COMPOSITIO MATHEMATICA

ZI-XIAO YANG
BU-XI LI

On restricted derivative approximation

Compositio Mathematica, tome 74, nº 3 (1990), p. 327-331
<http://www.numdam.org/item?id=CM_1990__74_3_327_0>

Compositio Mathematica **74**: 327–331, 1990.

On restricted derivative approximation

YANG ZI-XIAO & LI BU-XI
Department of Mathematics, Shan Xi University, Taiyuan, Shan Xi, China

Received 11 April 1988; accepted in revised form 20 September 1989

Abstract. In this paper we will show that the condition that f be $2k$ continuously differentiable is not necessary in order to guarantee the same order of approximation for both the restricted and the nonrestricted cases. Thus, we strengthen a result of J.A. Roulier [1].

1. Introduction

Let $0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_m$ be fixed integers and let v_i and μ_i, $i = 1, 2, \ldots, m$, be fixed extended real valued functions on $[-1, 1]$ which satisfy the following conditions:

(i) $v_i(x) < +\infty$, $\mu_i(x) > -\infty$ and $v_i(x) < \mu_i(x)$, $i = 1, 2, \ldots, m$ for all $-1 \leqslant x \leqslant 1$;
(ii) $X_i^- = \{x\colon v_i(x) = -\infty\}$ and $X_i^+ = \{x\colon \mu_i(x) = +\infty\}$ are open in $[-1, 1]$, $i = 1, 2, \ldots, m$;
(iii) v_i is continuous on $[-1, 1]\backslash X_i^-$ and μ_i is continuous on $[-1, 1]\backslash X_i^+$, $i = 1, 2, \ldots, m$.

Roulier [1] has proved the following

THEOREM 1.1. *Let $0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_m$ be fixed non-negative integers as above and let v_i and μ_i, $i = 1, 2, \ldots, m$ be extended real valued functions as above. Let $f \in C^{2k_m}[-1, 1]$ and let P_n be the algebraic polynomial of degree n of best approximation to f on $[-1, 1]$. Assume that for all x in $[-1, 1]$ and all $1 \leqslant i \leqslant m$ we have*

$$v_i(x) < f^{(k_i)}(x) < \mu_i(x). \tag{1.1}$$

Then for n sufficiently large we have

$$v_i(x) < P_n^{(k_i)}(x) < \mu_i(x) \tag{1.2}$$

for all $-1 \leqslant x \leqslant 1$ and all $1 \leqslant i \leqslant m$.

Theorem 1.1 means that if $f \in C^{2k_m}[-1, 1]$ satisfies (1.1) then the rate of the restricted derivate approximation to f on $[-1, 1]$ is the same as that of the nonrestricted approximation.

From Theorem 1.1, Roulier [1] also obtained the following

COROLLARY 1.2. *Let $f \in C^2[-1, 1]$ and assume $f'(x) \geqslant \delta > 0$ on $[-1, 1]$. Then for n sufficiently large the algebraic polynomial of degree n of best approximation to f is increasing on $[-1, 1]$.*

It is not known whether the condition that f be $2k_m$ continuously differentiable is necessary in order to guarantee the same order of approximation for both the restricted and nonrestricted cases. In particular, is the above corollary true if we only give $f \in C^1[-1, 1]$?

In this paper we will prove that the condition of Theorem 1.1 is unnecessarily strong and the result of the corollary 1.2 is true if we assume $f \in C^1[-1, 1]$ and $\lim_{n\to\infty} nE_n(f') = 0$.

2. Convergence of the sequence of derivatives of the polynomials of best approximation

In this section we study

$$\lim_{n\to\infty} \|f^{(k)} - P_n^{(k)}\| = 0 \quad k = 1, 2, \ldots$$

as well as the corresponding speed of the convergence, where P_n is the polynomial of degree n of best approximation to $f \in C^k[-1, 1]$.

Let $C[-1, 1]$ be the space of continuous real valued functions defined on the compact interval $[-1, 1]$, endowed with supremum norm denoted by $\|\cdot\|$. Let P_n be the algebraic polynomial of degree at most n of best approximation to $f \in C[-1, 1]$.

We state the theorem on which our study relies. Let $f \in C^r[-1, 1]$, the subspace of $C[-1, 1]$ of r-times continuously differentiable functions. Let $E_n(f) = \|f - P_n\|$.

THEOREM 2.1. [2. p. 39] *There exists a constant C_k such that, if $f \in C^k[-1, 1]$, $k \geqslant 1$ and $n > k$,*

$$E_n(f) \leqslant C_k n^{-k} E_{n-k}(f^{(k)}).$$

THEOREM 2.2. [3] *Let $f \in C^r[-1, 1]$ and $n \geqslant r + 1$. Then there exists a polynomial $\mathfrak{p}_n$ of degree $\leqslant n$ such that for $k = 0, 1, \ldots, r$.*

$$\|f^{(k)} - \mathfrak{p}_n^{(k)}\| \leqslant C_r n^{k-r} E_{n-r} f^{(k)}.$$

Now we prove the desired theorem.

THEOREM 2.3. *Let* $f \in C^k[-1,1]$ *and*

$$\lim_{n\to\infty} n^k E_{n-k}(f^{(k)}) = 0.$$

Then $\lim_{n\to\infty} \| f^{(k)} - P_n^{(k)} \| = 0$, *where* P_n *is the polynomial of best approximation to* f.

Proof. First of all there exists a polynomial $\mathfrak{p}_n$ of degree $\leqslant n$ such that

$$\| f^{(k)} - \mathfrak{p}_n^{(k)} \| \leqslant C_k E_{n-k}(f^{(k)})$$

by Theorem 2.2 for $k \geqslant 1$.

Again, applying Markov's inequality and Theorem 2.1, we obtain

$$\begin{aligned}
\| f^{(k)} - P_n^{(k)} \| &\leqslant \| f^{(k)} - \mathfrak{p}_n^{(k)} \| + \| \mathfrak{p}_n^{(k)} - P_n^{(k)} \| \\
&\leqslant C_k E_{n-k}(f^{(k)}) + n^{2k} \| \mathfrak{p}_n - P_n \| \\
&\leqslant C_k E_{n-k}(f^{(k)}) + n^{2k} \{ \| f - \mathfrak{p}_n \| + \| f - P_n \| \} \\
&\leqslant C_k E_{n-k}(f^{(k)}) + n^{2k} \{ C_k n^{-k} E_{n-k}(f^{(k)}) + C_k n^{-k} E_{n-k}(f^{(k)}) \} \\
&\leqslant C_k E_{n-k}(f^{(k)}) + C_k n^k E_{n-k}(f^{(k)}).
\end{aligned}$$

Here C_k is a constant depending on k, but not necessarily the same on each occurance. Thus, we have $\lim_{n\to\infty} \| f^{(k)} - P_n^{(k)} \| = 0$.

3. Main result

Let us call a pair (v, μ) of functions $[-1,1] \to [-\infty, \infty]$ "admissible" if it satisfies certain conditions similar to (i), (ii), (iii) as above.

Let $f \in C[-1,1]$ and let P_n be the algebraic polynomial of degree not exceeding n of best approximation to f.

Considering the following

PROPOSITION 3.1. *Suppose* k *is a nonnegative integer, and* $f \in C^{2k}[-1,1]$. *Let* (v, μ) *be admissible and*

$$v(x) < f^{(k)}(x) < \mu(x) \quad \text{for all } x \in [-1,1].$$

Then for n sufficiently large we have

$$v(x) < P_n^{(k)}(x) < \mu(x) \quad \text{for all } x \in [-1,1].$$

It is clear that Proposition 3.1 and Theorem 1.1 are equivalent statements, but also that Proposition 3.1 is easier to understand.

According to the explanation, we state our main result as follows.

THEOREM 3.2. *Suppose k is a nonnegative integer. Let $f \in C^k[-1,1]$ and $\lim_{n\to\infty} n^k E_{n-k}(f^{(k)}) = 0$. Assume that (ν, μ) be admissible and*

$$\nu(x) < f^{(k)}(x) < \mu(x) \quad \text{for all } x \in [-1,1] \tag{3.1}$$

Then for n sufficiently large, we have

$$\nu(x) < P_n^{(k)}(x) < \mu(x) \quad \text{for all } x \in [-1,1] \tag{3.2}$$

where P_n is the algebraic polynomial of degree n of best approximation to f on $[-1,1]$.

Proof. It is easy to see by (3.1) that there exists a constant $\delta > 0$ such that for $-1 \leqslant x \leqslant 1$,

$$\min\{\mu(x) - f^{(k)}(x), f^{(k)}(x) - \nu(x)\} \geqslant \delta. \tag{3.3}$$

By Theorem 2.3, we have

$$\| f^{(k)} - P_n^{(k)} \| < \delta$$

for n sufficiently large.

So, for n sufficiently large, we obtain by (3.3)

$$\begin{aligned}
\mu(x) - P_n^{(k)}(x) &= \mu(x) - f^{(k)}(x) + f^{(k)}(x) - P^{(k)}(x) \\
&\geqslant \mu(x) - f^{(k)}(x) - \| f^{(k)}(x) - P^{(k)}(x) \| \\
&\geqslant \delta - \| f^{(k)}(x) - P_n^{(k)}(x) \| \\
&> \delta - \delta = 0,
\end{aligned}$$

that is

$$\mu(x) > P_n^{(k)}(x) \quad -1 \leqslant x \leqslant 1. \tag{3.4}$$

Similarly, for n sufficiently large, we have by (3.3)

$$\begin{aligned}
P_n^{(k)}(x) - \nu(x) &= P_n^{(k)}(x) - f^{(k)}(x) + f^{(k)}(x) - \nu(x) \\
&\geqslant P_n^{(k)}(x) - f^{(k)}(x) - | f^{(k)}(x) - \nu(x) | \\
&\geqslant P_n^{(k)}(x) - f^{(k)}(x) - \| f^{(k)}(x) - \nu(x) \| \\
&\geqslant \delta - \| f^{(k)}(x) - \nu(x) \| \\
&> \delta - \delta = 0,
\end{aligned}$$

that is

$$P_n^{(k)} > \nu(x) \quad -1 \leqslant x \leqslant 1. \tag{3.5}$$

Thus, for n sufficiently large, we have by (3.4) and (3.5)

$$\nu(x) < P_n^{(k)}(x) < \mu(x) \quad -1 \leqslant x \leqslant 1.$$

This completes the proof of Theorem 3.2.

COROLLARY 3.3. *Let* $f \in C^1[-1, 1]$ *and* $\lim_{n\to\infty} nE_n(f') = 0$. *Assume that* $f(x) \geqslant \delta > 0$ *on* $[-1, 1]$. *Then for n sufficiently large the algebraic polynomial of degree n of best approximation to f is increasing on* $[-1, 1]$.

References

[1] Roulier, J.A., Best Approximation to Functions with Restricted derivate, *J. Approximation Theory* 17 (1976), 344–347.

[2] Feinerman, R.P. and Newman, D.J., "Polynomial Approximation", Williams and Wilkins, Baltimore 1974.

[3] Leviatan, D., The behavior of the derivatives of the algebraic polynomials of best approximation, *J. Approximation Theory* 35 (1982), 169–176.