINITION. Let $M_1, M_2 \subset \mathbb{N}^*$ be finite subsets with $|M_1| = |M_2|$. We define $\mathscr{S}(M_1, M_2)$ as follows: let q_1, q_2 be the projections of $M_1 \times M_2 \times (\pm 1\}$ onto the first and second factors. Then $\mathscr{S}(M_1, M_2)$ is the set of all $w \subset M_1 \times M_2 \times \{\pm 1\}$ such that $q_1|_w$ and $q_2|_w$ are bijections onto M_1 and M_2, respectively. If $n \in \mathbb{N}^*$ we write $\mathscr{S}(n, n)$ for $\mathscr{S}(\{1, \ldots, n\}, \{1, \ldots, n\})$.

EXAMPLE. Let $M_1 = \{1, 3, 5\}, M_2 = \{2, 3, 7\}$. Let $w = \{(1, 3, -1), (3, 7, 1), (5, 2, -1)\}$. Then $w \in \mathscr{S}(M_1, M_2)$.

(1.1.3) DEFINITION. Let $N = \{1, \ldots, n\}$. We define $\delta \colon W_n \to \mathscr{S}(n, n)$ by $\delta(w) = \{(k, \sigma(k), \varepsilon_k)\}$ where $w(e_k) = \varepsilon_k e_{\sigma(k)}$. Then δ is a bijection.

(1.1.4) DEFINITION. For $w \in \mathscr{S}(M_1, M_2)$, $m = |M_1|$, let $e = \sup M_1$, $u = \sup M_2$. Then $\{(e, f, \varepsilon_e)\} \in w$ and $\{(v, u, \varepsilon_v)\} \in w$ for some $f \in M_2, v \in M_1, \varepsilon_e, \varepsilon_v \in \{\pm 1\}$. We define ${}_m w = w \backslash \{(e, f, \varepsilon_e)\}$ and $w_m = w \backslash \{(v, u, \varepsilon_v)\}$. We will write, for example, $w_{m,m-1,m-2}$ for $((w_m)_{m-1})_{m-2}$.

EXAMPLE. Let w be as in the example in (1.1.2). Then ${}_3 w = \{(1, 3, -1), (3, 7, 1)\}$, $w_3 = \{(1, 3, -1), (5, 2, -1)\}, w_{3,2} = \{(5, 2, -1)\}$, and ${}_2(w_3) = {}_2 w_3 = \{(1, 3, -1)\}$.

(1.1.5) DEFINITION. For $w \in \mathscr{S}(M_1, M_2)$, $w = \{(l_i, r_i, \varepsilon_i)\}$, we define $w^{-1} \in \mathscr{S}(M_2, M_1)$ by $w^{-1} = \{(r_i, l_i, \varepsilon_i)\}$. Note that $y \in W_n, \delta(y^{-1}) = \delta(y)^{-1}$.

(1.1.6) REMARK. Let s_i be the simple reflection corresponding to the simple root α_i. Let $w \in W_n$, $\delta(w) = \{(i, \sigma(i), \varepsilon_i)\}_{1 \leqslant i \leqslant n}$. Then

(a) $\delta(ws_1)$ (resp. $\delta(s_1 w)$) is obtained from $\delta(w)$ by multiplying ε_1 (resp. ε_k where $k = \sigma^{-1}(1)$) by -1.

(b) For $i \geqslant 2$, $\delta(ws_i)$ (resp. $\delta(s_i w)$) is obtained from $\delta(w)$ by interchanging i and $i - 1$ in the first (resp. second) position of the triples.

(c) Let w_0 be the long element of W_n. Then $\delta(ww_0) = \delta(w_0 w) = \{(i, \sigma(i), -\varepsilon_i)\}$.

EXAMPLE. Consider $\delta \colon W_3 \to \mathscr{S}(3, 3)$. (Let $e \in W_3$ be the identity element.) Then

$$\delta(e) = \{(1, 1, 1), (2, 2, 1), (3, 3, 1)\}, \quad \delta(s_1) = \{(1, 1, -1), (2, 2, 1), (3, 3, 1)\},$$

$$\delta(s_3) = \{(1, 1, 1), (2, 3, 1), (3, 2, 1)\}, \quad \delta(s_2 s_3) = \{(1, 2, 1), (2, 3, 1), (3, 1, 1)\},$$

and

$$\delta(s_2 s_1 s_3 s_2 s_1) = \{(1, 3, -1), (2, 2, -1), (3, 1, 1)\}.$$

(1.1.7) DEFINITION. (1) Let $\mathscr{F} = \{S_{ij}\}_{i\geq 1, j\geq 1}$, where the S_{ij} are symbols and $S_{ij} = S_{kl} \Leftrightarrow i = k$ and $j = l$. Similarly let $\mathscr{F}^0 = \{S_{ij}\}_{i\geq 0, j\geq 0}$; note that $\mathscr{F} \subset \mathscr{F}^0$. The elements of $\mathscr{F}$ (but not of $\mathscr{F}^0 \backslash \mathscr{F}$) are called squares.

(2) Let $J \subset \mathscr{F}$. We say J satisfies condition Y if J is a finite subset and if for every $S_{ij} \in \mathscr{F}$ such that $S_{ij} \notin J$ we have $S_{i,j+1} \notin J$ and $S_{i+1,j} \notin J$.

(3) If $J \subset \mathscr{F}$ satisfies condition Y let $\rho_i(J) = 0$ if $S_{i1} \notin J$, otherwise $\rho_i(J) = \sup\{j \mid S_{ij} \in J\}$; similarly, let $\kappa_j(J) = 0$ if $S_{1j} \notin J$, otherwise $\kappa_j(J) = \sup\{i \mid S_{ij} \in J\}$.

(1.1.8) DEFINITION. Let $M \subset \mathbb{N}^*$ be a finite subset. We define two sets, $\mathscr{T}(M)$ and $\mathscr{T}^0(M)$, whose elements are called domino tableaux, as follows: let p_1 and p_2 be the projections from $\mathbb{N} \times \mathscr{F}$ onto the first and second factors.

$\mathscr{T}(M)$ is the set of all $T \subset M \times \mathscr{F}$ satisfying:

(1) $p_2|_T$ is injective and $p_2(T)$ satisfies condition Y.

(2) $p_1|_T$ is two to one.

(3) If $k \in M$ then for some $S_{ij} \in \mathscr{F}$ we have either $(k, S_{ij}) \in T$ and $(k, S_{i,j+1}) \in T$ or $(k, S_{ij}) \in T$ and $(k, S_{i+1,j}) \in T$.

(4) Suppose $(k, S_{ij}) \in T$. If $(k_1, S_{i,j+1}) \in T$ (resp. $(k_2, S_{i+1,j}) \in T$) then $k_1 \geq k$ (resp. $k_2 \geq k$).

$\mathscr{T}^0(M)$ is the set of all $T \subseteq (M \cup \{0\}) \times \mathscr{F}$ satisfying (1), (3), (4) as above, and

(2′) $(0, S_{11}) \in T$, $(0, S_{ij}) \notin T$ if $S_{ij} \neq S_{11}$, and $p_1|_{T \cap (M \times \mathscr{F})}$ is two to one.

EXAMPLE. See (1.4.1).

(1.1.9). NOTATION (a) For T a domino tableau let Shape$(T) = p_2(T)$.

(b) For T a domino tableau, set $\rho_i(T) = \rho_i(\text{Shape}(T))$, $\kappa_i(T) = \kappa_i(\text{Shape}(T))$.

(c) For T a domino tableau and $S \in \mathscr{F}$ we say S is filled in T if $S \in \text{Shape}(T)$, otherwise S is empty in T.

(d) For T a domino tableau define $\mathscr{N}(T) \subset \mathbb{N}^*$ by $\mathscr{N}(T) = M \Leftrightarrow T \in \mathscr{T}(M)$ or $\mathscr{T}^0(M)$.

(e) For $T \in \mathscr{T}(M)$, or $T \in \mathscr{T}^0(M)$, $k \in M$, let $D(k, T) = T \cap (\{k\} \times \mathscr{F})$.

(f) For $T \in \mathscr{T}(M)$ or $T \in \mathscr{T}^0(M)$, $k \in M$, let $P(k, T) = \{S_{ij} | (k, S_{ij}) \in T\}$.

(g) Let T be a domino tableau. Define $N_T : \mathscr{F}^0 \to \mathbb{N} \cup \{\infty\}$ by

$$N_T(S_{ij}) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ k & \text{if } (k, S_{ij}) \in T \\ \infty & \text{if } S_{ij} \in \mathscr{F} \backslash \text{Shape}(T). \end{cases}$$

(h) For $M_1, M_2 \subset \mathbb{N}^*$ finite subsets with $|M_1| = |M_2|$ let $\mathscr{T}(M_1, M_2) = \{(T_1, T_2) \mid T_i \in \mathscr{T}(M_i)$ for $i = 1, 2$ and Shape$(T_1) =$ Shape$(T_2)\}$. Similarly define $\mathscr{T}^\circ(M_1, M_2)$.

(i) A domino tableau T is called standard, of order n, if $\mathscr{N}(T) = \{1, \ldots, n\}$.

(j) If $n \in \mathbb{N}^*$ we write $\mathscr{T}(n)$ for $\mathscr{T}(\{1, \ldots, n\})$; similarly $\mathscr{T}^0(n)$, $\mathscr{T}(n, n)$, and $\mathscr{T}^0(n, n)$.

(1.1.10) PROPOSITION. *Let* $T \in \mathcal{T}(M)$ (resp. $T \in \mathcal{T}^0(M)$), *and let* $e = \sup M$. *Then* $T \backslash D(e, T) \in \mathcal{T}(M \backslash \{e\})$ (resp. $T \backslash D(e, T) \in \mathcal{T}^0(M \backslash \{e\})$), *and* $\text{Shape}(T \backslash D(e, T)) = \text{Shape}(T) \backslash P(e, T)$.

We now introduce some preliminaries for the definition of A.

(1.1.11) DEFINITION. Let $J \subset \mathcal{F}$ satisfy condition Y. Let $P = \{S_{ij}, S_{i,j+1}\}$ (resp. $\{S_{ij}, S_{i+1,j}\}$). We say P is an extremal position in J if $\rho_i(J) = j + 1$ and $\rho_{i+1}(J) \leqslant j - 1$ (resp. $\kappa_j(J) = i + 1$ and $\kappa_{j+1}(J) \leqslant i - 1$). If T is a domino tableau we say P is an extremal position in T if P is an extremal position in Shape(T).

REMARK. If $T \in \mathcal{T}(M)$ or $T \in \mathcal{T}^0(M)$ and $e = \sup M$ then $P(e, T)$ is an extremal position in T. Note, however, that P an extremal position in T does not imply that $P = P(k, T)$ for some $k \in M$. For example, if T is as in (1.4.1) then $\{S_{31}, S_{32}\}$ is an extremal position in T.

(1.1.12) DEFINITION. Let $T \in \mathcal{T}(M)$ or $T \in \mathcal{T}^0(M)$, $e \in \mathbb{N}^*$, and $P = \{S_{ij}, S_{i,j+1}\}$ or $P = \{S_{ij}, S_{i+1,j}\}$ for some (i, j). We say the pair (e, P) is adjoinable to T whenever the following hold:
(1) $e > \sup M$
(2) $\text{Shape}(T) \cap P = \emptyset$ and $\text{Shape}(T) \cup P$ satisfies condition Y.
Then P is an extremal position in $\text{Shape}(T) \cup P$.

(1.1.13) DEFINITION. Let T' be a domino tableau and suppose (e, P) is adjoinable to T'. Then let $\text{Adj}(T', P, e) = T' \cup \{(e, S_{ij}) \mid S_{ij} \in P\}$.

(1.1.14) PROPOSITION. Let T', e, P *be as in Definition* (1.1.13). *Then* $\text{Adj}(T', P, e)$ *is a domino tableau with* $\text{Shape}(T) = \text{Shape}(T') \cup P$.

REMARK. (1) If $T = \text{Adj}(T', P, e)$ then $T \backslash D(e, T) = T'$ and $P = P(e, T)$.
(2) If $T \in \mathcal{T}(M)$ or $\mathcal{T}^0(M)$ with $e = \sup M$ then $(e, P(e, T))$ is adjoinable to $T \backslash D(e, T)$ and $\text{Adj}(T \backslash D(e, T), P(e, T), e) = T$.

(1.1.15) DEFINITION. Let $J \subset \mathcal{F}$ satisfy condition Y, and let $P_1, P_2 \subset \mathcal{F}$. We say (P_1, P_2) is an adjoinable pair to J if for $i = 1, 2$, $P_i \cap J = \emptyset$, $P_i \cup J$ satisfies condition Y, and P_i is an extremal position in $P_i \cup J$.

(1.1.16) PROPOSITION. *Let* (P_1, P_2) *be an adjoinable pair to* J. *Then either*
(i) $P_1 \cap P_2 = \emptyset$,
(ii) $P_1 \cap P_2 = \{S_{ij}\}$ *where* $j = \rho_i(J) + 1, i = \kappa_j(J) + 1$, *or*
(iii) $P_1 = P_2$.

(1.1.17) DEFINITION. Let (P_1, P_2) be an adjoinable pair to J. We define $P_1^A(J, P_1, P_2)$ and $P_2^A(J, P_1, P_2)$ according to the three cases of Proposition (1.1.16), as follows.

In case (i), let $P_1^A(J, P_1, P_2) = P_1, P_2^A(J, P_1, P_2) = P_2$.

In case (ii) let $P_k^A(J, P_1, P_2) = (P_k \cup \{S_{i+1,j+1}\}) \backslash \{S_{ij}\}$ for $k = 1, 2$.

In case (iii), suppose $P_1 = P_2 = \{S_{ij}, S_{i,j+1}\}$ (resp. $\{S_{ij}, S_{i+1,j}\}$.) Then let

$$P_1^A(J, P_1, P_2) = P_2^A(J, P_1, P_2) = \{S_{i+1,r}, S_{i+1,r+1}\} \text{ (resp. } \{S_{r,j+1}, S_{r+1,j+1}\})$$

where

$$r = \rho_{i+1}(J) + 1 \text{ (resp. } r = \kappa_{j+1}(J) + 1).$$

We then define

$$J^A(J, P_1, P_2) = J \cup P_1 \cup P_2^A(J, P_1, P_2) = J \cup P_2 \cup P_1^A(J, P_1, P_2),$$

all unions being disjoint.

REMARK. Suppose (P, Q) is an adjoinable pair to J. Then (Q, P) is an adjoinable pair to J and $P_1^A(J, Q, P) = P_2^A(J, P, Q)$.

EXAMPLE. Let T, T' be as in (1.4.2). Let $r \geqslant 3$. Let $J = \text{Shape}(T(r))$ (see 1.3.1), let $P_1 = P(r+1, T')$, and let $P_2 = \text{Shape}(T(r))\backslash\text{Shape}(T'(r))$. Then (P_1, P_2) is an adjoinable pair to J. We have $P_1^A(J, P_1, P_2) = P(r+1, T)$,

$$P_2^A(J, P_1, P_2) = \text{Shape}(T(r+1))\backslash\text{Shape}(T'(r+1)),$$

and

$$J^A(J, P_1, P_2) = \text{Shape}(T(r+1)).$$

(1.1.18) DEFINITION. Let $J \subset \mathscr{F}$ satisfy condition Y, and let $P_1, P_2 \subset \mathscr{F}$. We say (P_1, P_2) is a removable pair for J if P_1 and P_2 are extremal positions in J. If $P_1 = P_2 = \{S_{1r}, S_{1,r+1}\}$ or if $P_1 = P_2 = \{S_{r1}, S_{r+1,1}\}$ for some r we say (P_1, P_2) is a minimal removable pair for J; otherwise (P_1, P_2) is a standard removable pair for J.

(1.1.19) PROPOSITION. *Let (P_1, P_2) be a removable pair for J. Then either*

(i) $P_1 \cap P_2 = \emptyset$,

(ii) $P_1 \cap P_2 = \{S_{ij}\}$ *where* $i \geqslant 2, j \geqslant 2$, *and* $i = \kappa_j(J), j = \rho_i(J)$, *or*

(iii) $P_1 = P_2$.

(1.1.20) DEFINITION. Let (P_1, P_2) be a standard removable pair for J. We define $P_1^R(J, P_1, P_2)$ and $P_2^R(J, P_1, P_2)$ according to the three cases of Proposition (1.1.19) as follows.

In case (i), let $P_1^R(J, P_1, P_2) = P_1, P_2^R(J, P_1, P_2) = P_2$.

In case (ii), let $P_k^R(J, P_1, P_2) = (P_k \cup \{S_{i-1,j-1}\})\backslash\{S_{ij}\}$ for $k = 1, 2$.

In case (iii) suppose $P_1 = P_2 = \{S_{ij}, S_{i,j+1}\}$ (resp. $\{S_{ij}, S_{i+1,j}\}$). Then let

$$P_1^R(J, P_1, P_2) = P_2^R(J, P_1, P_2) = \{S_{i-1,r-1}, S_{i-1,r}\} \text{ (resp. } \{S_{r-1,j-1}, S_{r,j}\})$$

where

$$r = \rho_{i-1}(J) \quad (\text{resp. } r = \kappa_{j-1}(J)).$$

We then define

$$J^R(J, P_1, P_2) = J \backslash (P_1^R(J, P_1, P_2) \cup P_2) = J \backslash (P_1 \cup P_2^R(J, P_1, P_2)).$$

Here the unions are disjoint and

$$P_1^R(J, P_1, P_2) \cup P_2 \subseteq J, P_1 \cup P_2^R(J, P_1, P_2) \subseteq J.$$

(1.1.21) PROPOSITION. (a) *Let* (Q_1, Q_2) *be an adjoinable pair to* I. *Let*

$$P_1 = P_1^A(I, Q_1, Q_2), \qquad P_2 = P_2^A(I, Q_1, Q_2), \qquad J = J^A(I, Q_1, Q_2).$$

Then (P_1, P_2) *is a standard removable pair for* J *and*

$$P_1^R(J, P_1, P_2) = Q_1, \qquad P_2^R(J, P_1, P_2) = Q_2, \qquad J^R(J, P_1, P_2) = I.$$

(b) *Let* (Q_1, Q_2) *be a standard removable pair for* I. *Let*

$$P_1 = P_1^R(I, Q_1, Q_2), \qquad P_2 = P_2^R(I, Q_1, Q_2), \qquad J = J^R(I, Q_1, Q_2).$$

Then (P_1, P_2) *is an adjoinable pair to* J *and*

$$P_1^A(I, P_1, P_2) = Q_1, \qquad P_2^A(I, P_1, P_2) = Q_2, \qquad J^A(J, P_1, P_2) = I.$$

(1.1.22) PROPOSITION. *Let* (Q_1, Q_2) *be an adjoinable pair to* I, *and let* $P_i = P_i^A(I, Q_1, Q_2)$, $i = 1, 2$, *and* $J = J^A(I, Q_1, Q_2)$. *Suppose further that we have* $T \in \mathscr{T}(M)$ *or* $T \in \mathscr{T}^0(M)$, $e \in \mathbb{N}^*$ *with* $e = \sup M$, *and* $\text{Shape}(T) = I \cup Q_1$ (*resp.* $\text{Shape}(T) = I \cup Q_2$). *Then* (e, P_2) (*resp.* (e, P_1)) *is adjoinable to* T *and* $\text{Shape}(\text{Adj}(T, P_2, e)) = J$ (*resp.* $\text{Shape}(\text{Adj}(T, P_1, e)) = J$).

Section 2

In this section we will give two definitions of A and show they are equivalent. The first definition, (1.2.1), is useful for showing the symmetry between left and right, (1.2.3). The second Definition, (1.2.7), is more useful for computations. We will show in (1.2.9) that A has an inverse, B.

(1.2.1) DEFINITION. For each $M_1, M_2 \subset \mathbb{N}^*$ with $|M_1| = |M_2| < \infty$ we define a map $A = A(M_1, M_2)$, $A: \mathscr{S}(M_1, M_2) \to \mathscr{T}(M_1, M_2)$. For $w \in \mathscr{S}(M_1, M_2)$

we write $A(w) = (L(w), R(w))$. If $|M_1| = 0$ we define $A(M_1, M_2)$ to be the unique map between the one-element sets $\mathscr{S}(\emptyset, \emptyset) = \{\emptyset\}$ and $\mathscr{T}(\emptyset, \emptyset) = \{(\emptyset, \emptyset)\}$. For $|M_1| \geqslant 1$, to define $A(M_1, M_2)$ we assume by induction that $A(M'_1, M'_2)$ is defined whenever $|M'_1| < |M_1|$, and that we have available Proposition (1.2.2) when $1 \leqslant |M'_1| < |M_1|$. Let $m = |M_1|$, $e = \sup M_1$, $u = \sup M_2$. There are two cases.

Case 1. $w_m = {}_m w$. Let $\varepsilon \in \{\pm 1\}$ be such that $w \backslash w_m = \{(e, u, \varepsilon)\}$. Then we define $L(w) = \mathrm{Adj}(L({}_m w), P, e)$ and $R(w) = \mathrm{Adj}(R(w_m), P, u)$, where $P = \{S_{1r}, S_{1,r+1}\}$ with $r = \rho_1(L({}_m w)) + 1$ when $\varepsilon = 1$ and $P = \{S_{r1}, S_{r+1,1}\}$ with $r = \kappa_1(R(w_m)) + 1$ when $\varepsilon = -1$.

Case 2. $w_m \neq {}_m w$. Let $\bar{w} = {}_{m-1}(w_m) = ({}_m w)_{m-1}$. Note that since ${}_m w \neq w_m$ we have $L(w_m) \in \mathscr{T}(M_1 \backslash \{f\})$ for some $f < e$ and $R({}_m w) \in \mathscr{T}(M_2 \backslash \{v\})$ for some $v < u$. Let $Q_1 = P(e, L(w_m))$, $Q_2 = P(u, R({}_m w))$. Let $I = \mathrm{Shape}(L(\bar{w})) = \mathrm{Shape}(R(\bar{w}))$. By Proposition (1.2.2) applied to w_m and to ${}_m w$, (Q_1, Q_2) is an adjoinable pair to I. We now define $L(w) = \mathrm{Adj}(L({}_m w), P^A_1(I, Q_1, Q_2), e)$ and $R(w) = \mathrm{Adj}(R(w_m), P^A_2(I, Q_1, Q_2), u)$. (By Proposition (1.1.22) this is possible, and $\mathrm{Shape}(L(w)) = \mathrm{Shape}(R(w)) = J^A(I, Q_1, Q_2)$ so $(L(w), R(w)) \in \mathscr{T}(M_1, M_2)$.)

(1.2.2) PROPOSITION. *Let* $w \in \mathscr{S}(M_1, M_2)$, $m = |M_1| \geqslant 1$, $e = \sup M_1$, $u = \sup M_2$. *Then* $L({}_m w) = L(w) \backslash D(e, L(w))$ *and* $R(w_m) = R(w) \backslash D(u, R(w))$.

Proof. This is clear from the definition. □

We now prove the relation between L and R.

(1.2.3) PROPOSITION. *Let* $w \in \mathscr{S}(M_1, M_2)$. *Then* $A(w^{-1}) = (R(w), L(w))$.

Proof. We assume by induction that the proposition is true for $y \in \mathscr{S}(M'_1, M'_2)$ whenever $|M'_1| < |M_1|$, the case $|M_1| = 0$ being obvious. Let $m = |M_1|$, $e = \sup M_1$, $u = \sup M_2$.

Note that $(w_m)^{-1} = {}_m(w^{-1})$ and $({}_m w)^{-1} = (w^{-1})_m$. It follows that w^{-1} satisfies the hypothesis of case 1 of Definition (1.2.1) if and only if w satisfies the hypothesis of case 1 of Definition (1.2.1).

Assume first that $w_m = {}_m w$, that is, w and w^{-1} satisfy the hypothesis of case 1 of Definition (1.2.1). Let $w \backslash w_m = \{(e, u, \varepsilon)\}$. Then $w^{-1} \backslash (w^{-1})_m = \{(u, e, \varepsilon)\}$ and the proposition is clearly true in this case.

Assume now that $w_m \neq {}_m w$, that is w and w^{-1} satisfy the hypothesis of case 2 of Definition (1.2.1). Let

$$\bar{w} = {}_{m-1}(w_m) = ({}_m w)_{m-1}, \quad \text{and} \quad \overline{w^{-1}} = {}_{m-1}((w^{-1})_m) = ({}_m(w^{-1}))_{m-1}.$$

Then $\overline{w^{-1}} = (\bar{w})^{-1}$. By induction we have

$$(L(\overline{w^{-1}}), R(\overline{w^{-1}})) = A(\overline{w^{-1}}) = A((\bar{w})^{-1}) = (R(\bar{w}), L(\bar{w})),$$
$$(L(w^{-1})_m), R((w^{-1})_m) = A((w^{-1})_m) = A(({}_m w)^{-1}) = (R({}_m w), L({}_m w))$$

and

$$(L({}_m(w^{-1})), R({}_m(w^{-1}))) = A({}_m(w^{-1})) = A((w_m)^{-1}) = (R(w_m), L(w_m)).$$

Let

$$I = \mathrm{Shape}(L(\bar{w})), \qquad Q_1 = P(e, L(w_m)), \qquad Q_2 = P(u, R({}_m w)),$$

and let

$$I' = \mathrm{Shape}(L(\overline{w^{-1}})), \qquad Q_1' = P(u, L((w^{-1})_m)), \qquad Q_2' = P(e, R({}_m(w^{-1}))).$$

Then $I = I', Q_1 = Q_2', Q_2 = Q_1'$. By Remark (1.1.17) $P_1^A(I', Q_1', Q_2') = P_2^A(I, Q_1, Q_2)$. Thus

$$L(w^{-1}) = \mathrm{Adj}(L({}_m(w^{-1})), P_1^A(I', Q_1', Q_2'), u) = \mathrm{Adj}(R(w_m), P_2^A(I, Q_1, Q_2), u) = R(w),$$

and similarly $R(w^{-1}) = L(w)$. □

We now introduce some preliminaries for the second definition of the algorithm A. This definition makes use of the map α, defined in (1.2.5), mentioned in the introduction.

(1.2.4) DEFINITION. Let $M \subset \mathbb{N}^*$, $|M| < \infty$.
(a) Let $\mathscr{C}(M) = \{(T, v, \varepsilon) \mid v \in M,\ T \in \mathscr{T}(M \backslash \{v\}),\ \varepsilon \in \{\pm 1\}\}$.
(b) Let $\mathscr{D}(M) = \{(T, P) \mid T \in \mathscr{T}(M)$ and P is an extremal position in $T\}$.

(1.2.5) DEFINITION. Let $M \subset \mathbb{N}^*$, $|M| < \infty$. We define a map $\alpha = \alpha(M)$, $\alpha(M): \mathscr{C}(M) \to \mathscr{D}(M)$. To define α, if $|M| \geqslant 2$ we assume by induction that $\alpha(M')$ is defined for all M' with $1 \leqslant |M'| < |M|$ and that we have available Proposition (1.2.6) for this situation (for $|M| = 1$ we are in case 1 of this definition, and this case does not require induction.) Let $e = \sup M$ and suppose $(T', v, \varepsilon) \in \mathscr{C}(M)$. There are two cases.

Case 1. $v = e$. If $\varepsilon = 1$ let $P = \{S_{1,r}, S_{1,r+1}\}$ where $r = \rho_1(T') + 1$, if $\varepsilon = -1$ let $P = \{S_{r,1}, S_{r+1,1}\}$ where $r = \kappa_1(T') + 1$. Let $T = \mathrm{Adj}(T', P, e)$. Then we define $\alpha((T', v, \varepsilon)) = (T, P)$.

Case 2. $v \neq e$. Let $(T'', P') = \alpha((T' \backslash D(e, T'), v, \varepsilon))$. Let $Q_1 = P(e, T')$, $Q_2 = P'$, $I = \mathrm{Shape}(T' \backslash D(e, T'))$. Then by Proposition (1.2.6) (Q_1, Q_2) is an adjoinable pair for I, so let $P_1 = P_1^A(I, Q_1, Q_2)$, $P_2 = P_2^A(I, Q_1, Q_2)$, $J = J^A(I, Q_1, Q_2)$. By Proposition (1.2.6) $\mathrm{Shape}(T'') = \mathrm{Shape}(T' \backslash D(e, T')) \cup P' = I \cup Q_2$. Then by Proposition (1.1.22) (e, P_1) is adjoinable to T''. Let $T = \mathrm{Adj}(T'', P_1, e)$, and let $P = P_2$. Then we define $\alpha((T', v, \varepsilon)) = (T, P)$. (By Proposition (1.1.22) $\mathrm{Shape}(T) = J$, so by Proposition (1.1.21) $P = P_2$ is an extremal position in T, so $(T, P) \in \mathscr{D}(M)$.)

(1.2.6) PROPOSITION. *Let* $(T', v, \varepsilon) \in \mathscr{C}(M)$ *and let* $(T, P) = \alpha((T', v, \varepsilon))$. *Then* $\text{Shape}(T') \cap P = \emptyset$, $\text{Shape}(T) = \text{Shape}(T') \cup P$, *and P is an extremal position in T.*

Proof. If (T', v, ε) is in case 1 of Definition (1.2.5) this is clear.In case 2 we have $\text{Shape}(T) = J = I \cup Q_1 \cup P_2$, these unions being disjoint. Now $\text{Shape}(T') = I \cup Q_1$ and $P = P_2$, which gives the first two statements of the proposition. The third statement follows from Proposition (1.1.21(a)). □

We now make a second definition of the algorithm, provisionally called A' instead of A until shown to be equivalent to (1.2.1), in (1.2.8). The map α is the basic building block. The map A' is obtained by repeated applications of α, analogously to the Robinson-Schensted algorithm, as mentioned in the introduction.

(1.2.7) DEFINITION. For each $M_1, M_2 \subset \mathbb{N}^*$ with $|M_1| = |M_2| < \infty$ we define a map $A' = A'(M_1, M_2)$, $A' : \mathscr{S}(M_1, M_2) \to \mathscr{T}(M_1, M_2)$. To define A' we assume by induction that we have defined $A'(M'_1, M'_2)$ whenever $|M'_1| < |M_1|$ (for $|M_1| = 0$ A' is the unique map from $\mathscr{S}(\emptyset, \emptyset)$ to $\mathscr{T}(\emptyset, \emptyset)$, that is $A'(\emptyset) = (\emptyset, \emptyset)$.) Let $m = |M_1|$. For $w \in \mathscr{S}(M_1, M_2)$ let $\{(v, u, \varepsilon)\} = w \backslash w_m$ (so $u = \sup M_2$), let $(T'_1, T'_2) = A'(w_m) \in \mathscr{T}(M_1 \backslash \{v\}, M_2 \backslash \{u\})$, and let $(T_1, P) = \alpha((T'_1, v, \varepsilon))$.

Now by Proposition (1.2.6) P is an extremal position in $\text{Shape}(T_1) = \text{Shape}(T'_1) \cup P$, this union being disjoint. Since $\text{Shape}(T'_2) = \text{Shape}(T'_1)$, and $u > \sup(M_2 \backslash \{u\})$, (u, P) is adjoinable to T'_2. Let $T_2 = \text{Adj}(T'_2, P, u)$. We define $A'(w) = (T_1, T_2)$.

(1.2.8) PROPOSITION. *Let* $M_1, M_2 \subset \mathbb{N}^*$ *with* $|M_1| = |M_2| < \infty$. *Then* $A(M_1, M_2) = A'(M_1, M_2)$.

LEMMA. *Let* $w \in \mathscr{S}(M_1, M_2)$, $m = |M_1|$, *and let* $\{(v, u, \varepsilon)\} = w \backslash w_m$. *Let* $(T, P) = \alpha((L(w_m), v, \varepsilon))$. *Then* $T = L(w)$.

Proof. Let $e = \sup M_1$. Now $e = v$ if and only if $w_m = {}_m w$ so $(L(w_m), v, \varepsilon)$ satisfies the hypothesis of case 1 of Definition (1.2.5) if and only if w satisfies the hypothesis of case 1 of Definition (1.2.1). In this case (that is $e = v$, case 1 of both definitions) the lemma is clear from the definitions.

Assume then $e \neq v$. We will assume by induction that the lemma holds for any $y \in \mathscr{S}(M'_1, M'_2)$ with $|M'_1| < |M_1|$ (when $|M_1| = 1$ any $w \in \mathscr{S}(M_1, M_2)$ satisfies the hypothesis of case 1 of Definition (1.2.5), and we have already proved the lemma in this case). Let $\bar{w} = {}_{m-1}(w_m) = ({}_m w)_{m-1}$. Recall from Definition (1.2.5) that T is obtained as follows: let

$$(T'', P') = \alpha((L(w_m) \backslash D(e, L(w_m)), v, \varepsilon)), \qquad Q_1 = P(e, L(w_m)),$$

$$Q_2 = P', \qquad I = \text{Shape}(L(w_m) \backslash D(e, L(w_m))), \qquad P_1 = P_1^A(I, Q_1, Q_2),$$

then $T = \text{Adj}(T'', P_1, e)$. On the other hand, recall from Definition (1.2.1) that we obtain $L(w)$ as follows: let

$$Q'_1 = P(e, L(w_m)), \qquad Q'_2 = P(u, R({}_m w)),$$
$$I' = \text{Shape}(L(\bar{w})), \qquad P'_1 = P_1^A(I', Q'_1, Q'_2),$$

then $L(w) = \text{Adj}(L({}_m w), P'_1, e)$. Then to show that $L(w) = T$ it suffices to show that

(i) $T'' = L({}_m w)$,

(ii) $I = I'$,

(iii) $Q_1 = Q'_1$, and

(iv) $Q_2 = Q'_2$.

By Proposition (1.2.2), $L(w_m)\backslash D(e, L(w_m)) = L({}_{m-1}(w_m)) = L(\bar{w})$, which gives (ii). By induction we can apply the lemma to $L({}_m w)$, to obtain

$$\alpha(L(w_m)\backslash D(e, L(w_m)), v, \varepsilon) = \alpha(L(\bar{w}), v, \varepsilon) = (L({}_m w), P''),$$

where $P'' = \text{Shape}(L({}_m w))\backslash\text{Shape}(L(\bar{w}))$.This gives (i) and also $P'' = P' = Q_2$. Now (iii) is by definition, so there remains only (iv), that is, we have to show that $P' = P(u, R({}_m w))$. Now $P' = P'' = \text{Shape}(L({}_m w))\backslash\text{Shape}(L(\bar{w})) = \text{Shape}(R({}_m w))\backslash \text{Shape}(R(\bar{w})) = P(u, R({}_m w))$, the last equality since $R(\bar{w}) = R({}_m w)\backslash D(u, R({}_m w))$ by Proposition (1.2.2). This completes the proof of the lemma.

Proof of Proposition (*1.2.8*). We will assume by induction that $A(M'_1, M'_2) = A'(M'_1, M'_2)$ whenever $|M'_1| < |M_1|$ (when $|M_1| = 0$ both $A(M_1, M_2)$ and $A'(M_1, M_2)$ are the unique map from $\mathscr{S}(\emptyset, \emptyset)$ to $\mathscr{T}(\emptyset, \emptyset)$). For $w \in \mathscr{S}(M_1, M_2)$ let $(T_1, T_2) = A'(w)$. We have to show $T_1 = L(w)$ and $T_2 = R(w)$.

Let $(T'_1, T'_2) = A'(w_m)$ and let $\{(v, u, \varepsilon)\} = w\backslash w_m$. By induction $(T'_1, T'_2) = (L(w_m), R(w_m))$. Then by Lemma (1.2.8) $T_1 = \alpha((T'_1, v, \varepsilon)) = \alpha((L(w_m), v, \varepsilon)) = L(w)$. Now $T_2 = \text{Adj}(T'_2, P, u)$ and $R(w) = \text{Adj}(R(w_m), P', u)$, for some P, P'. Thus to show $T_2 = R(w)$ it suffices to show $P = P'$. We have

$$\begin{aligned} P &= \text{Shape}(T_2)\backslash\text{Shape}(T'_2) \\ &= \text{Shape}(T_1)\backslash\text{Shape}(T'_1) \\ &= \text{Shape}(L(w))\backslash\text{Shape}(L(w_m)) \\ &= \text{Shape}(R(w))\backslash\text{Shape}(R(w_m)) = P', \end{aligned}$$

proving the proposition.

REMARK. In light of Proposition (1.2.8) we will use the notation$A(M_1, M_2)$ indifferently for $A(M_1, M_2)$ or $A'(M_1, M_2)$.

We now begin to define an inverse to A. We first introduce β, the inverse to α (1.2.5).

(1.2.9) DEFINITION. Let $M \subset \mathbb{N}^*$, $|M| < \infty$. We define a map $\beta = \beta(M)$, $\beta(M)$: $\mathscr{D}(M) \to \mathscr{C}(M)$. To define β, if $|M| \geqslant 2$ we assume by induction that $\beta(M')$ is defined for all M' with $1 \leqslant |M'| < |M|$ and that we have available Proposition (1.2.10) for this situation (for $|M| = 1$ we are in case 1 of this definition, and this case does not require induction.) Let $e = \sup M$. Suppose $(T, P) \in \mathscr{D}(M)$. Let $Q_1 = P(e, T)$, $Q_2 = P$ and let $I = \text{Shape}(T)$. Then (Q_1, Q_2) is a removable pair for I. There are two cases.

Case 1. If (Q_1, Q_2) is a minimal removable pair for I and $Q_1 = Q_2 = \{S_{1,p}, S_{1,p+1}\}$ (resp. $\{S_{p,1}, S_{p+1,1}\}$) for some p we define $\beta((T, P)) = (T \backslash D(e, T), e, 1)$ (resp. $\beta((T, P)) = (T \backslash D(e, T), e, -1)$).

Case 2. If (Q_1, Q_2) is a standard removable pair for I let $P_1 = P_1^R(I, Q_1, Q_2)$, $P_2 = P_2^R(I, Q_1, Q_2)$, $J = J^R(I, Q_1, Q_2)$. Then $\text{Shape}(T \backslash D(e, T)) = I \backslash Q_1 = J \cup P_2$ so by Proposition (1.1.21(b)) $(T \backslash D(e, T), P_2) \in \mathscr{D}(M \backslash \{e\})$. Let $(T'', v, \varepsilon) = \beta((T \backslash D(e, T), P_2))$. Then by Proposition (1.2.10)

$$\text{Shape}(T'') = \text{Shape}(T \backslash D(e, T)) \backslash P_2 = (J \backslash Q_1) \backslash P_2 = I.$$

Thus (e, P_1) is adjoinable to T''. Let $T' = \text{Adj}(T'', P_1, e)$. We define $\beta((T, P)) = (T', v, \varepsilon)$.

(1.2.10) PROPOSITION. *Let* $(T, P) \in \mathscr{D}(M)$ *and let* $(T', v, \varepsilon) = \beta((T, P))$. *Then* $\text{Shape}(T) \backslash P = \text{Shape}(T')$.

Proof. If (T, P) satisfies case 1 of Definition (1.2.9) this is clear; if it satisfies case 2 we have, in the notation of case 2, $\text{Shape}(T) = I$, $P = Q_2$ and $\text{Shape}(T') = \text{Shape}(T'') \cup P_1 = J \cup P_1 = I \backslash Q_2$, which proves the proposition. □

We now define what will be shown to be an inverse to A.

(1.2.11) DEFINITION. For each $M_1, M_2 \subset \mathbb{N}^*$ with $|M_1| = |M_2| < \infty$ we define a map $B = B(M_1, M_2)$, $B(M_1, M_2)$: $\mathscr{T}(M_1, M_2) \to \mathscr{S}(M_1, M_2)$. To define B we assume by induction that we have defined $B(M_1', M_2')$ whenever $|M_1'| < |M_1|$ (for $|M_1| = 0$, B is the unique map from $\mathscr{T}(\emptyset, \emptyset)$ to $\mathscr{S}(\emptyset, \emptyset)$). Let $u = \sup M_2$. For $(T_1, T_2) \in \mathscr{T}(M_1, M_2)$ let $P = P(u, T_2)$, let $(T_1', v, \varepsilon) = \beta((T_1, P))$, and let $T_2' = T_2 \backslash D(u, T_2)$. Then we define $B((T_1, T_2)) = B((T_1', T_2')) \cup \{(v, u, \varepsilon)\}$.

The following proposition is the main point needed to show that B is an inverse to A.

(1.2.12) PROPOSITION. *Let* $M \subset \mathbb{N}^*$, $|M| < \infty$. *Then* $\alpha(M)$ *is a bijection from* $\mathscr{C}(M)$ *to* $\mathscr{D}(M)$ *with inverse* $\beta(M)$.

Proof. We show first that $\beta \circ \alpha$ is the identity on $\mathscr{C}(M)$. Suppose $(T', v, \varepsilon) \in \mathscr{C}(M)$ and let $(T, P) = \alpha((T', v, \varepsilon))$. Assume first (T', v, ε) satisfies the hypothesis of case 1 of Definition (1.2.5). Then it is clear that (T, P) satisfies case 1 of Definition (1.2.9) and that $\beta(T, P)) = (T', v, \varepsilon)$.

So assume (T', v, ε) satisfies the hypothesis of case 2 of Definition (1.2.5). Then (T, P) is obtained as follows: let $e = \sup M$, $(T'', P') = \alpha((T'\backslash D(e, T'), v, \varepsilon))$, $Q_1 = P(e, T'), Q_2 = P'$, $I = \text{Shape}(T'\backslash D(e, T'))$, $P_1 = P_1^A(I, Q_1, Q_2), P_2 = P_2^A(I, Q_1, Q_2), J = J^A(I, Q_1, Q_2)$, then $T = \text{Adj}(T'', P_1, e)$ and $P = P_2$. Proposition (1.2.21(a)) says that (T, P) satisfies case 2 of Definition (1.2.9), so $\beta((T, P))$ is obtained as follows: we have $P_1 = P(e, T)$, $P_2 = P$, and $J = \text{Shape}(T)$, so let $\tilde{Q}_1 = P_1^R(J, P_1, P_2)$, $\tilde{Q}_2 = P_2^R(J, P_1, P_2)$, $\tilde{I} = J^R(J, P_1, P_2)$. Let $(T''', \tilde{v}, \tilde{\varepsilon}) = \beta((T\backslash D(e, T), \tilde{Q}_2))$ and let $\tilde{T}' = \text{Adj}(T''', \tilde{Q}_1, e)$. Then $\beta((T, P)) = (\tilde{T}', \tilde{v}, \tilde{\varepsilon})$.

We need to show $(\tilde{T}', \tilde{v}, \tilde{\varepsilon}) = (T', v, \varepsilon)$. We will assume by induction that the $\beta(M') \circ \alpha(M')$ is the identity on $\mathscr{C}(M')$ whenever $|M'| < |M|$ (when $|M| = 1$, any element of $\mathscr{C}(M)$ satisfies the hypothesis of case 1 of Definition (1.2.5), and we have already proved that $\beta \circ \alpha$ is the identity on $\mathscr{C}(M)$ in this case.) By Proposition (1.1.21(a)) $\tilde{Q}_1 = Q_1, \tilde{Q}_2 = Q_2$ and $\tilde{I} = I$. We have by induction

$$\begin{aligned}(T''', \tilde{v}, \tilde{\varepsilon}) &= \beta((T\backslash D(e, T), \tilde{Q}_2)) = \beta((T'', Q_2)) = \beta((T'', P')) \\ &= \beta(\alpha((T'\backslash D(e, T'), v, \varepsilon))) = (T'\backslash D(e, T'), v, \varepsilon).\end{aligned}$$

So $\tilde{v} = v, \tilde{\varepsilon} = \varepsilon$, and $T''' = T'\backslash D(e, T')$.

It remains to show that $\tilde{T}' = T$. We have

$$\tilde{T}' = \text{Adj}(T''', \tilde{Q}_1, e) = \text{Adj}(T'\backslash D(e, T'), Q_1, e) = \text{Adj}(T'\backslash D(e, T'), P(e, T'), e) = T'.$$

This completes the proof that $\beta \circ \alpha$ is the identity on $\mathscr{C}(M)$.

We show next that $\alpha \circ \beta$ is the identity on $\mathscr{D}(M)$. Suppose $(T, P) \in \mathscr{D}(M)$. Assume first (T, P) satisfies the hypothesis of case 1 of Definition (1.2.9). Then it is clear that $\beta((T, P))$ satisfies case 1 of Definition (1.2.5) and that $\alpha(\beta((T, P))) = (T, P)$.

So assume (T, P) satisfies the hypothesis of case 2 of Definition (1.2.9). Then $\beta((T, P))$ is obtained as follows: let

$$\begin{aligned}&e = \sup M, \qquad Q_1 = P(e, T), \qquad Q_2 = P, \\ &I = \text{Shape}(T), \qquad P_1 = P_1^R(I, Q_1, Q_2), \qquad P_2 = P_2^R(I, Q_1, Q_2), \\ &J = J^R(I, Q_1, Q_2), \qquad (T'', v, \varepsilon) = \beta((T\backslash D(e, T), P_2)),\end{aligned}$$

and

$$T' = \text{Adj}(T'', P_1, e).$$

Then $\beta((T, P)) = (T', v, \varepsilon)$. We next compute $\alpha((T', v, \varepsilon))$. Since $v \neq e$, (T', v, ε) satisfies case 2 of Definition (1.2.5), so $\alpha((T', v, \varepsilon))$ is obtained as follows. Note that $T'' = T'\backslash D(e, T')$, $P_1 = P(e, T')$, and $J = \text{Shape}(T'')$.

Let

$$(T''', P') = \alpha((T'', v, \varepsilon)), \qquad \tilde{P}_2 = P',$$
$$\tilde{Q}_1 = P_1^A(J, P_1, \tilde{P}_2), \qquad \tilde{Q}_2 = P_2^A(J, P_1, \tilde{P}_2),$$
$$\tilde{I} = J^A(J, P_1, \tilde{P}_2), \qquad \tilde{T} = \mathrm{Adj}(T''', \tilde{Q}_1, e) \quad \text{and} \quad \tilde{P} = \tilde{Q}_2.$$

Then $\alpha((T', v, \varepsilon)) = (\tilde{T}, \tilde{P})$.

We need to show $(\tilde{T}, \tilde{P}) = (T, P)$. We will assume by induction that $\alpha(M') \circ \beta(M')$ is the identity on $\mathscr{D}(M')$ whenever $|M'| < |M|$ (when $|M| = 1$ any element of $\mathscr{D}(M)$ satisfies case 1 of Definition (1.2.9), and we have already proved that $\alpha \circ \beta$ is the identity on $\mathscr{D}(M)$ in this case.) We have by induction

$$(T''', P') = \alpha((T'', v, \varepsilon)) = \alpha(\beta((T \backslash D(e, T), P_2))) = (T \backslash D(e, T), P_2),$$

that is $T''' = T \backslash D(e, T)$ and $P_2 = P' = \tilde{P}_2$. Then by Proposition (1.1.21(b)) $\tilde{Q}_1 = Q_1$, $\tilde{Q}_2 = Q_2$, $\tilde{I} = I$. So

$$\tilde{P} = \tilde{Q}_2 = Q_2 = P \quad \text{and} \quad \tilde{T} = \mathrm{Adj}(T''', \tilde{Q}_1, e) = \mathrm{Adj}(T \backslash D(e, T), Q_1, e)$$
$$= \mathrm{Adj}(T \backslash D(e, T), P(e, T), e) = T.$$

This completes the proof that $\alpha \circ \beta$ is the identity on $\mathscr{D}(M)$. □

(1.2.13) THEOREM. *Let* $M_1, M_2 \subset \mathbb{N}^*$ *with* $|M_1| = |M_2| < \infty$. *Then* $A(M_1, M_2)$ *is a bijection from* $\mathscr{S}(M_1, M_2)$ *to* $\mathscr{T}(M_1, M_2)$ *with inverse* $B(M_1, M_2)$.

Proof. The proof is by induction, that is we assume the theorem holds for $A(M'_1, M'_2)$ whenever $|M'_1| < |M_1|$. (When $|M_1| = 0$, $A(M_1, M_2)$ and $B(M_1, M_2)$ are the unique maps between the one-element sets in question.) In light of Proposition (1.2.8) we will use Definition (1.2.7) for $A(M_1, M_2)$. Let $m = |M_1|$ and let $u = \sup M_2$.

We show first that $B \circ A$ is the identity on $\mathscr{S}(M_1, M_2)$. Given $w \in \mathscr{S}(M_1, M_2)$ we compute $B(A(w))$. Let

$$\{(v, u, \varepsilon)\} = w \backslash w_m, \qquad A(w_m) = (T'_1, T'_2),$$
$$(T_1, P) = \alpha((T'_1, v, \varepsilon)), \quad \text{and} \quad T_2 = \mathrm{Adj}(T'_2, P, u).$$

Then $A(w) = (T_1, T_2)$. To compute $B((T_1, T_2))$, note that $P = P(u, T_2)$ and $T'_2 = T_2 \backslash D(u, T_2)$. Let $(\tilde{T}'_1, \tilde{v}, \tilde{\varepsilon}) = \beta((T_1, P))$. Then $B((T_1, T_2)) = B((\tilde{T}'_1, T'_2)) \cup \{(\tilde{v}, u, \tilde{\varepsilon})\}$.

We need to show that $w = B((\tilde{T}'_1, T'_2)) \cup \{(\tilde{v}, u, \tilde{\varepsilon})\}$. We will show that

(i) $B((T'_1, T'_2)) = w_m$ and

(ii) $\{(\tilde{v}, u, \tilde{\varepsilon})\} = \{(v, u, \varepsilon)\}$ which is $w \backslash w_m$.

Proposition (1.2.12) says that $(\tilde{T}'_1, \tilde{v}, \tilde{\varepsilon}) = \beta((T_1, P)) = \beta(\alpha((T'_1, v, \varepsilon))) = (T'_1, v, \varepsilon)$. So we have (ii) and also $\tilde{T}'_1 = T'_1$. By induction we have $B((T'_1, T'_2)) = B((T'_1, T'_2)) = B(A(w_m)) = w_m$. This completes the proof that $B \circ A$ is the identity on $\mathscr{S}(M_1, M_2)$.

We show next that $A \circ B$ is the identity on $\mathscr{T}(M_1, M_1)$. Given $(T_1, T_2) \in \mathscr{T}(M_1, M_2)$ we compute $A(B((T_1, T_2)))$. Let $P = P(u, T_2)$, $(T'_1, v, \varepsilon) = \beta((T_1, P))$ and $T'_2 = T_2 \backslash D(u, T_2)$. Then $B((T_1, T_2)) = B((T'_1, T'_2)) \cup \{(v, u, \varepsilon)\}$. Let $w = B((T_1, T_2))$. To compute $A(w)$, note that since $u = \sup M_2$, $w \backslash w_m = \{(v, u, \varepsilon)\}$. Then $w_m = B((T'_1, T'_2))$ so by induction $A(w_m) = A(B((T'_1, T'_2))) = (T'_1, T'_2)$. Let $(\tilde{T}_1, \tilde{P}) = \alpha((T'_1, v, \varepsilon))$ and $\tilde{T}_2 = \mathrm{Adj}(T'_2, \tilde{P}, u)$. Then $A(w) = (\tilde{T}_1, \tilde{T}_2)$.

We need to show $(\tilde{T}_1, \tilde{T}_2) = (T_1, T_2)$. We have $(\tilde{T}_1, \tilde{P}) = \alpha((T'_1, v, \varepsilon)) = \alpha(\beta(T_1, P))) = (T_1, P)$ (the last equality by Proposition (1.2.12)), so $\tilde{T}_1 = T_1$ and $\tilde{P} = P$. Finally $\tilde{T}_2 = \mathrm{Adj}(\tilde{T}'_2, \tilde{P}, u) = \mathrm{Adj}(T'_2, P, u) = \mathrm{Adj}(T_2 \backslash D(u, T_2)$, $P(u, T_2)$, $u) = T_2$. This completes the proof of the theorem. □

(1.2.14) REMARK. All the definitions in this section can be made for tableaux in $\mathscr{T}^0(M)$ as well. That is, we can define $A^0 = A^0(M_1, M_2)$ where $A^0(M_1, M_2)$: $\mathscr{S}(M_1, M_2) \to \mathscr{T}^0(M_1, M_2)$. This definition is identical to Definition (1.2.1), except that for $|M_1| = 0$ we have $A^0(\emptyset) = (\{(0, S_{11})\}, \{(0, S_{11})\}) \in \mathscr{T}^0(\emptyset, \emptyset)$. Similarly we define $\mathscr{C}^0(M)$, $\mathscr{D}^0(M)$, $\alpha^0(M)$, $(A^0)'(M_1, M_2)$, $\beta^0(M)$, and $B^0(M_1, M_2)$. The analogues of Propositions (1.2.2), (1.2.3), (1.2.6), (1.2.8), (1.2.10), and (1.2.12), and of Theorem (1.2.13) hold in this situation as well.

Section 3

In this section we prove some propositions about the maps A and B which are useful for applications. We will, in Section 4, illustrate how they are used in computations. We accomplish this by giving useful characterizations of α and β.

(1.3.1) DEFINITION. Let $T \in \mathscr{T}(M)$ or $\mathscr{T}^0(M)$. Let $M = \{e_1, \ldots, e_m\}$ with $e_1 < e_2 < \cdots < e_m$. Define

$$T(j) = T \Bigg\backslash \left(\bigcup_{k=j+1}^{m} D(e_k, T) \right).$$

(1.3.2) DEFINITION. Let $J, P \subseteq \mathscr{F}$ be such that J satisfies condition Y and $P = \{S_{ij}, S_{i,j+1}\}$ (resp. $\{S_{ij}, S_{i+1,j}\}$) for some (i,j). We define $P^A(J, P)$ in any one of the following situations:

(i) If $j = \rho_i(J) + 1$ (resp. $i = \kappa_j(J) + 1$) we define $P^A(J, P) = P$.

(ii) If $P \cap J = S_{ij}$ and $\rho_{i+1}(J) = j - 1$ (resp. $\kappa_{j+1}(J) = i - 1$) define $P^A(J, P) = (P \backslash \{S_{ij}\}) \cup \{S_{i+1,j+1}\}$.

(iii) If $j+1=\rho_i(J)$ (resp. $i+1=\kappa_j(J)$) we define $P^A(J,P)=\{S_{i+1,r},S_{i+1,r+1}\}$ (resp. $\{S_{r,j+1},S_{r+1,j+1}\}$) where $r=\rho_{i+1}(J)+1$ (resp. $r=\kappa_{j+1}(J)+1$).

EXAMPLE. See (1.4.2). Each of the five tableaux with a domino above it yields an example: that is, J is the shape of the tableau, P is the position occupied by the domino above it, the shaded area in the domino is $P\cap J$, (which is empty in case (i) of the definition) and $P^A(J,P)$ is the position in the next tableau occupied by the domino with the same number.

(1.3.3) PROPOSITION. *Suppose (Q_1,Q_2) is an adjoinable pair to I. Then $P^A(I\cup Q_2,Q_1)$ (resp. $P^A(I\cup Q_1,Q_2)$) is defined and $P^A(I\cup Q_2,Q_1)=P_1^A(I,Q_1,Q_2)$ (resp. $P^A(I\cup Q_1,Q_2)=P_2^A(I,Q_1,Q_2)$).*

The following proposition is designed to give a better grasp of α.

(1.3.4) PROPOSITION. *Let $M\subset\mathbb{N}^*$, $|M|<\infty$. Let $m=|M|$ and let $M=\{e_1,\dots,e_m\}$ with $e_1<\cdots<e_m$. Suppose $(T',v,\varepsilon)\in\mathscr{C}(M)$, and let $(T,P)=\alpha((T',v,\varepsilon))$. Let $1\leqslant k\leqslant m$ be such that $v=e_k$. Then the following statements hold and characterize T uniquely.*

(1) *For $j<k$ we have $T(j)=T'(j)$, that is $P(e_j,T')=P(e_j,T)$.*
(2) *If $\varepsilon=1$ then $P(e_k,T)=\{S_{1,r},S_{1,r+1}\}$ where $r=\rho_1(T(k-1))+1$; if $\varepsilon=-1$ then $P(e_k,T)=\{S_{r1},S_{r+1,1}\}$ where $r=\kappa_1(T(k-1))+1$.*
(3) *For $j>k$ we have $P(e_j,T)=P^A(T(j-1),P(e_j,T'))$.*

Proof. It is clear that statements (1)–(3) determine $P(e_j,T)$ for all $1\leqslant j\leqslant m$, and thus determine T. To see that they hold, suppose first that $k=m$, that is, that (T',v,ε) satisfies the hypothesis of case 1 of Definition (1.2.5). In this case it is clear that statements (1) and (2) are true and that (3) does not apply.

Suppose then that $k\neq m$, that is (T',v,ε) satisfies the hypothesis of case 2 of Definition (1.2.5). We assume by induction that the proposition is true for elements of any $\mathscr{C}(M')$ with $1\leqslant|M'|<|M|$ (when $|M|=1$ we must have $k=m=1$, and this case is proved already.) Let $T''\in\mathscr{C}(M\backslash\{e_m\})$ be such that $(T'',P)=\alpha((T'\backslash D(e_m,T'),v,\varepsilon))$. Then by Definition (1.2.5) $T''=T(m-1)$, so $T''(j)=T(j)$ for $1\leqslant j\leqslant m-1$. Since also $P(e_j,T'\backslash D(e_m,T'))=P(e_j,T')$ for $1\leqslant j\leqslant m-1$, we have by induction that statements (1) and (2) hold, and (3) holds for $k<j\leqslant m-1$. For $j=m$, statement (3) follows from the definition of α and from Proposition (1.3.3).

REMARK. The proposition (we conserve all its notation) says that we can form T by the following procedure: first write down the tableau formed with the numbers $e_1,\dots,e_{k-1}$, in the same position as they appear in T'. (This is the tableau called $T(k-1)$.) Then add $v=e_k$ to this tableau as a horizontal domino at the end of the first row if $\varepsilon=1$ or as a vertical domino at the end of the first column if $\varepsilon=-1$. (We now have $T(k)$.) Finally, add successively the e_j, $j>k$, in increasing order as follows: to add e_j to the tableau that we have obtained from the $e_i, i < j$, (this tableau is called $T(j-1)$) we look at the intersection of this tableau with the position occupied by e_j in T' (that is, $T(j-1) \cap P(e_j, T')$.) We then add e_j to the tableau in the position described in Definition (1.3.2), which depends on the position of e_j in T' and the intersection of this position with $T(j-1)$ (that is, we add e_j to $T(j-1)$ in the position $P^A(T(j-1), P(e_j, T'))$ of Definition (1.3.2): the result is $T(j)$). This procedure is illustrated in Example (1.4.2).

Note that $P = \text{Shape}(T)\backslash\text{Shape}(T')$ and is thus determined once T is known. Now we begin to develop a better characterization of β.

(1.3.5) DEFINITION. Let $I \subset \mathscr{F}$ satisfy condition Y, and let $Q_1, Q_2 \subset \mathscr{F}$ with Q_1 and $Q_2 \cap I$ extremal positions in I. If $(Q_1, Q_2 \cap I)$ is a standard removable pair for I, we define $P_1^R(I, Q_1, Q_2) = P_1^R(I, Q_1, Q_2 \cap I)$.

(1.3.6) PROPOSITION. *Let $M \subset \mathbb{N}^*$, $|M| < \infty$, let $m = |M|$ and let $M = \{e_1, \ldots, e_m\}$ with $e_1 < \cdots < e_m$. Suppose $(T, P) \in \mathscr{D}(M)$, and let $(T', v, \varepsilon) = \beta((T, P))$. Let*

$$D_j((T,P)) = P \cup \bigcup_{l=j+1}^{m} P(e_l, T').$$

Let $1 \leqslant k \leqslant m$ be such that $v = e_k$. Then the following statements hold and characterize (T', v, ε) uniquely. (In particular they determine k.)

(1) *For $k < j \leqslant m$ $P_1^R(\text{Shape}(T(j)), P(e_j, T), D_j((T,P)))$ is defined and is equal to $P(e_j, T')$.*

(2) *Let $1 \leqslant j \leqslant m$. Then (a) $j = k$ and $\varepsilon = 1$ if and only if $P(e_j, T) \cap D_j((T,P)) = \{S_{1r}, S_{1,r+1}\}$ for some r, and (b) $j = k$ and $\varepsilon = -1$ if and only if $P(e_j, T) \cap D_j((T,P)) = \{S_{r1}, S_{r+1,1}\}$ for some r.*

(3) *For $1 \leqslant j < k$ we have $P(e_j, T') = P(e_j, T)$.*

Proof. To see that if true statements (1)–(3) determine (T', v, ε) we need to show that for each $1 \leqslant j \leqslant m$ statements (1)–(3) first determine whether $j = k$, and if $j = k$ that they determine ε, and if $j \neq k$ that they determine $P(e_j, T')$. It is clear that for $j = m$ statements (1) and (2) do this. Now let $1 \leqslant j < m$. By induction we can assume that for $j + 1 \leqslant l \leqslant m$ statements (1)–(3) have determined whether $l = k$, and if $l \neq k$ that they have determined $P(e_l, T')$. Then if $j + 1 \leqslant l \leqslant m$ statement (3) determines $P(e_j, T')$, and if not by induction $P(e_l, T')$ is determined for $j + 1 \leqslant l \leqslant m$, thus so is $D_j((T,P))$. Then it is clear that statements (1) and (2) determine whether $j = k$, and if so that they determine ε and if not they determine $P(e_j, T')$.

To see that statements (1)–(3) hold, note first that if $k = m$ (case 1 of Definition (1.2.9)) then statement (1) is irrelevant and statements (2) and (3) follow directly from Definition (1.2.9). So assume $k \neq m$. We assume by induction that the proposition is true for $\beta(M')$ when $1 \leqslant |M'| < |M|$ (the case $|M| = 1$ also satisfies $k = |M| = 1$, which is treated above). We note that $D_m((T, P)) = P$, thus statements (1) and (2) hold when $j = m$ (and statement (3) cannot apply when $j = m$) by the definition of β. Let $Q_1 = P(e_m, T)$, $Q_2 = P$, $I = \text{Shape}(T)$, $P_1 = P_1^R(I, Q_1, Q_2) = P(m, T')$, $P_2 = P_2^R(I, Q_1, Q_2)$, and let $(T'', v, \varepsilon) = \beta((T \backslash D(e_m, T), P_2))$.

It remains to show that statements (1)–(3) hold for $1 \leqslant j \leqslant m-1$. By induction we can assume that they hold true for $\beta((T \backslash D(e_m, T), P_2))$. Using this and the facts that for $1 \leqslant j \leqslant m-1$ we have $P(e_j, T) = P(e_j, T \backslash D(e_m, T))$ (that is $T(j) = (T \backslash D(e_m, T))\ (j)$) and by the definition of β, $P(e_j, T') = P(e_j, T'')$ for $j \neq k$, $1 \leqslant j \leqslant m-1$, statements (1)–(3) follow from the following claim:

Claim. For $k \leqslant j \leqslant m-1$ we have $D_j((T, P)) \cap T(j) = D_j((T \backslash D(e_m, T), P_2)) \cap T(j)$.

Proof of the claim.

$$D_j((T, P)) = P \cup \bigcup_{i=j+1}^{m} P(e_i, T')$$

$$= P \cup P(e_m, T') \cup \bigcup_{i=j+1}^{m-1} P(e_i, T')$$

and

$$D_j((T \backslash D(e_m, T), P_2)) = P_2 \cup \bigcup_{i=j+1}^{m-1} P(e_i, T'')$$

$$= P_2 \cup \bigcup_{i=j+1}^{m} P(e_i, T').$$

So it suffices to show that $(P \cup P(e_m, T')) \cap T(j) = P_2 \cap T(j)$. But by the definition of P_1^R and P_2^R, $P \cup P(e_m, T') = P_2 \cup P(e_m, T)$, so this is clear.

Section 4

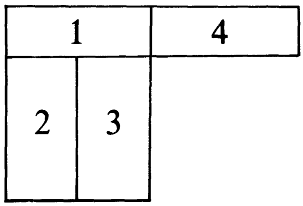
In this section we illustrate the preceding with some examples, showing how α, β, A, and B operate as algorithms. Thus we use the descriptions of α and β given in Section 3. The similarity to the Robinson-Schensted algorithm will be visible.

(1.4.1) REMARK. We think of the S_{ij} as squares in an array. If T is a domino tableau an element of T corresponds to the assignment of a number to a square.

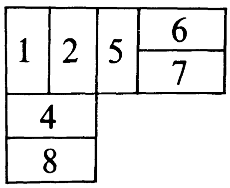
We picture the number as occupying the square. For example, the tableau $T = \{(1, S_{11}), (1, S_{12}), (2, S_{21}), (2, S_{31}), (3, S_{22}), (3, S_{32}), (4, S_{13}), (4, S_{14})\}$ looks like

1	1	4	4
2	3		
2	3		

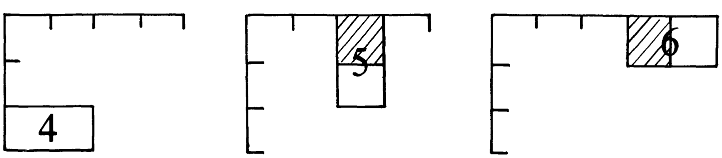
Usually, however, we display T as a tableau made of dominos, as

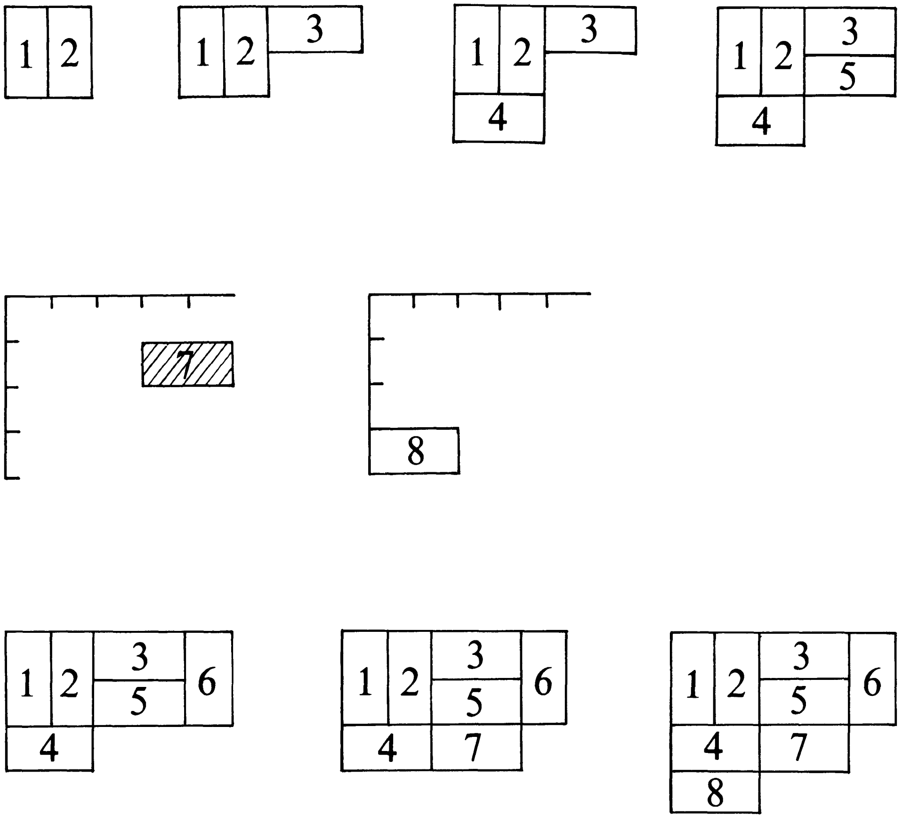


(1.4.2) EXAMPLE. Let T' be the domino tableau displayed as:

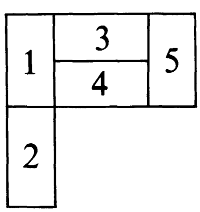


Let $(T, P) = \alpha((T', 3, 1))$. We show how to find T. We use Proposition (1.3.4) and its notation; see especially the remark after this proposition. (Since $M = \{1, 2, \ldots, 8\}$ we have $e_j = j \, \forall 1 \leqslant j \leqslant 8$.) Then $k = 3$ so $T(2) = T'(2)$. We start with this and display successively $T(2), T(3), \ldots, T(8) = T$. Above each tableau $T(j - 1)$, $j \geqslant 4$, we display $D(j, T')$. The set $T(j - 1) \cap P(j, T')$ is indicated by the shaded area. This is used, by Proposition (1.3.4), statement (3), to find $P(j, T)$.

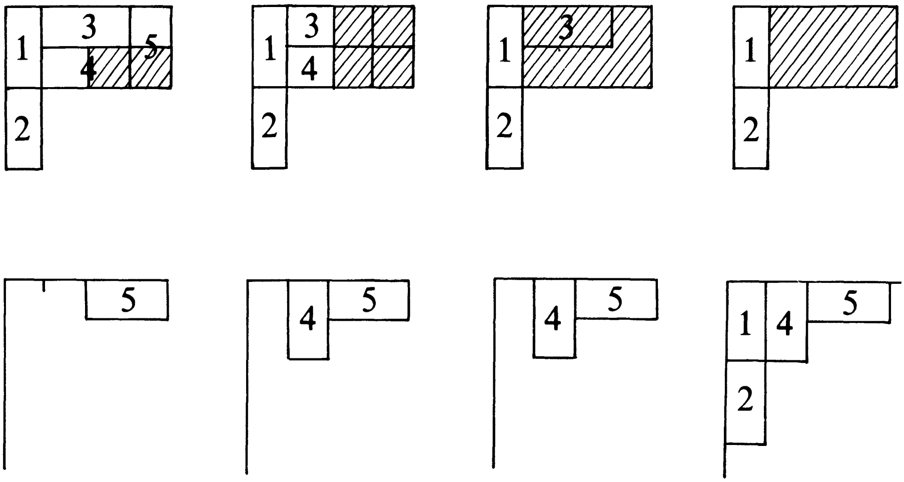




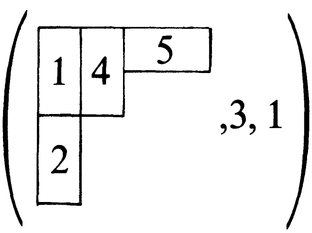
(1.4.3) EXAMPLE. Let T be the domino tableau displayed as



and let $P = \{S_{23}, S_{24}\}$. Let $(T', v, \varepsilon) = \beta((T, P))$. We show how to find (T', v, ε). We use Proposition (1.3.6), and its notation. We display, on the top row, $T(5)$, $T(4), \ldots,$ $T(k-1)$. (It develops that $k = 3$.) The shaded area for each $T(j)$ is $D(j)$. On the next row we display the information obtained about (T', v, ε) at each stage, using statements (1)–(3) of Proposition (1.3.6).

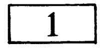
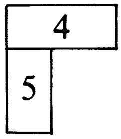
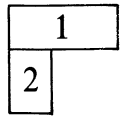
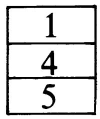
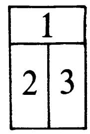
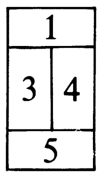
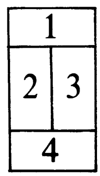
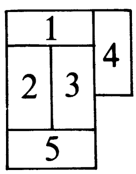
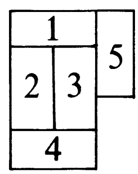


statement (1) gives $P(5, T')$	statement (1) gives $P(4, T')$	statement (2) gives $v = 3, \varepsilon = 1$	statement (3) gives the remaining information about T'.

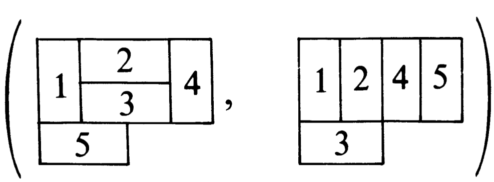
Thus $(T', v, \varepsilon) =$ 

(1.4.4) EXAMPLE. Let $w \in \mathscr{S}(\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5\})$, $w = \{(4, 1, 1), (5, 2, -1), (1, 3, 1), (3, 4, -1), (2, 5, -1)\}$. We show the steps in finding $A(w)$, using Definition (1.2.7).

Definition (1.2.7) says that for $k = 1, 2, \ldots, 5$, we obtain $(L(w_{5,\ldots,k+1}), R(w_{5,\ldots,k+1}))$ from $(L(w_{5,\ldots,k}), R(w_{5,\ldots,k}))$ using the fact that $(L(w_{5,\ldots,k+1}), P(k, R(w_{5,\ldots,k+1})) = \alpha((L(w_{5,\ldots,k}), v, \varepsilon))$ where $(v, k, \varepsilon) \in w$, and that $R(w_{5,\ldots,k+1}) = \operatorname{Adj}(R(w_{5,\ldots,k}), P(k, R(w_{5,\ldots,k+1})), k)$. We display here the result of each step.

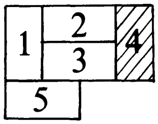
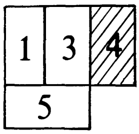
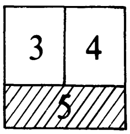
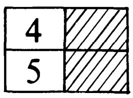
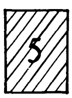
y	$L(y)$	$R(y)$
$w_{5,4,3,2}$		
$w_{5,4,3}$		
$w_{5,4}$		
w_5		
w		

(1.4.5) EXAMPLE. Let $(T_1, T_2) \in \mathscr{T}(\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5\})$,

$(T_1, T_2) =$ 

We show how to find $w = B((T_1, T_2))$. Note that $w = \bigcup_{k=1}^{5} (w_{5,\ldots,k+1} \backslash w_{5,\ldots,k})$. Definition (1.2.5) says that for $k = 5, \ldots, 1$ we obtain $w_{5,\ldots,k+1} \backslash w_{5,\ldots,k}$ using the fact that $\beta((L(w_{5,\ldots,k+1}), P(k, T_2))) = (L(w_{5,\ldots,k}), v, \varepsilon)$ where $w_{5,\ldots,k+1} \backslash w_{5,\ldots,k} = (v, k, \varepsilon)$.

(We are using here Theorem (1.2.13) to identify (T_1, T_2) with $(L(w), R(w))$ and to identify the various subtableaux which occur in computing $B((T_1, T_2))$.) We display each step. For each k the area shaded in the tableau on the same row is $P(k, T_2)$. Thus if T is a tableau in the second column and P the area shaded in it and $(T', v, \varepsilon) = \beta((T, P))$ then v and ε are the first and third entries, respectively, in the triple in the same row as T, and T' is the tableau directly below T.

k	$L(w_{5,\ldots,k+1})$	$w_{5,\ldots,k+1} \backslash w_{5,\ldots,k}$
5		$\{(2, 5, 1)\}$
4		$\{(1, 4, -1)\}$
3		$\{(3, 3, -1)\}$
2		$\{(4, 2, 1)\}$
1		$\{(5, 1, -1)\}$

So $w = \{(2, 5, 1), (1, 4, -1), (3, 3, -1), (4, 2, 1), (5, 1, -1)\}$.

Section 5

In this section we introduce the second algorithm mentioned in the introduction, S, which given any domino tableau (in the applications, $L(w)$) produces a domino tableau with special shape.

We first introduce the notion of special shape, (1.5.6), and its preliminaries. The notion of special representation of a Weyl group was introduced by Lusztig in [8] (in case A_n, every representation is special). Then the symbol of such a representation is also called special, and was explicitly characterized by Lusztig in [8]. In [1], Barbasch-Vogan use shapes of domino tableaux to parametrize representations of the Weyl groups of types B_n, C_n and D_n. One can easily show that the following definitions leading up to Definition (1.5.6) characterize the shapes of domino tableaux which parametrize the special representations (and so we call such shapes, special shapes).

(1.5.1) DEFINITION. A grid is a map ϕ from $\mathscr{F}$ into a four-element set denoted $\{X, Y, Z, W\}$, satisfying

(i) $\phi(S_{ij}) = X \Rightarrow \phi(S_{i,j+1}) = Y$ and $\phi(S_{i+1,j}) = Z$
(ii) $\phi(S_{ij}) = Y \Rightarrow \phi(S_{i,j+1}) = X$ and $\phi(S_{i+1,j}) = W$
(iii) $\phi(S_{ij}) = Z \Rightarrow \phi(S_{i,j+1}) = W$ and $\phi(S_{i+1,j}) = X$
(iv) $\phi(S_{ij}) = W \Rightarrow \phi(S_{i,j+1}) = Z$ and $\phi(S_{i+1,j}) = Y$

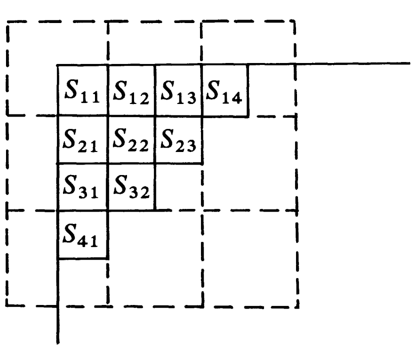
In particular we note that ϕ is determined by $\phi(S_{11})$.

(1.5.2) DEFINITION. We define the grids ϕ_B, ϕ_C, ϕ_D, and ϕ'_D by $\phi_B(S_{11}) = W, \phi_C(S_{11}) = X, \phi_D(S_{11}) = Y, \phi'_D(S_{11}) = Z$.

REMARK. We think of a grid as placing $\mathscr{F}$ (or any subset thereof) in an array of 2×2 boxes (see (1.5.7)). The squares within any 2×2 box are labelled X, Y, Z, or W as follows:

X	Y
Z	W

Thus each square of $\mathscr{F}$ (or a subset thereof) is labelled by an element of the set $\{X, Y, Z, W\}$. For example, the grid ϕ_B places $\mathscr{F}$ in an array of 2×2 boxes as pictured below:



Thus, for example, $\phi_B(S_{45}) = Y$.

(1.5.3) DEFINITION. (1) A tableau with grid is a pair (T, ϕ) where ϕ is a grid and T is contained in either $\mathscr{T}(M)$ or $\mathscr{T}^0(M)$ for some M.

(2) We are interested in three types of tableaux with grid, corresponding to $\mathfrak{g}$ of type B_n, C_n, and D_n. Let $\mathscr{T}_B(M) = \{(T, \phi_B) \mid T \in \mathscr{T}^0(M)\}$, let $\mathscr{T}_C(M) = \{(T, \phi_C) \mid T \in \mathscr{T}(M)\}$, and let $\mathscr{T}_D(M) = \{(T, \phi_D) \mid T \in \mathscr{T}(M)\}$. For $n \in \mathbb{N}^*$ we write $\mathscr{T}_B(n)$ for $\mathscr{T}_B(\{1, \ldots, n\})$; similarly $\mathscr{T}_C(n)$, $\mathscr{T}_D(n)$.

EXAMPLE. See (1.6.2)(i), (1.6.3)(i), and (1.6.4)(i).

(1.5.4) DEFINITION. Let ϕ be a grid. A square $S_{ij} \in \mathscr{F}$ is called ϕ-fixed if $\phi(S_{ij}) \in \{Y, Z\}$, otherwise it is called ϕ-variable. (If ϕ is understood, S_{ij} is called simply fixed or variable.)

(1.5.5) DEFINITION. Let $F \subset \mathscr{F}$ satisfy condition Y, and let ϕ be a grid.

(1) A square $S_{ij} \in \mathscr{F}$ is called a corner of F with respect to ϕ (or a ϕ-corner of F, or a corner of F, if ϕ is understood) if $\phi(S_{ij}) = X$, $(S_{i,j-1}, S_{i-1,j}\} \subset F \cup (\mathscr{F}^0 \backslash \mathscr{F})$, and $\{S_{i,j+1}, S_{i+1,j}\} \cap F = \emptyset$.

(2) A square $S_{ij} \in \mathscr{F}$ is called a hole of F with respect to ϕ if $\phi(S_{ij}) = W$, $\{S_{i,j-1}, S_{i-1,j}\} \subset F \cup (\mathscr{F}^0 \backslash \mathscr{F})$, and $\{S_{i,j+1}, S_{i+1,j}\} \cap F = \emptyset$.

Note that a corner or hole of F need not be contained in F. If (T, ϕ) is a tableau with grid we say $S_{ij} \in \mathscr{F}$ is a corner or hole of (T, ϕ) if it is a corner or hole of Shape(T) with respect to ϕ.

EXAMPLES. If **T** is as in (1.6.2) then S_{42} is an (empty) corner of **T** and S_{33} and S_{51} are (filled) holes of **T**. If **T** is as in (1.6.3) then S_{33} is a (filled) corner of **T**, S_{51} is an (empty) corner of **T** and S_{42} is an (empty) hole of **T**.

We now give the definition of the notion of a domino tableau's having special shape.

(1.5.6) DEFINITION. Let $F \subset \mathscr{F}$ satisfy condition Y, and let ϕ be a grid. F is called special with respect to ϕ (or ϕ-special) if F has no filled ϕ-corners or empty ϕ-holes. A tableau T is called special with respect to ϕ if Shape(T) is special with respect to ϕ. A tableau with grid (T, ϕ) is called special if T is special with respect to ϕ.

We now introduce some preliminaries for the definition of the algorithm S, which will be given in (1.5.34).

(1.5.7) DEFINITION. Let ϕ be a grid. A set $B \subset \mathscr{F}^0$ is called a box for ϕ (or a ϕ-box) if $B = \{S_{ij}, S_{i,j+1}, S_{i+1,j}, S_{i+1,j+1}\}$ for some S_{ij} with $\phi(S_{ij}) = X$. (We extend ϕ to $\mathscr{F}^0$ in the obvious way.) Let $P \subset \mathscr{F}$. P is called boxed (with respect to ϕ) if P is contained in some ϕ-box B, otherwise P is unboxed.

REMARK. It is easy to see that a set $F \subset \mathscr{F}$ satisfying condition Y is special with respect to a grid ϕ if and only if for every ϕ-box B, the set $(F \cup (\mathscr{F}^0 \backslash \mathscr{F})) \cap B$ consists of 0, 2, or 4 elements.

One of the main ingredients for the definition of S is the following.

(1.5.8) DEFINITION. Let $\mathbf{T} = (T, \phi)$ be a tableau with grid, $M = \mathcal{N}(T)$. Let $k \in M$. We define $P'(k, \mathbf{T})$ as follows: if $P(k, T) = \{S_{i,j-1}, S_{ij}\}$ or $\{S_{ij}, S_{i+1,j}\}$ with S_{ij} a fixed square, set $r = N_T(S_{i-1,j+1})$. If $r > k$ let $P'(k, \mathbf{T}) = \{S_{i-1,j}, S_{ij}\}$; if $r < k$ let $P'(k, \mathbf{T}) = \{S_{ij}, S_{i,j+1}\}$. If instead $P(k, T) = \{S_{ij}, S_{i,j+1}\}$ or $\{S_{i-1,j}, S_{ij}\}$ with S_{ij} a fixed square, let $r = N_T(S_{i+1,j-1})$. If $r > k$ let $P'(k, \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}$; if $r < k$ let $P'(k, \mathbf{T}) = \{S_{ij}, S_{i+1,j}\}$.

EXAMPLE. Let $\mathbf{T}$ be as in (1.6.2). Then $P'(1, \mathbf{T}) = \{S_{11}, S_{21}\}, P'(2, \mathbf{T}) = \{S_{12}, S_{13}\}, P'(3, \mathbf{T}) = \{S_{22}, S_{23}\}, P'(4, \mathbf{T}) = \{S_{31}, S_{41}\}$, and $P'(5, \mathbf{T}) = \{S_{32}, S_{42}\}$.

(1.5.9) REMARK. Let $\mathbf{T} = (T, \phi)$ be a tableau with grid, $M = \mathcal{N}(T)$. Let $k \in M$. Then $P(k, T)$ contains one ϕ-fixed and one ϕ-variable square, as does $P'(k, \mathbf{T})$, and $P(k, T) \cap P'(k, \mathbf{T}) = \{S_{ij}\}$ where S_{ij} is the ϕ-fixed square in $P(k, T)$. Furthermore, $P'(k, \mathbf{T})$ is boxed $\Leftrightarrow$ $P(k, T)$ is unboxed.

The following proposition will be used later, after the definition of S, in order to show that $S(\mathbf{T})$ is a domino tableau, provided $\mathbf{T}$ is.

(1.5.10) PROPOSITION. *Let* $\mathbf{T} = (T, \phi)$ *be a tableau with grid*, $M = \mathcal{N}(T)$, $k \in M$, *and let* $S_{ij} \in P'(k, \mathbf{T})$ *be* ϕ*-variable. Then* $k \geqslant N_T(S_{i-1,j})$, $k \geqslant N_T(S_{i,j-1})$, $k \leqslant N_T(S_{i,j+1})$, $k \leqslant N_T(S_{i+1,j})$.

Proof. We have to check separately the four cases in the definition of $P'(k, \mathbf{T})$, but in each case each of the four statements of the proposition follow from either the definition of $P'(k, \mathbf{T})$ or of domino tableaux.

The following definition is needed in order to define the process of moving a tableau, $\mathbf{T}$, through a cycle, c, which will be denoted $\mathbf{E}(\mathbf{T}, c)$ and defined in (1.5.26) (and "cycle" will be defined in (1.5.18)).

(1.5.11) DEFINITION. Let ϕ be a grid. Define $\phi^*: \mathcal{F} \to \{X, Y, Z, W\}$ by

$$\phi^*(S_{ij}) = X \Leftrightarrow \phi(S_{ij}) = W, \qquad \phi^*(S_{ij}) = Y \Leftrightarrow \phi(S_{ij}) = Z,$$
$$\phi^*(S_{ij}) = Z \Leftrightarrow \phi(S_{ij}) = Y, \qquad \phi^*(S_{ij}) = W \Leftrightarrow \phi(S_{ij}) = X.$$

Then ϕ^* is a grid, in particular $\phi_B^* = \phi_C, \phi_C^* = \phi_B, \phi_D^* = \phi'_D$, and $(\phi'_D)^* = \phi_D$.

(1.5.12) PROPOSITION. *Let T be a domino tableau and let* $M = \mathcal{N}(T)$. *Let* ϕ *be a grid. Let* $k \in M$.

(1) *$P(k, T)$ is boxed with respect to* $\phi \Leftrightarrow P(k, T)$ *is unboxed with respect to* ϕ^*.

(2) $P'(k, (T, \phi)) = P'(k, (T, \phi^*))$.

(3) *Let* $S \in \mathcal{F}$. *S is a corner of T for* $\phi \Leftrightarrow S$ *is a hole of T for* ϕ^*.

Proof. (1) and (3) are clear. (2) follows from Definition (1.5.8) and the fact that if $S \in \mathcal{F}$ then S is ϕ-fixed $\Leftrightarrow$ S is ϕ^*-fixed.

We will define the algorithm S with the following underlying notations and assumptions.

For the remainder of section 1.5 $\mathbf{T} = (T, \phi)$ will be a tableau with grid contained in either $\mathscr{T}_B(M)$, $\mathscr{T}_C(M)$, or $\mathscr{T}_D(M)$, for some $M \neq \emptyset$. Variable, boxed, etc., will be with respect to ϕ, except where specified otherwise. Propositions (1.5.13), (1.5.15), and (1.5.17) are the basis of what follows. The proofs of Propositions (1.5.13), (1.5.15), and (1.5.17) are similar; we prove (1.5.17)–(1) as a sample thereof.

(1.5.13) PROPOSITION. *Let $k \in M$ and let S be the variable square in $P'(k, \mathbf{T})$. Then exactly one of the following situations hold.*

(i) *There is a $k' \in M$ with $S \in P(k', T)$. $P(k', T)$ is boxed $\Leftrightarrow P(k, T)$ is boxed.*
(ii) *$P(k, T)$ is unboxed and S is an empty hole of T.*
(iii) *$P(k, T)$ is boxed and S is an empty corner of T.*
(iv) *$\mathbf{T} \in \mathscr{T}_B(M)$, $P(k, T)$ is boxed and $S = S_{11}$.*

(1.5.14) DEFINITION. Let $k \in M$. Define $N_f(k, \mathbf{T})$ according to the four situations of Proposition (1.5.13): if k satisfies (i) let $N_f(k, \mathbf{T}) = k'$; if k satisfies (ii) or (iii) let $N_f(k, \mathbf{T}) = \infty$; if k satisfies (iv) let $N_f(k, \mathbf{T}) = 0$.

EXAMPLE. With $\mathbf{T}$ as in (1.6.2) we have $N_f(1, \mathbf{T}) = 0$, $N_f(2, \mathbf{T}) = 3$, $N_f(3, \mathbf{T}) = 2$, $N_f(4, \mathbf{T}) = 1$, and $N_f(5, \mathbf{T}) = \infty$.

(1.5.15) PROPOSITION. *Let $k \in M$ and let S be the variable square in $P(k, T)$. Then exactly one of the following situations hold:*

(i) *There is a unique $k' \in M$ with $S \in P'(k', \mathbf{T})$. $P(k', T)$ is boxed $\Leftrightarrow P(k, T)$ is boxed.*
(ii) *$P(k, T)$ is unboxed and S is a filled corner of T.*
(iii) *$P(k, T)$ is boxed and S is a filled hole of T.*
(iv) *$\mathbf{T} \in \mathscr{T}_C(M)$, $P(k, T)$ is boxed, and $S = S_{11}$.*

(1.5.16) DEFINITION. Let $k \in M$. Define $N_b(k, \mathbf{T})$ according to the four situations of Proposition (1.5.15): if k satisfies (i) let $N_b(k, \mathbf{T}) = k'$; if k satisfies (ii) or (iii) let $N_b(k, \mathbf{T}) = \infty$; if k satisfies (iv) let $N_b(k, \mathbf{T}) = 0$.

(1.5.16a) REMARK. Let $k \in M$ be such that $N_b(k, \mathbf{T}) \notin \{0, \infty\}$ (resp. $N_f(k, \mathbf{T}) \notin \{0, \infty\}$.) Then $N_f(N_b(k, \mathbf{T}), \mathbf{T}) = k$ (resp. $N_b(N_f(k, \mathbf{T}), \mathbf{T}) = k$.)

(1.5.17) PROPOSITION. (1) *Let S be an empty hole in T. Then there is a unique $k \in M$ such that $S \in P'(k, \mathbf{T})$. $P(k, T)$ is unboxed.*

(2) *Let S be an empty corner in T, or let $T \in \mathscr{T}_B(M)$ and $S = S_{11}$. Then there is a unique $k \in M$ such that $S \in P'(k, \mathbf{T})$. $P(k, T)$ is boxed.*

Proof of (1.5.17)–(1). Let $S_{ij} \subset \mathscr{F}$ be an empty hole in T. Let $k = N_T(S_{i-1,j})$ and let $k' = N_T(S_{i,j-1})$. Since neither $S_{i,j+1}$ nor $S_{i+1,j}$ are contained in Shape(T), if $S_{ij} \in P'(k'', \mathbf{T})$ then either $k'' = k$ or $k'' = k'$. Suppose first $S_{i-1,j-1} \in P(k, T)$. We note first that the variable square in $P'(k, \mathbf{T})$ is either $S_{i-2,j}$ or $S_{i-1,j+1}$ and not S_{ij}. Furthermore, we have $k' > k$ so $P'(k', \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}$. Since $S_{i-1,j-1} \notin P(k, T)$, $P(k, T)$ is unboxed. A similar argument holds if $S_{i-1,j-1} \in P(k', T)$. Suppose then that either $k = 0$ or $k' = 0$ or $S_{i-1,j-1} \notin P(k, T) \cup P(k', T)$. Then we have $S_{ij} \in P'(k, \mathbf{T}) \Leftrightarrow k > k'$ and $S_{ij} \in P'(k, \mathbf{T}) \Leftrightarrow k' > k$. In either case the relevant subset of $\mathscr{F}$ is unboxed.

We now define the concept of a cycle of a domino tableau.

(1.5.18) DEFINITION. We define an equivalence relation (call it $\sim$) on M, relative to $\mathbf{T}$, by setting $\sim$ to be the equivalence relation generated by $k \sim N_f(k, \mathbf{T})$ when $N_f(k, \mathbf{T}) \notin \{0, \infty)$ and $k \sim N_b(k, \mathbf{T})$ when $N_b(k, \mathbf{T}) \notin \{0, \infty\}$. The equivalence classes of $\sim$ are called cycles in $\mathbf{T}$. For $k \in M$ the equivalence class containing k is denoted $c(k, \mathbf{T})$.

EXAMPLES. See (1.6.2)(iii), (1.6.3)(iii), and (1.6.4)(iii).

(1.5.19) DEFINITION. Let c be a cycle in $\mathbf{T}$. We say c is closed if $\forall r \in c$, $N_b(r, \mathbf{T}) \notin \{0, \infty)$ and $N_f(r, \mathbf{T}) \notin \{0, \infty\}$, otherwise c is called open. We write $\mathrm{OC}(\mathbf{T})$ for the set of open cycles in $\mathbf{T}$.

(1.5.20) PROPOSITION. (1) *If c is a closed cycle in $\mathbf{T}$ then $c = \{r_1, \dots, r_n\}$ where $N_f(r_i, \mathbf{T}) = r_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $N_f(r_n, \mathbf{T}) = r_1$.*

(2) *If c is an open cycle in $\mathbf{T}$ then $c = \{r_1, \dots, r_n\}$ where $N_f(r_i, \mathbf{T}) = r_{i+1}$ for $1 \leqslant i \leqslant n-1$, $N_f(r_n, \mathbf{T}) \in \{0, \infty\}$ and $N_b(r_1, \mathbf{T}) \in \{0, \infty\}$.*

Proof. This follows from Propositions (1.5.13) and (1.5.15).

(1.5.21) DEFINITION. Let c be an open cycle in $\mathbf{T}$. Let r_1, r_n be as in Proposition (1.5.20)-(2). We define $S_f(c) = S_f(c, \mathbf{T})$ to be the variable square in $P'(r_n, \mathbf{T})$ and $S_b(c) = S_b(c, \mathbf{T})$ to be the variable square in $P(r_1, T)$.

EXAMPLE. Let $\mathbf{T}$ be as in (1.6.3). If $c = \{2, 4, 5\}$ then $S_f(c) = S_{42}$ and $S_b(c) = S_{33}$. If $c = \{1, 3\}$ then $S_f(c) = S_{51}$ and $S_b(c) = S_{11}$.

(1.5.21a) PROPOSITION. *Let $S \in \mathscr{F}$. If S is either an empty hole or an empty corner in $\mathbf{T}$ then there exists a unique $c \in \mathrm{OC}(\mathbf{T})$* with $S_f(c) = S$.

Proof. This follows from Proposition (1.5.17).

NOTATION. Let S be as in Proposition (1.5.21a). We write $c(S, \mathbf{T})$, the cycle through S in $\mathbf{T}$, for the c of that proposition.

(1.5.22) PROPOSITION. *Let c be a cycle in $\mathbf{T}$; let $k, k' \in c$. Then $P(k, T)$ is boxed $\Leftrightarrow P(k', T)$ is boxed.*

Proof. This follows from Propositions (1.5.13) and (1.5.20).

(1.5.23) DEFINITION. Let c be a cycle in $\mathbf{T}$. If $\forall k \in c$, $P(k, T)$ is boxed we say c is boxed; otherwise c is unboxed.

The following proposition describes the different types of open cycles. In the process of making **T** have special shape, however, only case (i) ever occurs.

(1.5.24) PROPOSITION. *Let c be an open cycle in* **T**. *Then one of the following four situations obtains:*

(i) *c is unboxed,* $S_b(c)$ *is a filled corner in* **T**, *and* $S_f(c)$ *is an empty hole of* **T**.
(ii) *c is boxed,* $S_b(c)$ *is a filled hole in* **T**, *and* $S_f(c)$ *is an empty corner of* **T**.
(iii) $\mathbf{T} \in \mathscr{T}_B(M)$, *c is boxed,* $S_b(c)$ *is a filled hole in* **T**, *and* $S_f(c) = S_{11}$.
(iv) $\mathbf{T} \in \mathscr{T}_C(M)$, *c is boxed,* $S_b(c) = S_{11}$, *and* $S_f(c)$ *is an empty corner in* **T**.

Proof. This follows from Propositions (1.5.13), (1.5.15), and (1.5.20).

(1.5.25) DEFINITION. Let $k \in M$. Let $D'(k, \mathbf{T}) = \{(k, S_{ij}) \mid S_{ij} \in P'(k, \mathbf{T})\}$.

The following defines the process of moving through a cycle. Note that the grid of the tableau will change if cases (2) or (3) hold and we move it through the cycle c. Case (1) of (1.5.26) corresponds to either (i), (ii) of (1.5.24) or c's being closed: case (3), to case (ii) of (1.5.24): and case (2), to case (iv) of (1.5.24). For the definition of S, only case (1) is needed. However, we will need cases (2) and (3) for the applications to Harish-Chandra modules. (The grid changes because we will need to work on both sides of the character-multiplicity duality of Vogan, [11].) In fact, for the definition of S, the subcase of case (1) which is c's being closed, never arises either (as remarked previously to (1.5.24)). However, this possibility must be taken into account in Part II of this paper, in order to describe the effect of $T_{\alpha\beta}$ (see Vogan, [10], 3.4) on tableaux.

(1.5.26) DEFINITION. Let c be a cycle in $\mathbf{T} = (T, \phi)$.
(1) If either c is closed or both c is open and $S_b(c) \neq S_{11}, S_f(c) \neq S_{11}$, let

$$\mathbf{E}(\mathbf{T}, c) = \left(\left(T \setminus \bigcup_{k \in c} D(k, T)\right) \cup \bigcup_{k \in c} D'(k, \mathbf{T}), \phi\right).$$

(2) If $S_b(c) = S_{11}$ let

$$\mathbf{E}(\mathbf{T}, c) = \left(\left(T \setminus \bigcup_{k \in c} D(k, T)\right) \cup \bigcup_{k \in c} D'(k, \mathbf{T}) \cup \{(0, S_{11})\}, \phi_B\right).$$

(3) If $S_f(c) = S_{11}$ let

$$\mathbf{E}(\mathbf{T}, c) = \left(\left(T \setminus \left(\bigcup_{k \in c} D(k, T) \cup \{(0, S_{11})\}\right)\right) \cup \bigcup_{k \in c} D'(k, \mathbf{T}), \phi_c\right).$$

EXAMPLES. See (1.6.2)(v) and (vi), (1.6.3)(v) and (vi), and (1.6.4)(iv) and (v) for examples of the first case of the definition; see (1.6.3)(iv) for an example of the second case; see (1.6.2)(iv) for an example of the third case.

We now have to check that this procedure results in a domino tableau.

(1.5.27) PROPOSITION. *Let c be a cycle in* $\mathbf{T}$.

(1) *Suppose either c is closed or both c is open and $S_b(c) \neq S_{11}, S_f(c) \neq S_{11}$. If $\mathbf{T} \in \mathcal{T}$ where $\mathcal{T}$ is either $\mathcal{T}_B(M), \mathcal{T}_C(M)$, or $\mathcal{T}_D(M)$, then $\mathbf{E}(\mathbf{T}, c) \in \mathcal{T}$.*

(2) *Suppose $S_b(c) = S_{11}$ (so $\mathbf{T} \in \mathcal{T}_C(M)$). Then $\mathbf{E}(\mathbf{T}, c) \in \mathcal{T}_B(M)$.*

(3) *Suppose $S_f(c) = S_{11}$ (so $\mathbf{T} \in \mathcal{T}_B(M)$). Then $\mathbf{E}(\mathbf{T}, c) \in \mathcal{T}_C(M)$.*

Proof. This follows from the definitions and from Propositions (1.5.10) and (1.5.20).

We now need to show that in moving through several cycles consecutively, the order of these operations is irrelevant.

(1.5.28) PROPOSITION. *Let c be a cycle in $\mathbf{T} = (T, \phi)$, and let $k \in M$.*

(1) *If $k \notin c$ then $P'(k, \mathbf{E}(\mathbf{T}, c)) = P'(k, \mathbf{T})$.*

(2) *If $k \in c$ then $P'(k, \mathbf{E}(\mathbf{T}, c)) = P(k, T)$.*

Proof. Let $\mathbf{E}(\mathbf{T}, c) = (T', \phi')$. Then either $\phi' = \phi$ or $\phi' = \phi^*$. By Proposition (1.5.12–2) it suffices to prove the proposition with (T', ϕ) in place of $\mathbf{E}(\mathbf{T}, c)$. Part (1) of the proposition then follows from the fact that if $S \in \mathcal{F}$ is a fixed square or if $S \in \mathcal{F}^0 \backslash \mathcal{F}$ then $N_T(S) = N_{T'}(S)$ (see Remark (1.5.9)). For (2), suppose for example $P(k, T) = \{S_{ij}, S_{i,j+1}\}$ with S_{ij} fixed. Then $P(k, T') = P'(k, \mathbf{T})$ is either $\{S_{i,j-1}, S_{ij}\}$ or $\{S_{ij}, S_{i+1,j}\}$. Thus to find $P'(k, (T', \phi))$ we need to compare k with $r = N_{T'}(S_{i-1,j+1})$. Since $S_{i-1,j+1}$ is also fixed or in $\mathcal{F}^0 \backslash \mathcal{F}$, we have $r = N_T(S_{i-1,j+1})$, so $k > r$ and $P'(k, (T', \phi)) = P(k, T)$. The other three cases are similar.

(1.5.29) COROLLARY. *Let c be a cycle in $\mathbf{T}$. Let $c' \subseteq M$. Then c' is a cycle in $\mathbf{T} \Leftrightarrow c'$ is a cycle in $\mathbf{E}(\mathbf{T}, c)$.*

(1.5.30) DEFINITION. Let $c_1, \ldots, c_k$ be cycles in $\mathbf{T}$. For $k \geqslant 2$, define inductively,

$$\mathbf{E}(\mathbf{T}, c_1, \ldots, c_k) = \mathbf{E}(\mathbf{E}(\mathbf{T}, c_1, \ldots, c_{k-1}), c_k).$$

This definition makes sense by Corollary (1.5.29).

(1.5.31) PROPOSITION. $\mathbf{E}(\mathbf{T}, c_1, \ldots, c_k) = \mathbf{E}(\mathbf{T}, c_{\sigma(1)}, \ldots, c_{\sigma(k)})$ *for any $\sigma \in S_k$.*

Proof. It suffices to show that, for $c\,c'$ cycles in $\mathbf{T}$, $\mathbf{E}(\mathbf{T}, c, c) = \mathbf{T}$ and $\mathbf{E}(\mathbf{T}, c, c') = \mathbf{E}(\mathbf{T}, c', c)$. These follow from Proposition (1.5.28).

NOTATION. Let $c_1, \ldots, c_k$ be distinct cycles in $\mathbf{T}$. We write $\mathbf{E}(\mathbf{T}, \{c_1, \ldots, c_k\})$ for $\mathbf{E}(\mathbf{T}, c_1, \ldots, c_k)$; this is well-defined by the proposition.

(1.5.32) REMARK. (1) Let c be a cycle in $\mathbf{T}$ satisfying the hypotheses of case 1 of Definition (1.5.26). Then c is boxed in $\mathbf{E}(\mathbf{T}, c) \Leftrightarrow c$ is unboxed in $\mathbf{T}$. If c' is a cycle in $\mathbf{T}$, $c' \neq c$, then c' is boxed in $\mathbf{E}(\mathbf{T}, c) \Leftrightarrow c'$ is boxed in $\mathbf{T}$.

(2) Suppose c is a cycle in $\mathbf{T}$ where c and $\mathbf{T}$ satisfy the hypotheses either of case

(2) or of case (3) of Definition (1.5.26). Then c is boxed in both **T** and **E**(**T**, c). If c' is a cycle in **T**, $c' \neq c$ then c' is boxed in **E**(**T**, c) $\Leftrightarrow c'$ is unboxed in **T**. (These follow from Remark (1.5.9) and Proposition (1.5.12), recalling that $\phi_B = \phi_C^*, \phi_C = \phi_B^*$.)

The procedure **E**, of moving a tableau through a cycle, affects the shape as follows:

(1.5.33) PROPOSITION. *Let c be a cycle in* **T** *and write* **E**(**T**, c) $= (T', \phi')$.

(1) *If c is a closed cycle then* Shape(T') = Shape(T).

(2) *If c is open and $S_f(c) \neq S_{11}, S_b(c) \neq S_{11}$ then* Shape(T') = (Shape(T)$\backslash S_b(c)$) $\cup$ $S_f(c)$.

(3) *If c is open and $S_f(c) = S_{11}$ then* Shape(T') = Shape(T)$\backslash S_b(c)$.

(4) *If c is open and $S_b(c) = S_{11}$ then* Shape(T') = Shape(T) $\cup S_f(c)$.

Proof. These are clear from the definitions and from Proposition (1.5.20).

We now define the algorithm S, which changes a domino tableau into one of special shape.

(1.5.34) DEFINITION. Let $S_1, \ldots, S_k \in \mathscr{F}$ be the (distinct) filled corners of T. Let $c_i = c(S_i, \mathbf{T})$ for $1 \leqslant i \leqslant k$. Define **S**(**T**) = **E**(**T**, $c_1, \ldots, c_k$). **S**(**T**) is called the special tableau with grid associated to **T**.

EXAMPLES: See (1.6.2)(vii), (1.6.3)(v), and (1.6.4)(iv).

(1.5.35) THEOREM. **S**(**T**) *is special.*

Proof. It is clear that if c' is an open cycle in **T** then the choice of $r_1, \ldots, r_n$ satisfying Proposition (1.5.20)–(2) is unique. It follows from Proposition (1.5.28)–(1) that if c is any cycle in **T** and $c' \neq c$ is an open cycle in **T** then the $r_1, \ldots, r_n$ satisfying Proposition (1.5.20)–(2) for c' in **T** also satisfy it for c' in **E**(c, **T**). In particular, $S_b(c', \mathbf{E}(\mathbf{T}, c)) = S_b(c', \mathbf{T})$ and $S_f(c', \mathbf{E}(\mathbf{T}, c)) = S_f(c', \mathbf{T})$.

Let c_i, $1 \leqslant i \leqslant k$ be as in Definition (1.5.34). The c_i's are in situation (i) of Proposition (1.5.24) for **T**, and thus by the above, each c_i, $2 \leqslant i \leqslant k$, is in situation (i) of Proposition (1.5.24) for **E**(**T**, $c_1, \ldots, c_{i-1}$). Then by Proposition (1.5.33)–(2) **S**(**T**) has no filled corners. But Proposition (1.5.21a) and Proposition (1.5.24) show that a tableau with grid has the same number of empty holes and of filled corners. Thus **S**(**T**) is special.

(1.5.36) REMARK. If c_i is as in Definition (1.5.34) then as in the proof of Theorem (1.5.35), each c_i is in situation (i) of Proposition (1.5.24) for **E**(**T**, $c_1, \ldots, c_{i-1}$). Then if $\mathbf{T} \in \mathscr{T}_B(M)$ (resp. $\mathscr{T}_C(M)$, $\mathscr{T}_D(M)$) we have $\mathbf{S}(\mathbf{T}) \in \mathscr{T}_B(M)$ (resp. $\mathscr{T}_C(M)$, $\mathscr{T}_D(M)$).

(1.5.37) NOTATION. (a) Define S(**T**) by **S**(**T**) = (S(**T**), ϕ).

(b) Let $\mathscr{T}_B^S(M) = \{\mathbf{T} \in \mathscr{T}_B(M) | \mathbf{S}(\mathbf{T}) = \mathbf{T}\}$; similarly $\mathscr{T}_C^S(M)$, $\mathscr{T}_D^S(M)$.

(1.5.38) PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_R^S(M)$ *where $R = B, C$, or D. Let $H_1, \ldots, H_k$ be the filled holes of* **T**. *If $R = C$ or D let* $U = \{c(H_i, \mathbf{T}) \,|\, 1 \leqslant i \leqslant k\}$; *if $R =$*

B let $U = \{c(H_i, \mathbf{T}) \mid 1 \leqslant i \leqslant k\} \backslash \{c(S_{11}, \mathbf{T})\}$. *Then* $\{\mathbf{T}' \in \mathcal{T}_R(M) \mid \mathbf{S}(\mathbf{T}') = \mathbf{T}\} = \{\mathbf{E}(\mathbf{T}, V) \mid V \subseteq U\}$.

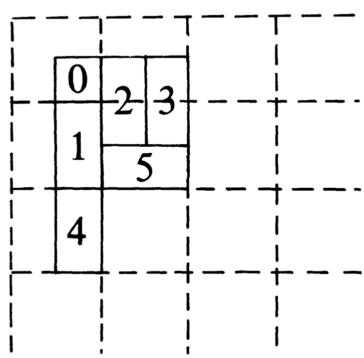
Proof. This follows from the fact that $\mathbf{E}(\mathbf{T}, c, c) = \mathbf{T}$ (see the proof of Proposition (1.5.31)), Proposition (1.5.24) and Proposition (1.5.33)–(2).

Section 6

In this section we illustrate the preceding section with some examples, showing how the result of the process of moving a tableau through a cycle, and the result of S, may be computed.

(1.6.1) We illustrate a tableau with grid (T, ϕ) by superimposing the tableau on an array of 2×2 boxes, in such a way that each square S which lies in the upper-left-hand corner of a box satisfies $\phi(S) = X$, etc. Note that this agrees with the definition of boxed (1.5.7), that is, a set $P \subset \mathcal{F}$ is boxed if and only if P is contained in one of the 2×2 boxes.

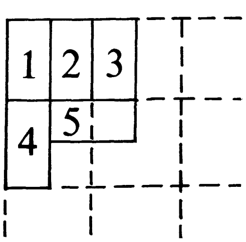
(1.6.2) (i) Let $\mathbf{T} \in \mathcal{T}_B(\{1, 2, 3, 4, 5\})$ be as displayed below:



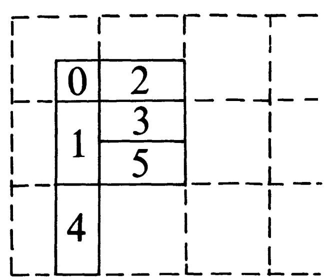
(ii) Then $P(k, T)$ is boxed if and only if $k \in \{1, 4, 5\}$.

(iii) The cycles in $\mathbf{T}$ are $\{1, 4\}$, $\{2, 3\}$, and $\{5\}$; the first and third are open. We have

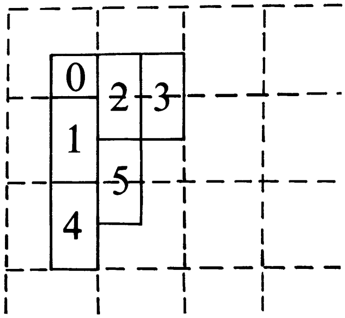
(iv) $\mathbf{E}(\mathbf{T}, \{1, 4\}) =$



(v) $\mathbf{E}(\mathbf{T}, \{2, 3\}) =$

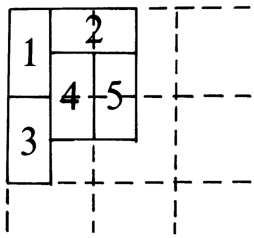


(vi) **E**(**T**, $\{5\}$) =



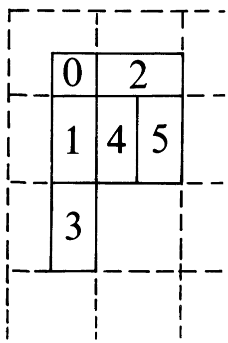
(vii) and **S**(**T**) = **T**.

(1.6.3) (i) Let $\mathbf{T} \in \mathscr{T}_C(\{1, 2, 3, 4, 5\})$ be as displayed below:

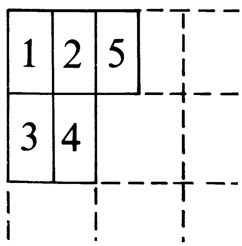


(ii) Then $P(k, T)$ is boxed if and only if $k \in \{1, 3\}$.
(iii) The cycles in **T** are $\{1, 3\}$ and $\{2, 4, 5\}$; both are open. We have:

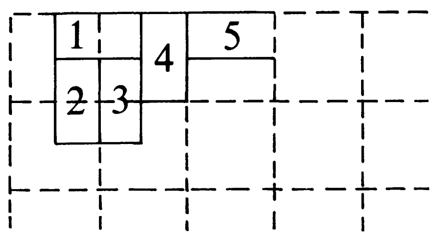
(iv) **E**(**T**, $\{1, 3\}$) =



(v) **S**(**T**) = **E**(**T**, $\{2, 4, 5\}$) =



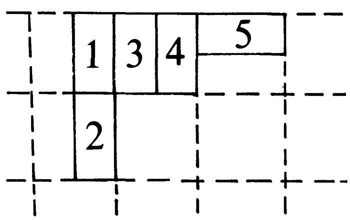
(1.6.4) (i) Let $\mathbf{T} \in \mathcal{T}_D(\{1, 2, 3, 4, 5\})$ be as displayed below:



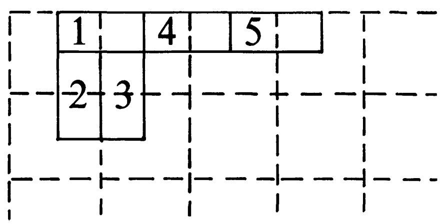
(ii) Then $P(k, T)$ is boxed if and only if $k \in \{4, 5\}$.

(iii) The cycles in **T** are $\{1, 2, 3\}$ and $\{4, 5\}$; both are open. We have

(iv) **S(T)**
$= \mathbf{E}(\mathbf{T}, \{1, 2, 3\}) =$



(v) $\mathbf{E}(\mathbf{T}, \{4, 5\}) =$



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