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HARALD BJAR
OLAV ARNFINN LAUDAL

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Deformation of Lie algebras and Lie algebras of deformations

HARALD BJAR & OLAV ARNFINN LAUDAL
Institute of Mathematics, University of Oslo, Oslo, Norway

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Introduction

This paper is a following up of some of the main ideas of the monograph [La-Pf], on which it depends notationally. The starting point is the fact, see (2.6) (iv) and (v) loc. cit., that the study of the local moduli for any k-scheme X naturally leads to the study of flat families of Lie algebras.

If $\pi: \mathbf{X} \to \mathbf{H} = \operatorname{Spec}(H)$ is a miniversal deformation of X, and if the sub Lie algebra $\mathbf{V}$ of $\operatorname{Der}_k(H)$ is the kernel of the Kodaira–Spencer map associated to π, then the prorepresenting substratum $\mathbf{H}_0 = \operatorname{Spec}(H_0)$ of $\mathbf{H}$ is, by definition, the complement of the support of $\mathbf{V}$, and $\Lambda^0 = \mathbf{V} \otimes_H H_0$, is a flat H_0-Lie algebra defining a deformation of the Lie algebra $L^0(X) = H^0(X, \theta_X)/A_\pi$, where A_π is the Lie ideal of those infinitesimal automorphisms of X that lifts to $\mathbf{X}^\wedge = \mathbf{X} \otimes_H H^\wedge$.

If $X = \operatorname{Spf}(k[[\boldsymbol{x}]]/(f))$ where $f \in k[\boldsymbol{x}] = k[x_1, \dots, x_n]$ is an isolated hypersurface singularity, then, putting $L^0(f) = L^0(X)$, we find (see Section 4. loc. cit.) that $L(f) = \operatorname{Der}(k[[\boldsymbol{x}]]/(f))/\operatorname{Der}_\pi$, where Der_π is the Lie ideal generated by the trivial deformations of the form, $E_{ij} \in \operatorname{Der}(k[[\boldsymbol{x}]]/(f))$, $E_{ij}(x_k) = 0$ for $k \neq i, j$ $E_{ij}(x_i) = \partial f/\partial x_j$, and $E_{ij}(x_j) = -\partial f/\partial x_i$. It is easy to see that $\dim_k L(f) = \dim_k k[[\boldsymbol{x}]]/(f, \partial f/\partial x_i)_i = \tau(f)$.

The family Λ^0 defines, in a natural way a map, $\mathbb{I}: \mathbf{H}_0 \to$ moduli space of Lie algebras of dimension $\tau(f)$, associating to the closed point t of $\mathbf{H}_0$ the class of the Lie algebra $\Lambda^0(t) = L^0(F_0(t))$, where F_0 is the restriction of the miniversal family F of f to $\mathbf{H}_0$.

Assume from now on that the field k is of characteristic 0. The main purpose of this paper is the proof of Theorem (5.9), which states that for the quasi-homogenous isolated plane curve singularity $f = x_1^k + x_2^l$, the map $\mathbb{I}$ is, "locally", an immersion, except for some very special cases.

To make this statement precise we first have to recall a few facts from the moduli theory of Lie algebras, and add a couple of rather easy consequences of the general theory of [La-Pf].

Moduli theory for Lie algebras has been studied by a number of mathematicians, for a long time, see for example [C], [C-D], [Fi], [K-N], [Mo], [Ra], [Ri], [Vi].

Identifying a Lie algebra, up to base change, with the set of structural constants $\{c_{ij}^k\}$ corresponding to some basis $\{x_i\}_i$, one sees immediately that the set of isomorphism classes of n-dimensional Lie algebras defined over a field k of characteristic 0, may be identified with a set of orbits $\mathbf{L}_n := \mathbf{Lie}_n/\mathrm{Gl}_n(k)$, where $\mathbf{Lie}_n$ is the closed subscheme of the affine space $\mathbf{A}^{1/2(n-1)n^2}$, of all systems of structural constants, defined by a set of quadratic equations deduced from the Jacobi identities, see Section 2.

Since $\mathrm{Gl}_n(k)$ is reductive there exists in the category of schemes a categorical quotient of $\mathbf{Lie}_n$ by $\mathrm{Gl}_n(k)$, see [M-F], (1.1). But since the action of $\mathrm{Gl}_n(k)$ is not closed, this quotient is not a geometric quotient, and the set of closed points cannot, in general, be identified with the set of orbits $\mathbf{L}_n$.

To formulate and prove the Theorem (5.9), refered to above, we therefore have to work a little, developing the deformation theory and the local moduli of Lie algebras along the lines of [La 1] and [La-Pf].

We start by taking another look at the cohomology of Lie algebras. This is the subject of Section 1. We then carry over to the Lie algebra case the obstruction calculus for the deformation functor, see [La 1], and many of the results of Sections 1,2,3, of [La-Pf]. In particular we shall consider the versal family $\mathbf{K}^n$ of Lie algebras defined on $\mathbf{Lie}_n$, and its Kodaira-Spencer map. Copying the proof of (3.18) of [La-Pf], we find that there exists, in the category of algebraic spaces, a good quotient $\mathbf{L}(\boldsymbol{h})$, $\boldsymbol{h} = (h_0, \ldots, h_n)$. of

$$\mathbf{Lie}_n(\boldsymbol{h}) := \{t \in \mathbf{Lie}_n \mid \dim_k H^i(\mathbf{K}^n(t), \mathbf{K}^n(t)) = h_i, i = 0, \ldots, n\}$$

by the action of $\mathrm{Gl}_n(k)$. Going back to the definition of the map $\mathbb{I}$, we observe that the family Λ^0 restricted to,

$$\mathbf{H}(\boldsymbol{h}) := \{t \in \mathbf{H}_0 \mid \dim_k H^i(\Lambda^0(t), \Lambda^0(t)) = h_i, i = 0, \ldots, \tau(f)\},$$

defines a morphism of algebraic spaces

$$I(h)\colon \mathbf{H}(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h}).$$

The main result now states that if f is the quasihomogenous isolated plane curve singularity $f = x_1^k + x_2^l$, then, in a neighbourhood of the base point $\boldsymbol{O}$ of $\mathbf{H}$, $I(\boldsymbol{h})$ is an immersion, except for some very special cases. It is easy to see that there are exceptions. In fact, an elementary computation shows that for $f(x_1, x_2) = x_1^4 + x_2^4$ the family Λ^0 is constant on $\mathbf{H}_0$.

In the process we are led to consider, for every hypersurface singularity f, a graded Lie algebra $L^*(f)$, and a corresponding map of the type $l(\boldsymbol{h})$ above, which we conjecture always is an immersion, see Section 3.

In [M-Y], and [Y], Mather and Yau consider the correspondences

$$f \dashrightarrow k[[\boldsymbol{x}]]/(f, \partial f/\partial x_i) =: A(f)$$
$$f \dashrightarrow \mathrm{Der}_k(A^1(f)).$$

They prove that $A(f)$, as a k-algebra, characterizes the singularity f, and they prove, in low dimensions, that $\mathrm{Der}_k(A(f))$ is a solvable Lie algebra.

Even for very simple singularities like E_6, our $L^0(f)$ is different from $\mathrm{Der}_k(A(f))$, and we don't see any immediate relationship between these two invariants. Notice also that for general singularities, A is not an algebra, therefore $\mathrm{Der}_k(A)$ is not defined. The Lie algebra $L^0(f)$, however, has an obvious generalization, see Sections2,3 of [La-Pf].

It is easy to see that when f is a quasihomogenous plane curve singularity, the Lie algebra $L^*(f)$, and except for some special cases, even $L^0(f)$, determines the Mather-Yau algebra, and therefore the singularity. Thus the morphism $l(\boldsymbol{h})$ induces an injective map from the set of isomorphism classes of singularities into the set of isomorphism classes of Lie algebras. Notice, however, that the modular stratum $\mathbf{H}_0$ is not a coarse moduli space. There is, in general, a nontrivial discrete group $\mathbf{G}_0$ acting on $\mathbf{H}_0$, identifying points with isomorphic fibres (see [La-M-Pf], p. 274), such that the family Λ^0 does not pass to the quotient. Theorem (5.9) therefore links the filtration $\{\mathbf{H}(\boldsymbol{h})\}_{\boldsymbol{h}}$ of $\mathbf{H}_0$ to the action of $\mathbf{G}_0$.

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1. Cohomology of Lie algebras

For the purpose of studying deformations of Lie algebras we need a cohomology theory and an obstruction calculus, see e.g. [La 1]. There is such a cohomology theory, due to Chevalley and Eilenberg [Ch-E], and one knows how to define the obstructions we need, see [Fi], [Ra], [Ri].

We shall, never the less, in this paragraph, define another set of cohomology groups, that fits more naturally into our development of the deformation theory, as described in [La 1]. Of course we shall prove that the new cohomology and the old one coincide, modulo a change of degree, and apart from the first two groups.

Let us first recall the Chevalley-Eilenberg-MacLane cohomology of a Lie algebra $\mathbf{g}$ defined on a field k, see [Ca-E]. Consider the functor,

$$H^0(\mathbf{g},\text{-}): \mathbf{g}\text{-mod} \to k\text{-mod}$$

defined by $H^0(\mathbf{g}, M) = M^{\mathbf{g}} := \{m \in M \mid x \in \mathbf{g}, x(m) = 0\}$.Then the cohomology of $\mathbf{g}$ with values in the $\mathbf{g}$-module M, $H^i(\mathbf{g}, M)$, is the ith derived of the above functor, applied to M. As usual there is an exact functor of complexes

$$C^*(\text{-}, M)\text{: Lie alg.} \to \text{complexes of } k\text{-mod}$$

with, in this case, $C^p(\mathbf{g}, \mathbf{M}) = \mathrm{Hom}_k(\Lambda^p \mathbf{g}, M)$, and differential $\mathrm{d}\colon C^p(\mathbf{g}, M) \to C^{p+1}(\mathbf{g}, M)$ defined by,

$$\mathrm{d}(f)(x_0 \Lambda \ldots \Lambda x_p) = \sum_i (-1)^i x_i f(x_0 \Lambda \ldots \Lambda \hat{x}_i \Lambda \ldots \Lambda x_p) + \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} f([x_i, x_j] \Lambda x_0 \Lambda \ldots \Lambda \hat{x}_i \Lambda \ldots \Lambda \hat{x}_j \Lambda \ldots \Lambda x_p),$$

such that $H^p(\mathbf{g}, M) = H^p(C^*(\mathbf{g}, M))$, $p \geqslant 0$.

Now, let S be any commutative ring with unit. We may consider the category S-**lie** of Lie algebras defined on S, and the full subcategory **free** of S-**lie** generated by the free S-Lie algebras, see [J]. Given any S-Lie algebra $\mathbf{g}_S$ and an S-module M, a $\mathbf{g}_S$-module structure on M is, of course, nothing but a homomorphism of S-Lie algebras $\mathbf{g}_S \to \mathrm{End}_S(M)$. We shall denote by $\mathbf{g}_S$-mod the category of $\mathbf{g}_S$-modules.

In the particular case where the S-Lie algebra $\mathbf{g}_S$ is free as an S-module, we shall extend the definition of the complex $C^*(\mathbf{g}_S, M)$ in the obvious way, and we shall denote by $H^*(\mathbf{g}_S, M)$ the resulting cohomology.

Associated to any S-Lie algebra $\mathbf{g}_S$ there is the category **free**/$\mathbf{g}_S$ of all morphisms of S-Lie algebras $\partial\colon \mathbf{F} \to \mathbf{g}_S$, where $\mathbf{F}$ is a free S-Lie algebra, and morphisms being morphisms between the free S-Lie algebras inducing a commutative diagram. Put, for any $\mathbf{g}_S$-module M,

$$\mathrm{Der}_S(\mathbf{g}_S, M) = \{D \in \mathrm{Hom}_S(\mathbf{g}_S, M) \mid D([x_1, x_2]) = x_1 D(x_2) - x_2 D(x_1)\}.$$

Obviously this defines a contravariant functor,

$$\mathrm{Der}_S(-, M)\colon \mathbf{free}/\mathbf{g}_S \to S\text{-mod},$$

and we may define, just like in [La 1], Chapter 2, the cohomology groups

$$A^i(S, \mathbf{g}_S; M) = \mathrm{limproj}^{(i)}\, \mathrm{Der}_S(-, M),$$

where $\text{limproj}^{(i)}$ is the ith. derived of the projective limit functor on the dual category of $\mathbf{free}/\mathbf{g}_S$. Notice that there is an S-module homomorphism $i: M \to \text{Der}_S(\mathbf{g}_S, \mathbf{M})$ defined by $i(m)(x) = xm$, $x \in \mathbf{g}_S$, $m \in M$. Notice also that if $\pi: R \to S$ is a homomorphism of commutative rings such that $(\ker \pi)^2 = 0$, and if there are given morphisms of R-flat R-Lie algebras $\psi_1, \psi_2 : \mathbf{g}' \to \mathbf{g}''$ such that $\psi_1 \otimes_R S = \psi_2 \otimes_R S$ then the R-module homomorphism $\psi_1 - \psi_2$ decomposes into the composition of $\mathbf{g}' \to \mathbf{g}_1 := \mathbf{g}' \otimes_R S$, a derivation D: $\mathbf{g}_1 \to \mathbf{g}_2 \otimes_S \ker \pi$, where $\mathbf{g}_2 := \mathbf{g}'' \otimes_R S$, and the obvious embedding $\mathbf{g}_2 \otimes_S \ker \pi \to \mathbf{g}''$.

With this done we may copy the procedure of [La 1] and obtain an obstruction calculus for the deformation functor of any S-Lie algebra, see Chapter 4, loc. cit. Before we sketch the results, let us prove the following,

THEOREM (1.1). Given any S-free S-Lie algebra $\mathbf{g}$, and any S-free $\mathbf{g}$-module M, there exists an exact sequence,

(i) $0 \to H^0(\mathbf{g}, M) \to M \to A^0(S, \mathbf{g}; \mathbf{M}) \to H^1(\mathbf{g}, M) \to 0,$

and there are isomorphisms,

(ii) $\theta_i: A^i(S, \mathbf{g}, M) \cong H^{i+1}(\mathbf{g}, M), \quad i \geqslant 1.$

Proof. (i) is the definition of $H^1(\mathbf{g}, M)$. To prove (ii), recall that $H^i(\mathbf{g}, -) = 0$ if $\mathbf{g}$ is a free S-Lie algebra, and consider the functor,

$$C^*(-, M): \mathbf{free}/\mathbf{g} \to \text{compl. of } S\text{-mod}.$$

defined by $C^*(\partial, M) = C^*(\mathbf{F}, M)$, for an object $\partial: \mathbf{F} \to \mathbf{g}$ of $\mathbf{free}/\mathbf{g}$.

We shall prove,

(1) $\text{limproj}^{(k)} C^*(-, M)$ is equal to 0 if $k \neq 0$, and to $C^*(\mathbf{g}, M)$ if $k = 0$. Assume for a moment that this is done, and let $C_1^*(-, \mathbf{M})$ be the subcomplex of $C^*(-, M)$ for which $C_1^0(-, M) = 0$, $C_1^i(-, M) = C^i(-, M)$ for $i \geqslant 1$. Notice that $H^1(C_1^*(-, M)) = \text{Der}_S(-, M)$, and that $H^i(\mathbf{F}, -) = 0$ for $\mathbf{F}$ a free S-Lie algebra and $i \geqslant 2$.

Consider the resolving complex $E^*(-)$ for the functor limproj, defined on $\mathbf{free}/\mathbf{g}$, and the double complex $E^*(C_1^*(-, M))$. It follows from (1) that the two spectral sequences of this double complex degenerate. Therefore $\text{limproj}^{(p)}$ $\text{Der}_S(-, M) \cong H^{p+1}(C_1^*(\mathbf{g}, M))$, which is (ii).

Now to prove (1), consider the forgetful functor

$$\mu: \mathbf{free}/\mathbf{g} \to S\text{-mod, defined by } \mu(\mathbf{F} \to \mathbf{g}) = \mathbf{F},$$

and the resolving complex $E_*(-)$ for the functor limind defined on $\mathbf{free}/\mathbf{g}$. Since $E^*(C^l(-, M)) = \text{Hom}_S(E_*(\Lambda^l \mu), M)$ for $l \geqslant 0$, (1) follows from the obvious spectral sequences if we prove,

(2) $\text{limind}_{(k)} \Lambda^l \mu$ is 0 for $k \neq 0$, and $\Lambda^l \mathbf{g}$ for $k = 0$.

Now, this follows easily from the Leray spectral sequence for the functor μ, see [La 1], Chapter 2. In fact, for any surjective morphism $\partial: \mathbf{F} \to \mathbf{g}$, the semi simplicial complex of S-modules

$$\to \mathbf{F} \times_{\mathbf{g}} \dots \times_{\mathbf{g}} \mathbf{F} \to \cdots \to \mathbf{F} \times_{\mathbf{g}} \mathbf{F} \to \mathbf{F} \to \mathbf{g} \to 0 \tag{3}$$

is acyclic. Notice that even though the exterior product Λ^l is not an additive functor, the semi simplicial complex

$$\to \Lambda^l(\mathbf{F} \times_{\mathbf{g}} \dots \times_{\mathbf{g}} \mathbf{F}) \to \cdots \to \Lambda^l(\mathbf{F} \times_{\mathbf{g}} \mathbf{F}) \to \Lambda^l \mathbf{F} \to \Lambda^l \mathbf{g} \to 0 \tag{4}$$

is still acyclic. To see this it suffices to pick an S-linear section $h: \mathbf{g} \to \mathbf{F}$ of ∂, and consider the contracting homotopy $\boldsymbol{h}$ of (3) defined by $\boldsymbol{h}(f_0, \dots, f_n) = (f_0, \dots. f_n, h(f_n))$. Put $\mathbf{F}_i = \mathbf{F} \times_{\mathbf{g}} \dots \times_{\mathbf{g}} \mathbf{F}$, $i+1$ factors, and denote by $\mathbf{F}_*$ the semi simplicial complex (3). The Leray spectral sequence referred to above has the form,

$$E^2_{rs} = H_r(\text{limind}_{*(s)} \Lambda^l \mu),$$

where limind_* is the inductive limit functor defined on the category **free**/$\mathbf{F}_*$. Moreover, it converges to $\text{limind}_{(r+s)} \Lambda^l \mu$. Since $\mathbf{F}_0$ is a free S-Lie algebra, we find just like in (2.1.5), loc. cit. that $E^2_{0,1} = 0$. Since moreover $E^2_{1,0} = H_1(\Lambda^l \mathbf{F}_*) = 0$ we conclude $\text{limind}_{(1)} \Lambda^l \mu = 0$, and by a standard technique, (2) follows. Q.E.D.

Now, using the Leray spectral sequence, see [La 1], (2.1.3) and the Remark 1, we may easily "compute" the first few cohomology groups of the S-Lie algebra $\mathbf{g}$, given its structural constants $c^k_{ij} \in S$, with respect to a basis $\{x_i\}, i = 1, \dots, n$. Consider the free S-Lie algebra $\mathbf{F}$ generated by the symbols $\mathbf{x}_i$, $i = 1, \dots, n$, and let $j: \mathbf{F} \to \mathbf{g}$ be the morphism of Lie algebras defined by $j(\mathbf{x}_i) = x_i$. Then the kernel of j is an ideal $\mathbf{J}$ of $\mathbf{F}$ generated by the elements

$$\mathbf{f}_{ij} = [\mathbf{x}_i, \mathbf{x}_j] - \sum_k c^k_{ij} \mathbf{x}_k, \quad 1 \leqslant i,j \leqslant n,$$

with ideal $\mathbf{I}$ of "linear" relations among the f_{ij}'s, $i < j = 1, \dots, n$, containing the elements

$$\mathbf{r}_{ijk} = [\mathbf{f}_{ij}, \mathbf{x}_k] + [\mathbf{f}_{jk}, \mathbf{x}_i] + [\mathbf{f}_{ki}, \mathbf{x}_j] + \sum_l c^l_{ij} \mathbf{f}_{lk} + \sum_l c^l_{jk} \mathbf{f}_{li} + \sum_l c^l_{ki} \mathbf{f}_{lj}.$$

PROPOSITION (1.2). *With the notations above we find,*

(i) $A^1(S, \mathbf{g}; M) \cong \text{Hom}_{\mathbf{F}}(\mathbf{J}, M)/\text{Der}$

(ii) $A^2(S, \mathbf{g}; M) \cong \text{Hom}_{\mathbf{F}}(\mathbf{I}, M)/\text{Der}$.

When $S = k$ *is a field, the isomorphisms* $\theta_1: A^1(k, \mathbf{g}; M) \cong H^2(\mathbf{g}, M)$ *and* $\theta_2: A^2(k, \mathbf{g}; M) \cong H^3(\mathbf{g}, M)$, *of* (1.1) *are given as follows:*

Let $\phi \in H^2(\mathbf{g}, M)$ *be represented by the cocycle* $f \in \mathrm{Hom}_k(\mathbf{g}\Lambda\mathbf{g}, M)$ *then the map* $\mathbf{f}_{ij} \to f(x_i \Lambda x_j)$ *extends to an* **F**-*linear map* $\mathbf{J} \to M$ *defining an element* $\xi \in A^1(k, \mathbf{g}; M)$, *such that* $\theta_1(\xi) = \phi$.

Let $\rho \in H^3(\mathbf{g}, M)$ *be represented by the cocycle* $r \in \mathrm{Hom}_k(\mathbf{g}\Lambda\mathbf{g}\Lambda\mathbf{g}, M)$, *then the map* $\mathbf{r}_{ijk} \to r(x_i \Lambda x_j \Lambda x_k)$ *extends to an* **F**-*linear map* $\mathbf{I} \to M$ *defining an element* $\zeta \in A^2(k, \mathbf{g}; M)$, *such that* $\theta_2(\zeta) = \rho$.

It is now easy to construct the obstructions we need for the "obstruction calculus". In fact if $\pi\colon R \to S$ is a surjective homomorphism of commutative rings with unit, such that $(\ker \pi)^2 = 0$, and if $\mathbf{g}$ is an S-Lie algebra, flat over S, given as above in terms of its structural constants $\{c_{ij}^k\}$, with respect to some basis $\{x_i\}_i$ of $\mathbf{g}$, any lifting $\mathbf{g}'$ of $\mathbf{g}$ to R, must necessarily have structural constants $\mathbf{c}_{ij}^k \in R$, with respect to some basis $\{\mathbf{x}_i\}_i$ of $\mathbf{g}'$, such that $\pi(\mathbf{x}_i) = x_i$ and $\pi(\mathbf{c}_{ij}^k) = c_{ij}^k$. To see whether there are such liftings or not, we pick $\mathbf{c}_{ij}^k \in R$ satisfying $\pi(\mathbf{c}_{ij}^k) = c_{ij}^k$ and consider the map

$$\mathbf{r}_{ijk} \to [[\mathbf{x}_i, \mathbf{x}_j], \mathbf{x}_k] + [[\mathbf{x}_j, \mathbf{x}_k], \mathbf{x}_i] + [[\mathbf{x}_k, \mathbf{x}_i], \mathbf{x}_j] =$$
$$\sum_m \left(\sum_l \mathbf{c}_{ij}^l \mathbf{c}_{lk}^m + \mathbf{c}_{jk}^l \mathbf{c}_{li}^m + \mathbf{c}_{ki}^l \mathbf{c}_{lj}^m \right) x_m \in \mathbf{g} \otimes_S \ker \pi.$$

It extends to a map $\mathbf{I} \to \mathbf{g} \otimes_S \ker \pi$ defining an element

$$\sigma(\pi, \mathbf{g}) \in A^2(S, \mathbf{g}; \mathbf{g} \otimes_S \ker \pi).$$

PROPOSITION (1.3). *With the notations above, there exists an obstruction*

$$\sigma(\pi, \mathbf{g}) \in A^2(S, \mathbf{g}; \mathbf{g} \otimes_S \ker \pi)$$

such that $\sigma(\pi, \mathbf{g}) = 0$ *if and only if there exists a lifting of* $\mathbf{g}$ *to* R, *in which case the set of isomorphism classes of such liftings is a principal homogeneous space, or torsor, over* $A^1(S, \mathbf{g}; \mathbf{g} \otimes_S \ker \pi)$.

Proof. This follows immediately from the definition of $\sigma(\pi, \mathbf{g})$, together with (1.2). Notice that we may also copy the proof from that of [La 1], (2.2.5).

Q.E.D.

Copying the definition of the cup product from (5.1.5) loc cit., we find a map,

$$v\colon A^1(k, \mathbf{g}, \mathbf{g}) \to A^2(k, \mathbf{g}, \mathbf{g})$$

defined as follows: Let $\xi \in A^1(k, \mathbf{g}, \mathbf{g})$ be given in terms of an **F**-linear map $h\colon \mathbf{J} \to \mathbf{g}$, where as above **J** is the kernel of a surjective morphism $\mathbf{F} \to \mathbf{g}$, **F** any free Lie algebra. Put $h(\mathbf{f}_{ij}) = \sum_k h_{ij}^k x_k$. Then $v\xi \in A^2(k, \mathbf{g}, \mathbf{g})$ is given in terms of the **F**-linear map $vh\colon \mathbf{I} \to \mathbf{g}$, defined by

$$\begin{aligned} vh(\mathbf{r}_{ijk}) &= h\Big(\sum_l h_{ij}^l [x_l, x_k] + \sum_l h_{jk}^l [x_l, x_i] + \sum_l h_{ki}^l [x_l, x_j] + \\ &\quad + \sum_{lm} c_{ij}^l h_{lk}^m x_m \sum_{lm} c_{jk}^l h_{li}^m x_m + \sum_{lm} c_{ki}^l h_{lj}^m x_m\Big) \\ &= \sum_{lm} (h_{ij}^l h_{lk}^m + h_{jk}^l h_{li}^m + h_{ki}^l h_{lj}^m) x_m \end{aligned}$$

which is nothing but the map θ of Rim, see [Ri].

EXAMPLE 1. If $\mathbf{g}$ is the abelian n-dimensional Lie algebra, then

$$\begin{aligned} A^1(k, \mathbf{g}; \mathbf{g}) &\cong \mathbf{g}^{n(n-1)/2} \cong k^{n^2(n-1)/2} \\ A^2(k, \mathbf{g}; \mathbf{g}) &\cong \mathbf{g}^N \cong k^{n.N}, \end{aligned}$$

where N is the number of generators $\mathbf{r}_{ijk}$ of the corresponding ideal $\mathbf{I}$. The map $v\colon A^1(k, \mathbf{g}, \mathbf{g}) \to A^2(k, \mathbf{g}, \mathbf{g})$ in this case is the obvious quadratic map,

$$v\{h_{ij}^k\}_{ijk} = \Big\{\sum_l (h_{ij}^l h_{lk}^m + h_{jk}^l h_{li}^m + h_{ki}^l h_{lj}^m)\Big\}_{ijkm}.$$

Notice the similarity with the quadratic forms defining the affine subscheme $\mathbf{Lie}_n = \mathrm{Spec}(\mathrm{Lie}_n)$ of $\mathbf{A}^{n^2(n-1)/2}$, deduced from the Jacobi identities. These are easily seen to be,

$$\sum_l (c_{ij}^l c_{jk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0, \quad \text{for all } i, j, k, m,$$

assuming of course that $c_{ij}^l = -c_{ji}^l$, and that char $k = 0$. This is, as we shall see a particular case of a general result, (2.1), about the structure of the formal moduli of any Lie algebra.

EXAMPLE 2. If $\mathbf{g}$ is semisimple it follows from (1.2) that $A^1(k, \mathbf{g}; \mathbf{g}) = 0$, which is a classical result.

2. Deformations of Lie algebras

As above we denote by k a fixed field, by R, S etc. commutative k-algebras. Let $\mathbf{l}$ be the category of local artinian k-algebras with residue field k. Given any k-Lie algebra $\mathbf{g}$, the deformation functor

$$\mathrm{Def}_{\mathbf{g}}:\mathbf{l}\to\mathbf{Sets}$$

is defined by,

$$\mathrm{Def}_{\mathbf{g}}(S) = \{\mathbf{g}_S \to \mathbf{g} \cong \mathbf{g}_S \otimes_S \mathbf{k} \text{ where } \mathbf{g}_S \text{ is an } S \text{ free } S\text{-lie algebra}\}/\sim$$

where $\sim$ is the equivalence relation defined by the S-isomorphisms of the liftings $\mathbf{g}_S \to \mathbf{g}$.

Using the obstructions $\sigma(\pi, \mathbf{g}_S)$ and the Proposition (1.3) we may proceed exactly as in the proof of [La 1], (4.2.4) to obtain the following, apparently well known result, see [Fi], [Ra].

THEOREM (2.1). *Let* $\mathbf{g}$ *be a Lie algebra of finite dimension, and put* $A^i = A^i(k, \mathbf{g}, \mathbf{g})$. *Let* $T^i = \mathrm{Sym}_k(A^{i*})^\wedge$. *Then there is a morphism of complete local k-algebras*

$$\sigma: T^2 \to T^1$$

such that

$$H(\mathbf{g})^\wedge = T^1 \otimes_{T^2} k$$

is a prorepresenting hull for the functor $\mathrm{Def}_{\mathbf{g}}$. *Moreover* σ *maps the maximal ideal* $\mathbf{m}_2$ of T^2 *into the square* $\mathbf{m}_1^2$ *of the maximal ideal of* T^1.

The dual of the resulting map $\mathbf{m}_2/\mathbf{m}_2^2 \to \mathbf{m}_1^2/\mathbf{m}_1^3$ *is the cup product*

$$v: A^1 \otimes A^1 \to A^2$$

deduced from the quadratic map $v: A^1 \to A^2$ *of Section 1.*

EXAMPLE 3. It follows from Example 1 and the above theorem, that the completion of $\mathbf{Lie}_n$ at the origin is isomorphic to the formal moduli $\mathbf{H}(k^n)^\wedge$ of the abelian Lie algebra k^n. In particular

$$\mathrm{Lie}_n = k[C_{ij}^k]/\mathbf{a}$$

where $\mathbf{a}$ is the ideal generated by the quadratic forms of Example 1.

Let $\mathbf{K}^n$ be the Lie_n-Lie algebra defined by the structural constants C_{ij}^k, or rather, the class of C_{ij}^k in Lie_n. Then $\mathbf{K}^n$ is an algebraization of the formal versal family defined on $H(k^n)^\wedge$.

Given any Lie algebra $\mathbf{g}$, there exists by (2.1) a formal versal family, i.e. an $H^\wedge$-Lie algebra $\mathbf{G}^\wedge$, flat as an $H^\wedge$-module, such that $\mathbf{G}^\wedge \otimes_H k = \mathbf{g}$, representing the smooth morphism

$$\mathrm{Mor}(H^\wedge, -) \to \mathrm{Def}_{\mathbf{g}}$$

defined on **l**.

Pick any $H^\wedge$-basis $\{x_i^\wedge\}_{i=1,\dots,n}$ of $\mathbf{G}^\wedge$, and consider the corresponding structural constants $\{c_{ij}^{\wedge k}\}$. Let H be the finitely generated k-subalgebra of $H^\wedge$ generated by the $c_{ij}^{\wedge k}$. By construction $H^\wedge/\mathbf{m}^2$ is generated as a k-algebra by the images of $c_{ij}^{\wedge k}$.

It follows readily that the completion of H w.r.t. the ideal $\mathbf{m}$ of H, generated by the $\hat{c}_{ij}^k$'s is $H^\wedge$. Let $\mathbf{G}$ be the H-Lie algebra defined by the structural constants $\mathbf{c}_{ij}^k = c_{ij}^{\wedge k}$, then $(H, \mathbf{G})$ is an algebraization of the formal versal family $(H^\wedge, \mathbf{g}^\wedge)$. We have proved the following,

LEMMA (2.2). *For every k-Lie algebra* $\mathbf{g}$ *there exists an algebraization* $(H, \mathbf{G})$ *of the formal versal family* $(H^\wedge, \mathbf{g}^\wedge)$, *and an embedding* $\mathbf{H} \to \mathbf{Lie}_n$ *compatible with the families* $\mathbf{G}$ *and* $\mathbf{K}^n$.

REMARK. Compare (2.2) to the condition (A_1) of Section 3 of [La-Pf].

We shall now apply the technique of [La-Pf], to the study of Lie algebras.

First we have to introduce the Kodaira–Spencer map of an S-flat S-Lie algebra $\mathbf{G}$. Following, word for word, the construction of the Kodaira–Spencer map of Section 3 loc. cit., we obtain an S-linear map

$$g\colon \mathrm{Der}_k(S) \to A^1(S, \mathbf{G}, \mathbf{G})$$

given explicitly by the following

LEMMA (2.3). (i) *Let* $\mathbf{c}_{ij}^k$ *be the structural constants of* $\mathbf{G}$ w.r.t. *some S-basis* $\{x_i\}$, *and let* $\partial\colon \mathbf{F} \to \mathbf{G}$ *be a surjective morphism of a free S*-Lie *algebra* $\mathbf{F}$ *onto* $\mathbf{G}$, *mapping the generators* $\mathbf{x}_i$ *onto* x_i. *Let* $\mathbf{F}_{ij} = [\mathbf{x}_i, \mathbf{x}_j] - \Sigma_k \mathbf{c}_{ij}^k \mathbf{x}_k \in \ker \partial$, *and let* $D \in \mathrm{Der}_k(S)$. *Then* $g(D)$ *is the element of* $A^1(S, \mathbf{G}, \mathbf{G})$ *determined by the element of*

$$\mathrm{Hom}_{\mathbf{F}}(\ker \partial, \mathbf{G}), \text{ mapping } \mathbf{F}_{ij} \text{ onto } -\Sigma_k D(\mathbf{c}_{ij}^k) x_k.$$

(ii) *Denote by* $\mathbf{G}(s)$ *the fiber of* $\mathbf{G}$ *at the closed point s of* $\mathbf{S} = Spec(S)$, *then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Der}_k(S) & \xrightarrow{g} & A^1(S, \mathbf{G}, \mathbf{G}) \\ \downarrow & & \downarrow \\ T_{s,S} & \xrightarrow{g(s)} & A^1(k, \mathbf{G}(s), \mathbf{G}(s)) \end{array}$$

Here $g(s)$ *is the canonical tangent map corresponding to the formal family* $(S_s^\wedge, \mathbf{G}_s^\wedge)$, *the completion of* $(S, \mathbf{G})$ *at s.*

If $S = H$ is an algebraization of the formal moduli of $\mathbf{g}$, and $\mathbf{G}$ is the corresponding versal deformation then we deduce as in (3.3) and (3.5) loc. cit., that the kernel $\mathbf{V}$ of the Kodaira–Spencer map is a sub k-Lie algebra of $\mathrm{Der}_k(H)$, and that

$$\mathbf{V} \otimes_H k \cong \mathrm{Der}_k(\mathbf{g})/\mathrm{Der}_\pi$$

where Der_π is the image of $\mathrm{Der}_H(\mathbf{G})$ in $\mathrm{Der}_k(\mathbf{g})$. If the generic fiber of some component of $\mathbf{H} = \mathrm{Spec}(H)$, is complete, it is easy to prove that $\mathrm{Der}_\pi = \mathbf{g}^{ad}$.

THEOREM (2.4). *Let* $\mathbf{g}$ *be any k-Lie algebra. Then the algebraization* $(H, \mathbf{G})$ *of the formal versal family* $(H^\wedge, \mathbf{g}^\wedge)$ *is locally formally versal, in the sense of* (3.6), [La-Pf].

Proof. We have to prove that there exists an open neighborhood U of the base point $o \in \mathbf{H}$, such that for every closed point $t \in U$ the map $g(t)\colon T_{t,\mathbf{H}} \to A^1(k, \mathbf{G}(t), \mathbf{G}(t))$ is surjective.

We know there exists an embedding $\alpha\colon \mathbf{H} \to \mathbf{Lie}_n$, such that the pull back of $\mathbf{K}^n$ is $\mathbf{G}$. Since $(H^\wedge, \mathbf{g}^\wedge)$ is formally versal, we find using M. Artins approximation theorem, an étale neighborhood E of $\alpha(o)$ in $\mathbf{Lie}_n$ and a diagram of morphisms

$$\begin{array}{ccc} E & \xrightarrow{\beta} & \mathbf{H} \\ {\scriptstyle i}\downarrow & \swarrow & \\ \mathbf{Lie}_n & & \end{array}$$

such that $U = \mathrm{im}\,\beta$ is a neighborhood of o in $\mathbf{H}$, and such that $i^*(\mathbf{K}^n) \cong \beta^*(\mathbf{G})$.

Let t be a closed point of U. Pick a point t' in E s.t. $\beta(t') = t$. Consider the Lie algebra $\mathbf{G}(t)$, and its formal moduli $H^\wedge(t)$. By definition of $H^\wedge(t)$, we know that $H^\wedge(t)/\mathbf{m}^2 \cong k \oplus A^1(k, \mathbf{G}(t), \mathbf{G}(t))^*$, where $\mathbf{m}$ is the maximal ideal of $H^\wedge(t)$.

Denote by $\mathbf{G}(t)_r$ the miniversal deformation of $\mathbf{G}(t)$ to $H(t)/\mathbf{m}^r$. There is a morphism $\gamma\colon \mathrm{Spec}(H^\wedge(t)/\mathbf{m}^r) \to \mathbf{Lie}_n$, mapping the closed point to $\alpha(t)$, such that $\gamma^*(\mathbf{K}^n) \cong \mathbf{G}(t)_r$. Since i is étale there is a morphism $\delta\colon \mathrm{Spec}(H^\wedge(t)/\mathbf{m}^r) \to E$ mapping the closed point to t', and such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Spec}(H^\wedge(t)/\mathbf{m}^r) & \xrightarrow{\partial} & E \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathbf{Lie}_n. & \xleftarrow{\alpha} & \mathbf{H} \end{array}$$

But this implies that $\delta^*\beta^*\mathbf{G} \cong \mathbf{G}(t)_r$. In particular, the tangent map $A^1(k, \mathbf{G}(t), \mathbf{G}(t)) \to T_{t,\mathbf{H}}$, induced by $\beta\delta$, must be a section of the functorial map $g(t)$, which is therefore surjective. Q.E.D.

The proof of (2.4) immediately implies,

COROLLARY (2.5). *In the situation above, there exists a neighborhood U of the base point of* $\mathbf{H}$ *such that for any closed point t of U, and any integer $r \geqslant 1$, the composition of the natural morphisms*

$$p: H^\wedge(t) \to H_t^\wedge$$
$$q: H_t^\wedge \to H(t)/\mathbf{m}(t)^r$$

is the canonical homomorphism

$$qp: H^\wedge(t) \to H^\wedge(t)/\mathbf{m}(t)^r.$$

It is now tempting to try to construct a *moduli suite* for the Lie algebra $\mathbf{g}$ copying the procedure of Section 3 [La–Pf].

However, the basic assumptions (V'), used there are not satisfied for Lie algebras. In particular $\mathbf{Lie}_n$ is far from non singular, there are many components of different dimensions, some of which are non reduced, see [Ra], [Ri]. For an exposition of the structure of $\mathbf{Lie}_n$ for small n, see [C-D], [K-N] and [N].

Recall that the constructions of (Section 3 [La-Pf]) are based on the existence of a locally closed subscheme $\mathbf{H}_0$ of $\mathbf{H}$ containing the base point, for which the formalization $\mathbf{H}_{0t}^\wedge$ of H_0 at every point t is isomorphic to the prorepresentable substratum of the corresponding formal moduli. In the case of Lie algebras there are no reasons to expect $\mathbf{H}_0$ to have this property.

Let for every Lie algebra $\mathbf{g}$, $h_i(\mathbf{g}) = \dim_k H^i(\mathbf{g}, \mathbf{g})$. If $\dim_k \mathbf{g} = n$ then $h_i(\mathbf{g}) = 0$ for $n + 1 \leqslant i$. Replace the filtration $\{S_\tau\}$ of $\mathbf{H}$ used in Section 3 loc. cit. by the filtration of $\mathbf{H}$ defined by,

$$\mathbf{H}(\boldsymbol{h}) = \mathbf{H}(h_0, h_1, \ldots, h_n) = \{t \in \mathbf{H} \mid \dim_k H^i(\mathbf{G}(t), \mathbf{G}(t)) = h_i, i = 0, 1, \ldots, n\},$$
$$\boldsymbol{h} \in \mathbf{Z}_+^{n+1}.$$

Notice that by (1.1), this is the flattening stratification of $\oplus_i H^i(H, \mathbf{G}, \mathbf{G})$.

Obviously $\mathbf{H}(h_0(\mathbf{g}), \ldots, h_n(\mathbf{g}))^\wedge$ is contained in the prorepresenting substratum $\mathbf{H}_0$ of $\mathbf{H}$. Observe also that for every closed point t in a neighborhood of the base point of $\mathbf{H}(h_0(\mathbf{g}), \ldots, h_n(\mathbf{g}))$, the tangent map of $\mathbf{H}_t^\wedge \to \mathbf{H}(\mathbf{G}(t))^\wedge$ is not only surjective, but, in fact, an isomorphism.

Therefore the map $p: H^\wedge(t) \to H_t^\wedge$ of Corollary (2.5) is surjective, and consequently an isomorphism. Summing up, we have proved,

COROLLARY (2.6). *There exists a neighborhood U of the base point of $H(h_0(\mathbf{g}), \ldots, h_n(\mathbf{g}))$, such that for every closed point t* of *U,*

$$\mathbf{H}_t^\wedge \cong \mathbf{H}(\mathbf{G}(t))^\wedge$$

and

$$\mathbf{H}(\mathbf{G}(t))(h_0(\mathbf{G}(t)),\ldots,h_n(\mathbf{G}(t))^\wedge \cong \mathbf{H}(h_0(\mathbf{g}),\ldots,h_n(\mathbf{g}))_t^\wedge.$$

Now, to show the existence of a moduli suite for Lie algebras, consider $\mathbf{Lie}_n = \mathbf{H}(k^n)$. For each $\boldsymbol{h} \in \mathbf{Z}^{n+1}$, $\mathbf{Lie}_n(\boldsymbol{h})$ is a locally closed subscheme of $\mathbf{Lie}_n$. Let $\mathbf{K}(\boldsymbol{h})$ be the restriction of $\mathbf{K}^n$ to $\mathbf{Lie}_n(\boldsymbol{h})$. If t is a closed point of $\mathbf{Lie}_n(\mathbf{h})$, corresponding to the Lie algebra $\mathbf{g}$, then there exists a unique morphism of proschemes

$$\mathbf{Lie}_n(\boldsymbol{h})_t^\wedge \to \mathbf{H}(\mathbf{g})(\boldsymbol{h})^\wedge$$

compatible with the families $\mathbf{K}(\boldsymbol{h})$ and $\mathbf{G}(\boldsymbol{h})$. This is exactly what we need to know, to be able to copy the proof of (3.16) of [La-Pf]. The result is the following,

THEOREM (2.7). *Let* $\boldsymbol{h} \in \mathbf{Z}^{n+1}$, *then there is a way of gluing together the subschemes* $\mathbf{H}(\boldsymbol{h})$ *of* $\mathbf{H}(\mathbf{g})$, *and the corresponding families of Lie algebras* $\mathbf{G}(\boldsymbol{h})$, $\mathbf{g}$ *running through* $\mathbf{Lie}_n(\boldsymbol{h})$, *to obtain an algebraic space* $\mathbf{L}(\boldsymbol{h})$, *and a family of Lie algebras* $\Lambda(\boldsymbol{h})$ *defined on* $\mathbf{L}(\boldsymbol{h})$. *Moreover there exists in the category of algebraic spaces a morphism*

$$\rho: \mathbf{Lie}_n(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h})$$

compatible with the families $\mathbf{K}(\boldsymbol{h})$ *and* $\Lambda(\boldsymbol{h})$, *such that*

$$\mathbf{L}(\boldsymbol{h}) \cong \mathbf{Lie}_n(\boldsymbol{h})/\mathrm{Gl}_n(k).$$

COROLLARY (2.8). *If* $k = \mathbf{C}$, *then* $\mathbf{L}(\boldsymbol{h})$, *with the family* $\Lambda(\boldsymbol{h})$, *is a fine moduli space in the category of analytic spaces.*

Proof. Since Gl_n is connected, and since the dimension of the fibers of ρ is equal to the dimension of the Gl_n orbits of $\mathbf{Lie}_n(\boldsymbol{h})$, (2.8) follows from the fact that $\mathbf{Lie}_n(\mathbf{C})/\mathrm{Gl}_n$ is the set of isomorphism classes of Lie algebras. Q.E.D.

As we mentioned above, the structure of $\mathbf{Lie}_n$, and of course, also the structure of the $\mathbf{L}(\boldsymbol{h})$'s, is very complicated. The dimensions of the components of $\mathbf{Lie}_n$ are known only for small n's, and there are few results for general n's, see [C-D] and [N].

The structure of the nontrivial $\mathbf{L}(\boldsymbol{h})$'s are, however, unknown.

Notice the rather trivial consequence of (3.10), [La-Pf],

PROPOSITION (2.9). *Let* $\mathbf{g}$ *be a Lie algebra, and let* $\mathbf{R}$ *be a component of* $\mathbf{H}(\mathbf{g})$ *such that the generic point corresponds to a rigid Lie algebra* $\mathbf{g}_0$, *then* $\dim \mathbf{R} = \dim_k \mathrm{Der}(\mathbf{g}) - \dim_k \mathrm{Der}(\mathbf{g}_0)$.

PROPOSITION (2.10). *Any component of* $\mathbf{L}(h)$ *containing the Lie algebra* $\mathbf{g}$ *has dimension less than or equal to,*

$$\dim_k H^0(\mathrm{Der}(\mathbf{g}), H^2(\mathbf{g}, \mathbf{g})).$$

Proof. We know that $\mathbf{H}(\mathbf{g})(h)^\wedge$ is contained in $\mathbf{H}_0(\mathbf{g})^\wedge$, and the tangent space of the latter is precisely $H^0(\mathrm{Der}(\mathbf{g}), H^2(\mathbf{g}, \mathbf{g}))$. Q.E.D.

3. Local moduli for isolated hypersurface singularities

In this paragraph we shall relate the local moduli of isolated hypersurface singularities to the local moduli of Lie algebras.

Consider an isolated hypersurface singularity, i.e. a complete local k-algebra of the form

$$k[[x_1, \ldots, x_n]]/(f)$$

where $f \in k[x_1, \ldots, x_n]$ is a hypersurface with an isolated singularity at the origin. Let us recall the following facts, see [La-Pf]. Put

$$A^1(f) = (x_1, \ldots, x_n)/(f, (x_i . \partial f/\partial x_j)_{ij})$$

and pick a monomial bases $\{x^\alpha\}_{\alpha \in l}$.for $A^1(f)$. Then the family,

$$F = f + \sum_\alpha t_\alpha x^\alpha \in k[t_\alpha][x_1, \ldots, x_n]$$

is a miniversal deformation of f as a singularity, with basis $H = k[t_\alpha]_{\alpha \in l}$.

The Kodaira-Spencer morphism

$$g\colon \mathrm{Der}(H) \to A^1(H, F) := (x_1, \ldots, x_n)H[[x]]/(F, (x_i\, \partial F/\partial x_j)_{ij})$$

is given by

$$g(\partial/\partial t_\alpha) = x^\alpha.$$

Put

$$\mathbf{V} = \ker g.$$

It is a sub k-Lie algebra of vectorfields on $\mathbf{H} = \mathrm{Spec}(H)$. Recall, Section 3 loc.

cit., that the family F is constant along a connected subscheme $\mathbf{Y}$ of $\mathbf{H}$ if and only if $\mathbf{Y}$ is contained in an integral submanifold of $\mathbf{V}$. Put

$$\tau(f) = \dim_k k[[x]]/(f, (\partial f/\partial x_i))_i$$
$$\tau.(f) = \dim_k A^1(f).$$

$\tau(f)$ is the Tjurina number of the singularity f.

Consider for every integer τ, the locally closed subscheme of $\mathbf{H}$ defined by

$$\mathbf{S}_\tau = \{t \in \mathbf{H} \mid \tau.(F(t)) = \tau\}.$$

This is simply the flattening stratification for the H-module $A^1(H, F)$. Put,

$$\mathbf{H}_0(f) = \mathbf{S}_{\tau.(f)}.$$

$\mathbf{V}$ operates on each $\mathbf{S}_\tau$ and, in the category of algebraic spaces, there exists a quotient of $\mathbf{S}_\tau$,

$$\mathbf{M}_\tau = \mathbf{S}_\tau/\mathbf{V},$$

and a family of singularities $\mathbf{F}_\tau$ defined on $\mathbf{M}_\tau$, such that the restriction of F to $\mathbf{S}_\tau$ is the pull-back of $\mathbf{F}_\tau$.

The collection $\{\mathbf{M}_\tau\}_\tau$ is what we have called the *local moduli suite* of the singularity f. Notice that there are examples of $\mathbf{M}_\tau$'s that are not scheme theoretic geometric quotients of $\mathbf{S}_\tau$, see Section 6 loc. cit.

To every isolated hypersurface singularity f we shall associate a graded Lie algebra,

$$L^*(f) = L^0(f) \oplus L^1(f).$$

where

$$L^0(f) = \mathrm{Der}_k(k[[x]]/(f))/\mathrm{Der}_\pi$$

Der_π being the Lie ideal of $\mathrm{Der}(f) := \mathrm{Der}_k(k[[x]]/(f))$ generated by the trivial derivations $E_{ij} \in \mathrm{Der}(f)$ defined by

$$E_{ij}(x_k) = 0 \text{ for } k \neq i, j, E_{ij}(x_i) = \partial f/\partial x_j, \quad \text{and } E_{ij}(x_j) = -\partial f/\partial x_i.$$

$L^1(f)$ is the vector space $A^1(f)$ considered as an $L^0(f)$-representation via the canonical action of $L^0(f) = \mathbf{V} \otimes_H k$ on the tangent space $A^1(f)$ of $\mathbf{H}$ at the origin.

One may check, see Section 4 loc. cit., that

$$L^0(f) \cong \ker\{\cdot f\colon k[[x]]/(\partial f/\partial x_i) \to k[[x]]/(\partial f/\partial x_i)\}.$$

With this identification an element d of $L^0(f)$ corresponding to an element q of $\ker(\cdot f)$ acts on an element ξ of $L^1(f)$ in the following way:

$$[d, \xi] = D(\xi) - q\xi,$$

where D is the derivation of $k[[x]]$ such that $q.f = \Sigma_i(\partial f/\partial x_i)D(x_i)$.

LEMMA (3.1) (i) $\dim_k L^0(f) = \tau(f)$
(ii) $\dim_k L^1(f) = \tau.(f) = \tau(f) + n - 1$.
(iii) $\dim_k L^*(f) = 2\tau.(f) + 1 - n$.

In [La-Pf], Sections 2 and 3, we prove that

$$\Lambda^0(f) := \mathbf{V} \otimes_H H_0$$

is a flat H_0-Lie algebra, the fibers of which are the Lie algebras $L^0(F(t))$, t running through the closed points of $\mathbf{H}_0$. $\Lambda^0(f)$ is thus a deformation of $L^0(f)$. More generally $\Lambda^0_\tau(f) := \ker\{\mathbf{V}|\mathbf{S}_\tau \to \theta_{\mathbf{S}_\tau}\}$ is a flat $O_{\mathbf{S}_\tau}$-Module, and a deformation of every $L^0(F(t))$, t running through the closed points of $\mathbf{S}_\tau$.

We also know that

$$\Lambda^1(f) := A^1(H_0, F_0) = (\mathbf{x})H_0[[\mathbf{x}]]/(F_0, (x_i.\partial F_0/\partial x_j)_{ij})$$

is a flat H_0-module and a $\Lambda^0(f)$-representation. Therefore,

$$\Lambda^*(f) := \Lambda^0(f) \oplus \Lambda^1(f)$$

is an H_0-flat graded Lie algebra, and a deformation of $L^*(f)$. More generally, considering $A^1(O_{\mathbf{S}_\tau}, F_\tau, F_\tau)$, we obtain a graded $O_{\mathbf{S}_\tau}$-flat $O_{\mathbf{S}_\tau}$-Lie algebra

$$\Lambda^*_\tau(f) := \Lambda^0_\tau(f) \oplus \Lambda^1_\tau(f).$$

PROPOSITION (3.2). (i) *Let* $\boldsymbol{h} = (h_0, \ldots, h_{2\tau+1-n}) \in \mathbf{Z}^{2\tau+2-n}$, *then the subsets*

$$\mathbf{S}(\boldsymbol{h}) = \{t \in \mathbf{S}_\tau \mid h_i(L^*(F(t)) = h_i, i = 0, 1, \ldots, 2\tau + 1 - n\}$$
$$\mathbf{M}(\boldsymbol{h}) = \{m \in \mathbf{M}_\tau \mid h_i(L^*(\mathbf{F}_\tau(m)) = h_i, i = 0, 1, \ldots, 2\tau + 1 - n\}$$

of $\mathbf{S}_\tau$ and $\mathbf{M}_\tau$ *respectively, are locally closed.*

(ii) *Let* $\boldsymbol{h} = (h_0, h_1, \ldots, h_\tau) \in \mathbf{Z}^{\tau+1}$, *then the subsets,*

$$\mathbf{S}^0(\boldsymbol{h}) = \{t \in \mathbf{S}_{\tau-1+n} \,|\, h_i(L^0(F(t)) = h_i, i = 0, 1, \ldots, \tau\}$$
$$\mathbf{M}^0(\boldsymbol{h}) = \{m \in \mathbf{M}_{\tau-1+n} \,|\, h_i(L^0(\mathbf{F}_\tau(m)) = h_i, i = 0, 1, \ldots, \tau\}$$

of $\mathbf{S}_{\tau-1+n}$ and $\mathbf{M}_{\tau-1+n}$ *are locally closed.*

Proof. We shall prove that $\mathbf{M}^0(\boldsymbol{h})$ is locally closed in $\mathbf{M}_{\tau-1+n}$. The rest follows immediately.

Let $m \in \mathbf{M}^0(\boldsymbol{h})$, and put $g = \mathbf{F}_{\tau-1+n}(m)$, then the prorepresentable substratum $\mathbf{H}_0(g)$ of $\mathbf{H}(g)$ is an open neighbourhood of m in $\mathbf{M}_{\tau-1+n}$. The corresponding $H_0(g)$-Lie algebra $\Lambda^0(g)$ is $H_0(g)$-flat. But then it follows that the subset

$$\mathbf{H}_0(g)(\boldsymbol{h}) = \{m \in \mathbf{H}_0(g) \,|\, h_i(\Lambda^0(g)) = h_i, i = 0, 1, \ldots, \tau(g)\}$$

is locally closed in $\mathbf{H}_0(g)$. Q.E.D.

Consider the family $\Lambda_\tau^*(f) = \Lambda_\tau^0(f) \oplus \Lambda_\tau^1(f)$ restricted to $\mathbf{S}(\boldsymbol{h})$, and the family $\Lambda_{\tau-1+n}^0(f)$ restricted to $\mathbf{S}^0(\boldsymbol{h})$. From what we have done above, we easily prove the following,

PROPOSITION (3.3). *The families* $\Lambda_\tau^*(f)$ *and* $\Lambda_{\tau-1+n}^0(f)$ *defines unique morphisms of algebraic spaces,*

$$l^*: \mathbf{M}(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h}), \quad \boldsymbol{h} = (h_1, \ldots, h_{2\tau+1-n}) \in \mathbf{Z}^{2\tau+2-n},$$
$$l^0: \mathbf{M}^0(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h}), \quad \boldsymbol{h} = (h_0, h_1, \ldots, h_\tau) \in \mathbf{Z}^{\tau+1}$$

compatible with the obvious families of Lie algebras.

Notice that, locally, the morphisms l^* and l^0 are morphismes of the form

$$l^*(f): \mathbf{H}_0(f)(h_0, \ldots, h_{\tau(f)+\tau.(f)}) \to \mathbf{L}(h_0, \ldots, h_{\tau(f)+\tau.(f)}).$$
$$l^0(f): \mathbf{H}_0(f)(h_0, \ldots, h_{\tau(f)}) \to \mathbf{L}(h_0, \ldots, h_{\tau(f)}).$$

We shall, in a later Section, study the tangent map of these morphisms, for quasi-homogenous singularities f. At this point we shall show that these tangent maps, which are nothing but the Kodaira–Spencer maps of the families $\Lambda^*(f)$ and $\Lambda^0(f)$, respectively, are related to a Massey-type product structure of $L^*(f)$. In fact there are partially defined products of the form,

$$\langle d_1, d_2, \ldots, d_{r+2} \rangle \in L^s(f), d_i \in L^{\deg d_i}, \quad i = 0, 1, \ldots, r+2,$$

where $s = \sum_i \deg d_i - r$. If $r = 0$, then $\langle d_1, d_2 \rangle = [d_1, d_2]$ is the ordinary Lie product. Recall (see Section 3. loc. cit.) that for $d \in L^0(f)$ and $\xi \in L^1(f)$, the Lie product $[d, \xi] \in L^1(f)$ is equal to the obstruction for lifting a derivation D of $\mathrm{Der}_k(f)$ representing d, to a derivation of $f + \varepsilon\xi \in k[\varepsilon][\boldsymbol{x}]$.

Now, let $d_i \in L^0(f)$, $i = 1, 2$, and let $d_3 = \xi \in L^1(f)$. Suppose $\langle d_1, d_2 \rangle = 0$ and $\langle d_i, d_3 \rangle = 0$ for $i = 1, 2$, then the first Massey product,

$$\langle d_1, d_2, \xi \rangle \in L^0(f),$$

is defined as follows: Represent d_i as a derivation D_i of $\mathrm{Der}_k(f)$, and consider the lifting $f + \varepsilon.\xi \in k[\varepsilon][\boldsymbol{x}]$. Since $\langle d_i, \xi \rangle = 0$ for $i = 1, 2$, we know, see Section 3 loc. cit., that D_i may be lifted to a derivation $\boldsymbol{D}_i \in \mathrm{Der}_{k[\varepsilon]}(f + \varepsilon\xi)$. Since $\langle d_1, d_2 \rangle = 0$ we find that $[\boldsymbol{D}_1, \boldsymbol{D}_2] \in \varepsilon \mathrm{Der}_k(f)$, and we define $\langle d_1, d_2, \xi \rangle$ by,

$$[\boldsymbol{D}_1, \boldsymbol{D}_2] = \varepsilon\langle d_1, d_2, \xi \rangle.$$

Suppose $d \in L^0(f)$, and $\xi_1, \xi_2 \in L^1(f)$ are such that $\langle d, \xi_i \rangle = 0$, $i = 1, 2$, then,

$$\langle d, \xi_1, \xi_2 \rangle \in L^1(f),$$

is defined as follows: Consider the lifting

$$f + t_1\xi_1 + t_2\xi_2 \in k[t_1, t_2]/(\boldsymbol{t})^2[\boldsymbol{x}].$$

The derivation D in $\mathrm{Der}_k(f)$ representing d, lifts to a derivation $\boldsymbol{D}$ in $\mathrm{Der}_{k[t_1,t_2]/(t)^2}(f + t_1\xi_1 + t_2\xi_2)$. Now $\langle d, \xi_1, \xi_2 \rangle$ is the obstruction for lifting $\boldsymbol{D}$ to a derivation of,

$$f + t_1\xi_1 + t_2\xi_2 \in k[t_1, t_2]/(t_1^2, t_2^2)[\boldsymbol{x}].$$

It is not difficult to see how to continue this process of defining higher and higher order Massey products. Moreover, if all Massey products are known, we may reconstruct the Kodaira–Spencer kernel $\mathbf{V}^\wedge$ of f, therefore its determinant $\Delta^\wedge = \det \mathbf{V}^\wedge$, i.e. the discriminant of f. It follows by a result of Brieskorn that this, in fact, determines the singularity f. This together with positive results in some special cases, see Section 5, lead us to formulate the following conjecture,

CONJECTURE (3.4). *For every* $\boldsymbol{h} \in \mathbf{Z}^{2\tau+2-n}$, *the morphism*

$$l^*: \mathbf{M}(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h})$$

is an immersion.

The relation between the Kodaira–Spencer map and these Massey products, is now given by the following,

PROPOSITION (3.5). *The Kodaira–Spencer map of* $\Lambda^0(f)$,

$$g_0\colon \mathrm{Der}_k(H_0(f)) \to A^1(H_0(f), F_0)$$

restricted to $\mathbf{H}_0(f)(\boldsymbol{h})$, *is exactly the tangent map of* $\mathrm{l}(f)$. *If* o *is the base point of* $\mathbf{H}_0(f)(\boldsymbol{h})$, *then the fiber of* g_0 *at* o, *i.e. the tangent map,*

$$T_{l(f)}\colon T_{0,\mathbf{H}_0(f)(\boldsymbol{h})} \to A^1(k, L^0(f), L^0(f))$$

is determined by the first Massey product of $L^*(f)$ *as follows:*

Let $\xi \in T_{0,\mathbf{H}_0(f)} = H^0(L^0(f), L^1(f))$, then $T_{l(f)}(\xi)$ *is represented by a 2-cocycle* $O \in \mathrm{Hom}_k(L^0(f)\Lambda L^0(f), L^0(f))$, *such that*

$$O(d_1 \Lambda d_2) = \langle d_1, d_2, \xi\rangle, \text{ whenever } [d_1, d_2] = 0.$$

Proof. This is just (2.3), (1.2), and the definition above. Q.E.D.

4. The Lie algebra $L^0(f)$. Rank and cohomology

In this paragraph we first determine, explicitly, a maximal torus T on the Lie algebra $L^0(f)$ for the case $f(x_1, \dots, x_k) = \Sigma_i x_i^{n_i}$.

We shall then continue our study of the Lie algebra $L^0(f)$, and of its deformation $\Lambda^0(f) = \mathbf{V} \otimes H_0$. This leads to a description of the cohomology group $A^1(k, L^0(t); L^0(t))$, where we have put $L^0(t) = L^0(F_0(t))$. For $f(x_1, x_2) = x_1^k + x_2^l$ we then compute the cohomology group $A^1(k, L^0(f); L^0(f))$ of the Lie algebra $L^0(f)$, which turns out to be determined by the dimension of the prorepresenting substratum $\mathbf{H}_0$ of the singularity f. For $f(x_1, \dots, x_k) = \Sigma_i x_i^{n_i}$ we also construct a deformation of $L^0(f)$ to $\mathrm{Sym}_k(T) = k[t_1, \dots, t_k]$, where T is the maximal torus.

DEFINITION (4.1). A torus T on a Lie algebra $\mathbf{g}$ is an abelian subalgebra of $\mathrm{Der}_k\, \mathbf{g}$ consisting of semisimple endomorphisms. T is a maximal torus if it is not contained in any torus $T' \neq T$.

THEOREM ([Mos]). *If* T, T' *are maximal tori on* $\mathbf{g}$, *then there exsts* $\theta \in \mathrm{Aut}_k\, \mathbf{g}$ *such that* $T' = \theta T \theta^{-1}$.

DEFINITION (4.2). If T is a maximal torus on $\mathbf{g}$, then $\dim_k T$ is called the rank of $\mathbf{g}$.

Whenever T is a maximal torus on $\mathbf{g}$, the elements of T are simultaneously diagonalizable (k being algebraically closed and T commutative). Hence $\mathbf{g}$ decomposes into a direct sum of root spaces

$$\mathbf{g} = \bigoplus_{\beta \in T^*} \mathbf{g}^\beta$$

where $\mathbf{g}^\beta = \{x \in \mathbf{g} \mid t(x) = \beta(t)x, \quad \forall t \in T\}$.

DEFINITION (4.3). The root system associated to T is

$$R(T) = \{\beta \in T^* \mid \mathbf{g}^\beta \neq (0)\}.$$

For the later calculations we note two simple lemmas:

LEMMA (4.3). *Let T_1 be a torus on $\mathbf{g}$, let T_2 be any other torus containing T_1. Then each root space of T_1 is a direct sum of root spaces for T_2.*

Proof. Any element β_2 of T_2^* restricts to an element β_1 of T_1^*. Obviously $\mathbf{g}_1^\beta \supseteq \mathbf{g}_2^\beta$, hence the result. Q.E.D.

LEMMA (4.4). *Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathbf{g}$ as a vector space, T a subalgebra of* $\mathrm{End}_k\, \mathbf{g}$ *such that*

$$t(x_i) = \beta_i(t)x_i \quad \forall t \in T, \quad i = 1, \ldots, n, \quad \beta_i \in T^*.$$

Then T is a torus iff $\beta_k = \beta_i + \beta_j$ whenever $c_{ij}^k \neq 0$.

Proof. T is a torus iff $t([x_i, x_j]) = [t(x_i), x_j] + [x_i, t(x_j)], \forall t \in T, i, j = 1, \ldots, n$. Expanding in terms of the β_i and c_{ij}^k we obtain the equivalent condition $\Sigma_k c_{ij}^k \beta_k(t) x_k = \Sigma_k c_{ij}^k(\beta_i(t) + \beta_j(t))x_k, \forall t \in T, i, j = 1, \ldots, n$, from which the lemma follows. Q.E.D.

We now recall a few simple facts about the Lie algebra $L^0(f)$, for $f = f(x_1, \ldots, x_k) = x_1^{n_1} + \cdots + x_k^{n_k}$ (cfr. [La-Pf]):

LEMMA (4.5). *Let f be as given above, let*

$$I = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k \mid 0 \leqslant \alpha_i \leqslant n_i - 2, i = 1, \ldots, k\}, I^* = I - \{0\}.$$

For $\boldsymbol{\alpha} \in \mathbf{Z}^k$, let $|\boldsymbol{\alpha}| = \Sigma_i \alpha_i / n_i$.

Then $\{d_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in I\}$ is a basis for $L^0(f)$. Furthermore, we have

$$[d_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}}] = \begin{cases} (|\boldsymbol{\beta} - \boldsymbol{\alpha}|) d_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\alpha} + \boldsymbol{\beta} \in I \\ 0 & \text{otherwise} \end{cases}$$

and $\{d_\alpha | \alpha \in I^*\}$ *is a basis for* $C^1L^0(f)$.

We also note the elementary observation:

COROLLARY (4.6). *Let* $\mathfrak{g}(f) = C^1L^0(f)$. *Then* $\mathfrak{g}(f)$ *is generated, as a Lie algebra, by* $\{d_\alpha | \alpha \in I^*, \Sigma_i \alpha_i \leqslant 2\}$.

We may now easily prove

THEOREM (4.7). *Let* $f(x_1, \ldots, x_k) = x_1^{n_1} + \cdots + x_k^{n_k}, 3 \leqslant n_1 \leqslant \cdots \leqslant n_k$. *Then there is a torus T on* $\mathfrak{g}(f)$ *generated by derivations* $t_1, \ldots, t_k$, *where* $t_i(d_\alpha) = \alpha_i d_\alpha, \alpha \in I^*, i = 1, \ldots, k$.

The root system associated to T is

$$R(T) = \{\beta_\alpha = \alpha_1 t_1^* + \cdots + \alpha_k t_k^* \,|\, \alpha \in I^*\},$$

and for $\alpha \in I^*$, *the root space* $\mathfrak{g}^{\beta_\alpha}$ *is generated by* d_α. *Here* $\{t_i^*\}$ *is the dual basis of* T^* *given by* $t_i^*(t_j) = \delta_{ij}$. *Furthermore, T is maximal, except when*

$$k \leqslant 4, n_1 = \cdots = n_k = 3 \text{ or}$$
$$k = 2, n_1 = n_2 = 4 \text{ or}$$
$$k = 1, n_1 \leqslant 6$$

in which cases T is contained in a torus of dimension $k + 1$.

Proof. Let $[d_\alpha, d_\beta] = \Sigma_{\gamma \in I^*} c_{\alpha\beta}^\gamma d_\gamma$. Then, by (4.5), if $c_{\alpha\beta}^\gamma \neq 0$ then $\gamma = \alpha + \beta$. Hence $c_{\alpha\beta}^\gamma \neq 0$ implies $\beta_\gamma(t_i) = \gamma_i = \alpha_i + \beta_i = (\beta_\alpha + \beta_\beta)(t_i)$.

Hence it follows from (4.4) that T is a torus with root spaces as asserted. Clearly $\dim_k T = k$. Thus, all that remains to prove is that if T' is a torus containing T, then $T' = T$. Since the root spaces of T are all of dimension 1, by (4.3) we have

LEMMA (4.8). *If* T' *contains T then the root spaces of* T' *are exactly the root spaces* $\mathfrak{g}\beta_\alpha$ of T.

Let $R(T') = \{\phi_\alpha \in T'^* \,|\, \alpha \in I^*\}$ be the root system of T'. By (4.4) and (4.5) we find,

$$\phi_\gamma = \phi_\alpha + \phi_\beta \quad \text{whenever} \quad \gamma = \alpha + \beta, |\alpha| \neq |\beta|.$$

Let $e_i = (0, \ldots, 1, \ldots, 0)$ be the ith unit vector of $\mathbf{Z}^k$. We shall prove,

LEMMA (4.9). *For* T' *and* $R(T')$ *as above, we have*

$$\phi_\gamma = \gamma_1 \phi_{e_1} + \cdots + \gamma_k \phi_{e_k}, \quad \gamma \in I^*,$$

except for the special cases listed in (4.7).

Proof. We proceed by induction on $\Sigma_i\gamma_i$, applying (4.4′) repeatedly. If $\gamma \in I^*, \Sigma_i\gamma_i \geqslant 3$, then for some j, $\boldsymbol{\beta} = \gamma - \boldsymbol{e}_j \in I^*$, $|\boldsymbol{\beta}| \neq |\boldsymbol{e}_j|$. Hence we need only prove that (4.9) holds when $\Sigma_i\gamma_i \leqslant 2$. Of course, when $\Sigma_i\gamma_i = 1$, (4.9) is trivial. To verify the formula for the remaining case, $\gamma = \boldsymbol{e}_i + \boldsymbol{e}_j \in I^*$, we write

$$f(x_1,\ldots,x_k) = x_1^{\nu_1} + \cdots + x_{k_1}^{\nu_1} + x_{k_1+1}^{\nu_2} + \cdots + x_{k_2}^{\nu_2} + \cdots + x_k^{\nu_r}$$

where $3 \leqslant \nu_1 < \nu_2 < \cdots < \nu_r$. Furthermore, let $S = \{1,\ldots,k\} = S_1 \cup \cdots \cup S_r$ where $S_i = \{k_{i-1}+1,\ldots,k_i\}$, $(k_0 = 0, k_r = k)$ so that the exponent n_t of the variable x_t is ν_i iff $t \in S_i$. We may now apply (4.4′) in a somewhat recursive fashion to finish the proof of (4.9). The steps are as follows (we omit the detailed calculations):

(4.10.1) If $s \in S_i, t \in S_j, i \neq j$ then $\phi_{(e_s+e_t)} = \phi_{e_s} + \phi_{e_t}$

(4.10.2) If $t \in S_i, i \geqslant 2$ then $\phi_{2e_t} = 2\phi_{e_t}$

(4.10.3) If $s, t \in S_i, i \geqslant 2$ then $\phi_{(e_s+e_t)} = \phi_{e_s} + \phi_{e_t}$

(4.10.4) Suppose S_2 is non-empty, $s, t \in S_1, (e_s + e_t) \in I^*$. Then $\phi_{(e_s+e_t)} = \phi_{e_s} + \phi_{e_t}$.

The only case left is:

(4.10.5) Suppose $f(x_1,\ldots,x_k) = x_1^n + \cdots + x_k^n$. Except for the special cases listed in (4.7) we have: If $\boldsymbol{e}_s + \boldsymbol{e}_t \in I^*$ then

$$\phi_{(e_s+e_t)} = \phi_{e_s} + \phi_{e_t}. \qquad \text{Q.E.D.}$$

The last part of (4.7) now follows from (4.9):

COROLLARY (4.11). If T' *is a torus containing* T, *then* $T' = T$.

Proof. $\phi_{e_1},\ldots,\phi_{e_k}$ generate $R(T')$, hence T'^*. Thus,

$$\dim_k T' = \dim_k T'^* \leqslant k = \dim_k T. \text{ Hence } T' = T. \qquad \text{Q.E.D.}$$

Notice that each derivation $t_i \in T$ extends to a derivation of $L^0(f)$, simply by putting $t_i(d_0) = 0$, $i = 1,\ldots,k$. Notice also that the inner derivation $\operatorname{ad}(d_0)$ is equal to $\Sigma_i t_i/n_i \in T$.

Given the torus T one may construct a Generalized Cartan Matrix that associates to f a Kac-Moody Lie algebra that might be of interest, see [San]. For a list of G.C.M.'s associated to some plane curve singularities, see [B-L].

We end this paragraph with a description of the deformation $\Lambda^0(f) = \mathbf{V} \otimes_H H_0$ of $L^0(f)$, and some general properties of the cohomology $A^1(k, L^0(f); L^0(f))$. For $f(x,y) = x^k + y^l, 10 \leqslant k \leqslant l$, we shall actually calculate this space.

We recall that if $f(x_1, \ldots, x_k) = \Sigma_i x_i^{n_i}$ and if

$$I = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k \mid 0 \leqslant \alpha_i \leqslant n_i - 2\},\ I_0 = \{\alpha \in I \mid |\alpha| = 1\},$$

then $\mathbf{F}_0 = f + \Sigma_{\alpha \in I_0} t_{\alpha,} \boldsymbol{x}^{\alpha}$ is the modular family defined on $\mathbf{H}_0$.

Let $\Lambda^0(f) = \mathbf{V} \otimes_H H_0$, then $\Lambda^0(f)$ is a locally free H_0-Lie algebra of rank $\tau(f)$ such that $\Lambda^0(f) \otimes_{H_0} k(t) \approx L^0(t)$ for each closed point $t \in \mathbf{H}_0$. On the other hand, since for $t \in \mathbf{H}_0, F_0(t)$ is quasihomogenous we have that

$$L^0(t) \approx k[[x_1, \ldots, x_k]]/(\partial F_0(t)/\partial x_1, \ldots, \partial F_0(t)/\partial x_k) = A(F_0(t))$$

where the Lie product on $A(F_0(t))$ is given by

$$[\boldsymbol{x}^{\alpha}, \boldsymbol{x}^{\beta}] = (|\boldsymbol{\beta} - \boldsymbol{\alpha}|)\boldsymbol{x}^{\alpha+\beta}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in I.$$

Hence we may consider the H_0-module

$$L^0(F_0) = H_0[[x_1, \ldots, x_k]]/(\partial F_0/\partial x_1, \ldots, \partial F_0/\partial x_k)$$

as an H_0-Lie algebra with the bracket $[\boldsymbol{x}^{\alpha}, \boldsymbol{x}^{\beta}] = (|\boldsymbol{\beta} - \boldsymbol{\alpha}|)\boldsymbol{x}^{\alpha+\beta}$.

Then $L^0(F_0)$ is locally free of rank $\tau(f)$, and the fibers are precisely $L^0(t)$. Thus we may conclude,

LEMMA (4.12). *The deformation* $\Lambda^0(f)$ *of* $L^0(f)$ *is given by*

$$\Lambda^0(f) \approx L^0(F_0) = H_0[[x_1, \ldots, x_k]]/(\partial F_0/\partial x_1, \ldots, \partial F_0/\partial x_k).$$

Now there is an open neighbourhood U_0 of the origin in $\mathbf{H}_0$ such that $B = \{\boldsymbol{x}^{\alpha} \mid \boldsymbol{\alpha} \in I\}$ is a basis for $L^0(F_0)$ on U_0. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in I$ we have

$$[\boldsymbol{x}^{\alpha}, \boldsymbol{x}^{\beta}] = \sum_{\gamma \in I} c_{\alpha\beta}^{\gamma} \boldsymbol{x}^{\gamma} \tag{4.13}$$

where the structural constants $c_{\alpha\beta}^{\gamma}$ are elements of a localization of H_0. Notice that, correspondingly, if $t \in U_0$ then,

$$\{d_{\alpha} \mid \boldsymbol{\alpha} \in I\} \quad \text{is a basis for } L^0(t), \tag{4.14}$$

and the structural constants with respect to this basis are simply the values, $c_{\alpha\beta}^{\gamma}(t)$, of $c_{\alpha\beta}^{\gamma}$ at the point t.

Notice also that, since $L^0(F_0)$ is a graded H_0-module, $c_{\alpha\beta}^{\gamma} = 0$ unless $|\gamma| = |\boldsymbol{\alpha} + \boldsymbol{\beta}|$. Equivalently, $L^0(t) \approx \bigoplus_{m \geqslant 0} L_m$ and $L_{m+m'} \supseteq [L_m, L_{m'}]$, where L_m is the subspace of $L^0(t)$ generated by $\{d_\alpha \mid |\alpha| = m\}$. In particular, if $\alpha + \beta \in I$ then

$$c^\gamma_{\alpha\beta} = \begin{cases} |\beta - \alpha| & \text{if } \gamma = \alpha + \beta \\ 0 & \text{otherwise} \end{cases}$$

Now, consider the cohomology $A^1(k, L^0(t); L^0(t))$. Recall that if $\mathbf{g}$ is a k-Lie algebra, $\mathbf{F}$ a free Lie algebra, and $\mathbf{J} = \ker(\mathbf{F} \to \mathbf{g})$ the kernel of a surjective homomorphism, then $A^1(k, \mathbf{g}; \mathbf{g})$ is the quotient of $\mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, \mathbf{g})$ by the image of $\mathrm{Der}_k(\mathbf{F}, \mathbf{g})$.

LEMMA (4.15). *Let* $\{x_1, \dots, x_n\}$ *be a basis for* $\mathbf{g}$, *and let* $\mathbf{F} = \mathbf{F}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ *be freely generated by n elements* $\mathbf{x}_1, \dots, \mathbf{x}_n$. *Let* $\mathbf{F} \to \mathbf{g}$ *be the homomorphism mapping* $\mathbf{x}_i$ *onto* x_i, $i = 1, \dots, n$. *If* $\mathbf{b}$ *is the kernel of the restriction map* $\mathrm{Der}_k(\mathbf{F}, \mathbf{g}) \to \mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, \mathbf{g})$ *then there is an isomorphism of vector spaces* $\mathbf{b} \approx \mathrm{Der}_k \mathbf{g}$.

Proof. If $\delta \in \mathrm{Der}_k \mathbf{g}$, then the corresponding element D of $\mathbf{b}$ is given by $D(\mathbf{x}_i) = \delta(x_i)$, $i = 1, \dots, n$. Q.E.D.

Now the ideal $\mathbf{J}$ in $\mathbf{F} = \mathbf{F}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is generated by

$$\left\{ f_{ij} = [\mathbf{x}_i, \mathbf{x}_j] - \sum_k c^k_{ij} \mathbf{x}_k \mid i, j = 1, \dots, n \right\}.$$

(Notice that $f_{ji} = -f_{ij}$).

Let W be the subspace of $\mathbf{J}$ generated as a vector space by the f_{ij}, let ϕ be a k-linear map of W into $\mathbf{g}$. Because of the Jacobi identity in $\mathbf{F}$, ϕ extends to an $\mathbf{F}$-homomorphism of $\mathbf{J}$ into $\mathbf{g}$ iff,

$$[x_i, \phi_{jk}] + [x_j, \phi_{ki}] + [x_k, \phi_{ij}] - \sum_s (c^s_{ij}\phi_{sk} + c^s_{jk}\phi_{si} + c^s_{ki}\phi_{sj}) = 0 \tag{4.16}$$

for $i, j, k = 1, \dots, n$, where $\phi_{ij} = \phi(f_{ij})$.

For $\mathbf{g} = L^0(t)$, let $\mathbf{F} = \mathbf{F}(\mathbf{d}_\alpha)_{\alpha \in I}$. Then the generators of $\mathbf{J}$ are

$$\left\{ f_{\alpha\beta} = [\mathbf{d}_\alpha, \mathbf{d}_\beta] - \sum_\gamma c^\gamma_{\alpha\beta} \mathbf{d}_\gamma \mid \alpha, \beta \in I \right\}.$$

The condition (4.16) then reads

$$[d_\alpha, \phi_{\beta\gamma}] + [d_\gamma, \phi_{\alpha\beta}] + [d_\beta, \phi_{\gamma\alpha}] - \sum_\delta (c^\delta_{\alpha\beta}\phi_{\delta\gamma} + c^\delta_{\beta\gamma}\phi_{\delta\alpha} + c^\delta_{\gamma\alpha}\phi_{\delta\beta}) = 0. \tag{4.16'}$$

Let $\phi_{\alpha\beta} = h_{\alpha\beta} + r_{\alpha\beta}$, where $h_{\alpha\beta} \in L_{|\alpha+\beta|}$, $r_{\alpha\beta} \in \bigoplus_{m \neq |\alpha+\beta|} L_m$.

LEMMA (4.17). *If* $\phi \in \mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, L^0(t))$ *then the maps h and* $r: \mathbf{J} \to L^0(t)$, *given by* $h(f_{\alpha\beta}) = h_{\alpha\beta}, r(f_{\alpha\beta}) = r_{\alpha\beta}$, *are* **F** *homomorphisms.*

Proof. This is immediate from (4.16′), keeping in mind that $c^{\gamma}_{\alpha\beta} = 0$ unless $|\gamma| = |\alpha + \beta|$. Q.E.D.

LEMMA (4.18). *Assume* $r \in \mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, L^0(t))$ *satisfy the condition*

$$r(f_{\alpha\beta}) = r_{\alpha\beta} \in \bigoplus_{m \neq |\alpha+\beta|} L_m$$

Then r is determined by the values $\{r_{0\beta} | \beta \in I\}$.

Any such homomorphism r is the restriction of a derivation.

Proof. Letting $\gamma = 0$ in (4.16′) we obtain

$$[d_0, r_{\alpha\beta}] - |\alpha + \beta| r_{\alpha\beta} = [d_\alpha, r_{0\beta}] - [d_\beta, r_{0\alpha}] - \sum_\delta c^\delta_{\alpha\beta} r_{0\delta}$$

and the first assertion follows, since $r_{\alpha\beta}$ contains no terms of weight $|\alpha + \beta|$. For the second half of the lemma, let $D \in \mathrm{Der}_k(\mathbf{F}, L^0(t))$, with $D(\mathbf{d}_\alpha) = \sum_\gamma s^\gamma_\alpha d_\gamma$. Then

$$\begin{aligned} D(f_{0\alpha}) &= [D(\mathbf{d}_0), d_\alpha] + [d_0, D(\mathbf{d}_\alpha)] - |\alpha| D(\mathbf{d}_\alpha) \\ &= \sum_\gamma s^\gamma_\alpha |\gamma - \alpha| d_\gamma + [D(\mathbf{d}_0), d_\alpha] \end{aligned}$$

and it is clear that D may be chosen so that $D(f_{0\alpha}) = r_{0\alpha}, \alpha \in I$. Q.E.D.

COROLLARY (4.19). *Let* G_t *be the subspace of* $\mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, L^0(t))$ *consisting of the homomorphisms h such that,*

$$h(f_{\alpha\beta}) = h_{\alpha\beta} \in L_{|\alpha+\beta|}.$$

Let B be the subspace of $\mathrm{Der}_k(\mathbf{F}, L^0(t))$ *of derivations D such that* $D(\mathbf{d}_\alpha) \in L_{|\alpha|}$, *and let* Q_t *be the subspace of* $\mathrm{Der}_k(L^0(t))$ *of derivations* δ *such that* $\delta(\mathbf{d}_\alpha) \in L_{|\alpha|}$. *Then there is an exact sequence*

$$0 \to Q_t \to B \to G_t \to A^1(k, L^0(t); L^0(t)) \to 0$$

Proof. This is clear from (4.17) and (4.18), since the restriction to **J** maps B into G_t. Q.E.D.

LEMMA (4.20). *Let* N_t *be the subspace of* G_t *consisting of the homomorphisms n such that* $n(f_{0\alpha}) = 0$ *for all* $\alpha \in I$, P_t *the homomorphisms p for which* $p(f_{\alpha\beta}) = 0$ *for*

α and β non-zero. Then $G_t \approx P_t \oplus N_t$, and P_t is isomorphic to the subspace Q_t of $\mathrm{Der}_k(L^0(t))$.

Proof. For any $h \in G_t$, consider the map $p: \mathbf{J} \to L^0(t)$ given by $p(f_{0\beta}) = -p(f_{\beta 0}) = h(f_{0\beta}), p(f_{\alpha\beta}) = 0$ for α and β non-zero. Then one easily checks that p is a homomorphism, and hence $h = p + n$, where $n := h - p \in N_t$. Let $p \in P_t$, then, putting $\gamma = \mathbf{0}$ in (4.16′), we find

$$[d_\alpha, p_{\beta 0}] + [p_{\alpha 0}, d_\beta] = \sum_\delta c^\delta_{\alpha\beta} p_{\delta 0}.$$

Hence the endomorphism q of $L^0(t)$ given by $q(d_\alpha) = p_{\alpha 0}$ is a derivation, and the correspondence $p \to q$ maps P_t isomorphically onto Q_t. Q.E.D.

PROPOSITION (4.21). *Let $G_t \approx P_t \oplus N_t$ be the decomposition given in* (4.20). *Then the image of P_t in $A^1(k, L^0(t); L^0(t))$ is isomorphic to $Q_t/(\mathrm{ad}(d_0))$. Moreover, if $t = \mathbf{0}$ then the torus T of* (4.7) *is contained in Q_t, and the infinitesimal deformations of $L^0(f)$ given by the image of T in $A^1(k, L^0(f); L^0(f))$ may be lifted to a deformation of $L^0(f)$ to $\mathrm{Sym}_k(T) \approx k[t_1, \ldots, t_k]$.*

Proof. Let $q \in Q_t$, suppose q is the restriction to $\mathbf{J}$ of a derivation D, with $D(\mathbf{d}_\alpha) = D_\alpha$. Then

$$q(f_{0\alpha}) = [D_0, d_\alpha] + [d_0, D_\alpha] - |\alpha| D_\alpha = [D_0, d_\alpha].$$

Hence, $q \approx \mathrm{ad}(D_0) \in (\mathrm{ad}(d_0))$.

To prove the second part of (4.21), let L_T be the free $k[t_1, \ldots, t_k]$-module generated by $\{d_\alpha \mid \alpha \in I\}$. Then it is easily verified that

$$[d_\alpha, d_\beta] = \begin{cases} \sum_i \beta_i (t_i + 1/n_i) d_\beta & \alpha = \mathbf{0} \\ |\beta - \alpha| d_{\alpha+\beta} & \alpha + \beta \in I, \quad \alpha, \beta \neq \mathbf{0} \\ 0 \quad \text{otherwise} & \end{cases}$$

defines a Lie product on L_T such that $L_T \otimes k(\mathbf{0}) \approx L^0(f)$. Q.E.D.

For the case $f(x_1, x_2) = x_1^k + x_2^l$ we shall show, cf. (5.23), that the subspace Q_t of (4.19) is just the torus T. We end this paragraph by describing the complementary subspace N_0 for this special case. Recall the isomorphism

$$L^0(f) \approx A(f) = k[[x_1, x_2]]/(\partial f/\partial x_i) = k[[x_1, x_2]]/(x_1^{k-1}, x_2^{l-1})$$

where the Lie product in $A(f)$ is given by

$$[x^{\alpha}, x^{\beta}] = |\beta - \alpha| x^{\alpha+\beta}.$$

Thus, if p and q are any quasihomogeneous elements of $L^0(f)$, we have

$$[p, q] = (n - m)pq, \qquad \text{where } m = wt(p), \quad n = wt(q), \tag{4.22}$$

since both sides are bilinear in p and q.

Let $\phi \in N$, such that $\phi(f_{0\beta}) = 0$, $\phi(f_{\alpha\beta}) = \phi_{\alpha\beta} \in L_{|\alpha+\beta|}$ $\quad \alpha, \beta \neq 0$.

Then, using (4.22), (4.16′) may be rewritten:

$$\begin{aligned} &|\beta + \gamma - \alpha| x^{\alpha}\phi_{\beta\gamma} + |\gamma + \alpha - \beta| x^{\beta}\phi_{\gamma\alpha} + |\alpha + \beta - \gamma| x^{\gamma}\phi_{\alpha\beta} \\ &\quad = \sum_{\delta} (c^{\delta}_{\alpha\beta}\phi_{\delta\gamma} + c^{\delta}_{\beta\gamma}\phi_{\delta\alpha} + c^{\delta}_{\gamma\alpha}\phi_{\delta\beta}) \end{aligned} \tag{4.23}$$

ϕ is an **F**-homomorphism iff this equation holds for all $\alpha, \beta, \gamma \in I^*$.

THEOREM (4.24). *Let* $f(x_1, x_2) = x_1^k + x_2^l, 10 \leqslant k \leqslant l$. *Then, modulo* $\mathrm{Der}_k(\mathbf{F}, L^0(f))$, *any element* ϕ *of the subspace* N_0 of G_0 *is congruent to a unique homomorphism of the form*

$$\phi_{PQ}(f_{\alpha\beta}) = \begin{cases} |\beta - \alpha| x_1^{1-k} x^{\alpha+\beta} P & \text{if } \alpha_1 + \beta_1 \geqslant k - 1 \\ |\beta - \alpha| x_2^{1-l} x^{\alpha+\beta} Q & \text{if } \alpha_2 + \beta_2 \geqslant l - 1 \\ 0 & \text{if } \alpha + \beta \in I \end{cases} \tag{4.24}$$

where P and Q may be any elements in $L^0(f) \approx A(f)$ *of weight* $(k - 1)/k, (l - 1)/l$, *respectively.*

Hence the image of N_0 *in* $A^1(k, L^0(f); L^0(f))$ *has dimension* $\dim_k(L_{(k-1)/k}) + \dim_k(L_{(l-1)/l}) = 2 \dim H_0 + \varepsilon$, *where* $\varepsilon = 2$ *if* $k = l$, 1 *if* $l = ak$, $a \geqslant 2$, *and* 0 *otherwise.*

Proof. We shall have to solve the system (4.23) with respect to $\phi_{\alpha\beta}, \alpha, \beta \in I$.

First, consider the system of equations derived from (4.23) by setting the left-hand terms equal to 0:

$$0 = \sum_{\delta} (c^{\delta}_{\alpha\beta}\phi_{\delta\gamma} + c^{\delta}_{\beta\gamma}\phi_{\delta\alpha} + c^{\delta}_{\gamma\alpha}\phi_{\delta\beta}) \quad \alpha, \beta, \gamma \in I^*. \tag{4.25}$$

Noting that $c^{\gamma}_{\alpha\beta} = |\beta - \alpha|$ if $\gamma = \alpha + \beta \in I$, and 0 otherwise, we may rewrite this in the form

$$|\boldsymbol{\beta} - \boldsymbol{\alpha}|\phi_{(\boldsymbol{\alpha}+\boldsymbol{\beta})\boldsymbol{\gamma}} + |\boldsymbol{\gamma} - \boldsymbol{\beta}|\phi_{(\boldsymbol{\beta}+\boldsymbol{\gamma})\boldsymbol{\alpha}} + |\boldsymbol{\alpha} - \boldsymbol{\gamma}|\phi_{(\boldsymbol{\alpha}+\boldsymbol{\gamma})\boldsymbol{\beta}} = 0, \tag{4.26}$$

where $\phi_{st} = 0$ if s or $t \notin \mathrm{I}$, $\phi_{ts} = -\phi_{st}$.

Now let $\boldsymbol{v} = \boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \in \mathbf{Z}^2$ be fixed, $v_1 + v_2 \geqslant 8$. Then it is rather easily verified that the resulting subsystem of (4.26) has a solution space spanned by one element $D_{\boldsymbol{v}}$, where

$$\phi_{st} = |\boldsymbol{t} - \boldsymbol{s}|D_{\boldsymbol{v}} \quad (\boldsymbol{s} + \boldsymbol{t} = \boldsymbol{v}). \tag{4.27}$$

Now, to prove (4.24) we make the following steps:

(i) Using the equations (4.23) and brute force, we show that if ϕ is any given homomorphism, then we may find a derivation $D_7\colon \mathbf{F} \to L^0(f)$ for which $D_7(d_\gamma) = 0$ if $\gamma_1 + \gamma_2 > 7$, and $\phi(f_{\alpha\beta}) = D_7(f_{\alpha\beta})$ for $\Sigma_i(\alpha_i + \beta_i) \leqslant 7$.

(ii) Consider the homomorphism $\phi_8 = \phi - D_7$. For this map, and $\Sigma_i(\alpha_i + \beta_i + \gamma_i) = 8$, the equations (4.23) reduce to (4.26). Hence, $\phi_8(f_{st}) = |\boldsymbol{t} - \boldsymbol{s}|D_{(\boldsymbol{s}+\boldsymbol{t})}$ for $\Sigma_i(s_i + t_i) = 8$. Define a derivation $D_8\colon \mathbf{F} \to L^0(f)$ by $D_8(d_{\boldsymbol{v}}) = D_{\boldsymbol{v}}$ if $v_1 + v_2 = 8$, 0 otherwise, then

$$(D_7 + D_8)(f_{st}) = \phi(f_{st}) \quad \text{for} \quad \sum_i (s_i + t_i) \leqslant 8.$$

(iii) Proceeding recursively as in (ii) we obtain a step-by-step reduction of ϕ: $\phi_j = \phi_{(j-1)} - D_{(j-1)}$ for $j = 9, \ldots, k-1$, such that $\phi_j(f_{st}) = 0$ if $\Sigma_i(s_i + t_i) < j$.

(iv) Assume first that $k = l$. Then, construct $D_{(k-1)}$ so that $D_{(k-1)}(f_{st}) = \phi_{(k-1)}(f_{st})$ for $|\boldsymbol{s} + \boldsymbol{t}| = (k-1)/k$, $\boldsymbol{s} + \boldsymbol{t} \neq (k-1, 0), (0, k-1)$. Let $P = [k/(k-3)]\phi_{(k-1)}(f_{(1,0)(k-2,0)})$, $Q = [k/(k-3)]\phi_{(k-1)}(f_{(0,1)(0,k-2)})$, and let $\phi_k = \phi_{(k-1)} - \phi_{PQ} - D_{(k-1)}$. Clearly, $\phi_k(f_{st}) = 0$ if $|\boldsymbol{s} + \boldsymbol{t}| < 1$. Furthermore, $\phi_k(f_{st}) = 0$ if $|\boldsymbol{s} + \boldsymbol{t}| = 1, \boldsymbol{s} + \boldsymbol{t} \notin I$, since, in this case, $\phi_k(f_{st}) = [|\boldsymbol{t} - \boldsymbol{s}|/(k-2)]\phi_k(f_{e_j(\boldsymbol{s}+\boldsymbol{t}-e_j)}) = 0$ for $j = 1$ or 2 (cf. 4.26).

The latter fact allows us to form D_k such that $D_k(f_{st}) = \phi_k(f_{st})$ for all $\boldsymbol{s}, \boldsymbol{t} \in I$, $|\boldsymbol{s} + \boldsymbol{t}| = 1$. This reduction also applies in degrees higher than 1, and we may conclude that

$$\phi = \sum_{i \geqslant 7} D_i + \phi_{PQ}.$$

If $k < l$, we may first form $\phi_k = \phi_{(k-1)} - \phi_{P0} - D_{(k-1)}$, then $\phi_l = \phi_{(l-1)} - \phi_{0Q} - D_{(l-1)}$. Since $\phi_{PQ} = \phi_{P0} + \phi_{0Q}$ the resulting reduction is the same.

Clearly, $(P, Q) \to \phi_{PQ}$ is a linear map of $L_{(k-1)/k} \oplus L_{(l-1)/l}$ into $A^1(k, L^0(f); L^0(f))$, and this map is easily seen to be injective. Q.E.D.

As a corollary of the immersion Theorem (5.9) we may compute the subspace Q_t of $\mathrm{Der}_k(L^0(t))$, thus completing the description of $A^1(k, L^0(f); L^0(f))$, see (5.23) and (5.27). It should be noticed that this description of $A^1(k, L^0(f); L^0(f))$ does not hold if $f(x, y) = x^k + y^l$ is a plane curve singularity of "small" Tjurina number $\tau(f)$. In such cases (4.24) only yields a lower bound for the dimension of this cohomology space.

5. The immersion theorem for quasihomogenous plane curve singularities

Let $f(x_1, x_2) = x_1^k + x_2^l$, let $\mathbf{V}$ denote the kernel of the Kodaira–Spencer map of the versal family F, and let $\Lambda^0(f) = \mathbf{V} \otimes_H H_0$. Then, for every closed point $t \in \mathbf{H}_0$, the Kodaira–Spencer map of the family $\Lambda^0(f)$ induces a commutative diagram

$$\begin{array}{rcl} \mathrm{Der}_k H_0 & \to & A^1(H_0, \Lambda^0(f); \Lambda^0(f)) \\ \downarrow & & \downarrow \\ \lambda(t): T_{t,H_0} & \to & A^1(k, L^0(t); L^0(t)). \end{array}$$

The main result of this paragraph (see (5.9)), asserts that on an open neighbourhood U of the origin in $\mathbf{H}_0$, the map $\lambda(t)$ is injective (except for some cases where $\dim \mathbf{H}_0 = 1$).

For a related result, recall the isomorphism $A(f) \approx L^0(f)$, where $[p, q] = (|q| - |p|)pq$ for p, q homogenous elements of $A(f)$. Hence the algebra structure on $A(f)$ can be deduced from the Lie product, provided that the Lie structure determines a monomial basis $\{X^iY^j\}$ for $A(f)$. In this case, it follows from a theorem of Mather and Yau, see [Ma-Y], that f determines the singularity f. It may easily be shown that this is the case except in a few cases where $\tau(f)$ is "small". As a counterexample, however, consider $L^0(t)$, where $F_0(x_1, x_2) = x_1^4 + x_2^4 + tx_1^2x_2^2$. Picking the basis B_t given by $1, x_1, x_2, x_1^2 + (t/2)x_2^2, x_2^2 + (t/2)x_1^2$, $[1 - (t/2)^2]x_1^ix_2^j$ for $(i, j) = (1, 1), (2, 1), (1, 2). (2, 2)$ one easily sees that the Lie algebras $L^0(t)$ are all isomorphic to $L^0(x_1^4 + x_2^4)$. For further examples, see [B-L].

The proof of (5.9) also allows us to determine the Lie algebra $\mathrm{Der}_k(L^0(t))$ for $L^0(t)$ as above, $t \in U$. This yields the cohomology group $H^1(L^0(t), L^0(t))$, and for $f(x_1, x_2) = x_1^k + x_2^l$ we may fill in the last details in our description of $A^1(k, L^0(f); L^0(f))$. Furthermore, for $t \in U - \{0\}$ we prove that $\mathrm{Der}_k(L^0(t))$ acts trivially on $A^1(k, L^0(t); L^0(t))$.

As in earlier paragraphs, the field k is assumed to be of characteristic 0.

Let $f(x_1, x_2) = x_1^k + x_2^l$, let $I = \{(\alpha_1, \alpha_2) \in \mathbf{Z}^2 \mid 0 \leqslant \alpha_1 \leqslant k - 2; 0 \leqslant \alpha_2 \leqslant l - 2\}$, and let $I_0 = \{(\alpha_1, \alpha_2) \in I \mid |\boldsymbol{\alpha}| = 1\}$, where $|\boldsymbol{\alpha}| = \alpha_1/k + \alpha_2/l$.

Then $F_0(x_1, x_2) = f + \Sigma_{\boldsymbol{\alpha} \in I_0} t_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$, and

$$\Lambda^0(f) \approx H_0[[x_1, x_2]]/(\partial F_0/\partial x_i) = L^0(F_0) \quad \text{(cf.(4.12))}.$$

Let U_0 be the neighbourhood of $\mathbf{0}$ in $\mathbf{H}_0$ on which

$$B = \{x_1^{\alpha_1} x_2^{\alpha_2} \mid \boldsymbol{\alpha} \in I\} \text{ is a basis for } L^0(F_0). \tag{5.1}$$

For $t \in U_0$ the Lie algebra $\Lambda^0(f) \otimes_{H_0} k(t) \approx L^0(t)$ is a graded Lie algebra, with basis $\{d_{\boldsymbol{\alpha}} | \boldsymbol{\alpha} \in I\}$. For such t we then have the following expression for the Kodaira–Spencer map

$$\lambda(t)\colon \mathrm{Der}_k(H_0, k(t)) \to A^1(k, L^0(t); L^0(t)):$$

If $\delta \in \mathrm{Der}_k(H_0, k(t))$ then $\lambda(t)(\delta)$ is represented by the $\mathbf{F}$-homomorphism given by

$$f_{\boldsymbol{\alpha\beta}} \to \lambda_{\boldsymbol{\alpha\beta}} = \sum_{\gamma} \delta(c_{\boldsymbol{\alpha\beta}}^{\gamma})(t) d_{\gamma} \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in I.$$

Since $c_{\boldsymbol{\alpha\beta}}^{\gamma} = 0$ unless $|\gamma| = |\boldsymbol{\alpha} + \boldsymbol{\beta}|$, $\lambda_{\boldsymbol{\alpha\beta}} \in L_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in I$, that is, in the notation of (4.19), $\lambda(t)$ maps $\mathrm{Der}_k(H_0, k(t))$ into G_t.

The image of δ is zero iff there exists a derivation $D\colon \mathbf{F} \to L^0(t)$ whose restriction to $\mathbf{J}$ is $\lambda(t)(\delta)$, that is

$$D(f_{\boldsymbol{\alpha\beta}}) = \lambda_{\boldsymbol{\alpha\beta}} \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in I. \tag{5.2}$$

By (4.19) we may assume that

$$D(\mathbf{d}_{\boldsymbol{\alpha}}) := v_{\boldsymbol{\alpha}} \in L_{|\boldsymbol{\alpha}|} \quad \text{for all } \boldsymbol{\alpha} \in I. \tag{5.3}$$

Substituting $[\mathbf{d}_{\boldsymbol{\alpha}}, \mathbf{d}_{\boldsymbol{\beta}}] - \Sigma_{\gamma} c_{\boldsymbol{\alpha\beta}}^{\gamma} \mathbf{d}_{\gamma}$ for $f_{\boldsymbol{\alpha\beta}}$ in (5.2) we find

$$[v_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}}] + [d_{\boldsymbol{\alpha}}, v_{\boldsymbol{\beta}}] - \sum_{\gamma} c_{\boldsymbol{\alpha\beta}}^{\gamma} v_{\gamma} = \sum_{\gamma} \delta(c_{\boldsymbol{\alpha\beta}}^{\gamma}) d_{\gamma} \tag{5.4}$$

Now it is in fact easily seen that if $t = \mathbf{0}$ this is impossible unless $\delta = 0$, that is,

PROPOSITION (5.5). *The map $\lambda(\mathbf{0})$ is injective.*

Proof. To see the injectivity, notice that $L^0(\mathbf{0}) \approx L^0(f)$, with Lie product given by

$$[d_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}}] = \begin{cases} |\boldsymbol{\beta} - \boldsymbol{\alpha}| d_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\alpha} + \boldsymbol{\beta} \in I \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Furthermore, write $F_0 = x_1^k + x_2^l + \Sigma_i t_i m_i$, where $m_i = x_1^{a_i} x_2^{b_i}$, $a_i/k + b_i/l = 1$, $(a_i, b_i) \in I$. Then $\partial F_0/\partial x_1 = k x_1^k + \Sigma_i a_i t_i (m_i/x_1)$ is zero in the H_0-Lie algebra $L^0(F_0)$. Hence

$$[x_1^c, x_1^d] = [(c-d)/k^2]\sum_i a_i t_i (m_i/x_1) \quad \text{whenever } c + d = k - 1.$$

In other words

$$c_{(c,0)(d,0)}^{(a_i-1,b_i)} = [(c-d)/k^2] a_i t_i \quad (\text{where } c + d = k - 1).$$

Let $\delta \in \mathrm{Der}_k(H_0, k(\mathit{0}))$, $\delta = \sum_j u_j \partial/\partial t_j$. Then

$$\lambda_{(c,0)(d,0)} = [(c-d)/k^2]\sum_i a_i u_i d_{(a_i-1,b_i)}. \tag{5.7}$$

Furthermore, since $[d_{(c,0)}, d_{(d,0)}] = 0$ in $L^0(f)$

$$D(f_{(c,0)(d,0)}) = [v_{(c,0)}, d_{(d,0)}] + [d_{(c,0)}, v_{(d,0)}]. \tag{5.8}$$

We claim that if $c + d = k - 1$, $c < d$, then $v_{(d,0)} \in (d_{(1,0)})$ (the ideal in $L^0(f)$ generated by $d_{(1,0)}$). If $k > 4$ this is clear: In this case

$$f_{(1,0)(d-1,0)} = [\mathbf{d}_{(1,0)}, \mathbf{d}_{(d-1,0)}] - (d-2)/k\mathbf{d}_{(d,0)}$$

while $\lambda_{(1,0)(d-1,0)} = 0$ (since all structural constants $c_{(1,0)(d-1,0)}^{\gamma}$ are independent of $\boldsymbol{t}$). Hence

$$D(f_{(1,0)(d-1,0)}) = [d_{(1,0)}, v_{(d-1,0)}] + [v_{(1,0)}, d_{(d-1,0)}] - \\ - (d-2)/kv_{(d,0)} = 0$$

which yields

$$v_{(d,0)} = k/(d-2)\{[d_{(1,0)}, v_{(d-1,0)}] + [v_{(1,0)}, d_{(d-1,0)}]\}$$

and the assertion follows (for the case $k = 4$, $l = 4n$, $n \geqslant 2$ a slightly modified argument is needed).

From (5.8) we then deduce that $D(f_{(c,0)(d,0)}) \in (d_{(c+1,0)})$, for $1 \leqslant c < d$. Hence, since

$$D(f_{(c,0)(d,0)}) = \lambda_{(c,0)(d,0)} = [(c-d)/k^2]\sum_i a_i u_i d_{(a_i-1,b_i)}$$

we conclude that $u_i = 0$ if $a_i - 1 < c + 1$.

Since this holds for any pair c, d such that $c + d = k - 1, c < d$, we may in fact assume that $c \geqslant (k-3)/2$.

Hence $u_i = 0$ unless $a_i - 1 \geqslant (k-3)/2 + 1$.

By symmetry in x_1 and x_2 we also have

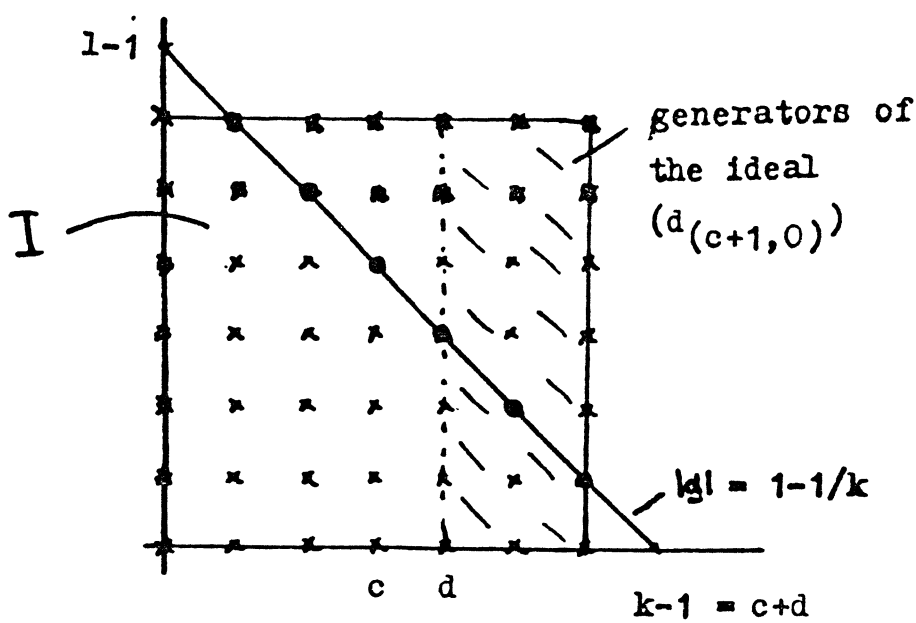
$u_i = 0$ unless $b_1 - 1 \geqslant (l-3)/2 + 1$.

However, if $a_i \geqslant (k+1)/2$, $b_i \geqslant (l+1)/2$ then

$l(a_i, b_i)| = a_i/k + b_i/l \geqslant 1 + 1/2k + 1/2l$

which is impossible, since, by assumption $|(a_i, b_i)| = 1$. Hence $u_i = 0$ for all i, that is, the derivation $\delta = \sum_j u_j \partial/\partial t_j$ is 0. Q.E.D.

The following figure gives the geometric idea behind this proof (each generator d_α is represented by the point $\alpha \in I$).



Notice that the Lie structure of $L^0(f)$ is easily read off from this diagram, the product $[d_\alpha, d_\beta]$ being represented by the point $\alpha + \beta$ (given the weight $|\beta - \alpha|$). If this point lies outside of I, then the product is zero.

It should be clear that the simplicity of the above proof depends heavily on the rather simple form of the Lie product (5.6). The reader should be warned that the calculations needed to prove that the map $\lambda(t)$ is generically injective are considerably more involved.

From now on, let $D: \mathbf{F} \to L^0(t)$, with $D(\mathbf{d}_\alpha) := v_\alpha \in L_{|\alpha|}$, $\alpha \in I$, satisfy (5.4).

We intend to demonstrate that, for any t in an open neighbourhood U of the origin in $\mathbf{H}_0$, such a derivation cannot exist unless $\delta = 0$. Thus we shall prove

THEOREM (5.9). *Let $f(x_1, x_2) = x_1^k + x_2^l$ be a quasihomogenous plane curve singularity such that the dimension of the prorepresenting substratum $\mathbf{H}_0$ is $\geqslant 2$. Let U_0 be the neighbourhood of $\mathbf{0}$ in $\mathbf{H}_0$ on which $\mathbf{B} = \{x^\alpha \mid \alpha \in I\}$ is a local basis for $L^0(F_0)$. Then the map $\lambda(t)$ is injective for each closed point t in a neighbourhood of the origin in U_0.*

Proof. We shall carry out the detailed computations only for the case $f(x_1, x_2) = x_1^n + x_2^n$, sketching which adjustments are needed to do the general case. For further details we refer to [B-L]. In any case, the trick is to observe that if (5.4) holds for D, then D is completely determined by its values v_α on the $\mathbf{d}_\alpha$'s of lowest weight. In fact we have

LEMMA (5.10). *Let $f(x_1, x_2) = x_1^n + x_2^n$, $n \geqslant 5$. For $\alpha = (\alpha_1, \alpha_2) \in I$, let $\alpha_+ = (\alpha_1 + 1, \alpha_2 - 1)$, $\alpha_- = (\alpha_1 - 1, \alpha_2 + 1)$. Suppose D satisfies* (5.4) *and assume that*
(i) $\alpha_+ \in I$ *or* $\alpha_2 = 0$ *and*
(ii) $\alpha_- \in I$ *or* $\alpha_1 = 0$.
Then

$$D(\mathbf{d}_\alpha) = v_\alpha = \alpha_2 a_{12} d_{\alpha_+} + (\alpha_2 a_{11} + \alpha_1 a_{22}) d_\alpha + \alpha_1 a_{21} d_{\alpha^-}$$

where the a_{ij} are fixed scalars.

Proof. It will suffice to prove the formula for $|\alpha| = 0, 1/n, 2/n$ since $\{d_\alpha \mid |\alpha| = 0, 1/n, 2/n\}$ generate $L^0(t)$ as a Lie algebra. If γ satisfies (i) and (ii), $|\gamma| \geqslant 3/n$, then $\beta = \gamma - e_j$ also satisfies (i) and (ii) for $j = 1$ or 2.

Then $f_{e_j\beta} = [\mathbf{d}_{e_j}, \mathbf{d}_\beta] - (|\gamma| - 2/n)\mathbf{d}_\gamma$, $\lambda_{e_j\beta} = 0$, hence

$$D(f_{e_j\beta}) = [v_{e_j}, d_\beta] + [d_{e_j}, v_\beta] - (|\gamma| - 2/n) v_\gamma = 0.$$

Thus, $v_\gamma = 1/(|\gamma| - 2/n)\{[v_{e_j}, d_\beta] + [d_{e_j}, v_\beta]\}$ and the formula (5.10) for v_γ is obtained by induction on $|\gamma|$.

To prove the formula for $|\alpha| = 0, 1/n, 2/n$ we first notice that $v_0 = 0$ follows trivially since $[v_0, d_\alpha] = 0$ for all $\alpha \in I$.

For $|\alpha| = 1/n, 2/n$ we need only consider pairs of equations that arise from (5.4) such as

$$[d_{(0,1)}, v_{(2,0)}] + [v_{(0,1)}, d_{(2,0)}] - (1/n)v_{(2,1)} = 0$$
$$[d_{(1,0)}, v_{(1,1)}] + [v_{(1,0)}, d_{(1,1)}] - (1/n)v_{(2,1)} = 0.$$

Writing everything out in terms of d_α, $|\alpha| = 1/n, 2/n$ we find ourselves left with the assertion (5.10). One should remark that the assumption $n \geqslant 5$ is crucial for this step. Q.E.D.

Now it is obvious that in order to prove (5.9) we shall have to consider equations of the form given in (5.4) where the right-hand side is non zero. In general, however, the structural constants in $L^0(F_0)$ are ugly rational functions of the t_i, and getting useful explicit expressions for these constants, seems practically impossible.

On the other hand, we find the values of the structural constants modulo $(t_1, \dots, t_d)^2$ by inspection. Luckily, it turns out that these values are all that we need for the proof.

In fact, applying (5.10), we find that the system of equations given in (5.4) turns out to be an enormous system of linear equations in the a_{ij} and $u_1, \dots, , u_d$ (cf. (5.7) and (5.10)). We are able to pick suitable $n \times n$-minors of the associated matrix, the entries of which are polynomials in the $c^\gamma_{\alpha\beta}$ and their partial derivatives. The value of these minors mod. $(t_1, \dots, t_d)^2$ are then easily computed.

First, notice that if $c^\gamma_{\alpha\beta}(t_1, \dots, t_d)$ are the structural constants in $L^0(F_0)$, then $c^\gamma_{\alpha\beta}(0, \dots, 0)$ are the structural constants in $L^0(F_0) \otimes_{H_0} k(\mathbf{0}) \approx L^0(f)$. By (5.6) we deduce that if $\alpha + \beta \notin I$ then $c^\gamma_{\alpha\beta} \in (t_1, \dots, t_d)$ for all $\gamma \in I$.

For convenience we change the indexation, letting

$$H_0 = k[t_2, \dots, t_{n-2}],$$

$F_0(x_1, x_2) = x_1^n + x_2^n + \sum_{i=2,\dots,n-2} t_i x_1^{n-i} x_2^i$. Then

$$\begin{aligned} &\text{(i) } (1/n)\partial F_0/\partial x_1 = x_1^{n-1} + \sum_i (n-i)/n\, t_i x_1^{n-i-1} x_2^i, \\ &\text{(ii) } (1/n)\partial F_0/\partial x_2 = x_2^{n-1} + \sum_i i/n\, t_i x_1^{n-1} x_2^{i-1}. \end{aligned} \tag{5.11}$$

By (5.1), any monomial of weight $(2n-5)/n$ may be expressed as an H_0-linear combination

$$x_1^j x_2^{2n-5-j} = P_j x_1^{n-2} x_2^{n-3} + Q_j x_1^{n-3} x_2^{n-2}, \quad j = 0, 1, \dots, 2n-5. \tag{5.12}$$

Recall that if $\boldsymbol{\alpha} \notin I$ then $\boldsymbol{x}^{\boldsymbol{\alpha}} \in (t_2, \ldots, t_{n-2})L^0(F_0)$. Thus, multiplying (5.11) through with $x_1^k x_2^{n-4-k}$, $k = 0, 1, \ldots, n-4$ and noting that $\partial F_0/\partial x_1 = \partial F_0/\partial x_2 = 0$ in $L^0(F_0)$ we find:

LEMMA (5.13). *Modulo* $(t_2, \ldots, t_{n-2})^2$ *the following holds:*

$$P_{2n-k} = -(k-2)/nt_{n+2-k}, \quad Q_{2n-k} = -(k-3)/nt_{n+3-k}, \quad k = 5, \ldots, n,$$
$$P_{n-1} = 0, \quad Q_{n-1} = -(n-2)/nt_2,$$
$$P_{n-4} = -(n-2)/nt_{n-2}, \quad Q_{n-4} = 0,$$
$$P_k = -(k+2)/nt_{k+2}, \quad Q_k = -(k+3)/nt_{k+3}, \quad k = 0, \ldots, n-5.$$

and trivially,

$$P_{n-2} = Q_{n-3} = 1, \quad P_{n-3} = Q_{n-2} = 0.$$

It is an elementary, but useful observation that from (5.13) we may also compute the constant term of the partial derivatives of P_k, Q_k.

Next, notice that, by (4.12), if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in I$, $|\boldsymbol{\alpha} + \boldsymbol{\beta}| = (2n-5)/n$, then

$$[\boldsymbol{x}^{\boldsymbol{\alpha}}, \boldsymbol{x}^{\boldsymbol{\beta}}] = |\boldsymbol{\beta} - \boldsymbol{\alpha}|\{P_{\alpha_1+\beta_1}x^{n-2}y^{n-3} + Q_{\alpha_1+\beta_1}x_1^{n-3}x_2^{n-2}\},$$

hence in this case

$$f_{\boldsymbol{\alpha}\boldsymbol{\beta}} = [\mathbf{d}_{\boldsymbol{\alpha}}, \mathbf{d}_{\boldsymbol{\beta}}] - |\boldsymbol{\beta} - \boldsymbol{\alpha}|\{P_{\alpha_1+\beta_1}\mathbf{d}_{(n-2,n-3)} + Q_{\alpha_1+\beta_1}\mathbf{d}_{(n-3,n-2)}\}. \tag{5.13}$$

From now on, let $\boldsymbol{A} = (n-2, n-3)$, $\boldsymbol{B} = (n-3, n-2) \in I$. We shall examine the equations $D(f_{\boldsymbol{\alpha}\boldsymbol{\beta}}) - \lambda_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$ for $|\boldsymbol{\alpha}| = (n-3)/n$, $|\boldsymbol{\beta}| = (n-2)/n$. That is, we consider

$$\boldsymbol{\alpha} = (n-3-j, j), \quad \boldsymbol{\beta} = (n-2-k, k) \quad \text{for} \quad j = 0, \ldots, n-3, \quad k = 0, \ldots, n-2.$$

Using (5.14), we obtain

$$D(f_{(n-3-j,j)(n-2-k,k)}) = [d_{(n-3-j,j)}, v_{(n-2-k,k)}] + [v_{(n-3-j,j)}, d_{(n-2-k,k)}] -$$
$$- (1/n)P_{(2n-5-j-k)}v_{\boldsymbol{A}} - (1/n)Q_{(2n-5-j-k)}v_{\boldsymbol{B}}.$$

Now, $v_{(n-3-j,j)}$ and $v_{(n-2-k,k)}$ are given by (5.10). Although the conditions (i) and

(ii) of (5.10) do not hold for A and B, we may still apply (5.4) to get the expressions

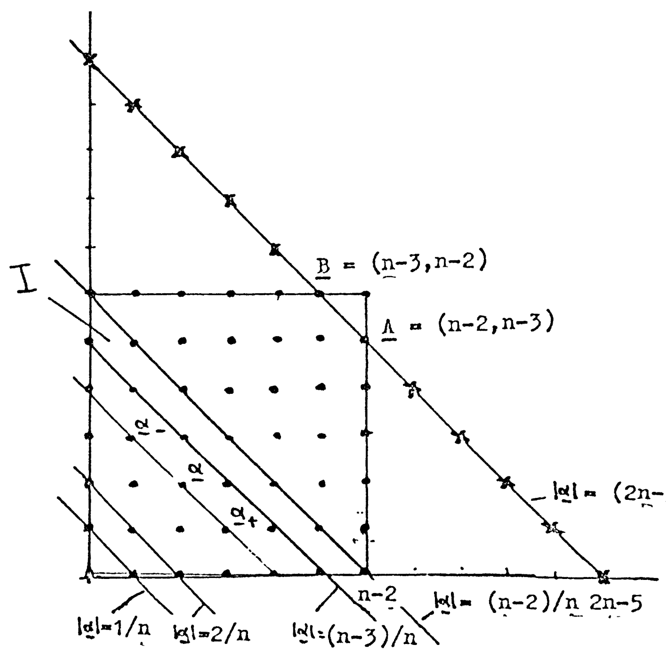
$$
\begin{aligned}
v_A &= \{(n-3)(a_{11} + P_{n-1}a_{12}) + (n-2)a_{22}\}d_A + \\
&\quad + \{(n-3)Q_{n-1}a_{12} + (n-2)a_{21}\}d_B \\
v_B &= \{(n-3)P_{n-4}a_{21} + (n-2)a_{12}\}d_A + \\
&\quad + \{(n-3)(a_{22} + Q_{n-4}a_{21}) + (n-2)a_{11}\}d_B.
\end{aligned}
$$

Expanding (5.15) we obtain, by another trivial calculation:

LEMMA (5.16) The left hand side of (5.15) is given by,

$$
\begin{aligned}
& nD(f_{(n-3-j,j)(n-2-k,k)}) \\
&\quad = \{(j+k+3-n)P_{2n-5-j-k}(a_{11} - a_{22}) + [(j+k)P_{2n-4-j-k} + \\
&\quad + (3-n)P_{n-1}P_{2n-5-j-k} + (2-n)Q_{2n-5-j-k}]a_{12} + \\
&\quad + [(2n-5-j-k)Q_{2n-6-j-k} + (3-n)P_{n-4}Q_{2n-5-j-k}]a_{21}\}d_A + \\
&\quad + \{(j+k+2-n)Q_{2n-5-j-k}(a_{11} - a_{22}) + [(k+l)Q_{2n-4-j-k} + \\
&\quad + (3-n)Q_{n-1}P_{2n-5-j-k}]a_{12} + [(2n-5-j-k)Q_{2n-6-j-k} + \\
&\quad + (2-n)P_{n-4} + (3-n)Q_{n-4}Q_{2n-5-j-k}]a_{21}\}d_B.
\end{aligned}
$$

At this stage, a simple sketch of $\mathbf{Z}^2 \supseteq I$ might be helpful to illustrate what we are really doing.



Recall the Lie product $[x^\alpha, x^\beta] = |\beta - \alpha| x^{\alpha+\beta}$ in the H_0-Lie algebra $L^0(F_0)$. If $\gamma \notin I$ then x^γ is an H_0-linear combination of the basis monomials of the same weight, and these relations induce the Lie structure on $L^0(t)$ for $t \in U_0$. Thus, in particular, if $|\alpha + \beta| = (2n-5)/n$, then $[d_\alpha, d_\beta]$ is represented by the integral point $\alpha + \beta$, and this product may be expressed by d_A, d_B.

Finally, we notice that if $\delta \in \mathrm{Der}_k(H_0, k(t))$, $\delta = \Sigma_i u_i \partial/\partial t_i$, then

$$n\lambda_{(n-3-j,j)(n-2-k,k)} = \left(\sum_i \partial P_{(2n-5-j-k)}/\partial t_i u_i\right) d_A + \left(\sum_i \partial Q_{(2n-5-j-k)}/\partial t_i u_i\right) d_B. \tag{5.17}$$

Consider now the system of linear equations in $(a_{11} - a_{22}), a_{12}, a_{21}, u_2, \ldots, u_{n-2}$ that results if we compare the respective coefficients in the following equations

$$\begin{aligned}
&n[D(f_{(n-3-j,j)(0,n-2)}) - \lambda_{(n-3-j,j)(0,n-2)}] = 0,\\
&\quad j = 1, 2, \ldots, n-3, \qquad d_A\text{-coefficient}\\
&n[D(f_{(0,n-3)(n-4,2)}) - \lambda_{(0,n-3)(n-4,2)}] = 0,\\
&\qquad\qquad\qquad\qquad d_B\text{-coefficient}\\
&n[D(f_{(n-3,0)(n-2-k,k)}) - \lambda_{(n-3,0)(n-2-k,k)}] = 0,\\
&\quad k = n-4, n-3, \ldots, 1, 0, \quad d_A\text{-coefficient}\\
&n[D(f_{(n-3,0)(n-2,0)}) - \lambda_{(n-3,0)(n-2,0)}] = 0,\\
&\qquad\qquad\qquad\qquad d_B\text{-coefficient.}
\end{aligned} \tag{5.18}$$

Then, from (5.16) it is immediately checked that the coefficient of $(a_{11} - a_{22})$ is zero modulo $(t_2, \ldots, t_{n-2})$ in each equation. Hence, if

$$M' = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \vdots \\ \phi_{N1} & \cdots & \phi_{Nn} \end{bmatrix} \quad (N = 2n-4)$$

is the matrix associated to this system, all $n \times n$-minors are 0 modulo $(t_2, \ldots, t_{n-2})$. On the other hand, the first order term of the minors are exactly the minors of the matrix

$$M' = \begin{bmatrix} \phi'_{11}\phi_{12}(0) & \cdots & \phi_{1n}(0) \\ \vdots & & \vdots \\ \phi'_{N1}\phi_{N2}(0) & \cdots & \phi_{Nn}(0) \end{bmatrix} \tag{5.19}$$

where ϕ'_{i1} = first-order term of ϕ_{i1}.

For $i = 1, \dots, n-3$ let M'_i be the $n \times n$-minor in which the first row is the ith row of M', and whose last $(n-1)$ rows are the last rows of M'. Then the first row of M'_i corresponds to the equation

$$[(i+1)(i+1-n)/n]t_{n-i-1}(a_{11} - a_{22}) + [(n-i-1)/n]u_{(n-i-1)} = 0 \quad (5.20)$$

and the last rows of M' are

$$\begin{bmatrix} 0 & (n-1) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & (n-1) & 0 & 0 & \dots & 0 & 0 & 0 \\ 2(n-2)/nt_2 & 0 & 0 & (n-2)/n & 0 & \dots & 0 & 0 & 0 \\ 3(n-3)/nt_3 & 0 & 0 & 0 & (n-3)/n & \dots & 0 & 0 & 0 \\ \cdots & & & & & & & & \\ (n-3)3/nt_{n-3} & 0 & 0 & 0 & 0 & \dots & 0 & 3/n & 0 \\ (n-2)2/nt_{n-2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2/n \end{bmatrix} \quad (5.21)$$

Hence we easily conclude that

$$\det(M'_i) = [(n-1)^2(n-2)!(i+1-n)/n^{n-3}]t_{n-i-1}, \quad i = 1, \dots, n-3. \quad (5.22)$$

Now, the homogenous system (5.18) only has the trivial solution $(a_{11} - a_{22}) = a_{12} = a_{21} = u_2 = \cdots = u_{n-2} = 0$, if the rank of the coefficient matrix M is n. Let $\mathbf{a}$ be the ideal in H_0 generated by all $n \times n$-minors of M. Then (5.22) shows that $\mathbf{a} + (t_2, \dots, t_{n-2})^2 = (t_2, \dots, t_{n-2})$.

Hence the set of zeros of $\mathbf{a}$ is a proper algebraic subset of U_0, in which $\mathbf{0}$ is an isolated point. By (5.5) the map $\lambda(\mathbf{0})$ is also one-one, hence $\lambda(t)$ is injective in an open set U in U_0 including the origin. Q.E.D.

To prove (5.9) for the general case $f(x_1, x_2) = x_1^k + x_2^l$ no new ideas are needed. The case $k = l$ is already done away with. For $k \neq l$ it is necessary to treat two cases separately:

(i) $F_0(x_1, x_2) = x_1^n + x_1^{an} + \sum_i t_i x_1^{n-i} x_2^{ai} \qquad n \geqslant 4, a \geqslant 2$

(ii) $F_0(x_1, x_2) = x_1^{an} + x_2^{bn} + \sum_i t_i x_1^{a(n-i)} x_2^{bi} \qquad n \geqslant 3, b > a \geqslant 2.$

The formula of (5.10) is somewhat simplified: in the first case we find

$$v_{\alpha} = (\alpha_1 a_{22} + \alpha_2 a_{11}) d_{\alpha} + \alpha_1 a_{21} d_{\alpha}, \quad \text{where, for } \boldsymbol{\alpha} = (\alpha_1, \alpha_2),$$
$$\alpha_- = (\alpha_1 - 1, \alpha_2 + a).$$

In case (ii) we simply obtain $v_{\alpha} = (\alpha_1 a_{22} + \alpha_2 a_{11}) d_{\alpha}$.

The proof of these formuli still runs by induction on $\alpha_1 + \alpha_2$, though, to get the induction started, some rather messy calculations are needed. Once again the assumption on $\dim \mathbf{H}_0$ turns out to be indispensable.

These calculations done, we may again deduce a system of homogenous equations in the a_{ij} and the u_i. In the first case, this system may be derived from the equations

$$D(f_{\alpha\beta}) = \lambda_{\alpha\beta}, \text{ where } |\boldsymbol{\alpha}| = (n-2)/n, |\boldsymbol{\beta}| = (n-1)/n.$$

In the second case, the two equations

$$D(f_{(1,0)(an-2,0)}) = \lambda_{(1,0)(an-2,0)}$$
$$D(f_{(0,1)(0,bn-2)}) = \lambda_{(0,1)(0,bn-2)}$$

will do the trick. The details may safely be left to the reader.

Furthermore, it should be remarked that if one considers the deformation $C^1L^0(F_0)$ of the nilpotent Lie algebra $C^1L^0(f)$ then (5.9) still holds. The proof is essentially a blueprint of the calculations above, except for the added complication that (5.3) must now be proved by calculations similar to those in the proof of (5.10).

As a further result of our proof of (5.9), we may determine the Lie algebra of derivations of $L^0(t)$.

THEOREM (5.23). *Let* $f(x_1, x_2) = x_1^k + x_2^l$, *and let* $L^0(t)$ *be as in* (5.9).

Then $\mathrm{Der}_k(L^0(t)) \approx T_t \oplus \mathrm{ad}(C^1L^0(t))$, *where, if* $t = \mathbf{0}$, T_t *is the torus* T *of* (4.7), *whereas if* $t \in U - \{\mathbf{0}\}$, T_t *is generated by* $\mathrm{ad}(d_0)$. *Hence, in the latter case* $\mathrm{Der}_k(L^0(t)) = \mathrm{ad}(L^0(t))$, *that is,* $H^1(L^0(t), L^0(t)) = (0)$.

Proof. Let $\delta \in \mathrm{Der}_k(L^0(t))$, $\delta(d_{\alpha}) = \delta_{\alpha}$. We claim that $\delta_{\alpha} \in L_{\geqslant |\alpha|}$ for all $\boldsymbol{\alpha} \in I$. To see this, notice that

$$|\boldsymbol{\alpha}|\delta_{\alpha} = \delta([d_0, d_{\alpha}]) = [\delta_0, d_{\alpha}] + [d_0, \delta_{\alpha}]$$

and the result follows from the fact that the spaces L_m are just the weight spaces of $\mathrm{ad}(d_0)$.

Thus, let $\delta_{\alpha} = \delta'_{\alpha} + \varepsilon_{\alpha}$, $\varepsilon_{\alpha} \in L_{>|\alpha|}$, $\delta'_{\alpha} \in L_{|\alpha|}$.

Since $L^0(t)$ is graded, the map δ' given by $d_\alpha \to \delta'_\alpha$ is a derivation of $L^0(t)$. By (4.14), δ' corresponds to a derivation $D \in B$ contained in $\mathrm{Der}_k(\mathbf{F}, L^0(t))$ (cf.(4.19)). By the proof of (5.9), $\delta'_\alpha = (\alpha_2 a_{11} + \alpha_1 a_{22})d_\alpha$.

Hence, if $t = \mathbf{0}$, then $\delta' = a_{11}t_2 + a_{22}t_1$ is an element of the torus T.

If $t \in U - \{\mathbf{0}\}$ then, in addition, $a_{11} - k/la_{22} = 0$, therefore

$$\delta = ka_{22}(1/lt_1 + 1/kt_2) = ka_{22}\mathrm{ad}(d_0).$$

To finish the proof, we must show that the derivation E given by $E(d_\alpha) = \varepsilon_\alpha$ is inner.

LEMMA (5.24). *Let* $E \in \mathrm{Der}_k(L^0(t)), E(d_\alpha) = \varepsilon_\alpha \in L_{>|\alpha|}$.

Then E *is uniquely determined by* ε_0.

Proof. $E([d_0, d_\alpha]) = [d_0, \varepsilon_\alpha] + [\varepsilon_0, d_\alpha] = |\alpha|\varepsilon_\alpha$, which yields

$$[d_0, \varepsilon_\alpha] - |\alpha|\varepsilon_\alpha = [d_\alpha, \varepsilon_0]$$

and the result is clear, since ε_α is a sum of eigenvectors for $\mathrm{ad}(d_0)$ with eigenvalues $>|\alpha|$. Q.E.D.

(5.23) Then follows from

LEMMA (5.25). *If* $\varepsilon_0 \in L_{>0} = C^1L^0(t)$, *then there exists a unique* $z \in C^1L^0(t)$ *such that* $\varepsilon_0 = [z, d_0]$.

Proof. If $\varepsilon_0 = \Sigma_{\gamma\in I^*} e_\gamma d_\gamma$, then $z = -\Sigma_{\gamma\in I^*}(e_\gamma/|\gamma|)d_\gamma$. Q.E.D.

COROLLARY (5.26). *The Lie algebra* $L^0(t)$ *is of rank* 1 *whenever* $t \in U - \{\mathbf{0}\}$.

COROLLARY (5.27). *Let* $f(x_1, x_2) = x_1^k + x_2^l$ *be a quasihomogenous plane curve singularity such that the conditions of* (4.24) *and* (5.9) *hold. Then*

$$\dim_k(A^1(k, L^0(f); L^0(f)) = 2\dim \mathbf{H}_0 + \delta,$$

where $\delta = 3$ *if* $k = l$, $\delta = 2$ *if* $l = ak$, $a \geqslant 2$, $\delta = 1$ *otherwise, and where* $\mathbf{H}_0$ *is the prorepresenting substratum of the versal basis* $\mathbf{H}$ *for* f.

Proof. This follows from (4.24) and (5.23). Q.E.D.

LEMMA (5.28). *There exists a neighbourhood of* $\mathbf{0}$ *in* U *such that for* $t \neq \mathbf{0}$

$$\dim_k(A^1(k, L^0(t); L^0(t)) \leqslant \dim_k(A^1(k, L^0(f); L^0(f)) - 2.$$

Proof. For $t \in U$ consider the exact sequence

$$0 \to T_t \to B \to P_t \oplus N_t \to A^1(k, L^0(t); L^0(t) \to 0$$

(cf. (4.19), (4.20) and (5.23)).

Now $\dim_k(B) = \Sigma_m(\dim_k L_m)^2$ is independent of t, whereas $\dim_k(T_t) = \dim_k(P_t) = 2$ if $t = \mathbf{0}$, and $=1$ if $t \neq \mathbf{0}$.

Hence (5.28) follows from the observation that N_t is the solution space of the homogenous system (4.23). Since the rank of this system is locally increasing, we have $\dim_k(N_t) \leqslant \dim_k(N_0)$ on a neighbourhood of 0. Q.E.D.

Notice also that, by the injectivity of (5.9), we have

$$\dim_k(A^1(k, L^0(t); L^0(t)) \geqslant \dim \mathbf{H}_0,$$

where $\mathbf{H}_0$ is the prorepresenting substratum of f.

In fact, further calculations indicate that the dimension of the space of **F**-homomorphisms N_t has constant dimension equal to $\dim_k N_0$. We therefore conjecture that the inequality in (5.28) is an equality:

CONJECTURE (5.29). For all $t \neq \mathbf{0}$ in a neighbourhood of $\mathbf{0}$.

$$\dim_k(A^1(k, L^0(t); L^0(t)) = \dim_k(A^1(k, L^0(f); L^0(f)) - 2.$$

We finally note a highly interesting consequence of (5.23):

COROLLARY (5.30). $\mathrm{Der}_k(L^0(t))$ *acts trivially on*

$$A^1(k, L^0(t); L^0(t)) \quad \text{for} \quad t \in U - \{\mathbf{0}\}.$$

Proof. Recall the action of $\mathrm{Der}_k\,\mathbf{g}$ on $A^1(k, \mathbf{g}; \mathbf{g})$, where $\mathbf{g}$ is any Lie algebra. Let $\phi \in \mathrm{Hom}_{\mathbf{F}}(\mathbf{J}, \mathbf{g})$, $D \in \mathrm{Der}_k\,\mathbf{g}$. Then

$$D.\phi = \phi D' - D\phi$$

where D' is the derivation D lifted to **F**, that is, the diagram

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{D'} & \mathbf{F} \\ \pi\downarrow & & \downarrow\pi \\ \mathbf{g} & \xrightarrow{D} & \mathbf{g} \end{array}$$

commutes. By (4.19), any element of $A^1(k, L^0(t); L^0(t))$ may be represented by an element of G_t.

One easily checks that $\mathrm{ad}(C^1L^0(t))$ acts trivially on $A^1(k, L^0(t); L^0(t))$. Furthermore, let $\mathrm{Ad}(d_0)$ be a lifting of $\mathrm{ad}(d_0)$ such that $\mathrm{Ad}(d_0)(\mathbf{d}_\alpha) = |\boldsymbol{\alpha}|\mathbf{d}_\alpha$. Then

$$\mathrm{Ad}(d_0)(f_{\alpha\beta}) = |\boldsymbol{\alpha} + \boldsymbol{\beta}| f_{\alpha\beta},$$

since $f_{\alpha\beta} = [\mathbf{d}_\alpha, \mathbf{d}_\beta] - \Sigma_{|\gamma| = |\alpha+\beta|} c^\gamma_{\alpha\beta}\mathbf{d}_\gamma$.

Let $\phi \in G_t$, then $\phi(f_{\alpha\beta}) = \phi_{\alpha\beta} \in L_{|\alpha+\beta|}$, hence

$$\mathrm{ad}(\mathbf{d}_0)(\phi_{\alpha\beta}) = |\boldsymbol{\alpha} + \boldsymbol{\beta}|\phi_{\alpha\beta} = \phi(\mathrm{Ad}(d_0)(\phi_{\alpha\beta})), \tag{5.31}$$

and consequently $\mathrm{ad}(d_0).\phi = 0$, for all $\phi \in G_t$. Q.E.D.

Recall that each Lie algebra $L^0(t)$ has the formal moduli $H^\wedge(L^0(t))$ containing the prorepresenting substratum $H^\wedge_0(L^0(t))$. (5.30) then states that, for any point t in the open set $U - \{0\}$, the tangent spaces of $H^\wedge(L^0(t))$ and $H^\wedge_0(L^0(t))$ are equal.

6. Conclusions

Summing up Section 5, we have proved, see (5.28) and (5.30), that for $f(x_1, x_2) = x_1^k + x_2^l$ the filtration $\{\mathbf{S}(\boldsymbol{h})\}_{\boldsymbol{h}}$ of $\mathbf{S}^0_{\tau.(f)} = \mathbf{H}^0(f)$, see (3.2) and (3.3), is non trivial. Notice that in this case $\mathbf{M}^0(\boldsymbol{h}) = \mathbf{S}^0(\boldsymbol{h})$ and $\mathbf{M}_{\tau.(f)} = \mathbf{M}_{\tau(f)+1} = \mathbf{H}_0(f)$. The main result, (5.9), therefore implies:

THEOREM (6.0). *Let* $f(x_1, x_2) = x_1^k + x_2^l$, *and assume the conditions of* (5.9) *are satisfied, then there exists an open neighborhood* $\mathbf{U}$ *of* $\mathbf{0}$ *in* $\mathbf{H}_0(f)$, *such that for every* $\boldsymbol{h} \in \mathbf{Z}^{\tau(f)+1}$, *the restriction of the morphism of algebraic spaces*

$$l^0 : \mathbf{M}^0(\boldsymbol{h}) \to \mathbf{L}(\boldsymbol{h})$$

to $\mathbf{U} \cap \mathbf{M}^0(\boldsymbol{h})$, *is an immersion.*

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