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Real algebraic curves in the moduli space of complex curves

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Abstract. The complex moduli space of real algebraic curves of genus g consists of complex isomorphism classes of genus g complex algebraic curves that are defined by real polynomials. In this paper[1] we study that moduli space. We show that it is a semialgebraic variety. In the cases of genus 1 or genus 2 curves we get also an explicit presentation for this moduli space.

1. Introduction

We shall study in this paper the set of complex isomorphism classes of such complex curves of a given genus g that are defined by real polynomials. This is the complex moduli space of real curves of genus g. This study of moduli problems of real algebraic curves was initiated already by Felix Klein.

The space of isomorphism classes of smooth complex algebraic curves of genus g, $\mathscr{M}^g$, is a quasiprojective algebraic variety and a normal complex space. It can be made compact adding points corresponding to isomorphism classes of stable genus g complex algebraic curves with nodes. A stable complex algebraic curve with nodes is simply a complex algebraic curve with a finite number of double-points such that each component of the complement of the double points has a negative Euler characteristic. The compactified moduli space $\bar{\mathscr{M}}^g$ is a projective variety and a normal complex space.

The algebraic structure of this compactified moduli space has been first constructed using concrete Geometric Invariants associated to stable curves ([13], [14] and [15], see also [17]). A concrete description of this projective structure is given in [18]. Later Wolpert gave an analytic proof for the existence of the projective structure on $\bar{\mathscr{M}}^g$ (cf. [28]). This latter approach is based on Teichmüller theory. It is common knowledge among experts that these two approaches lead to the same projective structure. In this paper we shall use that fact even though it has not been possible for me to find an explicit reference proving it.

We are interested in the subspace $\mathscr{M}^g_{\mathbf{R}}$ of $\mathscr{M}^g$ whose points are complex

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isomorphism classes of real algebraic curves of genus g. Equally interesting is the space $\bar{\mathcal{M}}_{\mathbf{R}}^{g}$ whose points are complex isomorphism classes of stable genus g real algebraic curves. $\bar{\mathcal{M}}_{\mathbf{R}}^{g}$ is a compactification of $\mathcal{M}_{\mathbf{R}}^{g}$. The space $\mathcal{M}_{\mathbf{R}}^{g}$ is a moduli space for real algebraic curves of genus g. It has been extensively studied by Klein (cf. [10], [11], [12]), who already understood the basic properties of the moduli space of real algebraic curves.

We shall study here topology and geometry of these moduli spaces of real algebraic curves. We shall show that *the moduli spaces $\mathcal{M}_{\mathbf{R}}^{1}$ and $\mathcal{M}_{\mathbf{R}}^{2}$ are connected.* Recall that in [21] we had shown that *the moduli spaces $\bar{\mathcal{M}}_{\mathbf{R}}^{g}$, $g > 1$, are connected.*

We shall also study geometry of the moduli spaces $\mathcal{M}_{\mathbf{R}}^{g}$ and $\bar{\mathcal{M}}_{\mathbf{R}}^{g}$. The main result (Theorem 10.2) of this paper describes the analytic structure of these moduli spaces. An important consequence of that result is that these moduli spaces are actually semialgebraic varieties.

We rely on explicit analytic and geometric methods. Smooth complex algebraic curves are compact Riemann surfaces. A complex algebraic curve is isomorphic to a curve defined by real polynomials if and only if the corresponding Riemann surface admits an antiholomorphic involution. Even though these basic facts are assumed known we shall briefly summarize some of them in Section 2.

Here we have defined the moduli space of real algebraic curves as the set of complex isomorphism classes of real algebraic curves of a given genus. It may seem strange that we have chosen to study the complex isomorphism classes instead of the real isomorphism classes of such curves. This definition and the problems of the present paper have their roots deep in the history: the study of this topic was initiated by Felix Klein already more than 100 years ago. This and the related papers [24], [23], [21] and [19] provide, among other things, proofs to many of the results stated by Klein in [10], [11] and in [12].

Equally interesting is also the space of real isomorphism classes of stable real algebraic curves of a given genus. This real moduli space for stable real curves forms a covering of our present complex moduli space for stable real curves. Similar results hold also for that moduli space. Most importantly, this real moduli space (of stable real algebraic curves of a given genus) is connected. It can be expressed as the closure of a union of real analytic spaces with a natural semialgebraic structure. This result has been shown by other methods in [20], see also [25].

This paper has matured a long time. The results presented here have been announced already five years ago in [22]. I take this opportunity to present my apology for the delay in the publication. A big part of the present work was done during my stay at the University of Regensburg. I would like to thank my colleagues there for their great hospitality. Finally I dedicate this paper to my wife Cici who has, in many ways, helped me in getting this work done.

2. Preliminaries

Let Σ be a fixed compact and oriented C^∞-surface of genus g. The surface Σ carries usually several complex structures X which are assumed to agree with the given orientation of Σ and with the C^∞-structure of Σ. Let $M(\Sigma)$ denote the space of all such complex structures of Σ.

The surface Σ together with a complex structure $X \in M(\Sigma)$ is a Riemann surface of genus g. We shall use the notation X for the Riemann surface (Σ, X) when there is no danger of confusion. So we shall study several Riemann surfaces at the same time but it is always assumed that the underlying topological space is the same.

A Riemann surface $X = (\Sigma, X)$ is also a complex projective curve. X can be embedded in a complex projective space $\mathbf{P}^N(\mathbf{C})$ in such a way that the image C of X in $\mathbf{P}^N(\mathbf{C})$ is defined by a finite number of polynomial equations, i.e., C is a complex algebraic curve.

In the sequel the notations $X, X', X_1, \ldots$ always refer to complex structures of Σ. The notations $C, C', C_1, \ldots$ are used for the corresponding algebraic curves. So that the Riemann surface X *is* the complex curve C. We shall use both these notations for the same object.

Assume now that the projective curve $C \in \mathbf{P}^N(\mathbf{C})$ is defined by polynomials having real coefficients. Then it is immediate that C remains invariant under the complex conjugation in $\mathbf{P}^N(\mathbf{C})$. The complex conjugation induces, therefore, an antiholomorphic involution of the corresponding Riemann surface X. Let us denote this involution by $\sigma: X \to X$.

Conversely, a compact Riemann surface X together with an antiholomorphic involution $\sigma: X \to X$ can be embedded in a projective space $\mathbf{P}^N(\mathbf{C})$ in such a way that the involution σ is the restriction to $X \in \mathbf{P}^N(\mathbf{C})$ of the complex conjugation in $\mathbf{P}^N(\mathbf{C})$. Therefore the Riemann surface X is actually a complex algebraic curve defined by real polynomials.

The above embedding of X into $\mathbf{P}^N(\mathbf{C})$ can be formed choosing a pluricanonical embedding of X in a suitable way. For more details we refer to [3].

We conclude that a projective real algebraic curve of genus g is simply a compact genus g Riemann surface X together with an antiholomorphic involution $\sigma: X \to X$.

An antiholomorphic involution $\sigma: X \to X$ is induced by an orientation reversing involution $\sigma: \Sigma \to \Sigma$ of the underlying topological surface.

It is now necessary to recall the topological classification of such involutions. Two involutions $\sigma_1: \Sigma \to \Sigma$ and $\sigma_2: \Sigma \to \Sigma$ are *of the same topological type* if there exists a homeomorphism $f: \Sigma \to \Sigma$ such that $\sigma_1 = f \circ \sigma_2 \circ f^{-1}$. Equivalently we may say that $\sigma_1: \Sigma \to \Sigma$ and $\sigma_2: \Sigma \to \Sigma$ are of the same topological type if the orbit surfaces $\Sigma/\langle\sigma_1\rangle$ and $\Sigma/\langle\sigma_2\rangle$ are homeomorphic to each other.

Here $\langle \sigma_j \rangle$ is the group generated by $\sigma_j: \Sigma \to \Sigma$.

Let Σ be an oriented topological surface of genus g and let $\sigma: \Sigma \to \Sigma$ be an orientation reversing involution. Topologically the pair (Σ, σ) is determined by the following invariants:

1. The genus g of Σ.
2. The number n of connected components of the fixed-point set Σ_σ of the mapping σ.
3. The index of orientability, $k = k(\sigma)$, which is defined setting $k = 2 -$ the number of connected components of $\Sigma \backslash \Sigma_\sigma$.

These invariants satisfy:

- $0 \leqslant n \leqslant g + 1$.
- For $k = 0$, $n > 0$ and $n \equiv g + 1 \pmod 2$.
- For $k = 1$, $0 \leqslant n \leqslant g$.

These are the only restrictions for topological types of involutions of a genus g surface Σ. One computes that there are $\lfloor (3g + 4)/2 \rfloor$ topological types of orientation reversing involutions of a genus g surface. This formula was shown by G. Weichhold, a student of F. Klein ([27], see also [10]).

There are many equivalent definitions for the Teichmüller space of an oriented surface Σ. One definition that is suitable for our considerations is the following.

Let Σ be an oriented and compact C^∞-surface of genus g. Let $M(\Sigma)$ denote the set of those complex structures of Σ that agree with the orientation and the differentiable structure of Σ. The group Diff(Σ) of diffeomorphic self-mappings of Σ acts on $M(\Sigma)$ by pull back. The action is defined in the following way. Any diffeomorphism $f: \Sigma \to \Sigma$ induces a mapping $f^*: M(\Sigma) \to M(\Sigma)$ which is defined setting, for any complex structure $X \in M(\Sigma)$, $f^*(X) \in M(\Sigma)$ to be that complex structure of Σ for which the mapping $f: (\Sigma, f^*(X)) \to (\Sigma, X)$ is either holomorphic or antiholomorphic depending on whether $f: \Sigma \to \Sigma$ is orientation preserving or not.

Let $\mathrm{Diff}_0(\Sigma)$ denote the subgroup of Diff(Σ) consisting of mappings homotopic to the identity mapping of Σ, and let $\mathrm{Diff}_+(\Sigma)$ be the subgroup of orientation preserving mappings.

The Teichmüller space $T(\Sigma)$ of the surface Σ is then defined as the quotient

$$T(\Sigma) = T^g = M(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

The moduli space $\mathcal{M}(\Sigma) = \mathcal{M}^g$ is the quotient

$$\mathcal{M}(\Sigma) = \mathcal{M}^g = M(\Sigma)/\mathrm{Diff}_+(\Sigma).$$

Here g is the genus of Σ.

The moduli space $\mathcal{M}^g$ is just the space of isomorphism classes of smooth complex projective curves of genus g. The space $\mathcal{M}^g$ is a quasiprojective variety and a normal complex space.

The Teichmüller space T^g carries also a natural complex structure. T^g together with its complex structure is a complex manifold which is homeomorphic to an euclidean space $\mathbf{R}^{6g-6}$ (cf. e.g. [16, V.5]).

The *modular group*

$$\Gamma(\Sigma) = \Gamma^g = \mathrm{Diff}_+(\Sigma)/\mathrm{Diff}_0(\Sigma)$$

acts on T^g and $\mathcal{M}^g = T^g/\Gamma^g$. This action is properly discontinuous and the elements of Γ^g are holomorphic automorphisms of T^g. In the general case Γ^g is the full group of holomorphic automorphisms of T^g.

In view of the remarks made in the beginning of this section, the moduli space, $\mathcal{M}^g_{\mathbf{R}}$, of real curves of genus g, is a subspace of $\mathcal{M}^g$ consisting of isomorphism classes of such genus g Riemann surfaces $X = (\Sigma, X)$ which admit an antiholomorphic involution $\sigma: (\Sigma, X) \to (\Sigma, X)$.

An orientation reversing involution $\sigma: \Sigma \to \Sigma$ induces an antiholomorphic involution $\sigma^*: T(\Sigma) \to T(\Sigma)$ ([24, 5.10]). If $X \in M(\Sigma)$ is such a complex structure that $\sigma: (\Sigma, X) \to (\Sigma, X)$ is antiholomorphic then, by the definition, the point $[X] \in T(\Sigma)$ remains fixed under the induced mapping $\sigma^*: T(\Sigma) \to T(\Sigma)$.

Conversely, if a point $p \in T(\Sigma)$ remains fixed under $\sigma^*: T(\Sigma) \to T(\Sigma)$ then we can choose a complex structure X such that $[X] = p$ and $\sigma: (\Sigma, X) \to (\Sigma, X)$ is antiholomorphic (cf. e.g. [24, Theorem 5.1, page 33]).

Let $\sigma_1, \ldots, \sigma_m, m = \lfloor (3g+4)/2 \rfloor$, be a list of different topological types of involutions of Σ. Denote

$$N = \bigcup_{j=1}^{m} T(\Sigma)_{\sigma_j^*}.$$

Let $\pi: T(\Sigma) \to \mathcal{M}^g$ be the projection. Then, by the previous considerations, $\pi(N) = \mathcal{M}^g_{\mathbf{R}}$.

Let $\sigma: \Sigma \to \Sigma$ and $\sigma': \Sigma \to \Sigma$ be orientation reversing involutions. Then $\sigma \circ \sigma' \in \mathrm{Diff}_+(\Sigma)$. Therefore, σ and σ' induce the same involution $\sigma^*: \mathcal{M}^g \to \mathcal{M}^g$. It follows that

$$\pi(N) = \mathcal{M}^g_{\mathbf{R}} \subset (\mathcal{M}^g)_{\sigma^*}. \tag{1}$$

Examples show that this inclusion is proper (cf. e.g. [6] and [19]) if $g > 1$. In the next section we shall show, however, that for $g = 1$ we have equality in (1).

3. Elementary cases

The moduli spaces $\mathcal{M}_{\mathbf{R}}^0$ and $\mathcal{M}_{\mathbf{R}}^1$ are rather particular. The space $\mathcal{M}^0$ consists of a single point only. Consequently, the space $\mathcal{M}_{\mathbf{R}}^0$ consists of one point only and $\mathcal{M}_{\mathbf{R}}^0 = \mathcal{M}^0$. This point corresponds to the Riemann sphere $\hat{\mathbf{C}}$ which admits two topologically different antiholomorphic involutions. They are $\sigma_1(z) = \bar{z}$ and $\sigma_2(z) = -1/\bar{z}$. The orbit surface $\hat{\mathbf{C}}/\langle\sigma_1\rangle$ is the upper half-plane U and $\hat{\mathbf{C}}/\langle\sigma_2\rangle$ is the real projective plane. Hence the case of genus 0 real curves is completely elementary.

The reminding part of this section is devoted to the genus 1 case. Let Σ be a compact and oriented surface of genus 1 and X a complex structure on Σ. The Riemann surface X is a torus. It can always be represented in the form

$$X = \mathbf{C}/L$$

where L is a lattice which may be taken to be $L = \mathbf{Z} + \mathbf{Z}\cdot\omega$, $\operatorname{Im}\omega > 0$.

Let $\sigma\colon \Sigma \to \Sigma$ be an orientation reversing involution, let X be a complex structure on Σ and let $L = \mathbf{Z} + \mathbf{Z}\cdot\omega$ be a lattice corresponding to X. Let $p\colon \mathbf{C} \to (\Sigma, X)$ be the corresponding covering map.

Tracing through all the definitions it follows then that the lattice $\bar{L} = \mathbf{Z} + \mathbf{Z}\cdot(-\bar{\omega})$ is a lattice corresponding to $(\Sigma, \sigma^*(X))$ and that the universal covering map of the Riemann surface $(\Sigma, \sigma^*(X))$ is $\kappa\circ p\circ\sigma\colon \mathbf{C} \to (\Sigma, \sigma^*(X))$ where $\kappa(z) = -\bar{z}$.

For a lattice $L = \mathbf{Z} + \mathbf{Z}\cdot\omega$ consider the numbers

$$g_2 = 60 \sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \frac{1}{\lambda^4}$$

$$g_3 = 140 \sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \frac{1}{\lambda^6}.$$

Let

$$j(X) = j(\omega) = \frac{1728\cdot g_2^3}{g_2^3 - 27g_3^2}$$

for a torus $X = \mathbf{C}/L$, $L = \mathbf{Z} + \mathbf{Z}\cdot\omega$.

The function $j(\omega)$ is the famous elliptic modular function. Its most important property is its invariance under $\mathrm{SL}(2, \mathbf{Z})$.

One can show that, for $\omega_1, \omega_2 \in U$, $j(\omega_1) = j(\omega_2)$ if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$$

such that

$$\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}.$$

One can further show that $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\cdot\omega_1) \approx \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\cdot\omega_2)$ if and only if $j(\omega_1) = j(\omega_2)$ [26, page 91].

It follows that j defines a mapping $j\colon \mathscr{M}^1 \to \mathbf{C}$ which is actually a bijection. We have, furthermore, the commutative diagram:

$$\begin{tikzcd} U \arrow[d, "\pi"] \arrow[dr, "j"] & \\ \mathscr{M}^1 \arrow[r, "j"] & \mathbf{C} \end{tikzcd} \tag{2}$$

This is the classical construction for the moduli space of tori. The mapping $j\colon U \to \mathbf{C}$ is a smooth covering map whose cover group is SL(2, **Z**). This is the elliptic modular group. Its fundamental domain is

$$W = \{\omega \in U \mid -\tfrac{1}{2} < \operatorname{Re}\omega \leqslant \tfrac{1}{2}, |\omega| \geqslant 1 \text{ and } |\omega| > 1 \text{ for } \operatorname{Re}\omega \leqslant 0\}.$$

Consider now the following mappings:

$$\kappa\colon U \to U, \kappa(\omega) = -\bar{\omega}$$
$$\sigma^*\colon \mathscr{M}^1 \to \mathscr{M}^1, [X] \mapsto [\sigma^*(X)]$$

and

$$\tau\colon \mathbf{C} \to \mathbf{C}, \tau(z) = \bar{z}.$$

On basis of the construction and the commutative diagram (2) we have: $\pi \circ \kappa = \sigma^* \circ \pi$ and $j \circ \kappa = \tau \circ j$.

We conclude therefore that the involution σ^* of $\mathscr{M}^1$ is simply complex conjugation on $j(\mathscr{M}^1)$. The fixed-point set of the complex conjugation is the real axes. Therefore

$$j(\mathscr{M}^1_{\mathbf{R}}) \subset \mathbf{R}. \tag{3}$$

Consider, on the other hand, the mapping $j: U \to \mathbf{C}$ and the corresponding mapping $\kappa: U \to U, \kappa(\omega) = -\bar{\omega}$. From the equation $j \circ \kappa = \tau \circ j$ we conclude that the inverse image of the real axes under the restriction $j: W \to \mathbf{C}$ of j to the closure of the fundamental domain W of the elliptic modular group $\mathrm{SL}(2, \mathbf{Z})$ is the following union

$$j^{-1}(\mathbf{R}) = \{\omega \mid \operatorname{Re}\omega = 0, \operatorname{Im} \geqslant 1\} \cup \{\omega \mid |\omega| = 1, 0 \leqslant \operatorname{Re}\omega \leqslant \tfrac{1}{2}\}$$
$$\cup \{\omega \mid \operatorname{Re}\omega = \tfrac{1}{2}, |\omega| \geqslant 1\}.$$

Alling and Greenleaf have computed ([3, pp. 57–66]), on the other hand, that every $\omega \in j^{-1}(\mathbf{R})$ gives rise to a torus $X = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\cdot\omega)$ which carries antiholomorphic involutions. We conclude therefore that we have equality in (1). This observation proves the following result:

THEOREM 3.1. *The moduli space of real algebraic curves of genus* 1, $\mathscr{M}^1_{\mathbf{R}}$, *is connected and equals the real part of the moduli space* $\mathscr{M}^1$.

Theorem 3.1 is actually well known in algebraic geometry and easy to prove. For $j \neq 1728$, the j-invariant of the curve $y^2 = 4x^3 - ax - a, a = 27j/(j - 1728)$, equals this given j. For $j = 1728$, take the curve $y^2 = 4x^3 - x$.

The case of real algebraic curves of genus 1 is therefore quite simple. Their moduli space is simply the real line which is, of course, also a real analytic space and a semialgebraic variety. Similar statements hold also for real curves of higher genera. They are not any more trivial or obvious as in the present case and they will be proven in the sequel.

4. Hyperelliptic real curves

Hyperelliptic algebraic curves form a particular case as well. First of all every curve of genus 2 is hyperelliptic. To avoid certain technical difficulties, let us consider first curves of genus 2.

The moduli space $\mathscr{M}^2$ has been completely analyzed by Igusa (cf. [9]). It is necessary to recall briefly his construction in order to see how that can be applied to the study of the moduli space of real curves of genus 2.

A complex algebraic curve C of genus 2 is a double covering of the Riemann sphere ramified at 6 points. That gives us a presentation of the curve C by an equation

$$C: y^2 = (x - \lambda_1)\cdot(x - \lambda_2)\cdot \cdots \cdot(x - \lambda_6). \tag{4}$$

Here the λ_j's are the branch points of the covering $C \to \hat{\mathbf{C}}$ and they have to be all distinct. The polynomial (4) is real if and only if the set of the λ_j's in (4) is invariant under the complex conjugation.

Gross and Harris ([8]) have shown, on the other hand, that every real curve of genus 2 is always complex isomorphic to a curve defined by a real polynomial of the type (4).

One should observe quite generally that any hyperelliptic real algebraic curve of genus $g, g > 1$, which has real points, is always *real* isomorphic to a curve defined by

$$y^2 = \pm(x - \lambda_1)\cdot(x - \lambda_2)\cdot\cdots\cdot(x - \lambda_{2g+2}), \tag{5}$$

where the set of the λ_j's is invariant under the complex conjugation ([8, Proposition 6.1, page 170]). The mapping $(x, y) \mapsto (x, y\sqrt{-1})$ defines an isomorphism between the curves $y^2 = +(x - \lambda_1)\cdot(x - \lambda_2)\cdot\cdots\cdot(x - \lambda_{2g+2})$ and $y^2 = -(x - \lambda_1)\cdot(x - \lambda_2)\cdot\cdots\cdot(x - \lambda_{2g+2})$. Therefore these two curves define always the same point in $\mathcal{M}^g_{\mathbf{R}}$ even though they are not real isomorphic to each other.

By (4) we get a continuous surjective mapping

$$\pi\colon (\hat{\mathbf{C}}^6 \setminus \{\text{all diagonals}\})/\{\text{permutations of coordinates}\} \to \mathcal{M}^2.$$

Use the notation

$$A = (\hat{\mathbf{C}}^5 \setminus \{\text{all diagonals}\})/\{\text{permutations of coordinates}\}.$$

Let $\tau\colon A \to A$ denote the mapping defined by taking the complex conjugates of the coordinates, and let A_τ be set of the fixed-points. On A we may consider the function

$$m\colon A \to \mathbf{Z}, m(\lambda_1, \ldots, \lambda_6) = \text{the number of real } \lambda_j\text{s}.$$

On A_τ the function m takes the values 0, 2, 4 and 6. It is clear that m is constant on components of A_τ and that it separates components of A_τ. We conclude that A_τ has four connected components; call them B_0, B_2, B_4 and B_6. Recall that there are five different topological types of real curves of genus 2.

The projection $\pi\colon A \to \mathcal{M}^2$ being continuous, the sets $\pi(B_j) \subset \mathcal{M}^2_{\mathbf{R}}$, $j = 0, 2, 4, 6$, are also connected. On the other hand, since $\mathcal{M}^2_{\mathbf{R}} = \bigcup_{k=0}^3 \pi(B_{2k})$ we conclude that $\mathcal{M}^2_{\mathbf{R}}$ has at most four connected components. This upper limit for the number of components of the moduli space $\mathcal{M}^2_{\mathbf{R}}$ reflects the fact that the points of $\mathcal{M}^g_{\mathbf{R}}$ are *complex* isomorphism classes of real curves. Actually we have:

THEOREM 4.1. *The moduli space of smooth real algebraic curves of genus* 2 *is connected.*

Proof. In odrer to show that $\mathcal{M}^2_{\mathbf{R}}$ is connected we shall construct points where the various parts $\pi(B_j)$ intersect.

To that end observe that $\pi(\{\lambda_1, \ldots, \lambda_6\}) = \pi(\{\lambda'_1, \ldots, \lambda'_6\})$ if and only if there exists an automorphism g of $\hat{\mathbf{C}}$ such that $g(\{\lambda_j\}) = \{\lambda'_j\}$.

After this observation the proof becomes rather mechanical. We divide it into three steps.

First step: $\pi(B_0) \cap \pi(B_6)$ is not empty. Take for instance

$$C_0\colon y^2 = (x - i)(x - 2i)(x - 3i)(x + i)(x + 2i)(x + 3i)$$

and

$$C_6\colon y^2 = (x - 1)(x - 2)(x - 3)(x + 1)(x + 2)(x + 3)$$

where i is the imaginary unit. Then C_k represents a point in $\pi(B_k)$ for $k = 0, 6$. The rotation of the complex plane by $\pi/2$ maps $\{1, 2, 3, -1, -2, -3\}$ onto $\{i, 2i, 3i, -i, -2i, -3i\}$. The curves C_0 and C_6 are therefore, isomorphic, proving that $\pi(B_0)$ and $\pi(B_6)$ intersect.

The two remaining steps are similar. The same argument proves that the curves

$$C_4\colon y^2 = (x - i)(x + i)(x - 1)(x - 2)(x + 1)(x + 2)$$

and

$$C_2\colon y^2 = (x + 1)(x - 1)(x - i)(x - 2i)(x + i)(x + 2i)$$

are isomorphic. Hence $\pi(B_2)$ and $\pi(B_4)$ intersect.

Finally we conclude that the curves

$$C_2\colon y^2 = (x - 1)(x - 1 + i)(x - 1 - i)(x + 1)(x + 1 + i)(x + 1 - i)$$

and

$$C_0\colon y^2 = (x - 1 + i)(x - 1 - i)(x + i)(x - i)(x + 1 + i)(x + 1 - i)$$

are isomorphic which then proves that $\pi(B_0)$ and $\pi(B_2)$ intersect. This concludes the proof.

A general hyperelliptic curve of genus g can be written in the form

$$C\colon y^2 = (x - \lambda_1)\cdot(x - \lambda_2)\cdot \cdots \cdot(x - \lambda_{2g+2}). \tag{6}$$

A real hyperelliptic curve *with real points* is always isomorphic to a curve of the form (6) where the set of λ_j's is invariant under the complex conjugation(cf. e.g. [8, Propositions 6.1 and 6.2]). A hyperelliptic real curve without real points is either isomorphic to a curve of the form (6) or to a curve of the type

$$\begin{aligned} u^2 + v^2 &= -1 \\ y^2 &= f(u, v), \quad \deg f = g + 1 \end{aligned}$$

where f is a real polynomial.

We may, nevertheless, repeat the above arguments for these general hyperelliptic real curves to prove:

THEOREM 4.2. *Real hyperelliptic curves form a connected subset of the moduli space of smooth complex curves of genus g, $g > 1$.*

The above theorem can be proven repeating the computations of Theorem 4.1. The considerations here are only a little bit more complicated and require some technical work.

In [8, Page 171] Gross and Harris make the remark that the *real* moduli space of real algebraic curves of genus f, $g > 3$, has whole components that do not contain hyperelliptic curves. In view of the above result, only one component of the *complex* moduli space $\mathscr{M}^g_{\mathbf{R}}$ of real algebraic curves has hyperelliptic curves.

Gross and Harris prove their observation by considering the possible topological types of real hyperelliptic curves. It turns out that only certain topological types of real curves can correspond to hyperelliptic curves. This argument goes actually back to Klein ([11, §5, pages 12–16]).

5. Strong deformation spaces

At this point it is necessary to review a construction presented by Lipman Bers in [5]. Recall first that *a surface with nodes* Σ is a Hausdorff space for which every point has a neighborhood homeomorphic either to the open disk in the complex plane or to

$$N = \{(z, w) \in \mathbf{C}^2 \mid zw = 0, |z| < 1, |w| < 1\}.$$

A point p of Σ is *a node* if every open neighborhood of p contains an open set homeomorphic to N. Component of the complement of the nodes of Σ is *a part* of Σ. *The arithmetic genus* of a compact surface with nodes Σ is the genus of the compact smooth surface obtained by thickening each node of Σ.

A stable surface with nodes is a compact surface with nodes for which every part has a negative Euler characteristic. *A stable Riemann surface with nodes* is simply a stable surface Σ together with a complex structure X for which each component X_j of the complement of the nodes of $X = (\Sigma, X)$ is obtained by deleting a certain number p_j points from a compact Riemann surface of genus g_j. These components X_j are *the parts* of $X = (\Sigma, X)$. The stability condition simply means that

$$2 - 2g_j - p_j < 0.$$

If X is a stable Riemann surface, then every part X_j of X is a hyperbolic Riemann surface, i.e., every X_j carries a canonical metric of constant curvature -1. This metric is obtained from the non-euclidean metric of the upper half-plane (or the unit disk) via uniformization. When we later speak of lengths of curves on parts of a stable Riemann surface, we always refer to this canonical hyperbolic metric.

A stable surface Σ of genus g can have at most $3g - 3$ nodes. We say that Σ is *terminal* if it has this maximal number of nodes.

A stable Riemann surface is a stable topological surface equipped with a complex structure as described above. When defining topological concepts for stable Riemann surfaces it is sometimes convenient simply to forget the complex structure and consider only the topological surface Σ instead of the stable Riemann surface $X = (\Sigma, X)$. That is done in the sequel.

A strong deformation of a surface with nodes Σ_1 onto a surface with nodes Σ_2 is a continuous surjection $\Sigma_1 \to \Sigma_2$ such that the following holds:

- the image of each node of Σ_1 is a node of Σ_2,
- the inverse image of a node of Σ_2 is either a node of Σ_1 or a simple closed curve on a part of Σ_1,
- the restriction of $\Sigma_1 \to \Sigma_2$ to the complement of the inverse image of the nodes of Σ_2 is an orientation preserving homeomorphism onto the complement of the nodes of Σ_2.

Let X and X' be stable Riemann surfaces and Σ a stable topological surface with nodes. We say that two strong deformations $f\colon X \to \Sigma$ and $f'\colon X' \to \Sigma$ are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \Sigma \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow{f'} & \Sigma \end{array}$$

where φ and ψ are bijective homomorphisms homotopic to an isomorphism and to the identity, respectively.

If $f\colon X \to \Sigma_1$ and $g\colon \Sigma_1 \to \Sigma$ are both strong deformations, then $g \circ f\colon X \to \Sigma$ is a strong deformation as well and the equivalence class of $g \circ f$ depends only on the classes of g and f.

Let now $\mathscr{D}(\Sigma)$ denote the set of equivalence classes of strong deformations $X \to \Sigma$ of Σ. This is *the strong deformation space* of Σ.

Next we have to introduce a topology on $\mathscr{D}(\Sigma)$. That is done by lengths of geodesic curves. Here we need more notations. Let X be a stable Riemann surface and let α be a closed curve that does not pass through nodes of X. Then we denote by $\ell_\alpha(X)$ the length of the geodesic curve on X homotopic to α. Lengths are measured in the hyperbolic metric of parts of X.

The topology on $\mathscr{D}(\Sigma)$ can now be defined declaring a set A *open* if, for each point $[f: X \to \Sigma] \in \mathscr{D}(\Sigma)$, there exists a finite set of closed curves $\alpha_1, \dots, \alpha_m$ on parts of X and an $\varepsilon > 0$ such that whenever the strong deformation $g: Y \to X$ satisfies

$$|\ell_{\alpha_j}(X) - \ell_{g^{-1}(\alpha_j)}(Y)| < \varepsilon \text{ for } j = 1, \dots, m$$
$$|\ell_{g^{-1}(N)}(Y)| < \varepsilon \text{ for each node } N \text{ of } X$$

then the point $[f \circ g: Y \to \Sigma]$ belongs to the set A. This is the definition given by Bers.

Let Σ be a stable topological surface and $f: \Sigma \to \Sigma$ a homeomorphism. If X is any stable Riemann surface, then $\bar{X}$ is the complex conjugate of X, i.e., $\bar{X}$ is the stable Riemann surface obtained from X replacing each holomorphic local variable $z(p)$ of X by its complex conjugate $\overline{z(p)}$. The identity mapping is then an antiholomorphic mapping $\kappa: \bar{X} \to X$.

The set $\mathrm{Homeo}_\pm(\Sigma)$ consists of homeomorphic self-mappings of Σ, that are either everywhere orientation preserving or everywhere orientation reversing. The group $\mathrm{Homeo}_\pm(\Sigma)$ acts on $\mathscr{D}(\Sigma)$ in the following way. An orientation preserving homeomorphism $h: \Sigma \to \Sigma$ induces the mapping

$$h^*: \mathscr{D}(\Sigma) \to \mathscr{D}(\Sigma),\ [f: X \to \Sigma] \mapsto [h \circ f: X \to \Sigma].$$

An orientation reversing $\sigma: \Sigma \to \Sigma$ induces a mapping

$$\sigma^*: \mathscr{D}(\Sigma) \to \mathscr{D}(\Sigma),\ [f: X \to \Sigma] \mapsto [\sigma \circ f \circ \kappa: \bar{X} \to \Sigma].$$

It is obvious by the definitions that all elements of $\mathrm{Homeo}_\pm(\Sigma)$ define homeomorphisms of $\mathscr{D}(\Sigma)$ onto itself.

6. Strong deformations of real curves

Let us now consider the situation where the stable surface Σ admits an orientation reversing involution σ. If (Σ, X) is a stable Riemann surface for which the mapping $\sigma: (\Sigma, X) \to (\Sigma, X)$ is antiholomorphic, then the stable Riemann surface (Σ, X) is actually a stable algebraic curve and it is isomorphic to a curve defined by real polynomials.

In this section our point of view is rather topological. We call the involution σ *a symmetry* of Σ, and we consider strong deformations of such symmetric surfaces. This is the same thing as considering strong deformations of stable real algebraic curves with nodes.

In Section 5 we defined the action of orientation reversing or preserving self-mappings of Σ on the strong deformation space. In particular, the symmetry σ acts on $\mathscr{D}(\Sigma)$ and we have:

THEOREM 6.1. *The fixed-point set of the action σ^* of the orientation reversing involution $\sigma\colon \Sigma \to \Sigma$ consists of strong deformations $[f\colon X \to \Sigma]$ where the Riemann surface X admits an orientation reversing antiholomorphic involution. In other words, the set $\mathscr{D}(\Sigma)_{\sigma^*}$ consists of real algebraic curves.*

Proof. To prove this result is a simple matter of tracing through all the definitions. Assume that a point $[f\colon X \to \Sigma]$ remains fixed under the action of σ^*. Recall that $\kappa\colon \bar{X} \to X$ is the antiholomorphic mapping induced by the identity mapping between the Riemann surface X and its complex conjugate $\bar{X}$.

We have now the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & \Sigma \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \bar{X} & \xrightarrow{\sigma\circ f\circ\kappa} & \Sigma \end{array} \tag{7}$$

where φ and ψ are bijective homeomorphisms homotopic to an isomorphism and to the identity, respectively.

It may be better to think of this diagram in the form

$$\begin{array}{ccc} X & \xrightarrow{f} & \Sigma \\ \downarrow{\scriptstyle\kappa\circ\varphi} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & \Sigma \end{array} \tag{8}$$

where $\xi = \kappa \circ \varphi$ is homotopic to an antiholomorphic mapping $\tau\colon X \to X$ and $s\colon \Sigma \to \Sigma$ is homotopic to the involution $\sigma\colon \Sigma \to \Sigma$. Theorem is shown if we prove that τ is an involution.

On basis of our construction, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \Sigma \\ \downarrow{\scriptstyle\xi^2} & & \downarrow{\scriptstyle s^2} \\ X & \xrightarrow{f} & \Sigma \end{array} \tag{9}$$

Here $s^2\colon \Sigma \to \Sigma$ is homotopic to the identity.

Let $N_1, \ldots, N_p$ be the nodes of Σ that get thickened in the strong deformation $f: X \to \Sigma$, i.e., we suppose that the inverse images of the nodes $N_j, j = 1, \ldots, p$, of Σ are simple closed curves on X and that the inverse images of all other nodes of Σ are nodes of X.

Let $d_1, \ldots, d_p$ be the Dehn twists around the simple curves

$$f^{-1}(N_1), \ldots, f^{-1}(N_p),$$

respectively. It follows, from the above commutative diagram, that τ^2 is homotopic to a mapping in the group $D = \langle d_1, \ldots, d_p \rangle$ generated by the Dehn twists (cf. e.g. [2, Theorem 2, page 93]).

The group D is freely generated by the Dehn twists d_j. Furthermore, D does not have torsion. Therefore D does not contain any elements of finite order, save the identity.

The mapping τ^2 is, on the other hand, a holomorphic mapping $X \to X$. We conclude that τ^2 is of finite order. Since τ^2 is homotopic to a mapping in D, we finally deduce that τ^2 must be the identity. This shows that the Riemann surface X admits antiholomorphic involutions and proves the theorem.

The above reasoning shows that for all points $[f: X \to \Sigma]$, which remain fixed under the involution σ^* of $\mathscr{D}(\Sigma)$, the Riemann surface X admits antiholomorphic involutions. The topological type of these involutions is not fixed, however.

Applying the arguments of [21] one can easily show the following fact. If the symmetry σ of the stable surface Σ fixes m of the nodes of Σ, then the fixed-point set $\mathscr{D}(\Sigma)_{\sigma^*}$ consists $m + 1$ different topological types of smooth real curves C.

The point here is that if a node N of Σ is fixed by σ, then we can thicken that node in such a way that the involution σ of the stable surface Σ induces an involution of the thickened surface such that this involution either fixes the curve corresponding to the node N pointwise or does not.

7. Degeneration of real curves

In this section we consider degenerations of smooth real curves. More precisely, we prove the following result.

THEOREM 7.1. *Let $C_j, j = 1, 2, \ldots,$ be smooth real curves of genus g. Let $p_j \in \mathscr{M}^g$ be the corresponding points in the moduli space. Assume that the sequence $p_j, j = 1, 2, \ldots$ converges to a point $p = [C] \in \bar{\mathscr{M}}^g$. Then C is isomorphic to a real curve.*

Proof. Let Σ be a smooth topological genus g surface. The curves C_j are real. This means that the corresponding Riemann surfaces (Σ, X_j) are symmetric. There are only finitely many different topological types of symmetries of

Riemann surfaces of genus g. Therefore, by passing to a subsequence if necessary, we may assume that all the Riemann surfaces (Σ, X_j) admit the same symmetry $\sigma\colon \Sigma \to \Sigma$. That is, for each index j, the mapping $\sigma\colon (\Sigma, X_j) \to (\Sigma, X_j)$ is antiholomorphic.

Let (Σ^*, X^*) be a stable Riemann surface representing the point $p = \lim_{j\to\infty} p_j \in \bar{\mathcal{M}}^g$. The strong deformation space $\mathcal{D}(\Sigma^*)$ is a manifold and a covering of an open neighborhood of the point $p = [(\Sigma^*, X^*)]$. The identity mapping of the topological surface Σ^* induces the strong deformation $h\colon X^* \to \Sigma^*$. The point $[h\colon X^* \to \Sigma^*] \in \mathcal{D}(\Sigma^*)$ lies over the point $p \in \bar{\mathcal{M}}^g$.

The strong deformation space and the moduli space are both locally compact, of course, and a compact subset $M(p)$ of $\mathcal{D}(\Sigma^*)$ covers a compact neighborhood of the point p.

Each strong deformation $f\colon \Sigma \to \Sigma^*$ induces a mapping $f^*\colon T(\Sigma) \to \mathcal{D}(\Sigma^*)$, $[X] \mapsto [f\colon X \to \Sigma^*]$.

Let $\pi^*\colon \mathcal{D}(\Sigma^*) \to \bar{\mathcal{M}})^g$ denote the projection that takes the point $[Y \to \Sigma^*]$ to the isomorphism class of the stable Riemann surface Y. Let, furthermore, $\pi\colon T(\Sigma) \to \bar{\mathcal{M}}^g$ be the corresponding projection from the Teichmüller space.

Then clearly, $\pi = \pi^* \circ f^*$ for any choice of the strong deformation f.

Since $\pi(X_j) \to p$ as $j \to \infty$, we can assume that the strong deformation $f\colon \Sigma \to \Sigma^*$ and the complex structures X_j are so chosen that $f^*(X_j) \in M(p)$ for sufficiently large indices j. This follows from the topological properties of the coverings $\mathcal{D}(\Sigma^*) \to \bar{\mathcal{M}}^g$ and $T(\Sigma) \xrightarrow{f^*} \mathcal{D}(\Sigma^*)$.

Since $M(p)$ is compact, we can suppose, by passing again to a subsequence if necessary, that the sequence $f^*([X_j])$ converges in $\mathcal{D}(\Sigma^*)$ to the point $[h\colon X^* \to \Sigma^*]$ where h is the identity mapping.

Recall that, for each j, the mapping $\sigma\colon (\Sigma, X_j) \to (\Sigma, X_j)$ is antiholomorphic. It is, in particular, an isometry of the corresponding hyperbolic metric.

Now, if the length of the geodesic curve homotopic to a curve α on (Σ, X_j) converges to 0 as $j \to \infty$ then the same holds also for the curve $\sigma(\alpha)$. We infer that if the curve α on Σ gets squeezed to a node as $j \to \infty$ then also the curve $\sigma(\alpha)$ gets squeezed to a node as $j \to \infty$.

We conclude, therefore, that the nodes of Σ^* are symmetric with respect to σ, or that the involution σ induces an involution $\sigma^*\colon \Sigma^* \to \Sigma^*$. The diagram

$$\begin{array}{ccc} (\Sigma, X_j) & \xrightarrow{f} & (\Sigma^*) \\ \downarrow{\sigma} & & \downarrow{\sigma^*} \\ (\Sigma, X_j) & \xrightarrow{f} & (\Sigma^*) \end{array}$$

is then commutative.

Each mapping $\sigma\colon (\Sigma, X_j) \to (\Sigma, X_j)$ is an isometry of the hyperbolic metric, $\sigma^* \circ f = f \circ \sigma$, and the sequence $[f\colon X_j \to \Sigma^*]$ converges to $[h\colon X^* \to \Sigma^*]$ as

$j \to \infty$. We conclude, by the definition of the topology of $\mathcal{D}(\Sigma^*)$, that the mapping $\tau = h^{-1} \circ \sigma \circ h\colon X^* \to X^*$ maps closed hyperbolic geodesics of X^* onto geodesics of the same length. This actually means that the above mapping τ is antiholomorphic. It is also an involution, since σ is an involution. This means that X^* is actually isomorphic to a real algebraic curve proving the theorem.

The above theorem simply means that a degenerating sequence of symmetric Riemann surfaces converges to a symmetric Riemann surface in the moduli space of stable Riemann surfaces.

The above argument is rather clumsy. Another way for proving this would be to consider decompositions of symmetric Riemann surfaces into pairs of pants. It turns out that symmetric Riemann surfaces admit always *symmetric* decompositions into pairs of pants by curves whose lengths are bounded by a constant depending only on the genus of the surface. This technical lemma allows one to use the Fenchel–Nielsen coordinates to study limits of converging sequences of symmetric Riemann surfaces. Details and a completely different proof for the above result can be found in [20].

Theorem 7.1 is a technical result that is necessary for the considerations in the sequel. It looks only a little bit more complicated than saying that if a sequence of real numbers converges in the complex plane, then also the limit must be real. And it can be shown by resorting to almost any parametrization of the Teichmüller space or the moduli space. The proof, however, is technically complicated in whatever setting we choose to work.

8. Real structure of the moduli spaces

All the moduli spaces of stable complex curves carry a canonical real structure. That is, they all have an antiholomorphic involution which is in some sense canonically defined.

That involution can most easily be defined in terms of algebraic geometry. The involution simply takes the isomorphism class of a complex curve onto that of its complex conjugate. One could resort to a parametrization of the moduli space $\bar{\mathcal{M}}^g$ given by Mumford in [18, Method II, page 30] to show that this mapping is actually antiholomorphic.

We may, indeed, embed the moduli space $\bar{\mathcal{M}}^g$ into a projective space in such a way that the above involution is simply the complex conjugation in the ambient projective space. This is, after all, quite natural, since we know, from Geometric Invariant Theory, that the moduli space $\bar{\mathcal{M}}^g$ is a projective variety defined over the field of rational numbers.

For our purposes, however, a more analytic approach is called for. To that end we have to consider first the moduli space of smooth Riemann surfaces of genus g, $\mathcal{M}^g$, and then extend the considerations to its compactification $\bar{\mathcal{M}}^g$.

Let Σ be a fixed oriented, compact and smooth topological surface of genus g. Let σ be an orientation reversing involution of Σ. As was remarked already in Section 2, the involution $\sigma\colon \Sigma \to \Sigma$ induces an antiholomorphic involution σ^* of the Teichmüller space T^g (a detailed description of this involution can be found in [24, 5.10]). The involution $\sigma^*\colon T^g \to T^g$ induces then an involution $\sigma^*\colon \mathscr{M}^g \to \mathscr{M}^g$ which is antiholomorphic as well. This involution of the moduli space does not depend on the choice of the particular involution $\sigma\colon \Sigma \to \Sigma$. In that sense this involution is canonical.

Next we have to extend this involution to an involution of the moduli space $\bar{\mathscr{M}}^g$ of stable Riemann surfaces of genus g. That can be done considering strong deformation spaces, for instance.

Now let $p = [(\Sigma^*, X^*)]$ be an arbitrary point in $\bar{\mathscr{M}}^g \backslash \mathscr{M}^g$. (Σ^*, X^*) is then a stable Riemann surface with some nodes. To extend the involution σ^* to this point p consider a strong deformation $f\colon \Sigma \to \Sigma^*$.

Form the stable surface with nodes Σ_σ^* in the following way. If the strong deformation f squeezes a closed curve α on Σ to a node, then to form the surface Σ_σ^*, squeeze the curve $\sigma(\alpha)$ on Σ to a node of Σ_σ^*. Do this to all closed curves of Σ that get mapped onto a node of Σ^* by the strong deformation f.

This is how you get the stable surface Σ_σ^*. The construction gives also immediately a strong deformation $g\colon\Sigma \to \Sigma_\sigma^*$ and an orientation reversing mapping $\sigma\colon \Sigma^* \to \Sigma_\sigma^*$ such that the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & \Sigma^* \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \Sigma & \xrightarrow{g} & \Sigma_\sigma^* \end{array} \tag{10}$$

commutes.

Another way to define the new objects for the diagram (10) would be the following. Let Σ_σ^* be the oriented stable surface with nodes which is obtained from the surface Σ^* by reversing the orientation. The identity mapping is an orientation reversing mapping $\Sigma^* \to \Sigma_\sigma^*$. Denote this mapping again by σ. Then $g = \sigma \circ f \circ \sigma^{-1}$ is a strong deformation $\Sigma \to \Sigma_\sigma^*$ and the diagram (10) commutes.

The mapping $\sigma\colon \Sigma^* \to \Sigma_\sigma^*$ induces now the mapping

$$\sigma^*\colon \mathscr{D}(\Sigma^*) \to \mathscr{D}(\Sigma_\sigma^*),\ [f\colon Y \to \Sigma^*] \mapsto [\sigma^*(Y) \xrightarrow{\sigma} Y \xrightarrow{f} \Sigma^* \xrightarrow{\sigma} \Sigma_\sigma^*], \tag{11}$$

which is clearly a homeomorphism. Recall that, for a complex structure Y of the surface Σ, the complex structure $\sigma^*(Y)$ of Σ is defined requiring that the mapping $\sigma\colon (\Sigma, \sigma^*(Y)) \to (\Sigma, Y)$ be antiholomorphic.

The extension of the involution $\sigma^*\colon \mathscr{M}^g \to \mathscr{M}^g$ to a self-mapping of the compactified moduli space $\bar{\mathcal{M}}^g$ is defined in terms of the equation (11) and in terms of the following commutative diagram.

$$\begin{array}{ccc} \mathcal{D}(\Sigma^*) & \xrightarrow{\sigma^*} & \mathcal{D}(\Sigma^*_\sigma) \\ \downarrow & & \downarrow \\ \bar{\mathcal{M}}^g & \xrightarrow{\sigma^*} & \bar{\mathcal{M}}^g \end{array}$$

It follows then, from the continuity of all mappings (11) that this extension of σ^* to a self-mapping of $\bar{\mathcal{M}}^g$ is continuous as well.

It is an exercise to prove that this extension of σ^* is injective and therefore a homeomorphism.

We have now defined a homeomorphic self-mapping $\sigma^*: \bar{\mathcal{M}}^g \to \bar{\mathcal{M}}^g$. We know that the restriction of this mapping to $\mathcal{M}^g \subset \bar{\mathcal{M}}^g$ is antiholomorphic. The set $\bar{\mathcal{M}}^g \backslash \mathcal{M}^g$ is a divisor. Therefore the possible singularities of $\sigma^*: \bar{\mathcal{M}}^g \to \bar{\mathcal{M}}^g$ at points of $\bar{\mathcal{M}}^g \backslash \mathcal{M}^g$ are removable. This follows from the Riemann Extension Theorem (cf. e.g. [1, Page 108]). We conclude that the mapping $\sigma^*: \bar{\mathcal{M}}^g \to \bar{\mathcal{M}}^g$ is an antiholomorphic involution.

An antiholomorphic involution of a complex space is called a real structure of that space. The above considerations show that the moduli space $\bar{\mathcal{M}}^g$ carries a canonical real structure. It seems likely that, at least in the general case, $\bar{\mathcal{M}}^g$ does not have any other real structures. I do not have, however, any proof for this.

9. Moduli of smooth real curves

In Section 8 we saw that all the moduli spaces $\bar{\mathcal{M}}^g$ carry real structures σ^*. It follows also that

$$\mathcal{M}^g_{\mathbf{R}} \subset (\mathcal{M}^g)_{\sigma^*} \quad \text{and} \quad \bar{\mathcal{M}}^g_{\mathbf{R}} \subset (\bar{\mathcal{M}}^g)_{\sigma^*}. \tag{12}$$

This is more or less immediate by the definitions. Here $(\mathcal{M}^g)_{\sigma^*}$ and $(\bar{\mathcal{M}}^g)_{\sigma^*}$ denote the corresponding sets of fixed points of the canonical involution σ^* of the moduli space. This inclusion is proper provided that $g > 1$. This was first noticed by C. Earle (cf. [6]).

In order to describe the moduli space $\mathcal{M}^g_{\mathbf{R}}$, of smooth real curves of genus g, consider a point $p = [X] \in \mathcal{M}^g$, which remains fixed under the involution σ^*. This means that there is an isomorphism of Riemann surfaces $\alpha: X \to \sigma^*(X)$. The mapping $\sigma: \sigma^*(X) \to X$ is, on the other hand, antiholomorphic by the definition of $\sigma^*(X)$. We conclude that the mapping $\tau = \sigma \circ \alpha: X \to X$ is antiholomorphic.

Then the mapping $g = \tau^2$ is a holomorphic self-mapping of X. If the point $p = [X]$ is a smooth point of the moduli space $\mathcal{M}^g$, then, assuming that $g > 3$, the Riemann surface X does not have non-trivial holomorphic self-mappings. We conclude that in such a case $g = \tau^2$ is the identity mapping of X. This means that X is actually a real algebraic curve.

We conclude by the preceding considerations that regular points of $(\mathcal{M}^g)_{\sigma^*}$ belong to the moduli space of real curves of genus g.

Refining the above argument we have shown in [23, Theorem 6, page 123] that $\mathcal{M}^g_{\mathbf{R}}$ is actually the closure of the regular part of $(\mathcal{M}^g)_{\sigma^*}$.

Now $\mathcal{M}^g$ is a complex space, $\sigma^*\colon \mathcal{M}^g \to \mathcal{M}^g$ is a *real structure* of that space and the closure of the regular part of $(\mathcal{M}^g)_{\sigma^*}$ is called *the quasiregular real part* of $\mathcal{M}^g$. This is the terminology of Aldo Andreotti and Per Holm ([4]). The quasiregular real part can be characterized as the set of those points of the real part where the local dimension of the real part is as large as possible.

The moduli space $\mathcal{M}^g$ is also a quasiprojective variety defined over the field of rational numbers. Its real part is, of course, a quasiprojective real algebraic variety. It is well known that the closure of the regular points of such a real algebraic variety is a semialgebraic variety (i.e. a set in a projective space defined in terms of polynomial equations and inequalities).

In [23, Theorem 5] we have also shown that $\mathcal{M}^g_{\mathbf{R}}$ is a real analytic space for all values of g. Therefore we have the following result.

THEOREM 9.1. *The moduli space of real algebraic curves of genus g, $\mathcal{M}^g_{\mathbf{R}}$, is a real analytic space. Provided that $g > 3$, $\mathcal{M}^g_{\mathbf{R}}$ is the quasiregular real part of $\mathcal{M}^g$ and hence also a semialgebraic variety.*

The case of genus 2 and genus 3 curves offer technical difficulties because in those cases the singularities of the moduli space $\mathcal{M}^g$ do not correspond to the Riemann surfaces with non-trivial automorphisms. I suppose that Theorem 9.1 holds even in that case but I have not worked out the details.

10. Moduli of stable real curves

In this section we shall extend the preceeding considerations to stable real curves. That extension is based on Theorem 7.1 and on a concrete continuous thickening of the nodes of a stable curve.

Let (Σ, X) be a stable Riemann surface that represents a point $p \in \bar{\mathcal{M}}^g$. Assume that the surface Σ has nodes, let them be $N_1, N_2, \ldots, N_m$. For our purposes it is necessary to obtain a concrete and continuous way to thicken these nodes. One such thickening has been given by Fay (cf. e.g. [7]).

Let us first describe this thickening at one node $N \in \Sigma$. Since N is a node of the stable Riemann surface X, we can take a neighborhood U of N such that $U \setminus \{N\}$ consists of two open sets U_1 and U_2 that are both holomorphically homeomorphic to the unit disk $\Delta^* \subset \mathbf{C}$ which is punctured at the origin. Let $\alpha_j \colon U_j \to \Delta^*$ be holomorphic homeomorphisms.

The thickening of the node N that we are presently describing depends the sets U_j and on the conformal mappings α_j. So we have to fix them first. Assume that they are now fixed.

The thickening of the node N will depend on one complex parameter $z \in \Delta = \{z \in \mathbf{C} \mid |z| < 1\}$. Let now $z \in \Delta$ be fixed. Let $\Delta_z = \{\zeta \in \mathbf{C} \mid 0 < |\zeta| < |z|\}$ be the disk of radius $|z|$ and center at the origin.

Delete, from the Riemann surface (Σ, X), the punctured disks $\alpha_j^{-1}(\Delta_z), j = 1, 2$, and the node N. In that way we get a Riemann surface with two boundary components γ_1 and γ_2 which correspond to the deleted punctured disks $\alpha_1^{-1}(\Delta_z)$ and $\alpha_2^{-1}(\Delta_z)$, respectively.

Let $\xi_z(w) = z|z|/w$. The mapping ξ_z is conformal and maps the inside of the punctured disk $\{w \mid |w| < |z|\}$ onto its outside. Therefore we obtain, identifying the point $p \in \gamma_1$ with the point $\alpha_2^{-1} \circ \xi_w \circ \alpha_1(p)$, a stable Riemann surface X_z such that the node N has been replaced by a handle. Rotating the point $z \in \Delta^*$ means simply a partial Dehn twist on the surface X_z around the simple closed curve of X_z that corresponds to the curves γ_1 and γ_2.

Important is that the mapping

$$\Delta \to \bar{\mathscr{M}}^g, [z] \mapsto [X_z] \tag{13}$$

is continuous. This is essentially the construction presented already by Fay ([7]) and then used by many other authors.

Actually the above construction could easily be modified to get a holomorphic thickening of the type (13). That is, actually, the way this is usually done. For our computations this mapping (13) is, however, more convenient.

The mapping (13) thickens only one node. If the stable surface X has m nodes $N_1, \ldots, M_m$, then we can repeat this construction and obtain a continuous mapping

$$\Delta^m \to \bar{\mathscr{M}}^g, \zeta \mapsto [X_\zeta] \tag{14}$$

to get a continuous mapping that thickens all the nodes simultaneously.

We use this thickening of the nodes to prove the following result (cf. [21, Proposition 3.1, page 90]).

THEOREM 10.1. *$\bar{\mathscr{M}}^g_{\mathbf{R}}$ is the closure of $\mathscr{M}^g_{\mathbf{R}}$ in $\bar{\mathscr{M}}^g$.*

Proof. By Theorem 7.1 the closure of $\mathscr{M}^g_{\mathbf{R}}$ in $\bar{\mathscr{M}}^g$ is contained in $\bar{\mathscr{M}}^g_{\mathbf{R}}$. It suffices, therefore, to show the converse inclusion. That we have actually shown in [21, Proposition 3.1]. This argument is rather important to the present discussion. We shall therefore repeat it here for the benefit of the reader.

To that end, let (Σ, X) be a stable genus g Riemann surface with nodes

$N_1, \ldots, N_m, m > 0$, representing a point of $\bar{\mathscr{M}}^g_{\mathbf{R}}$. We have to show that in any neighborhood of the point $[X] \in \bar{\mathscr{M}}^g$ there are points of $\mathscr{M}^g_{\mathbf{R}}$.

Since $[X] \in \bar{\mathscr{M}}^g_{\mathbf{R}}$, the Riemann surface (Σ, X) admits an antiholomorphic involution σ. To show the theorem we have to thicken the nodes in such a fashion which is compatible with the involution σ.

The mapping σ must map the set of nodes of (Σ, X) onto itself. To thicken the nodes of (Σ, X) we use the continuous thickening (14) with suitably chosen parameters $\zeta = (z_1, \ldots, z_m) \in \Delta^m$.

In order to see how we have to choose these parameters we need to divide the nodes N_j into different classes according to the behaviour of the mapping σ. So we assume that the nodes are numbered in such a fashion that the following holds.

1. Nodes $N_1, \ldots, N_{m_1}$ are kept pointwise fixed by σ in such a way that each node N_{m_j} has a neighborhood consisting of the node itself and of two punctured disks $U^1_{m_j}$ and $U^2_{m_j}$ which are both kept fixed by σ (as sets).
2. Nodes $N_{m_1+1}, \ldots, N_{m_2}$ are kept pointwise fixed by σ in such a way that they all have neighborhoods consisting of the nodes themselves and punctured disks $U^1_{m_k}$ and $U^2_{m_k}$ around the nodes such that $\sigma(U^1_{m_k}) = U^2_{m_k}$ and $\sigma(U^2_{m_k}) = U^1_{m_k}$.
3. Nodes $N_{m_2+1}, \ldots, N_{m_3}$ are mapped by σ onto the nodes $N_{m_3+1}, \ldots, N_m$ which are numbered in such a way that $\sigma(N_{m_2+k}) = N_{m_3+k}$ for all values of k.

The type of the node N_j determines how we have to choose the thickening parameter z_j in order to ensure that the involution σ induces an involution of the thickened surface $X_\zeta = X_{(z_j)}$.

First we choose disjoint neighborhoods $\{N_j\} \cup U^1_j \cup U^2_j$ of the nodes N_j such that each U^1_j and each U^2_j is holomorphically homeomorphic to the punctured disk, $U^1_j \cap U^2_j = \emptyset$ for each index j, and if $\sigma(N_j) = N_k$, then either $\sigma(U^1_j) = U^1_k$ and $\sigma(U^2_j) = U^2_k$ or $\sigma(U^1_j) = U^2_k$ and $\sigma(U^2_j) = U^1_k$.

Next we choose holomorphic homeomorphisms

$$\alpha^1_j : U^1_j \to \Delta^*, \quad \alpha^2_j : U^2_j \to \Delta^*$$

such that

$$\alpha^t_j \circ \sigma \circ (\alpha^s_k)^{-1}(z) = \bar{z} \tag{15}$$

whenever defined.

It is a simple matter to see that we can choose the mappings α^t_j: $U^t_j \to \Delta^*$ in such a manner that the equations (15) are satisfied. For any choice of holomorphic homeomorphisms α^t_j the mapping $\alpha^t_j \circ \sigma \circ (\alpha^s_k)^{-1}(z)$ is an antiholomorphic self-mapping of the unit disk which keeps the origin fixed whenever defined. Such a mapping is always conjugate to the complex conjugation.

After all these choices we can start thickening the nodes N_j. We have to perform this thickening in a way that is compatible with the involution σ. That imposes certain conditions on the coordinates z_j of the thickening parameter $\zeta \in \Delta^m$.

To shorten the notation, let X_{z_j} denote the deformed surface $X_{(0,\ldots,z_j,\ldots,0)}$ where we have thickened only the node N_j.

The nodes $N_1, \ldots, N_{m_1}$ of the type 1 impose conditions on the coordinates $z_1, \ldots, z_{m_1}$. A straightforward verification shows that, for $j = 1, \ldots, m_1$, the involution $\sigma: X \to X$ induces an antiholomorphic involution of X_{z_j} if and only if z_j is real.

In the same way the nodes $N_{m_1+1}, \ldots, N_{m_2}$ impose conditions on the thickening coordinates $z_{m_1+1}, \ldots, z_{m_2}$. If these coordinates z_k are real then the involution $\sigma: X \to X$ induces an antiholomorphic involution $X_{z_k} \to X_{z_k}$.

The remaining nodes impose a slightly different condition. Remember that for all values of $k, \sigma(N_{m_2+k}) = N_{m_3+k}$. Therefore, if we want to thicken these nodes in a way that the involution σ induces an antiholomorphic self-mapping of the deformed surface, we have to thicken the nodes N_{m_2+k} and N_{m_3+k} simultaneously for each value of k. Denote by $X_{z_{m_2+k}z_{m_3+k}}$ the surface obtained from X by thickening the nodes N_{m_2+k} and N_{m_3+k} according to the parameters z_{m_2+k} and z_{m_3+k}, respectively. Then a straightforward verification shows again that the involution $\sigma: X \to X$ induces an antiholomorphic mapping

$$\sigma: X_{z_{m_2+k}z_{m_3+k}} \to X_{z_{m_2+k}z_{m_3+k}}$$

if and only if $z_{m_2+k} = \bar{z}_{m_3+k}$. This condition is always verified if the parameters z_{m_2+k} and z_{m_3+k} are real and $z_{m_2+k} = z_{m_3+k}$.

For a real number e, $-1 < e < 1$, let $\varepsilon = (e, \ldots, e) \in \Delta^m$. We conclude that for each such ε, the antiholomorphic involution $\sigma: X \to X$ induces an antiholomorphic involution of deformed surfaces X_ε. The surface X_ε is, furthermore, a smooth Riemann surface of genus g.

From the continuity of the mapping (14) it follows then that each neighborhood of the point $[(\Sigma, X)] \in \bar{\mathscr{M}}^g_{\mathbf{R}} \setminus \mathscr{M}^g_{\mathbf{R}}$ contains points of $\mathscr{M}^g_{\mathbf{R}}$. We conclude therefore, that $\bar{\mathscr{M}}^g_{\mathbf{R}}$ is contained in the closure of $\mathscr{M}^g_{\mathbf{R}}$ and the proof is complete.

We finish this paper by the following result.

THEOREM 10.2. *The complex moduli space of stable real algebraic curves of genus* $g > 3$, $\bar{\mathscr{M}}^g_{\mathbf{R}}$, *is the quasiregular real part of the moduli space of complex algebraic curves equipped with its canonical real structure. The moduli space of stable real curves is also a semialgebraic variety.*

Proof. Instead of the original argument we present here a simplification which is due to the referee. Embed $\bar{\mathscr{M}}^g$ into a projective space $\mathbf{P}^n(\mathbf{C})$ in such a way that $(\bar{\mathscr{M}}^g)_{\sigma^*} = \bar{\mathscr{M}}^g \cap \mathbf{P}^n(\mathbf{R})$. This is possible by the considerations of Section 8.

In this context $\bar{\mathcal{M}}^g$ is the projective closure of $\mathcal{M}^g$. The same is true for $(\bar{\mathcal{M}}^g)_{\sigma*}$ and $(\mathcal{M}^g)_{\sigma*}$. This implies that the regular part of $(\mathcal{M}^g)_{\sigma*}$ is dense in the regular part of $(\bar{\mathcal{M}}^g)_{\sigma*}$. Hence the closure of the regular part of $(\mathcal{M}^g)_{\sigma*}$ in $\bar{\mathcal{M}}^g$ (or in $\mathbf{P}^n(\mathbf{R})$), is the same as the closure of the regular part of $(\bar{\mathcal{M}}^g)_{\sigma*}$.

This proves that the closure of $\mathcal{M}^g_{\mathbf{R}}$ in $\bar{\mathcal{M}}^g$ is the quasiregular real part $(\bar{\mathcal{M}}^g)^{qr}_{\sigma*}$. Hence the theorem is hereby proved, since, by Theorem 10.1, $\bar{\mathcal{M}}^g_{\mathbf{R}}$ is the closure of $\mathcal{M}^g_{\mathbf{R}}$.

References

[1] Abhayankar, S.S., "Local Analytic Geometry". Volume XIV of *Mathematics*, Academic Press, New York and London, 1964.

[2] Abikoff, William, "The Real Analytic Theory of Teichmüller Space". Volume 820 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin–Heidelberg–New York, 1980.

[3] Alling, Norman L. and Newcomb Greenleaf, "Foundations of the theory of Klein surfaces". Volume 219 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[4] Andreotti, Aldo and Per Holm, Quasianalytic and parametric spaces. In Per Holm, editor, *Real and complex singularities, Oslo 1976*, pages 13–97, Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands, 1977.

[5] Bers, Lipman, Finite dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc.*, 5(2):131–172, September 1981.

[6] Earle, C.J., On moduli of closed Riemann surfaces with symmetries. In "Advances in the Theory of Riemann Surfaces". Annals of Mathematics Studies 66, pages 119–130, Princeton University Press and University of Tokyo Press, Princeton, New Yersey, 1971.

[7] Fay, J., "Theta functions on Riemann surfaces". Volume 352 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin–Heidelberg–New York, 1973.

[8] Gross, B.H. and J. Harris, Real algebraic curves. *Ann. scient. Éc. Norm. Sup.* 4^e *série*, 14:157–182, 1981.

[9] Igusa, J.-J., "Theta Functions". Springer-Verlag, Berlin–Heidelberg–New York, 1972.

[10] Klein, Felix, Über eine neue Art von Riemannschen Flächen. *Math. Annalen*, 10, 1876.

[11] Klein, Felix, Über Realitätsverhältnisse bei der einem beliebigen Geschlechte zugehörigen Normalkurve der φ. *Math. Annalen*, 42, 1892.

[12] Klein, Felix, "Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale". – *Eine Ergänzung der gewöhnlichen Darstellungen*. B.G. Teubner, Leipzig, 1882.

[13] Knudsen, Finn and David Mumford, The projectivity of the moduli space of stable curves. i: preliminaries on "det" and "Div". *Math. Scand.*, 39:19–55, 1976.

[14] Knudsen, Finn F., The projectivity of the moduli space of stable curves. ii: the stacks *m sub*(g, n). *Math. Scand.*, 52:161–199, 1983.

[15] Knudsen, Finn F., The projectivity of the moduli space of stable curves. iii: the line bundles on *m sub*(g, n), and a proof of the projectivity of $m - sub(g, n)$ in characteristic 0. *Math. Scand.*, 52:200–212, 1983.

[16] Lehto, Olli, "Univalent functions and Teichmüller spaces". Volume 109 of *Graduate Texts in Mathematics*, Springer, Berlin–Heidelberg–New York, 1986.

[17] Mumford, D. and Fogarty, J., "Geometric Invariant Theory". Volume 34 of *Ergebnisse der Mathematik*, Springer-Verlag, Berlin–Heidelberg–New York, 1982.

[18] Mumford, David, "Curves and Their Jacobians". University of Michigan Press, Ann Arbor, Michigan, second printing 1976 edition, 1975.

[19] Seppälä, Mika, Complex algebraic curves with real moduli. *J. reine angew. Math.*, 387:209–220, 1988.

[20] Seppälä, Mika, Moduli spaces of real algebraic curves. July 1989. Preprint.

[21] Seppälä, Mika, On moduli of real curves. In Jean-Jacques Risler, editor, *Seminaire sur la Geometrie Algebrique Reelle*, pages 85–95, Publications Mathématiques de l'Université Paris VII, Paris VII, 1986.

[22] Seppälä, Mika, On moduli of real curves. *Rocky Mountain Journal of Mathematics*, 14(4), 1984.

[23] Seppälä, Mika, Quotients of complex manifolds and moduli spaces of Klein surfaces. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, 6:113–124, 1981.

[24] Seppälä, Mika, Teichmüller spaces of Klein surfaces. *Ann. Acad. Sci. Fenn. Ser. A. I. Mathematica Diss.*, 15:1–37, 1978.

[25] Seppälä, Mika and Robert Silhol, Moduli spaces for real algebraic curves and real abelian varieties. *Math. Z.*, 201:151–165, 1989.

[26] Siegel, Carl L., "Topics in Complex Function Theory". Volume 2 of *Wiley–Interscience*, John Wiley & Sons, Inc., New York–London–Sydney–Toronto, 1971.

[27] Weichhold, Guido, Über symmetrische Riemannsche Flächen und die Periodizitätsmodulen der zugehörigen Abelschen Normalintegrale erstes Gattung. *Leipziger Dissertation*, 1883.

[28] Wolpert, Scott A., The geometry of the moduli space of Riemann surfaces. *Bull. Am. Math. Soc., New Ser.*, 11:189–191, 1984.