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Formal cohomology, analytic cohomology and non-algebraic manifolds

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0. Introduction

In a recent paper [PS], compact complex manifolds containing $\mathbb{C}^n$ were considered under the special aspect of existence of meromorphic functions. This existence was reduced to a certain conjecture relating formal and analytic cohomology along hypersurfaces. One of the purposes of this paper is to study this relation further (and for instance it will turn out that the conjecture mentioned above needs additional assumptions).

Let X be a complex space, $Y \subset X$ a closed complex subspace and $\mathscr{F}$ an $\mathcal{O}_X$-module. Then there is a canonical homomorphism

$$r^k(\mathscr{F})\colon H^k(Y, \mathscr{F} \mid Y) \to H^k(\hat{X}, \hat{\mathscr{F}}), \tag{0.1}$$

called the *comparison map for* $\mathscr{F}$ (in degree k), where $\mathscr{F} \mid Y$ is the topological restriction and $\hat{X}$, $\hat{\mathscr{F}}$ are the formal completions of X and $\mathscr{F}$ along Y. In general, $r^k(\mathscr{F})$ is neither injective nor surjective, but under certain curvature conditions on the neighbourhood structure of Y in X, some results of this kind are known (see Section 2 and [Ko 1] (3.4)). This connection was already used in the book [Ha 2] of R. Hartshorne (p. 94 and p. 225) and recently by U. Karras in [Ka].

The purpose of this paper is on one hand to prove some comparison theorems between formal and convergent cohomology resp. local and moderate local cohomology, and to formulate a certain conjecture. On the other hand, there are interesting examples in the literature, concerning special kinds of neighbourhood structure of Y in X, which give immediately counterexamples to some natural questions. For instance if $n = \dim X$, then $r^{n-1}(\mathscr{F})$ is in general not injective for a locally free sheaf $\mathscr{F}$ on X, even if Y has a fundamental system of strictly pseudoconcave neighbourhoods in X.

We discuss especially a class of examples, due to A. Ogus ([Og] (4.17)), which is also connected to a problem of Hartshorne ([Ha 2] p. 235) and the formal principle for holomorphic embeddings. In all examples X is a compact surface and Y a smooth curve with $Y^2 = 0$.

Then we give a classification of all those compact surfaces X containing a curve C with $C^2 = 0$ such that $X \backslash C$ is Stein. More specifically, it is shown in Section 5 that X is either projective or a Hopf surface of algebraic dimension 0. And if X is projective and minimal, it has to be a certain distinguished $\mathbb{P}_1$-bundle over an elliptic curve (i.e. a ruled surface).

Results of this kind are applied in the last section to prove that any compact manifold X containing an irreducible divisor Y such that $X \backslash Y \simeq \mathbb{C}^3$ biholomorphically is Moišezon (hence projective) if it has at least one non-constant meromorphic function.

1. Local cohomology and duality

Let $Y \subset X$ be complex spaces as in the introduction and $\mathscr{F}, \mathscr{G}$ two $\mathcal{O}_X$-modules. Then there are two notions of local cohomology groups, or more generally local Ext-groups

$$\begin{aligned} &\operatorname{Ext}^k_Y(X; \mathscr{F}, \mathscr{G}) := H^k(R\Gamma_y(R\,\mathscr{H}om_X(\mathscr{F}, \mathscr{G}))), \\ &\operatorname{Ext}^k_{[Y]}(X; \mathscr{F}, \mathscr{G}) := \varinjlim_{\mathscr{I}} \operatorname{Ext}^k(X; \mathscr{F}/\mathscr{I}\mathscr{F}, \mathscr{G}). \end{aligned} \tag{1.1}$$

Here the limit is taken over all coherent ideal sheaves $\mathscr{I} \subset \mathcal{O}_X$ with $V(\mathscr{I}) = |Y|$. Obviously, both groups depend only on the reduced structure of Y. The second group in (1.1) is called the Ext-group of $\mathscr{F}$ and $\mathscr{G}$ with *moderate* (or "algebraic") *support* along Y. In the special case $\mathscr{F} = \mathcal{O}_X$, we also write

$$\begin{aligned} &H^k_Y(X, \mathscr{G}), \\ &H^k_{[Y]}(X, \mathscr{G}). \end{aligned}$$

By sheafification, we obtain $\mathcal{O}_X$-modules

$$\begin{aligned} &\mathscr{E}xt^k_Y(X; \mathscr{F}, \mathscr{G}), \\ &\mathscr{E}xt^k_{[Y]}(X; \mathscr{F}, \mathscr{G}) \end{aligned}$$

which are supported on Y and spectral sequences

$$\begin{aligned} &H^p(X, \mathscr{E}xt^q_Y(X; \mathscr{F}, \mathscr{G})) &&\Rightarrow \operatorname{Ext}^{p+q}_Y(X; \mathscr{F}, \mathscr{G}) \\ &H^p_Y(X, \mathscr{E}xt^q(X; \mathscr{F}, \mathscr{G})) &&\Rightarrow \operatorname{Ext}^{p+q}_Y(X; \mathscr{F}, \mathscr{G}) \\ &\varinjlim_{\mathscr{I}} H^p(X, \mathscr{E}xt^q(X; \mathscr{F}/\mathscr{I}\mathscr{F}, \mathscr{G})) &&\Rightarrow \operatorname{Ext}^{p+q}_{[Y]}(X; \mathscr{F}, \mathscr{G}) \end{aligned}$$

and for $\mathscr{F}$ flat:

$$H^p_{[Y]}(X, \mathscr{E}xt^q(X; \mathscr{F}, \mathscr{G})) \Rightarrow \mathrm{Ext}^{p+q}_{[Y]}(X; \mathscr{F}, \mathscr{G}).$$

Moreover, there is a canonical map

$$\rho^k(\mathscr{F}, \mathscr{G}): \mathrm{Ext}^k_{[Y]}(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k_Y(X; \mathscr{F}, \mathscr{G}) \tag{1.2}$$

for each k which is in general neither injective nor surjective. If we put $U := X \setminus Y$ and set

$$\mathrm{Ext}^k([U]; \mathscr{F}, \mathscr{G}) := \varinjlim_{\mathscr{I}} \mathrm{Ext}^k(X; \mathscr{I}\mathscr{F}, \mathscr{G}) \tag{1.3}$$

(Ext with *moderate growth* along Y), we get the following long exact cohomology sequences

$$\cdots \to \mathrm{Ext}^k_Y(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k(U; \mathscr{F}, \mathscr{G}) \to \cdots \tag{1.4.1}$$

$$\cdots \to \mathrm{Ext}^k_{[Y]}(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k([U]; \mathscr{F}, \mathscr{G}) \to \cdots \tag{1.4.2}$$

$$\cdots \to \mathrm{Ext}^k_c(U; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k_c(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k_c(X \mid Y; \mathscr{F}, \mathscr{G}) \to \cdots \tag{1.4.3}$$

where we assume in addition $\mathscr{F}$ to be coherent in (1.4.3). Here the subscript "c" denotes "compact support" and $X \mid Y$ is the ringed space $(Y, \mathcal{O}_X \mid Y)$. For the proof of (1.4.3) resp. (1.4.1), we can use for instance distinguished triangles in $D^+(X)$ (cp. [Bo])

$$\begin{tikzcd} j_! j^* \mathscr{M}^\bullet \arrow[rr] & & \mathscr{M}^\bullet, \arrow[dl] \\ & i_* i^{-1} \mathscr{M}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd} \qquad \begin{tikzcd} i_* R i^! \mathscr{M}^\bullet \arrow[rr] & & \mathscr{M}^\bullet \arrow[dl] \\ & R j_* j^* \mathscr{M}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $i\colon Y \to X$ and $j\colon U \to X$ are the canonical embeddings, $j_!$ is the proper direct image functor and $i^!(\_) := \mathscr{H}^0_Y(X, \_) \mid Y$. Note that $\mathrm{Ext}^k_c(X \mid Y; \mathscr{F}, \mathscr{G}) \simeq \mathrm{Ext}^k_c(X; \mathscr{F}, i_* i^{-1} \mathscr{G})$, since $\mathscr{F}$ is coherent.

Let W be an open neighbourhood of Y in X. Then the restriction mappings

$$\mathrm{Ext}^k_Y(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k_Y(W; \mathscr{F}, \mathscr{G}),$$

$$\mathrm{Ext}^k_{[Y]}(X; \mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^k_{[Y]}(W; \mathscr{F}, \mathscr{G})$$

are bijective for all k (excision property).

From now on we shall always assume that $\mathscr{F}$ and $\mathscr{G}$ are *coherent* $\mathcal{O}_X$-modules. Then the space $\operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})$ resp. $\operatorname{Ext}^k_{[Y]}(X;\mathscr{F},\mathscr{G})$ carries a natural QFN-structure resp. a LQFN-structure (QFN = "quotient-Fréchet-nuclear", L = "limit"). Via duality it is possible to relate moderate local cohomology to formal cohomology and usual local cohomology to convergent cohomology (see also [Ha 2] p. 95, p. 225 and [Ka] Section 3). We are going to make this precise in a more general setting.

Let K_X be the dualizing complex of X and assume $\mathscr{F}$ to be of finite Tor-dimension. Then there is a canonical pairing

$$\begin{array}{c} \operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G}) \times \operatorname{Ext}^{-k}_c(X\,|\,Y;\mathscr{G},\mathscr{F} \overset{L}{\otimes}_{\mathcal{O}_X} K_X) \\ \Big\downarrow \\ \operatorname{Ext}^0_{Y,c}(\mathscr{F},\mathscr{F} \overset{L}{\otimes}_{\mathcal{O}_X} K_X) \\ \Big\downarrow \text{Trace} \\ \mathbb{C} \end{array} \tag{1.5}$$

Observe that we have

$$\operatorname{Ext}^{\bullet}_c(X\,|\,Y;\mathscr{G},\mathscr{H}) \simeq \varinjlim_{W\supset Y} \operatorname{Ext}^{\bullet}_{\Phi(W)}(W;\mathscr{G},\mathscr{H})$$

for $\mathscr{H}$ in $D^+(X)$ and $\Phi(W) := \{A \subset W \text{ closed}, A \Subset X\}$ for $W \subset X$ open.

Especially, we get a morphism

$$\Phi^k(\mathscr{F},\mathscr{G})\colon \operatorname{Ext}^{-k}_c(X\,|\,Y;\mathscr{G},\mathscr{F} \overset{L}{\otimes}_{\mathcal{O}_X} K_X) \to \operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})^\vee \tag{1.6}$$

which factorizes over the topological dual space $\operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})'$.

Moreover, each $\operatorname{Ext}^\nu_c(X\,|\,Y;\mathscr{G},\mathscr{F} \otimes^L_{\mathcal{O}_X} K_X)$ has a natural QDFN-structure ("quotient-dual-Fréchet-nuclear"), which will be specified in a moment, and $\Phi^k(\mathscr{F},\mathscr{G})$ is continuous. We have

(1.7) DUALITY THEOREM. *For $\mathscr{F},\mathscr{G} \in \operatorname{Coh}(X)$ and $\mathscr{F}$ of finite Tor-dimension, the pairing* (1.5) *gives a topological perfect duality between the associated separated topological vector spaces. Moreover,* $\operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})$ *is separated, iff*

$$\operatorname{Ext}^{1-k}_c(X\,|\,Y;\mathscr{G},\mathscr{F} \otimes^L_{\mathcal{O}_X} K_X) \textit{ is.}$$

Proof. First we consider the case $\mathscr{F} = \mathcal{O}_X$ and use the following fact

$$\operatorname{Ext}_c^0(W; \mathscr{G}, K_X)' \simeq \Gamma(W, \mathscr{G}) \tag{1.7.1}$$

for an open Stein subset $W \Subset X$. We fix a countable open Stein covering $\mathscr{U} = (U_r)_{r \in R}$ of X which is a basis for the topology. If $\mathscr{E}$ is the pre-cosheaf $\mathscr{E}xt_c^0(X; \mathscr{G}, K_X)$ on X and $\mathscr{U}(U) := \{U_r \colon U_r \subset U, r \in R\}$, we get a short exact sequence of Čech-cocomplexes of DFN-spaces (see also [Bn] Chap. VII)

$$0 \leftarrow C_\cdot(\mathscr{U}^Y, \mathscr{E}) \leftarrow C_\cdot(\mathscr{U}, \mathscr{E}) \leftarrow C_\cdot(\mathscr{U}(U), \mathscr{E}) \leftarrow 0 \tag{1.7.2}$$

where the left hand side is by definition the cokernel of the third map. This sequence splits canonically in each degree. Taking the topological dual, we obtain the following short exact sequence of complexes of FN-spaces from (1.7.1)

$$0 \to C^\cdot(\mathscr{U}^Y, \mathscr{G}) \to C^\cdot(\mathscr{U}, \mathscr{G}) \to C^\cdot(\mathscr{U}(U), \mathscr{G}) \to 0. \tag{1.7.3}$$

If we show that we can identify

$$H_Y^p(X, \mathscr{G}) \simeq H^p(C^\cdot(\mathscr{U}^Y, \mathscr{G})), \tag{1.7.4}$$

$$\operatorname{Ext}_c^q(X \mid Y; \mathscr{G}, K_X) \simeq H_{-q}(C_\cdot(\mathscr{U}^Y, \mathscr{E})) \tag{1.7.5}$$

we are done by using the lemmata in section 1 of [R-R1]. Replacing $\mathscr{G}$ by its Godement resolution $\mathscr{W}^\cdot(\mathscr{G})$, we see immediately that $C^\cdot(\mathscr{U}^Y, \mathscr{G})$ is quasi-isomorphic to $\Gamma_Y(X, \mathscr{W}^\cdot(\mathscr{G}))$.
The proof of (1.7.5) is somewhat more complicated. We begin by recalling a few standard notions. For an abelian sheaf on X and an open embedding $\alpha \colon W \to X$, we define

$$\mathscr{M}_W := \alpha_! \alpha^* \mathscr{M}$$

where $\alpha_!$ is the proper direct image functor (or the trivial extension here) and set

$$\mathscr{C}_q(\mathscr{U}, \mathscr{M}) := \coprod_{(i_0, \ldots, i_q) \in \operatorname{Ner}(\mathscr{U})} \mathscr{M}_{U_{i_0} \cap \cdots \cap U_{i_q}}.$$

With the obvious notations, there is the following (augmented) exact sequence of

Čech-cocomplexes

$$\begin{array}{ccccccc} 0 \leftarrow \mathscr{C}.(\mathscr{U}^Y, \mathscr{M}) & \leftarrow & \mathscr{C}.(\mathscr{U}, \mathscr{M}) & \leftarrow & \mathscr{C}.(\mathscr{U}(U), \mathscr{M}) & \leftarrow 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \leftarrow i_* i^{-1}\mathscr{M} & \longleftarrow & \mathscr{M} & \longleftarrow & j_! j^* \mathscr{M} & \longleftarrow 0. \end{array} \tag{1.7.6}$$

Here $i\colon Y \to X$ and $j\colon U \to X$ are the inclusions. If $\mathscr{M}$ is flabby then the vertical arrows are all quasi-isomorphisms. For the second and the third one, this follows from [Bn] Chap VII (3.9).

We fix an injective resolution $K_X \to \mathscr{I}^\bullet$ of K_X in $D^+(X)$ and put

$$\mathscr{M}^\nu := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I}^\nu), \quad \nu \in \mathbb{Z}.$$

Then each $\mathscr{M}^\nu$ is flabby. Especially

$$\mathscr{C}.(\mathscr{U}^Y, \mathscr{M}^\bullet) \to i_* i^{-1}\mathscr{M}^\bullet$$

is a quasi-isomorphism. Note that both sides are complexes of c-soft sheaves and $i^{-1}\mathscr{M}^\nu$, $i_* i^{-1}\mathscr{M}^\nu$ are flabby. Since $C.(\mathscr{U}, \mathscr{M}^\bullet_c) \to \Gamma_c(X, \mathscr{M}^\bullet)$ and $C.(\mathscr{U}(U), \mathscr{M}^\bullet_c) \to \Gamma_c(U, \mathscr{M}^\bullet)$ are quasi-isomorphisms, we get by (1.7.6) that also

$$C.(\mathscr{U}^Y, \mathscr{M}^\bullet_c) = \Gamma_c(X, \mathscr{C}.(\mathscr{U}^Y, \mathscr{M}^\bullet)) \to \Gamma_c(X, i_* i^{-1}\mathscr{M}^\bullet) = \Gamma_c(Y, i^{-1}\mathscr{M}^\bullet)$$

is a quasi-isomorphism. It suffices therefore to prove

$$H^q(\Gamma_c(Y, i^{-1}\mathscr{M}^\bullet)) \simeq \operatorname{Ext}^q_c(X \mid Y; \mathscr{G}, K_X),$$

but this follows from

$$i^{-1} R\,\mathscr{H}om_X(\mathscr{G}, \mathscr{I}^\bullet) \simeq R\,\mathscr{H}om_{i^{-1}\mathscr{O}_X}(i^{-1}\mathscr{G}, i^{-1}\mathscr{I}^\bullet).$$

This shows the case $\mathscr{F} = \mathscr{O}_X$. In the general situation, we first observe that the above argument can be immediately generalized to $\mathscr{G} \in D^b_{\mathrm{coh}}(X)$, using for instance the simplicial technique of [Ba] p. 115.

Moreover, we have (topological) isomorphisms

$$\operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G}) \simeq H^k_Y(X, R\,\mathscr{H}om_X(\mathscr{F},\mathscr{G})),$$

$$\operatorname{Ext}^l_c(X\mid Y;\mathscr{G},\mathscr{F}\overset{L}{\otimes}_{\mathcal{O}_X}K_X) \simeq \operatorname{Ext}^l_c(X\mid Y; R\,\mathscr{H}om_X(\mathscr{F},\mathscr{G}),K_X),$$

c.f. [Ba] p. 118, ... So the general case can be deduced from the special one by applying a truncation procedure, see loc. cit. □

We obtain now the following commutative diagram with exact lines ($\mathscr{F},\mathscr{G}\in \operatorname{Coh}(X)$ with $\mathscr{F}$ of finite Tor-dimension)

$$\begin{array}{ccccccc}
\cdots \to \operatorname{Ext}^{-k}_c(U;\mathscr{G},\mathscr{F}\overset{L}{\otimes}K_X) & \to & \operatorname{Ext}^{-k}_c(X;\mathscr{G},\mathscr{F}\overset{L}{\otimes}K_X) & \to & \operatorname{Ext}^{-k}_c(X\mid Y;\mathscr{G},\mathscr{F}\overset{L}{\otimes}K_X) & \to \cdots \\
\downarrow & & \downarrow & & \Big\downarrow{\scriptstyle \Phi^k(\mathscr{F},\mathscr{G})} & \\
\cdots \to \operatorname{Ext}^k(U;\mathscr{F},\mathscr{G})^\vee & \longrightarrow & \operatorname{Ext}^k(X;\mathscr{F},\mathscr{G})^\vee & \longrightarrow & \operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})^\vee & \longrightarrow \cdots
\end{array} \tag{1.8}$$

where the vertical arrows are induced by the duality pairings on U and X. From (1.7) we get especially

(1.9) REMARK. If $\operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})$ is finite dimensional, then $\Phi^k(\mathscr{F},\mathscr{G})$ is surjective and $\Phi^{k-1}(\mathscr{F},\mathscr{G})$ is injective.

Next, we shall prove a "formal" version of (1.7).

(1.10) THEOREM. *Let Y be compact with ideal sheaf $\mathscr{I}$ and $\mathscr{F}_m := \mathscr{F}/\mathscr{I}^{m+1}\mathscr{F}$ for $m\in\mathbb{N}$. We assume that $\mathscr{F},\mathscr{G}\in\operatorname{Coh}(X)$ and $\mathscr{F},\mathscr{F}_m$ are of finite* Tor*-dimension for all m. Then there is a natural isomorphism*

$$\Psi^k(\mathscr{F},\mathscr{G})\colon \operatorname{Ext}^{-k}(\hat{X};\hat{\mathscr{G}},\hat{\mathscr{F}}\overset{L}{\otimes}_{\mathcal{O}_X}K_X) \to \operatorname{Ext}^k_{[Y]}(X;\mathscr{F},\mathscr{G})^\vee$$

such that the diagram

$$\begin{array}{ccc}
\operatorname{Ext}^{-k}(X\mid Y;\mathscr{G},\mathscr{F}\overset{L}{\otimes}_{\mathcal{O}_X}K_X) & \xrightarrow{\Phi^k(\mathscr{F},\mathscr{G})} & \operatorname{Ext}^k_Y(X;\mathscr{F},\mathscr{G})^\vee \\
\downarrow & & \Big\downarrow{\scriptstyle \rho^k(\mathscr{F},\mathscr{G})^\cdot} \\
\operatorname{Ext}^{-k}(\hat{X};\hat{\mathscr{G}},\hat{\mathscr{F}}\overset{L}{\otimes}_{\mathcal{O}_X}K_X) & \xrightarrow[\Psi^k(\mathscr{F},\mathscr{G})]{} & \operatorname{Ext}^k_{[Y]}(X;\mathscr{F},\mathscr{G})^\vee
\end{array}$$

commutes, where the left vertical arrow is "completion along Y".

Proof. We consider first the pairing between finite dimensional $\mathbb{C}$-vector spaces

$$\begin{array}{c}
\mathrm{Ext}^k(X;\mathscr{F}_m,\mathscr{G}) \times \mathrm{Ext}^{-k}(\hat{X};\hat{\mathscr{G}},\mathscr{F}_m \overset{L}{\bigotimes}_{\mathcal{O}_X} K_X) \\
\big\downarrow \wr \\
\mathrm{Ext}^k(X;\mathscr{F}_m,\mathscr{G}) \times \mathrm{Ext}^{-k}(X;\mathscr{G},\mathscr{F}_m \overset{L}{\bigotimes}_{\mathcal{O}_X} K_X) \\
\big\downarrow \\
\mathrm{Ext}^0(X;\mathscr{F}_m,\mathscr{F}_m \overset{L}{\bigotimes}_{\mathcal{O}_X} K_X) \\
\big\downarrow \\
\mathbb{C}
\end{array}$$

which is perfect. This follows for instance from [Ba] Theorem 5.3. Since now

$$\mathrm{Ext}^l(\hat{X};\hat{\mathscr{G}},\hat{\mathscr{F}} \overset{L}{\bigotimes}_{\mathcal{O}_X} K_X) \simeq \varprojlim_m \mathrm{Ext}^l(\hat{X};\hat{\mathscr{G}},\mathscr{F}_m \overset{L}{\bigotimes}_{\mathcal{O}_X} K_X),$$

the construction of $\Psi^k(\mathscr{F},\mathscr{G})$ is established. By perfectness and functoriality of the usual duality pairing, $\Psi^k(\mathscr{F},\mathscr{G})$ is bijective and the diagram in (1.10) commutes. □

For the rest of this section we assume $\mathscr{F} = \mathcal{O}_X$ and $\mathscr{G}$ locally free. Furthermore, *one* of the following conditions should be satisfied

(i) *X is smooth of dimension n,*

(ii) *X is Cohen–Macaulay of pure dimension n and Y is a locally complete intersection in X.*

Then we have the commutative diagram

$$\begin{array}{ccc}
H^{n-k}(X \mid Y,\mathscr{G}^\vee \bigotimes_{\mathcal{O}_X} \omega_X) & \xrightarrow{\Phi^k(\mathscr{G})} & H^k_Y(X,\mathscr{G})^\vee \\
{\scriptstyle r^{n-k}(\mathscr{G}^\vee \otimes \omega_X)}\big\downarrow & & \big\downarrow{\scriptstyle \rho^k(\mathscr{G})^\vee} \\
H^{n-k}(\hat{X},(\mathscr{G}^\vee \bigotimes_{\mathcal{O}_X} \omega_X)^\wedge) & \xrightarrow[\Psi^k(\mathscr{G})]{\sim} & H^k_{[Y]}(X,\mathscr{G})^\vee
\end{array} \qquad (1.11)$$

with $\Phi^k(\mathscr{G}) := \Phi^k(\mathcal{O}_X,\mathscr{G})$, $\Psi^k(\mathscr{G}) := \Psi^k(\mathcal{O}_X,\mathscr{G})$ and $\rho^k(\mathscr{G}) := \rho^k(\mathcal{O}_X,\mathscr{G})$.

This gives the desired connection between convergent (formal) cohomology and cohomology with (moderate) support on Y. Moreover, there is the following topological version of (1.11)

$$\begin{tikzcd} H^{n-k}(X \mid Y, \mathscr{G}^\vee \bigotimes_{\mathcal{O}_X} \omega_X)_{\mathrm{sep}} \arrow[r, "\Phi^k(\mathscr{G})_{\mathrm{sep}}", "\sim"'] \arrow[d, "r^{n-k}(\mathscr{G}^\vee \otimes \omega_X)_{\mathrm{sep}}"'] & (H^k_Y(X, \mathscr{G})_{\mathrm{sep}})' \arrow[d, "(\rho^k(\mathscr{G})_{\mathrm{sep}}^\vee"] \\ H^{n-k}(\hat{X}, (\mathscr{G}^\vee \bigotimes_{\mathcal{O}_X} \omega_X)^\wedge) \arrow[r, "\sim", "\Psi^k(\mathscr{G})"'] & H^k_{[Y]}(X, \mathscr{G})^\vee \end{tikzcd} \tag{1.11$'$}$$

by using (1.7). Here "sep" means the associated separated topological vector space (resp. map).

(1.12) REMARK. (1) $r^0(\mathscr{G}^\vee \otimes \omega_X)$ and $\Phi^n(\mathscr{G})$ are injective
(2) if $r^{n-k}(\mathscr{G}^\vee \otimes \omega_X)$ is surjective, then $\rho^k(\mathscr{G})$ is injective
(3) if $r^{n-k}(\mathscr{G}^\vee \otimes \omega_X)$ is bijective and $H^k_Y(X, \mathscr{G})$ finite-dimensional, then $\rho^k(\mathscr{G})$ is bijective.

2. Some finiteness and comparison theorems

We fix the following situation: Let X be a separated complex space, $Y \subset X$ a compact complex subspace, defined by the ideal sheaf $\mathscr{I}$ and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module.

(2.1) THEOREM. *Assume in addition that $Y \subset X$ is a locally complete intersection and that $\mathscr{I}$ is locally generated by a regular $\mathscr{F}$-sequence. Let $N_{Y|X}$ be the normal bundle of Y in X, then*

$$r^l(\mathscr{F})\colon H^l(X \mid Y, \mathscr{F}) \to H^l(\hat{X}, \hat{\mathscr{F}})$$

is an isomorphism of finite dimensional $\mathbb{C}$-vector spaces in the following cases
(1) *$N_{Y|X}$ is a p-convex bundle* and $l \geqslant p$*
(2) *$N_{Y|X}$ is a q-concave bundle* and $l + 1 < \operatorname{depth}_X \mathscr{F} - q$.*

(2.2) REMARK. Under the assumption of (2), the map $r^l(\mathscr{F})$ is injective for $l < \operatorname{depth}_X \mathscr{F} - q$.

The proof of (2.1) and (2.2) is an easy corollary of the vanishing theorem [Ko 1] (4.3).

The estimate in (2.1) (2) should not be optimal, in fact we have the

* In the sense of [Ko 1] (2.8).

(2.3) CONJECTURE. *If $N_{Y|X}$ is a q-concave bundle, then $r^l(\mathscr{F})$ is bijective for $l < \operatorname{depth}_X \mathscr{F} - q$ and injective for $l = \operatorname{depth}_X \mathscr{F} - q$.*

(2.4) REMARK. There is some evidence for this conjecture, because by [Ko 2] (3.6) it is true in the case where $X = N_{Y|X}$ and $\mathscr{F}$ is the pullback of a coherent sheaf on Y. Note also that an analogous conjecture is *not* true, if one only assumes the existence of a q-concave neighbourhood system of $Y \subset X$. This will be shown by an example in section 4. A sketch of a possible approach to (2.3) is given in Section 3.

(2.5) THEOREM. *Let $\mathscr{G}$ be a locally free sheaf on X. Then the map $\rho^k(\mathscr{G})$: $H^k_{[Y]}(X, \mathscr{G}) \to H^k_Y(X, \mathscr{G})$ is an isomorphism of finite dimensional $\mathbb{C}$-vector spaces and, moreover, all maps in diagram* (1.11), *if one of the following conditions is satisfied*

(1) *X is Cohen-Macaulay of dimension n, $Y \subset X$ exceptional and $k \leqslant n - 1$*
(2) *X is Gorenstein of dimension n, $Y \subset X$ a locally complete intersection, $N_{Y|X}$ p-convex and $k \leqslant n - p$*
(3) *X is Gorenstein, $Y \subset X$ a locally complete intersection, $N_{Y|X}$ q-concave and $k > q + 1$.*

(2.6) REMARK.

(1) The first assertion has also been proved by Karras in [Ka].
(2) In the situation of (2.5) (1), the $\mathbb{C}$-vector space $H^n_Y(X, \mathscr{G})$ is separated and $\rho^n(\mathscr{G})$ has *dense* image.
(3) In (2.5) (3), $H^{q+1}_Y(X, \mathscr{G})$ is finite dimensional, $\Phi^{q+1}(\mathscr{G})$ is bijective and $\rho^{q+1}(\mathscr{G})$ is surjective.
(4) If the Conjecture (2.3) is true, then the estimate in (2.5) (3) may be sharpened to $k > q$.
(5) For similar results in the algebraic category see [Ha 2].

Proof of (2.5). (1) First we note that $H^{n-k}(X \mid Y, \mathscr{G}^\vee \otimes \omega_X)$ is finite dimensional since $n - k \geqslant 1$. Moreover, $H^k_Y(X, \mathscr{G})$ is separated by the criterion (1.7). Now diagram (1.11)′ gives the assertion, because $r^{n-k}(\mathscr{G}^\vee \otimes \omega_X)$ is an isomorphism for $n - k \geqslant 1$ (see [Ko 1] (3.4)).

(2) Again, we show that $H^{n-k}(X \mid Y, \mathscr{G}^\vee \otimes \omega_X)$ is finite dimensional and $H^k_Y(X, \mathscr{G})$ separated for $n - k \geqslant p$.

This follows from the fact that there exists a continuous exhaustion function $\varphi: W \to [0, \infty[$ which is differentiable and strictly p-convex on $W \backslash Y$ and $\varphi^{-1}(0) = Y$ (use the construction of K. Fritzsche in [F] §5). Now, we apply (2.1) (1) and (1.11)′.

(3) Can be obtained by the same method and (2.1) (2), (1.11)′. □

Proof of (2.6). For abbreviation we set $\mathscr{F} := \mathscr{G}^\vee \otimes \omega_X$.

(2) By (1.7), $H^n_Y(X,\mathscr{G})$ is separated iff $H^1(X\mid Y,\mathscr{F})$ is. But this space is finite dimensional. So using (1.11)′, (1.7)

$$H^n_Y(X,\mathscr{G})' \to H^n_{[Y]}(X,\mathscr{G})^\vee$$

is injective. This gives (2) by an Hahn-Banach argument.

(3) By q-concavity of the neighbourhood structure of Y in X, $H^{n-q}(X\mid Y,\mathscr{F})$ is separated and therefore $H^{q+1}_Y(X,\mathscr{G})$ too. Moreover, $\dim H^{n-q-1}(X\mid Y,\mathscr{F}) < \infty$, so $\Phi^{q+1}(\mathscr{G})$ is bijective. By (2.2), we see that $\rho^{q+1}(\mathscr{G})^\vee$ is injective and, consequently, $\rho^{q+1}(\mathscr{G})$ is surjective.

(4) The same argument as in (3) shows that $\rho^{q+1}(\mathscr{G})^\vee$ is a bijection of finite dimensional $\mathbb{C}$-vector spaces. □

(2.7) PROPOSITION. *Let X be Gorenstein of pure dimension n, $Y\subset X$ a compact hypersurface and $\mathscr{G}\in \mathrm{Coh}(X)$ locally free. If Y has a fundamental system of strictly pseudoconcave (= 1-concave) neighbourhoods, then $H^1_Y(X,\mathscr{G})$ is separated and $\Phi^1(\mathscr{G})$: $H^{n-1}(X\mid Y,\mathscr{G}^\vee\otimes\omega_X)\to H^1_Y(X,\mathscr{G})'$ is an isomorphism. If in addition the normal bundle $N_{Y|X}$ is 1-concave and* (2.3) *holds, then*

$$\rho^1(\mathscr{G})\colon\ H^1_{[Y]}(X,\mathscr{G})\to H^1_Y(X,\mathscr{G})$$

is injective with dense image.

Proof: From (1.7), we see that $H^1_Y(X,\mathscr{G})$ is separated. Now

$$H^{n-1}(X\mid Y,\mathscr{G}^\vee\otimes\omega_X)$$

is, by 1-concavity, separated too. This gives the first part of (2.7).

Under the additional assumptions, $r^{n-1}(\mathscr{G}^\vee\otimes\omega_X)$ is injective. By Hahn-Banach (and 1.11)′), $\mathrm{Im}(\rho^1(\mathscr{G}))$is dense. The injectivity of $\rho^1(\mathscr{G})$ follows from that of $\Gamma([U],\mathscr{G})\to\Gamma(U,\mathscr{G})$, $U := X\setminus Y$, and the comparison of the exact sequences (1.4.1), (1.4.2) with $\mathscr{F} = \mathscr{O}_X$. □

(2.8) REMARK. The additional assumption in (2 .7) is indeed crucial, as the example in (4.1) shows.

Let X be a complex space, $Y\subset X$ a closed complex subspace and $\mathscr{F},\mathscr{G}\in \mathrm{Coh}(X)$. We denote by

(2.9) $\sigma^k(\mathscr{F},\mathscr{G})$: $\mathrm{Ext}^k([U];\mathscr{F},\mathscr{G})\to\mathrm{Ext}^k(U;\mathscr{F},\mathscr{G})$ the canonical map, where $U := X\setminus Y$. With this notation we have

(2.10) LEMMA. (1) *$\rho^k(\mathscr{F},\mathscr{G})$, $\rho^{k+1}(\mathscr{F},\mathscr{G})$ bijective $\Rightarrow \sigma^k(\mathscr{F},\mathscr{G})$ bijective*

(2) $\rho^k(\mathscr{F},\mathscr{G})$ *surjective,* $\rho^{k+1}(\mathscr{F},\mathscr{G})$ *injective* $\Rightarrow \sigma^k(\mathscr{F},\mathscr{G})$ *injective*
(3) X *compact,* $\rho^{k+1}(\mathscr{F},\mathscr{G})$ *dense image* $\Rightarrow \sigma^k(\mathscr{F},\mathscr{G})$ *has dense image.*

The proof of (1) and (2) follows immediately from (1.4.1), (1.4.2) and the five-lemma whilst (3) is a consequence of (2.12). □

(2.11) COROLLARY. *In the appropriate situations of theorem* (2.5), *we have (with* $\sigma^k(\mathscr{G}) := \sigma^k(\mathscr{O}_X, \mathscr{G})$*)*
(1) $\sigma^k(\mathscr{G})$ *is bijective for* $k < n-1$ *and has dense image for* $k = n-1$, *if* X *is compact.*
(2) $\sigma^k(\mathscr{G})$ *is bijective for* $k < n-p$
(3) $\sigma^k(\mathscr{G})$ *is bijective for* $k \geqslant q+1$.

If X *is compact, then in the situation of the second part of* (2.7), *we have*
(4) $\sigma^0(\mathscr{G})$ *is injective with dense image, especially*

$$\dim_{\mathbb{C}} \Gamma([U],\mathscr{G}) = \infty$$

if $\mathscr{G} \neq 0$. □

The following lemma was used above.
Given a commutative diagram of $\mathbb{C}$-vector spaces

$$\begin{tikzcd} E_1 \arrow[r, "\alpha_1"] \arrow[d, "e", "\wr"'] & F_1 \arrow[r, "\beta_1"] \arrow[d, "f"] & G_1 \arrow[r, "\gamma_1"] \arrow[d, "g"] & H_1 \arrow[d, "h", "\wr"'] \\ E_2 \arrow[r, "\alpha_2"'] & F_2 \arrow[r, "\beta_2"'] & G_2 \arrow[r, "\gamma_2"'] & H_2 \end{tikzcd}$$

where both lines are exact. We assume that the bottom line is a complex of topological vector spaces with QF-spaces F_2, G_2 and H_2 finite dimensional and separated.

(2.12) LEMMA. *Assume e and h to be bijective. If g has dense image, so does f.*

Proof. We put $I_\nu := \operatorname{Im}(\gamma_\nu)$. Then, obviously, the diagram

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d, "\wr"] & 0 \\ F_2 \arrow[r] & G_2 \arrow[r] & I_2 \arrow[r] & 0 \end{tikzcd}$$

has exact lines and $I_1 \to I_2$ is an isomorphism. We fix a section of $G_1 \to I_1$ which gives also a continuous section of $G_2 \to I_2$. By putting $K_\nu := \operatorname{Ker}(\gamma_\nu)$, we are reduced to the case $H_1 = H_2 = 0$ (observe that K_2 is again a QF-space and $g: K_1 \to K_2$ has dense image).

Now take $x \in F_2$ and a sequence $(y_i)_{i \in \mathbb{N}}$ in G_1 with

$$g(y_i) \to \beta_2(x).$$

We choose $x'_i \in F_1$ with $y_i = \beta_1(x_i)$. Then

$$\beta_2 f(x'_i) \to \beta_2(x)$$

so

$$\beta_2(x - f(x'_i)) \to 0.$$

Since β_2 is an epimorphism of QF-spaces, there is a sequence $(w_i)_{i \in \mathbb{N}}$ in F_2 with

$$w_i \to 0,$$
$$\beta_2(w_i) = \beta_2(x - f(x'_i)).$$

We fix $z_i \in E_1$ such that

$$\begin{aligned} x - f(x'_i) - w_i &= \alpha_2 e(z_i) \\ &= f\alpha_1(z_i) \end{aligned}$$

Setting $x_i := x'_i + \alpha_1(z_i)$, we obtain

$$f(x_i) \to x,$$

so f has dense image. $\square$

3. An approach to the conjecture (2.3)

We start by giving an equivalent reformulation of (2.3).

(3.1) LEMMA. *The conjecture is equivalent to*

$$\varprojlim_m H^k(X \mid Y, \mathscr{I}^m \mathscr{F}) = 0, \quad \text{for } k \leqslant \operatorname{depth}_X \mathscr{F} - q.$$

Proof. We consider the following short exact sequences, coming from the long exact cohomology sequence for $0 \to \mathscr{I}^{m+1}\mathscr{F} \to \mathscr{F} \to \mathscr{F}_m \to 0$

$$
\begin{aligned}
&0 \to K_m^{k-1} \to H^{k-1}(X \mid Y, \mathscr{F}) \to B_m^{k-1} \to 0,\\
&0 \to B_m^{k-1} \to H^{k-1}(Y, \mathscr{F}_m) \to W_m^k \to 0,\\
&0 \to W_m^k \to H^k(X \mid Y, \mathscr{I}^{m+1}\mathscr{F}) \to K_m^k \to 0,\\
&0 \to K_m^k \to H^k(X \mid Y, \mathscr{F}) \to H^k(Y, \mathscr{F}_m).
\end{aligned}
\tag{3.11}
$$

Since $k - 1 < \operatorname{depth}_X \mathscr{F} - q$, we get by the q-concavity of the neighbourhood structure of $Y \subset X$ (and [$A - G$]) that $\dim_{\mathbb{C}} K_m^{k-1} < \infty$ for all m. Moreover, B_m^{k-1}, W_m^k are finite dimensional and

$$H^\nu(\hat{X}, \hat{\mathscr{F}}) = \varprojlim_m H^\nu(Y, \mathscr{F}_m), \quad \nu \in \mathbb{N}$$

because Y is compact. Therefore we get the short exact sequence

$$0 \to \varprojlim_m W_m^k \to \varprojlim_m H^k(X \mid Y, \mathscr{I}^{m+1}\mathscr{F}) \to \varprojlim_m K_m^k \to 0 \tag{3.1.2}$$

and

$$\operatorname{Ker}(r^\nu(F)) = \varprojlim_m K_m^\nu, \quad \text{for all } \nu$$

$$
\begin{aligned}
\operatorname{Ker}(\varprojlim_m H^k(X \mid Y, \mathscr{I}^m\mathscr{F}) \to H^k(X \mid Y, \mathscr{F})) &= \varprojlim_m W_m^k\\
&= \operatorname{Coker}(r^{k-1}(\mathscr{F})).
\end{aligned}
$$

This gives the assertion. □

To prove (2.3), it is sufficient to treat the case $k = \operatorname{depth}_X \mathscr{F} - q$ by [Ko 1] (4.3). With the above notations, we first indicate how to prove $\varprojlim_m K_m^k = 0$.

(3.2) LEMMA. *Let V be an open neighbourhood of Y in X. Then*

$$\operatorname{Ker}(H^\nu(V, \mathscr{F}) \to H^\nu(\hat{V}, \hat{\mathscr{F}})) = \bigcap_m \operatorname{Im}(H^\nu(V, \mathscr{I}^{m+1}\mathscr{F}) \to H^\nu(V, \mathscr{F}))$$

for all ν.

The proof is clear by using the appropriate version of the last two short exact sequences in (3.1.1).

(3.3) We fix an open neighbourhood V of $Y \subset X$ and consider the short exact sequence of Čech-complexes

$$0 \to C^\cdot(V, \mathscr{I}^{m+1}\mathscr{F}) \xrightarrow{\alpha_m} C^\cdot(V, \mathscr{F}) \to C^\cdot(V, \mathscr{F}_m) \to 0. \tag{3.3.1}$$

From now on, we assume for simplicity that X, Y are smooth and $\mathscr{F}$ is locally free, $n := \dim X, k := n - q$. We take $z \in Z^k(V, \mathscr{F})$, such that there exist $w_m \in Z^k(V, \mathscr{I}^{m+1}\mathscr{F})$ and $c_m \in C^{k-1}(V, \mathscr{F})$ for any m with

$$z = \alpha_m(w_m) + \delta(c_m). \tag{3.3.2}$$

Here δ denotes the Čech-coboundary. We fix shrinkings $V_2 \Subset V_1 \Subset V$ containing Y and *assume* that there are constants $M_1, C > 0$ with

$$|w_m|_{V_1} \leqslant M_1 C^{m+1} |z|_V \quad \text{for all } m \tag{3.3.3}$$

where $|\cdot\cdot|_{..}$ are some suitable square-integrable norms on the Čech-complex with respect to a suitable covering. By Schwarz'-Lemma, there is a $\rho \in]0,1[$ (essentially the "shrinking factor" from V_1 to V_2) and a constant $M_2 > 0$, depending only on $\mathscr{F}$ and V_1, such that we have

$$\begin{aligned} |z - \delta(c_m)|_{V_2} &\leqslant M_2 \rho^{m+1} |w_m|_{V_1} \\ &\leqslant M_1 M_2 (\rho C)^{m+1} |z|_V. \end{aligned}$$

If the shrinking V_2 is chosen sufficiently small, we find an $\varepsilon \in]0,1[$ with

$$|z - \delta(c_m)|_{V_2} \leqslant \varepsilon^m |z|_v \quad \text{for all } m. \tag{3.3.4}$$

Consider the canonical commutative diagram

$$\begin{tikzcd} H^k(V,\mathscr{F}) \arrow[r] \arrow[dr] & H^k(V_2,\mathscr{F}) \arrow[r] & H^k(V_2,\mathscr{F})_{\text{sep}} \\ & H^k_b(V_2,\mathscr{F}) \arrow[r] \arrow[u] & H^k_b(V_2,\mathscr{F})_{\text{sep}} \arrow[u] \end{tikzcd}$$

where "b" means bounded cohomology. By the estimate (3.3.4), we see that the image of z in $H^k_b(V_2, \mathscr{F})_{\text{sep}}$, which is a Banach space, is zero. If we choose V_2 to be q-concave, then $H^k(V_2, \mathscr{F})$ is separated, so $z | V_2$ is zero in $H^k(V_2, \mathscr{F})$.

Now the essential step is to prove an estimate as in (3.3.3). For this we consider the commutative diagram

$$\begin{tikzcd} C^{k-1}(V,\mathscr{F}) \arrow[r, two heads] \arrow[d, "\delta"'] & C^{k-1}(V,\mathscr{F}_m) \arrow[l, bend right, dashed, "s_m"'] \\ Z^k(V,\mathscr{F}) \arrow[r] & Z^k(V,\mathscr{F}_m) \arrow[u, "\delta"'] \arrow[u, bend right, dashed, "r_m"'] \end{tikzcd}$$

We would like to have a section s_m and a section r_m on $B^k(V, \mathscr{F}_m)$ together with the estimates

$$|r_m| \leqslant C_1^{m+1}, \tag{3.3.5}$$

$$|s_m| \leqslant C_2^{m+1} \tag{3.3.6}$$

where C_1, C_2 are constants not depending on m. In our situation, (3.3.6) can be easily established. Note that it is sufficient to allow a shrinking V_1 of V for the construction of s_m and r_m (but *simultaneously* for all m). The estimate (3.3.5) is much more difficult to obtain since here the q-concavity of the normal bundle comes in.

Let $\pi\colon N_{Y|X} \to Y$ be the bundle projection, $\mathscr{J}$ the ideal sheaf of the zero-section, $\mathscr{M} := \pi^*\mathscr{F}_0$ and $\mathscr{M}_m := \mathscr{M}/\mathscr{J}^{m+1}\mathscr{M}$. By using the deformation to the normal bundle and a general argument on parametrized complexes (essentially that the splitting property is an open condition), we should seek a splitting h_m of

$$\delta\colon C_b^{k-1}(\mathscr{U}, \mathscr{M}_m) \to C_b^k(\mathscr{U}, \mathscr{M}_m) \tag{3.3.7}$$

where $\mathscr{U}$ is some fixed finite suitable cover of Y, such that

$$|h_m| \leqslant C_1^{m+1} \quad \text{for all } m. \tag{3.3.8}$$

Now, if $\mathscr{U}_1 \ll \mathscr{U}$ is a small shrinking of $\mathscr{U}$, it is not difficult to find a map

$$\tilde{h}_m : B_b^k(\mathscr{U}, \mathscr{M}_m) \to C_b^{k-1}(\mathscr{U}_1, \mathscr{M}_m)$$

such that an estimate as in (3.3.8) holds and $\delta\tilde{h}_m$ = restriction. This follows from the two facts:

(i) $H^k(W, \mathscr{M})$ is separated for a q-concave neighbourhood W of the zero-section,

(ii) $H^k(Y, \mathscr{M}_m)$ is in a canonical way a direct summand of $H^k(W, \mathscr{M})$ by "expansion" along the fibres of π.

For a similar argument, compare also [B-K] Kap. II (1.5) and [Ko 3] (7.2)(1).

Next we need to avoid the shrinking from $\mathscr{U}$ to $\mathscr{U}_1$. For this purpose we must have a smoothing technique (in the L^2-sense for instance). Probably we can achieve this by $\bar{\partial}$-methods (and Hörmander's theorem). The point here is that the smoothing operator can be estimated by C'^{m+1} for each m and a constant C' independent of m. This seems quite plausible by Hörmander's method. Applying the analogous argument as in the proof of [B-K] (II 1.6), we are able to obtain a map

$$h_m\colon B_b^k(\mathscr{U}, \mathscr{M}_m) \to C_b^{k-1}(\mathscr{U}, \mathscr{M}_m)$$

with $\delta h_m = \mathrm{id}$ and such that (3.3.8) holds (being more precise, we should replace $\mathscr{U}$ by $\mathscr{U}_1$). Since we use preferably L^2-norms here and $B_b^k(\mathscr{U}, \mathscr{M}_m)$ is closed in $C_b^k(\mathscr{U}, \mathscr{M}_m)$, we take the orthogonal projection on $B_b^k(\mathscr{U}, \mathscr{M}_m)$ and compose with h_m. This gives us the desired splitting.

(3.4) Finally, we want to sketch how to obtain the vanishing in (3.1) for $k = \mathrm{depth}_X \mathscr{F} - q$. First note that we have

$$\begin{aligned} H^{k-1}(Y, \mathscr{I}^m \mathscr{F}/\mathscr{I}^{m+1}\mathscr{F}) &= H^{k-1}(Y, \mathscr{I}^m/\mathscr{I}^{m+1} \otimes_{\mathcal{O}_Y} \mathscr{F}_0) \\ &= 0 \quad \text{for } m \gg 0. \end{aligned}$$

Especially

$$H^k(X \mid Y, \mathscr{I}^{mo+m}\mathscr{F}) \to H^k(X \mid Y, \mathscr{I}^{mo}\mathscr{F})$$

is injective for $m_0 \gg 0, m \geqslant 0$. Now, if we can replace $\mathscr{F}$ by $\mathscr{I}^{mo}\mathscr{F}$ in (3.3), which seems to be possible, we get from (3.1.2)

$$\varprojlim_m H^k(X \mid Y, \mathscr{I}^{mo+1+m}\mathscr{F}) \xrightarrow{\sim} \varprojlim_m K_m^k(\mathscr{I}^{mo}\mathscr{F})$$

and so the vanishing in (3.1) could be achieved.

4. Some examples of neighbourhood structures

(4.1) We first show that an analogous conjecture (2.3) for the injectivity of r^{n-1} is *not* true in the case when $Y \subset X$ has only a fundamental system of strongly pseudoconcave neighbourhoods.

In our case, X is a compact algebraic surface and $Y \subset X$ an elliptic curve with the following properties

(1) $N_{Y|X}$ is trivial
(2) there is a projection $\pi\colon X \to Y$ such that $\pi \mid Y = \mathrm{id}_Y$
(3) $U = X \setminus Y$ is Stein, but not affine algebraic.

Such a surface has been constructed by Serre, see [Ha 2] p. 232–234 or (5.2).

Now we take $\mathscr{L} \in \operatorname{Pic}(Y)^0$, non-torsion, and put

$$\mathscr{G} := \pi^* \mathscr{L}$$
$$\mathscr{F} := \mathscr{G}^\vee \otimes_{\mathcal{O}_X} \omega_X.$$

We show the following

(4.2) LEMMA. (1) $\dim H^1(\hat{X}, \hat{\mathscr{G}}) < \infty$
(2) $\dim H^1(X \mid Y, \mathscr{G}) = \infty$, $\dim H^1_Y(X, \mathscr{F}) = \infty$
(3) $\dim H^0([U], \mathscr{F}) < \infty$, $\dim H^1{}_{[Y]}(X, \mathscr{F}) < \infty$.

Proof. (1) We have the short exact sequences

$$0 \to \mathscr{I}^k\mathscr{G}/\mathscr{I}^{k+1}\mathscr{G} \to \mathscr{G}_k \to \mathscr{G}_{k-1} \to 0,$$

and, moreover,

$$\mathscr{I}^k\mathscr{G}/\mathscr{I}^{k+1}\mathscr{G} \simeq \mathscr{L}.$$

Since $H^1(Y, \mathscr{L}) = 0$, we get $H^1(X, \mathscr{G}_k) \xrightarrow{\sim} H^1(X, \mathscr{G}_{k-1})$, $k \geqslant 1$, which proves (1).
(2) By using the long exact sequence (1.4.3)

$$\cdots \to H^i_c(U, \mathscr{G}) \to H^i(X, \mathscr{G}) \to H^i(X \mid Y, \mathscr{G}) \to \cdots$$

it is sufficient to show for the first part that

$$\dim H^2_c(U, \mathscr{G}) = \infty.$$

But this holds by duality and because U is Stein.
Now, $H^1(X \mid Y, \mathscr{G})$ and $H^1_Y(X, \mathscr{F})$ are separated. For the last space, this follows from the criterion in (1.7) and Siu's vanishing theorem, and for the first one because of

$$H^1(X \mid Y, \mathscr{G}) = \varinjlim_{V \supset Y} H^1(V, \mathscr{G})$$

and $H^1(V, \mathscr{G})$ is separated if V is chosen to be strongly pseudoconcave. So the second part of (2) is a consequence of (1.7).
(3) This follows immediately from the moderate local cohomology sequence (1.4.2) and (1.11), together with (1). □

(4.3) REMARK. The example above also disproves the conjecture of [P-S] (1.7). As an additional assumption we should add $Y^n \neq 0$ in (1.7).

(4.4) Next, we are going to discuss an example, treated also by A. Ogus in [Og] (4.17). Here X is the compact complex surface which is obtained by blowing up 9 points in $\mathbb{P}^2_{\mathbb{C}}$, lying on an elliptic curve $Y' \subset \mathbb{P}^2_{\mathbb{C}}$, and $Y \subset X$ is the strict transform of Y'. Moreover, the normal bundle $N = N_{Y|X}$ is in $\mathrm{Pic}(Y)^0$ and non-torsion. We have the following properties

(1) $H^0(X, \Omega^1_X) = 0$

(2) $\dim_{\mathbb{C}} H^0(X \mid Y, \Omega^1_X) \leqslant 1$

(3) there is a unique isomorphism $f\colon \hat{X} \xrightarrow{\sim} \hat{N}$ with $f \mid Y = \mathrm{id}_Y$

(4) there is a unique section $g\colon \hat{X} \to Y$ of $Y \hookrightarrow \hat{X}$

(5) g is convergent iff $\dim_{\mathbb{C}} H^0(X \backslash Y, \Omega^1_X) = 1$.

Especially, if f is convergent, so is g.

By (1), the sequence

$$0 \to H^0(X \mid Y, \Omega^1_X) \to H^1_c(U, \Omega^1_X) \to H^1(X, \Omega^1_X)$$

is exact and $H^1_c(U, \Omega^1_X)$ is separated (use the criterion in [R−R1] and Siu's vanishing theorem), so

$$H^1_c(U, \Omega^1_X) \xrightarrow{\sim} (H^1(U, \Omega^1_X)_{\mathrm{sep}})'$$

is bijective by duality theory. If we know, for instance, that U is Stein, then $H^0(X \mid Y, \Omega^1_X) = 0$ and g cannot be convergent.

There is now the following theorem which follows from the results of V.I. Arnold [A] and the paper [I-P1] on the neighbourhood structure of embedded elliptic curves (or more generally complex tori).

(4.5) THEOREM. *There exist configurations of 9 points in $\mathbb{P}^2_{\mathbb{C}}$, lying on an elliptic curve, such that f is convergent.*

On the other hand, we have

(4.6) THEOREM. *If the points $p_1, \ldots, p_9 \in Y'$ blown up are in general position, Y has a neighbourhood which is both pseudoconvex and pseudoconcave, but Y has no strongly pseudoconcave neighbourhood; in particular $U = X \backslash Y$ is not Stein.*

Proof. "General position" means that $p_1, \ldots, p_9$ are chosen in such a way that the normal bundle $N = N_{Y|X}$ is not contained in some subset of $\mathrm{Pic}^0(Y)$ of measure 0 (with respect to an invariant measure) to be specified in a moment. By Ueda [U, Section 4 Theorem 3 and Section 5] there is a set S of measure 0 in $\mathrm{Pic}^0(Y)$ such that if $N \in S$, then our claim holds or C has finite order in the sense of Ueda. But of course, if we enlarge S, we may assume that N is non-torsion and hence C is of order ∞ ([U, p. 595/596]).

(4.7) REMARK. It is open what happens if $p_1, \ldots, p_9$ are in special position. The reason is the following problem. Suppose we have a regular family of projective surfaces (X_t) over the unit disc Δ. Suppose further that we have a family (C_t) of smooth curves over Δ, such that $C_t \subset X_t$, $C_t^2 = 0$. Let $X_0 \backslash C_0$ be Stein. Under which conditions is $X_t \backslash C_t$ Stein for $t \neq 0$, t near 0?

5. Stein complements in compact surfaces

In this section we deal with the following problem: Given a compact surface X (i.e. a 2-dimensional connected compact manifold) and an irreducible curve $C \subset X$ such that $C^2 = 0$ and $X \backslash C$ is Stein. What pairs (X, C) can really occur?

(5.1) THEOREM. *Let X and C be as described above. Then either X is algebraic or X is a Hopf surface of algebraic dimension* 0 *with exactly one curve. In the latter case, C is elliptic and $X \backslash C \simeq \mathbb{C}^* \times \mathbb{C}^*$.*

Proof. As usual we denote by $a(X)$ the algebraic dimension of X. X is algebraic iff $a(X) = 2$. So let $a(X) < 2$.

(1) Assume $a(X) = 1$. Let X_0 be the minimal model of X. Then X_0 is an elliptic surface, and we obtain a surjective map $\tau: X \to C_0$ onto a compact Riemann surface C_0 (see e.g. [BPV]).

Assume $\tau(C) = C_0$. Denoting by F a general smooth fiber of τ, this means $(C.F) > 0$. But $(C + F)^2 = 2(C.F) > 0$, hence X is algebraic (Kodaira). So $\tau(C)$ is a point. But then $X \backslash C$ contains compact curves, hence cannot be Stein.

(2) Now let $a(X) = 0$, i.e. there are no non-constant meromorphic functions on X.

(a) First let X be Kähler. Then (by the Kodaira classification) X_0 (the minimal model) is either a torus or a $K3$ surface. If X_0 is a torus, by homogenity, X_0 contains no curve at all ([BPV, p. 129]). On the other hand – denoting by $\pi: X \to X_0$ the blow-down – π cannot contract C except $C = \mathbb{P}_1$. But then X is algebraic by [BPV, p. 142]. So the torus case is excluded. Now X_0 is a $K3$ surface. If $X \neq X_0$, we find a (-1)-curve C' such that $C \cap C' \neq \emptyset$. Hence $(nC + C')^2 = n(C.C') - 1 > 0$ for $n \geqslant 2$ and X is algebraic again. So $X = X_0$.

Since $H^1(\mathcal{O}_X) = 0$ and $\omega_X \simeq \mathcal{O}_X$, we compute from

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

$$h^0(\mathcal{O}_X(C)) = h^2(\mathcal{O}_X(-C)) = 1 + h^1(\mathcal{O}_C).$$

Since $a(X) = 0, h^0(\mathcal{O}_X(C)) \leqslant 1$, so $h^1(\mathcal{O}_C) = 0$, i.e. $C \simeq \mathbb{P}_1$. This contradicts the adjunction formula.

(b) Finally let X be non-Kähler. Then by the Kodaira classification X is of type VII.

Arguing as in the $K3$-case, we may assume X minimal. First let $b_2(X) > 0$. Now we use Enoki's classification ([E]) of surfaces of type VII with $b_2 > 0$, containing a curve and containing a divisor $D \neq 0$ with $D^2 = 0$. Applying his structure theorems (with $D = C$), we conclude that $X \backslash C \simeq Y \backslash C'$ where Y is a ruled surface over an elliptic curve with "eccentricity" $e = b_2(X)$ (see Hartshorne [Ha 3]) such that $C'^2 = -e$. So $X \backslash C$ is not Stein. We are thus reduced to $b_2(X) = 0$. By Kodaira ([K2], [E]), X is a Hopf surface.

Such a Hopf surface either has exactly one or two curves. If X has two curves, call the second C', then $C \cap C' = \emptyset$ contradicting $X \backslash C$ Stein. So X has one curve C and, by Kodaira ([K1]), $X \backslash C \simeq \mathbb{C}^* \times \mathbb{C}^*$ (in fact $X \backslash C \simeq Y \backslash C'$ where Y is the ruled surface given by the bundle $0 \to \mathcal{O}_{C_0} \to \mathcal{E} \to \mathcal{O}_{C_0} \to 0$ over an elliptic curve and $C'^2 = 0$; this is just the Serre example discussed in Section 4). This ends the proof. □

(5.2) Now we turn to the projective case and doing this, we restrict ourselves to elliptic C. We call a pair (X, C) as above the Serre example, if $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the rank-2 bundle $\mathcal{E}$ over an elliptic curve C_0 given by the non-splitting extension

$$0 \to \mathcal{O}_{C_0} \to \mathcal{E} \to \mathcal{O}_{C_0} \to 0,$$

and where C is the uniquely determined section with $C^2 = 0$. Then $X \backslash C \simeq \mathbb{C}^* \times \mathbb{C}^*$.

(5.3) THEOREM. *If X is a minimal projective surface, $C \subset X$ an elliptic curve with $C^2 = 0$, such that $X \backslash C$ is Stein then (X, C) is the Serre example.*

Proof. (*a*) First we show $\kappa(X) = -\infty$.

First let $\kappa(X) = 2$. We obtain by the adjunction formula $(\omega_X . C) = 0$. But this is known to be possible only if C is a (-2)-curve (see [BPV, Chapter 7]). If $\kappa(X) = 1$, ω_X^μ is globally generated if $\mu \gg 0$. We consider the associated map $\phi: X \to \mathbb{P}(H^0(X, \omega_X^\mu))$ whose image is a curve. ϕ contracts exactly the curves C' with $(\omega_X . C') = 0$. Since $(\omega_X . C) = 0$, ϕ contracts C, contradicting $X \backslash C$ Stein.

So assume finally $\kappa(X) = 0$. If X is a torus, we obtain (using an automorphism) an elliptic $C' \neq C$, $C' \sim C$. Since $C \cap C' \neq \emptyset$, $(C . C') > 0$, so $C^2 > 0$, contradiction. If X is $K3$, then we compute as in (5.1): $h^0(\mathcal{O}_X(C)) = 2$ and obtain a curve C' as above. If X is an Enriques surface, we have – using [BPV, VIII.17] – an elliptic fibration $f: X \to \mathbb{P}_1$ contracting C which is impossible.

Finally, if X is hyperelliptic then there are two elliptic curves E_1, E_2 such that $X = (E_1 \times E_2)_G$, quotient by some finite group (see [BPV, Chapter V]). Denote by $\rho: E_1 \times E_2 \to X$ the projection map. Clearly $(E_1 \times E_2) \backslash \rho^{-1}(C)$ is Stein, hence $\rho^{-1}(C)$ is connected. Now $\rho^{-1}(C)^2 = 0$, so if $\rho^{-1}(C)$ is reducible, it must contain an exceptional curve in $E_1 \times E_2$. This does not exist, so $\rho^{-1}(C)$ is irreducible. So $\rho^{-1}(C) = \{x\} \times E_2$ or $E_1 \times \{y\}$, which contradicts $(E_1 \times E_2) \backslash \rho^{-1}(C)$ Stein.

(b) Now we know that $\kappa(X) = -\infty$, X is either $\mathbb{P}_2$ or ruled. Clearly $X \neq \mathbb{P}_2$. Let $p: X \to C_0$ be a ruling. Since $g(C) = 1$ and $\dim p(C) = 1$, we obtain $g(C_0) \leqslant 1$. If $C_0 = \mathbb{P}_1, C^2 = 0$ implies first $X \neq \mathbb{P}_1 \times \mathbb{P}_1$. So $X = \Sigma_m = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$,

$m \in \mathbb{N}$. But any curve with self-intersection 0 in Σ_m is known to be a fiber of the ruling, see [Ha 3], contradiction.

So $g(C_0) = 1$. Let F be a fiber of the ruling, $\tilde{C} \subset X$ a curve of minimal $\tilde{C}^2$. Let $e := -\tilde{C}^2$. Write for numerical equivalence $C \sim a\tilde{C} + bF$. Then $C^2 = 0$ just says $b = \frac{1}{2}ae$.

Furthermore $(C.\tilde{C}) = b - ae = -\frac{1}{2}ae$. Since $(C.\tilde{C}) \geqslant 0$, we conclude $ae = 0$. Since $C \neq F, a > 0$, so $e = 0$. Hence either $X = C_0 \times \mathbb{P}_1$ – which is clearly impossible – or $X = \mathbb{P}(\mathscr{E})$ as in the Serre example. Observe that $C = \tilde{C}$ is uniquely determined in X by $C^2 = 0$.

(5.4) REMARK. (1) If we don't assume X to be minimal in (5.3), we take a minimal model $p\colon X \to X_0$ and conclude $(\omega_{X_0}.p(C_0)) < 0$. Hence it is a priori clear that $\kappa(X) = -\infty$ and moreover that X_0 is $\mathbb{P}_2$, a rational ruled surface or a ruled surface over an elliptic curve.

(2) Neeman [N] has also investigated the problem of classification of the pairs (X, C) as above. He additionally assumes that the "Ueda class" of C is non-zero.

For application in the next section, we state the following version of (5.1):

(5.5) THEOREM. *Let X be a compact surface, D a non-zero divisor on X such that $D^2 = 0$ and $X \backslash \mathrm{supp}(D)$ is Stein. Then either X is algebraic or X is a Hopf surface with $a(X) = 0$ having just one curve (hence $X \backslash \mathrm{supp}(D) = X \backslash C \simeq \mathbb{C}^* \times \mathbb{C}^*$).*

The proof is along the lines of (5.1) and thus omitted. Observe that we need (5.5) only for X minimal; then it is almost word by word the same proof as in (5.1).

6. On Compactifications of $\mathbb{C}^3$

In relation with the problems considered in Section 5 we now look at compactifications of $\mathbb{C}^3$.

This means that we consider pairs (X, Y) where X is a compact manifold, $Y \subset X$ a hypersurface and $X \backslash Y \simeq \mathbb{C}^3$ biholomorphically. We always assume $b_2(X) = 1$ which is equivalent to saying that Y is irreducible. We let $\mathcal{O}_X(1)$ denote the line bundle on X given by Y. Then $c_1(\mathcal{O}_X(1))$ generates $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$.

(6.1) LEMMA. *If the algebraic dimension $a(X) \leqslant 2$, then $Y^3 = c_1(\mathcal{O}_X(1))^3 < 0$.*

Proof. We have seen in [PS] that $Y^3 > 0$ implies X projective. So we must only exclude $Y^3 = 0$. Assume $Y^3 = 0$, we have also $Y^2 = 0$ (since the intersection form $H^2 \times H^4 \to H^6$ is non degenerate). But $Y^2 = c_1(N_{Y|X})$ and since $H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$ via restriction, $c_1(N_{Y|X})$ generates $H^2(Y, \mathbb{Z})$, so $Y^2 \neq 0$. □

(6.2) THEOREM. $a(X) \neq 1$.

Combining with the results of [PS], we obtain:

(6.3) COROLLARY. *If X is not projective, any meromorphic function on X is constant* ($a(X) = 0$).

Proof of (6.2). Let

$$\begin{array}{ccc} \hat{X} & \xrightarrow{q} & Z \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

be an algebraic reduction of X. Then Z is a compact Riemann surface and all fibers of q are connected (see Ue]). p can be taken as sequence of blow-up's with smooth centers. Since $b_1(X) = 0$, we have $Z \simeq \mathbb{P}_1$. We put $\hat{X}_s = q^{-1}(s)$, $X_s = p(\hat{X}_s)$, $s \in \mathbb{P}_1$. Furthermore let $\mathscr{F} := (p_* q^* \mathcal{O}_Z(1))^{**}$. $\mathscr{F}$ is a line bundle on X. We claim:

(1) $c_1(\mathscr{F}) = 0$.

Proof. Let f: $\tilde{Y} \to Y$ be the normalization of Y (observe that by [PS] Y must be non-normal!) and π: $\hat{Y} \to \tilde{Y}$ a minimal desingularization, σ: $\hat{Y} \to Y_m$ a minimal model. By [PS, proof of 1.5], $a(Y) = 1$.

Let τ: $Y_m \to C$ be an algebraic reduction which is here an elliptic fiber space (cp. [BPV]). Clearly, there is no curve $C_0 \subset Y_m$ such that $\tau(C_0) = C$ (otherwise $(C_0 + n \cdot \text{fiber})^2 > 0$ for n large and Y_m would be algebraic). Hence $\dim \tau\sigma(Z(\pi)) = 0$, $Z(\pi)$ the center of π. Let E be the non-normal locus of Y, $f^{-1}(E) = \tilde{E}$. We know $E \neq \emptyset$, so E is purely 1-dimensional (Y being Gorenstein). We conclude as for $Z(\pi)$: $\dim \tau\sigma\pi^{-1}(\tilde{E}) = 0$. Let $\hat{l} = \sigma^{-1}\tau^{-1}(x)$, $\tilde{l} = \pi(\hat{l})$, $l = l(x) = f(\tilde{l})$ for $x \in C$. Then by our consideration, we see that $l \cap S(Y) = \emptyset$ for x generic, $S(Y)$ being the singular locus of Y. In particular, $l(x)$ is a Cartier divisor on Y, thus

$$c_1(\mathcal{O}_Y(l(x))) = c_1(\mathcal{O}_Y(k)), \quad k \in \mathbb{Z}.$$

(where $\mathcal{O}_Y(1) := \mathcal{O}_X(1) \mid Y$).

Now $l(x)^2 = 0$, hence by (6.1) we conclude $k = 0$; in particular $l(x) \sim 0$ in homology. Since all $l(x)$ are homologous, we obtain $l(x) \sim 0$ for all $x \in C$.

Now we compute $(Y . X_s)$. Since $\mathcal{O}_X(X_s) \simeq \mathscr{F}$, we have just to show $(Y . X_s) = 0$ in order to prove (1). Write $(Y . X_s) = [\Sigma \alpha_i C_i]$ in $H_4(X, \mathbb{R})$.

The curves C_i are contained in Y, so $C_i \subset l(x_i)$ for some x_i. Now for any x, either $l(x) \subset X_s$ or $l(x) \cap X_s = \emptyset$ (since $(l(X) . X_s) = 0$!).

So in fact $(Y . X_s) = [\Sigma \alpha_i l(x_i)] = 0$. This proves (1).

If we knew $H^1(X, \mathcal{O}) = 0$, $\mathscr{F}$ would have to be trivial and so X could not exist. But this vanishing is not at all obvious.

(2) Let $Z(p)$ be the center of p in $\hat{X}$. Then

$$p(Z(p)) \subset Y.$$

Proof. Clearly we may assume $p(Z(p)) \subset \bigcap_{s \in Z} X_s$.

Choose $s, s' \in Z, s \neq s'$. Then $(X_s . X_{s'}) = [\sum \alpha_i l(x_i)] + [\sum \beta_j B_j]$ where the B_j are the components of $X_s \cap X_{s'}$ which are not contained in Y (clearly $\dim B_j = 1$). By (1), $[\sum \beta_j B_j] = 0$. Since $\beta_j > 0$ and $B_j \cap Y \neq \emptyset$ ($X \setminus Y$ is Stein), we would have $(Y . \sum \beta_j B_j) > 0$ if there is some B_j. So $X_s \cap X_{s'} \subset Y$, hence $p(Z(p)) \subset Y$.

(3) It is sufficient to prove projectivity for some $\hat{X}_s$. In fact, then we find a curve $C' \subset X_s$ such that $C' \cap Y$ is finite. Hence $(Y . C') > 0$. On the other hand $C' \sim 0$ in X, since $X_s \sim 0$; hence the non-existence of X is proved.

(4) So it remains to show $a(\hat{X}_s) = 2$. We have $\hat{X}_s \cap p^{-1}(Y) = C_1 \cup \cdots \cup C_q =: D$, $\hat{X}_s \setminus D$ is Stein as closed subvariety of $\mathbb{C}^3$. We want to apply (5.5). First let us convince ourselves that D contains an elliptic curve. In fact, $q(\bar{Y}) = Z$, $\bar{Y}$ the strict transform of Y in $\hat{X}$ (otherwise any meromorphic function on $\hat{X}$ would be constant on $\bar{Y}$; on the other hand there are clearly meromorphic functions on X which are not constant on Y). Hence for general s, $\bar{Y} \cap \hat{X}_s$ contains an elliptic curve (since $a(\bar{Y}) = 1$). So D contains an elliptic curve.

For reasons of minimality we have to check that there is no (-1)-curve in $\hat{X}_s$ not contained in D. In fact, if there is some, say C', then let $C'' = p(C')$ and we would have $(Y . C'') > 0$, contradicting $C'' \sim 0$.

In order to apply (5.5) we have to find a divisor $0 \neq \tilde{D}$ with $\operatorname{supp}(\tilde{D}) = D$, such that $\tilde{D}^2 = 0$. Assume that for any such $\tilde{D}$ we have $\tilde{D}^2 < 0$. Then for all $u = (u_1, \ldots, u_q) \in \mathbb{Z}^q \setminus \{0\}$:

$$\sum C_i C_j u_i u_j < 0.$$

Hence $(C_i . C_j)$ is negative definite and D thus exceptional. But this is impossible since $\hat{X}_s \setminus D$ is Stein!

So we are now in position to apply (5.5) and hence $\hat{X}_s$ is either algebraic or its minimal model is a Hopf surface with one curve.

(5) In order to finish the proof we have to exclude the latter. From our discussion in (1), concerning curves in Y it is clear that for general x

$$l(x) \cap Z(p) = \emptyset.$$

$X_s \setminus (X_s \cap Y)$ being Stein, $X_s \cap Y$ must be connected. Hence for general s: $X_s \cap Y$ is of the form $l(x)$. Now $l(x) \cap l(x') = \emptyset$ in general, so for general s, X_s does not meet any $X_{s'}$. But this means that $\mathscr{F}$ is globally generated and we may take

$\hat{X} = X^1$! Observe that the smooth surfaces X_s have to be minimal! Let $C_s = X_s \cap Y$. So for general s, C_s is an elliptic curve, in fact the uniquely determined curve in X_s.

Now we are in position to prove:

$$\chi(\mathcal{O}_X(1)) = \chi(\mathcal{O}_X). \tag{$*$}$$

First we observe: $R^2q_*(\mathcal{O}_X) = 0$.

To see this, we must show $h^2(\mathcal{O}_{X_s}) = 0$ for all s, q being flat. Equivalently, $h^0(\omega_{X_s}) = 0$. For general s this is true. So $q_*(\omega_X)$ is torsion. Since $h^0(\omega_X) = 0$ (by a theorem of Kodaira [K3]), $q_*(\omega_X) = 0$, hence $h^0(\omega_{X_s}) = 0$ for all s. Since $q_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}_1}$, the invariance of $\chi(\mathcal{O}_{X_s})$ gives $R^1q_*(\mathcal{O}_X) \in \operatorname{Pic}(\mathbb{P}_1)$, even $R^1q_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}_1}(a)$, $a \geqslant 0$ (since $h^1(\mathcal{O}_X) > 0$).

We consider the exact sequence

$$\begin{aligned}
0 &\to q_*(\mathcal{O}_X) \to q_*(\mathcal{O}_X(1)) \to q_*(\mathcal{O}_Y(1)) \to R^1q_*(\mathcal{O}_X) \\
&\to R^1q_*(\mathcal{O}_X(1)) \to R^1q_*(\mathcal{O}_Y(1)) \to 0 = R^2q_*(\mathcal{O}_X) \to \\
&\quad R^2q_*(\mathcal{O}_X(1)) \to 0.
\end{aligned}$$

So $R^2q_*(\mathcal{O}_X(1)) = 0$.

Generically $h^0(\mathcal{O}_{X_s}(1)) = h^0(\mathcal{O}_{X_s}(C_s)) = 1$ and also $h^1(\mathcal{O}_{X_s}(C_s)) = 1$ (Riemann-Roch).

So $q_*(\mathcal{O}_X(1))$ and $R^1q_*(\mathcal{O}_X(1))$ have generically rank 1. If $q_*(\mathcal{O}_X(1))$ has a torsion part, we conclude $h^0(\mathcal{O}_X(1)) \geqslant 2$, hence $\kappa(X, \mathcal{O}_X(1)) \geqslant 1$. But in [PS] it was proved that then X is algebraic. So $q_*(\mathcal{O}_X(1)) \in \operatorname{Pic}(\mathbb{P}_1)$ and by the same argument: $q_*(\mathcal{O}_X(1)) = \mathcal{O}_{\mathbb{P}\mathbb{C}}{}^1$. Hence $R^1q_*(\mathcal{O}_X(1)) \in \operatorname{Pic}(\mathbb{P}_1)$ too. For our Hopf surface X_s we have:

$$\omega_{X_s} \simeq \mathcal{O}_{X_s}(-mC_s), \quad m \geqslant 2$$

([K1]). First let $m \geqslant 3$. Then the normal bundle $N_{C_s|X_s}$ is topologically trivial, but not trivial. Consequently, $q_*(\mathcal{O}_Y(1))$ is torsion as well as $R^1q_*(\mathcal{O}_Y(1))$[2]. By the above exact sequence, $R^1q_*(\mathcal{O}_X(1)) \simeq R^1q_*(\mathcal{O}_X)$. Now we exclude $m = 2$. Assuming $m = 2$, $N_{C_s|X_s} \simeq \mathcal{O}_{C_s}$ and so $q_*(\mathcal{O}_Y(1))$ and $R^1q_*(\mathcal{O}_Y(1))$ would have rank 1 generically and hence everywhere by the exact sequence. Hence $q_*(\mathcal{O}_Y(1)) \simeq R^1q_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}_1}(a)$, $a \geqslant 0$.

So $h^0(\mathcal{O}_Y(1)) > 0$. Since $\mathcal{O}_Y(1) \not\simeq \mathcal{F} \mid Y$ and $\mathcal{F} \mid Y \simeq \mathcal{O}_Y(C_s)$, we get a curve

[1] This means that, instead of the algebraic reduction, we just consider the map $X \to \mathbb{P}(H^0(X, \mathcal{F}))$.
[2] Since $\mathcal{O}_Y(1)|C_s \simeq N_{C_s|X_s}$.

$C_0 \subset Y$ not contracted by $q \mid Y$. But then Y must be algebraic (similar arguments have been used earlier), contradiction (we know $a(Y) \leqslant 1$).

Now we know: $q_*(\mathcal{O}_X(1)) = \mathcal{O}_{\mathbb{P}_1}$; $R^i q_*(\mathcal{O}_X(1) = R^i q_*(\mathcal{O}_X)$, $i > 0$. So by the Leray spectral sequence, we obtain easily: $\chi(\mathcal{O}_X(1)) = \chi(\mathcal{O}_X)$, proving (*). Now Riemann-Roch for $\chi(\mathcal{O}_X(1))$ gives – setting $c_1(X) = kY$, $c_2(X) = \lambda Y^2$ and using

$$\frac{c_1(X)c_2(X)}{24} = \chi(\mathcal{O}_X): \quad \frac{Y^3}{6} + \tfrac{1}{4}kY^3 + \frac{k^2Y^2 + \lambda Y^2}{12} Y = 0,$$

hence (since $Y^3 \neq 0$):

$$k^2 + 3k + \lambda + 2 = 0. \tag{+}$$

With the same methods we obtain also:

$$\chi(\mathcal{O}_X(-1)) = 0 \quad \text{and} \quad \chi(\mathcal{O}_X(-2)) = 0$$

(since $R^i q_*(\mathcal{O}_X(-\nu)) = 0, \nu = 1, 2, i = 0, 1, 2,$ for $\nu = 2$ use that $m \neq 2$ where $\omega_{X_s} = \mathcal{O}_{X_s}(-mC_s)$, see proof of (+)).

Now by Riemann-Roch we obtain:

$$k^2 - (\tfrac{1}{2}\lambda + 3)k + \lambda + 2 = 0 \tag{++}$$

$$k^2 - (\tfrac{1}{4}\lambda + 6)k + \lambda + 8 = 0. \tag{+++}$$

Altogether, the three equations (+), (++), (+++) have no common solution, hence the proof is finished. □

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