JING YU

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JING YU*

School of Mathematics, The Institute for Advanced Study, Princeton, NJ 08540, USA
and
Academia Sinica, Taipei, Taiwan, R.O.C.

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Introduction

Let $\mathscr{C}$ be a smooth projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$, $q = p^n$. We fix a closed point ∞ on $\mathscr{C}$, and consider the ring A of functions on $\mathscr{C}$ regular away from ∞. We set k to be the function field of $\mathscr{C}$ and k_∞ its completion at ∞. After taking algebraic closure, we obtain the field $\bar{k}_\infty$ whose elements will be called "numbers". We fix an embedding $\bar{k} \subset \bar{k}_\infty$ throughout.

We are interested in transcendental numbers (i.e. elements in $\bar{k}_\infty$ transcendental over k) which arise naturally from algebro-geometric objects defined over $\bar{k}$. Thus our aim is to develop a theory in characteristic p which is analogous to the classical transcendence theory of abelian integrals. The algebro-geometric objects we have in mind are the Drinfeld A-modules (elliptic modules) introduced by V.G. Drinfeld in [5], 1973. One can associate periods to such Drinfeld A-modules of characteristic ∞, and we have shown in [10] that if a given Drinfeld A-module is defined over $\bar{k}$, then all its periods are transcendental. This result is parallel to the well-known theorem of Siegel-Schneider, on elliptic integrals of first kind.

Our purpose here is twofold. First, we shall extend our previous work to deal with higher-dimension Drinfeld modules. More specifically, we shall study the transcendence properties of the abelian t-modules. We shall prove in particular that, for period vectors of abelian t-modules defined over $\bar{k}$, at least one coordinate component is transcendental.

The second purpose is to extend transcendence theory to periods of the second kind. Just recently, basing on an idea of P. Deligne, a very interesting theory of quasi-periods for Drinfeld modules emerges from the work of G. Anderson [2]. With this we shall prove that all quasi-periods are transcendental, once the (dimension one) Drinfeld A-module in question is defined over $\bar{k}$. This parallels completely the classical work of Schneider on elliptic integrals of the second kind.

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For brevity, we shall restrict ourselves here only to the case of dimension one quasi-periods theory.

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1. Background of Drinfeld modules

Following [10], we first introduce the concept of E_q-functions. We denote by $d(a)$ the additive valuation of $a \in k$ which equals the order of pole of a at ∞ times the degree of ∞. As usual we extend this valuation to $\bar{k}_\infty$. If $\alpha \in \bar{k} \subset \bar{k}_\infty$, the maximum of the valuations of all the conjugates of α is said to be the size of α, noted by $\overline{\alpha}$.

Let K be a finite extension of k. We say that an entire function $f: \bar{k}_\infty \to \bar{k}_\infty$ is an E_q-function with respect to K if it has the following properties:

(i) It is additive and it has the form

$$f(z) = \sum_{h=0}^{\infty} b_h z^{q^h}, \quad b_h \in K \quad \text{with } \overline{b_h} \leqslant 1.$$

(ii) It has finite growth order, i.e. there exists real $\rho > 0$ such that

$$\max_h (d(b_h) + q^h r) \leqslant q^{\rho r} \quad \text{for all rationals } r \text{ large.}$$

(iii) There exists a sequence (a_h) in A satisfying

(1) $d(a_h) \leqslant hq^h$
(2) For all $j \leqslant h$, $a_h b_j$ are integral over A.
(3) If $q^{h_1} + \cdots + q^{h_s} < q^N$, then $a_{h_1} \ldots a_{h_s} \mid a_N$.

These E_q-functions behave like classical functions satisfying algebraic differential equations, and we have proved the following basic theorem in [10]:

THEOREM 1.1. *Let K/k be a finite extension. Let f_1, f_2 be E_q-functions with respect to K which are algebraically independent over $\bar{k}$. Then there are only finitely many points at which f_1, f_2, simultaneously assume values in K.*

All interesting examples of E_q-functions are related to Drinfeld's theory. Let τ be the Frobenius map $X \mapsto X^q$. Let $\bar{k}_\infty\{\tau\}$ be the non-commutative polynomial ring generated by τ over $\bar{k}_\infty$ under composition (i.e. the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_a$). Recall that a Drinfeld A-module ϕ is a $\mathbb{F}_q$-linear ring homomorphism from the Dedekind ring A into $\bar{k}_\infty\{\tau\}$ such that for a suitable positive integer n and all $a \neq 0$ in A

$$\phi(a) = a\tau^0 + \sum_{j=1}^{nd(a)} \phi(a)_j \tau^j, \quad \phi(a)_{nd(a)} \neq 0.$$

The integer n is said to be the rank of ϕ. What makes such a homomorphism more significant is the fact that $\mathbb{G}_a$, together with the A-action given by ϕ, can be parametrized by an unique entire exponential function e_ϕ, in the sense that the following identities are satisfied:

$$e_\phi(az) = \phi(a)(e_\phi(z)), \quad \text{for all } a \in A$$

$$e'_\phi(z) \equiv 1.$$

One can deduce from here that once the Drinfeld A-module ϕ is defined over $\bar{k}$ (i.e. all the coefficients $\phi(a)_j$ lie in $\bar{k} \subset \bar{k}_\infty$), then $e_\phi: \bar{k}_\infty \to \bar{k}_\infty$ is a E_q-function with respect to some finite extension of k, cf. Theorem 3.3 in [10].

Let L_ϕ be the zero set of the exponential function e_ϕ. This is always a finitely generated discrete A-submodule of $\bar{k}_\infty$ (considered as Lie $\mathbb{G}_a$). Its projective A-rank equals the rank of the Drinfeld A-module ϕ. We call L_ϕ the period lattice, and any non-zero element in it is called a period of the Drinfeld A-module ϕ. By applying Theorem 1.1, we have shown in [10] that all the periods are transcendental if ϕ is defined over $\bar{k}$.

As an illustration, we shall extract one more application of Theorem 1.1 to periods. Recall that if ϕ_1, ϕ_2 are two Drinfeld A-modules, a morphism from ϕ_1 to ϕ_2 is an element $P \in \bar{k}_\infty\{\tau\}$ satisfying $P \circ \phi_1(a) = \phi_2(a) \circ P$, for all $a \in A$. A non-zero morphism is called an isogeny. If there exists isogeny from ϕ_1 to ϕ_2, there also exists isogeny from ϕ_2 to ϕ_1, and we say ϕ_1 is isogenous to ϕ_2. Given isogenous Drinfeld A-modules ϕ_1 and ϕ_2, they must have the same rank. If both of them are defined over $\bar{k}$, then one can always find an isogeny P with coefficients in $\bar{k}$. It follows $P' \in \bar{k} = \bar{k}\tau^0$, $P' \neq 0$ and $P' L_{\phi_1} \subset L_{\phi_2}$. Thus given any period ω_1 of ϕ_1, there exists period ω_2 of ϕ_2 such that the ratio ω_1/ω_2 is algebraic. This, however, will never happen if ϕ_1 is not isogenous to ϕ_2.

THEOREM 1.2. *Let ϕ_1 and ϕ_2 be Drinfeld A-modules defined over $\bar{k}$. Suppose there exists $\omega_1 \in L_{\phi_1} - \{0\}$ and $\omega_2 \in L_{\phi_2} - \{0\}$ such that $\omega_1/\omega_2 \in \bar{k}$. Then ϕ_1 is isogenous to ϕ_2.*

Proof. Let K be a common field of definition for ϕ_1 and ϕ_2, finite over k. Let $\omega_2 = \lambda\omega_1$. Then the functions $f_1(z) = e_{\phi_1}(z), f_2(z) = e_{\phi_2}(\lambda z)$ are E_q-functions with respect to $K(\lambda)$. Since

$$f_1(a\omega_1) = f_2(a\omega_1) = 0 \in K(\lambda), \quad \text{for all } a \in A,$$

Theorem 1.1 implies that $e_{\phi_1}(z)$ and $e_{\phi_2}(\lambda z)$ are algebraically dependent functions over $\bar{k}$.

By a well-known theorem of E. Artin (cf. [7], Chap. VIII), one can then find non-trivial algebraic relations of the form

$$\sum_{i=0}^{l} \alpha_i e_{\phi_1}(z)^{p^i} + \sum_{j=0}^{m} \beta_j e_{\phi_2}(\lambda z)^{p^j} \equiv 0.$$

Thus, if $\omega \in L_{\phi_1}, \omega_1 \neq 0$ and $a \in A$, all the values $e_{\phi_2}(\lambda a \omega)$ must be among the finitely many roots of the additive equation

$$\sum_{j=0}^{m} \beta_j X^{p^j} = 0.$$

Hence there exists $a \neq 0$ in A such that $a\lambda\omega \in L_{\phi_2}$. Let ω run over a finite set of generators of L_{ϕ_1}. We then get $a_0 \in A$, $a_0 \neq 0$ such that $a_0 \lambda L_{\phi_1} \subset L_{\phi_2}$. Similarly, one can also get $a_1 \neq 0$ in A such that $a_1 \lambda^{-1} L_{\phi_2} \subset L_{\phi_1}$. This shows that the two Drinfeld A-modules ϕ_1 and ϕ_2 have the same rank. Also, multiplication by $a_1 \lambda$ induces an isogeny from ϕ_1 to ϕ_2. □

The above proof actually leads to a more general theorem.

THEOREM 1.3. *Let ϕ_1 and ϕ_2 be non-isogenous Drinfeld A-modules defined over $\bar{k}$. Let $u_1, u_2 \in \bar{k}_\infty - \{0\}$ satisfying $e_{\phi_1}(u_1) \in \bar{k}$ and $e_{\phi_2}(u_2) \in \bar{k}$. Then u_1/u_2 is transcendental.*

2. Abelian t-modules and transcendence

We shall consider abelian t-modules introduced by G. Anderson in [1]. Let T be a non-constant element in A. Let $\tilde{K}$ be either $\bar{k}$ or $\bar{k}_\infty$, viewed as $\mathbb{F}_q[t]$-algebra via $t \mapsto T$. By a t-module defined over $\tilde{K}$, we mean a pair consisting of an algebraic group E defined over $\tilde{K}$ and an $\mathbb{F}_q$-linear ring homomorphism $\phi: \mathbb{F}_q[t] \to \mathrm{End}_{\mathbb{F}_q} E$ such that the following properties are satisfied:

(i) There is an isomorphism of E onto $\mathbb{G}_a^d$ which identifies $\phi(\mathbb{F}_q)$ with scalar multiplications on $\mathbb{G}_a^d$.

(ii) $(\phi(t)_* - TI)^N \mathrm{Lie}(E) = 0$ for some integer $N > 0$.

We let $\mathbb{F}_q$ act on E by $\phi(\mathbb{F}_q)$, and let $\mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ be the $\tilde{K}$-vector space of $\mathbb{F}_q$-linear algebraic group homomorphisms over $\tilde{K}$. We say that the t-module (E, ϕ) is an abelian t-module if there exists a finite-dimensional subspace W in $\mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ such that

$$\mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a) = \sum_{j=0}^{\infty} W \circ \phi(t^j).$$

Let $(E_1, \phi_1), (E_2, \phi_2)$ be two t-modules. A $\mathbb{F}_q$-linear morphism $f\colon E_1 \to E_2$ which commutes with the t-action is said to be a morphism of the t-modules. To each t-module $E = (E, \phi)$, one can associate functorially an exponential map

$$\exp_E\colon \operatorname{Lie} E(\bar{k}_\infty) \to E(\bar{k}_\infty).$$

Expressed in terms of a given coordinate system (i.e. fixed isomorphism of E onto $\mathbb{G}_a^d$ over $\tilde{K}$ identifying $\phi(\mathbb{F}_q)$ with scalars), this exponential map becomes an entire $\mathbb{F}_q$-linear map e_E from $\bar{k}_\infty^d$ to $\bar{k}_\infty^d$ satisfying the equation

$$e_E(\phi(t)_*(z)) = \phi(t)(e_E(z)).$$

We let t act on $\operatorname{Lie} E(\bar{k}_\infty)$ via $\phi(t)_*$. The $\ker(\exp_E)$ is always a discrete $\mathbb{F}_q[t]$-submodule in $\operatorname{Lie} E(\bar{k}_\infty)$. We call $\ker(\exp_E)$ the period lattice of the t-module $E = (E, \phi)$, and any non-zero element in it is called a period vector of E. If (E, ϕ) is abelian, then its period lattice is always free of finite rank over $\mathbb{F}_q[t]$ (cf. Anderson [1], Lemma 2.4.1).

EXAMPLES:

(I) The trivial t-module. Let $E = \mathbb{G}_a$ and let t act as scalar multiplication by T. This is not an abelian t-module. The exponential here is just $e(z) = z$.

(II) Any Drinfeld A-module can be considered as abelian t-module with $E = \mathbb{G}_a$ and Drinfeld's exponential as the exponential map. In fact, all one-dimensional abelian t-modules arise in this way.

(III) A very interesting class of higher dimensional abelian t-modules is given by the tensor powers of the Carlitz module $E_c^{\otimes m}$. The underlying algebraic group of $E_c^{\otimes m}$ is $\mathbb{G}_a^m$. The homomorphism ϕ is given by

$$\phi(t)\colon \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \mapsto \begin{pmatrix} T & 1 & 0 \\ & \ddots & 1 \\ 0 & & T \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X_1^q \\ \vdots \\ X_m^q \end{pmatrix},$$

where the first square matrix on the right-hand side is the standard Jordan block, the second square matrix is the elementary one with the lower left corner equal to 1. The exponential map for $E = E_c^{\otimes m}$ is thus characterized by the condition

$$e_E\left(\begin{pmatrix} T & 1 & 0 \\ & \ddots & 1 \\ 0 & & T \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}\right) = \phi(t)(e_E(z)).$$

By a Theorem of Anderson-Thakur [3], the period lattice of $E_c^{\otimes m}$ is a rank one $\mathbb{F}_q[t]$-module generated by $\omega(m)$, with the last coordinate component of $\omega(m)$ equal to $\tilde{\pi}^m$, where $\tilde{\pi}$ is the period of the Carlitz module given by

$$\tilde{\pi} = (T - T^q)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \frac{T^{q^i} - T}{T^{q^{i+1}} - T}\right).$$

If m is a power of p, then one simply has $\omega(m) = (0, \ldots, 0, \tilde{\pi}^m)$.

To study general abelian t-modules, we consider the $\tilde{K}$-vector space $\mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ as finitely generated $\tilde{K}[t]$-module, via the t-action $f \mapsto f \circ \phi(t)$. It is also finitely generated as $\tilde{K}[t^n]$-module, for any $n > 0$.

LEMMA 2.1. *Let (E, ϕ) be an abelian t-module over $\tilde{K}$. Let $n > 0$ be an integer. Let $E_1 \neq \{0\}$ be a connected algebraic subgroup of E over $\tilde{K}$. Suppose E_1 is invariant under $\phi(\mathbb{F}_q[t^n])$. Then $(E_1, \phi|_{\mathbb{F}_q[t^n]})$ is an abelian t^n-module over $\tilde{K}$.*

Proof. Since E_1 is connected, we can always find $\mathbb{F}_q$-linear isomorphism between E_1 and $\mathbb{G}_a^d$ for some $d > 0$. It remains only to show that $\mathrm{Hom}_{\mathbb{F}_q}(E_1, \mathbb{G}_a)$ is a quotient $\tilde{K}[t^n]$-module of $\mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$. This follows from the fact that any $\mathbb{F}_q$-linear homomorphism from E_1 to $\mathbb{G}_a$ can be extended to a $\mathbb{F}_q$-linear homomorphism from E to $\mathbb{G}_a$ (cf. [11], Lemma 5.2). □

The exponential maps associated to t-modules also give E_q-functions:

LEMMA 2.2. *Let (E, ϕ) be a t-module of dimension n defined over $\bar{k}$. Let $e_E(z) = (e^{(1)}(z), \ldots, e^{(n)}(z))$ be the associated exponential map with respect to a fixed coordinate system. Let $V \in \bar{k}^n$, and let $f_i(y) = e^{(i)}(yV)$, for $i = 1, \ldots, n$ and $y \in \bar{k}_\infty$. Then the functions $f_i\colon \bar{k}_\infty \to \bar{k}_\infty, i = 1, \ldots, n$, are E_q-functions relative to some finite extension field over k.*

Proof. Let s be an integer such that $p^s \geqslant n$. Then $\phi(t^{p^s})_*$ is the scalar multiplication T^{p^s} on $\mathrm{Lie}(E)$. Hence one has a functional equation for the exponential which is of the form

$$e_E(T^{p^s}) = \sum_{j=0}^{l} G_j \begin{pmatrix} e^{(1)}(z)^{q^j} \\ \vdots \\ e^{(n)}(z)^{q^j} \end{pmatrix},$$

where $G_0 = T^{p^s} I, G_1, \ldots, G_l$ are $n \times n$ matrices with entries in a suitable finite extension field K/k.

Write

$$e_E(yV) = \sum_{h=0}^{\infty} y^{q^h} b_h, \quad \text{with } b_h \in \bar{k}^n.$$

We can solve the vector Taylor coefficients b_h recursively from the formula

$$[(T^{p^s})^{q^h} - T^{p^s}]b_h = \sum_{j=1}^{\inf(h,l)} G_j b_{h-j}^{q^j},$$

where $b_{h-j}^{q^j}$ denotes the column vector obtained from b_{h-j} by raising all coordinate components to its q^j-th power. From this recursive formula, it is rather easy to see that the functions $f_i(y), i = 1, \ldots, n$ are E_q-functions with respect to K. □

Now we come to the main point. Let $E = (E, \phi)$ be an abelian t-module defined over $\bar{k}$. Let V be any period vector of E. We contend that at least one coordinate component of V is transcendental. This is special case of the following

THEOREM 2.3. *Let E be an abelian t-module of dimension n defined over $\bar{k}$. Let $e_E(z)$ be the associated exponential map with respect to a fixed coordinate system. Let $V \in \bar{k}_\infty^n$ such that $V \neq 0$ and $e_E(V) \in \bar{k}^n$. Then at least one coordinate component of V is transcendental.*

Proof. We first verify that the one-parameter map $y \mapsto e_E(yV)$ is not a polynomial map. Let s be an integer such that $\phi(t^{p^s})_*$ acts as scalar multiplication on Lie E.

Suppose $y \mapsto e_E(yV)$ is polynomial. Let Z be the image of $\bar{k}_\infty$ under this map in $\bar{k}_\infty^n \simeq E(\bar{k}_\infty)$. Then the connected component of the Zariski closure of Z is an one-dimensional algebraic subgroup E_1. By Lemma 2.1 $(E_1, \phi\,|_{\mathbb{F}_q[t^{p^s}]})$ is an abelian t^{p^s}-module over $\bar{k}_\infty$.

Identify Lie $E_1(\bar{k}_\infty)$ inside Lie $E(\bar{k}_\infty)$, and regard $\exp_{E_1}$ as a restriction of $\exp_E$. Under our chosen coordinate system, Lie $E_1(\bar{k}_\infty)$ coincides with $\bar{k}_\infty V$, because of the inverse mapping theorem. This implies that the abelian t^{p^s}-module E_1 has a polynomial exponential, which is impossible.

We may then write $e_E(yV) = (f_1(y), \ldots, f_n(y))$ and assume $f_1(y)$ is not a polynomial in y. Suppose $V \in \bar{k}_n$. Then the functions $f_i(y), i = 1, \ldots, n$ are E_q-functions with respect to some finite extension K/k. We apply Theorem 1.1 to the two E_q-functions, $f_1(y)$ and $f(y) = y$. By Artin's theorem we then have non-trivial additive relation of the form

$$\sum_{l=0}^{m_1} \alpha_l (f_1(y))^{p^l} + \sum_{j=0}^{m_2} \beta_j y^{p^j} \equiv 0.$$

Since $f_1(y)$ is entire but not polynomial, it has an infinite number of zeros. Hence all β_j are zero. This is clearly impossible. Therefore, we have $V \notin \bar{k}^n$. □

Finally, we note that we have proved the stronger result in [11] for those abelian t-modules over $\bar{k}$ which admit sufficiently many "real" endomorphisms (i.e.

Hilbert-Blumenthal abelian t-modules). In that case, if $V \neq 0$ and $\exp_E(V) \in E(\bar{k})$, then all coordinate components of V with respect to suitably normalized coordinate system are transcendental.

3. Quasi-periodic functions and transcendence

To introduce quasi-periodic functions into Drinfeld's theory, we first recall some facts from classical function theory.

(I) Let $E_1 = \mathbb{G}_m$. The exponential function e^z gives complex analytic isomorphism $\mathbb{C}/2\pi i\mathbb{Z} \simeq E_1(\mathbb{C})$, where e^z is a solution of the algebraic differential equation $f'(z) = f(z)$, and $2\pi i\mathbb{Z}$ is the period lattice.

(II) Let L be a rank two lattice in $\mathbb{C}$. The periodic Weierstrass function $\wp_L(z)$ leads to complex analytic isomorphism from $\mathbb{C}/L$ onto $E_2(\mathbb{C})$, where E_2 is the elliptic curve associated to L. In this connection, one also has quasi-periodic Weierstrass function $\zeta_L(z)$. Both $\wp_L(z)$ and $\zeta_L(z)$ are solutions of suitable algebraic differential equations.

$$[\wp_L'(z)]^2 = 4\wp_L(z)^3 - g_2(L)\wp_L(z) - g_3(L)$$

$$\zeta_L'(z) = -\wp_L(z).$$

Write $L = \langle \omega_1, \omega_2 \rangle$, with $\mathrm{Im}(\omega_1/\omega_2) > 0$. Let $\eta_i = 2\zeta_L(\frac{1}{2}\omega_i)$, for $i = 1, 2$. Then one has the Legendre's relation connecting (I) and (II),

$$\begin{vmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{vmatrix} = 2\pi i.$$

We regard the entries ω_i, η_i as elliptic integrals of the first and second kind respectively, then the non-vanishing of this determinant gives the de Rham isomorphism theorem for the elliptic curve E_2.

In Drinfeld's theory, one starts with lattices $L \subset \bar{k}_\infty$ (i.e. finitely generated discrete A-submodules). One can associate to given lattice L a Drinfeld A-module $\phi = \phi_L \colon A \to \mathrm{End}_{\mathbb{F}_q}\mathbb{G}_a$. We let $E_\phi = \mathbb{G}_a$, equipped with the A-action given by ϕ. Drinfeld's exponential function $e_\phi(z)$ then gives an analytic A-module isomorphism $\bar{k}_\infty/L \simeq E_\phi(\bar{k}_\infty)$. Fix any non-constant a in A, the function $e_\phi(z)$ is a solution of the "algebraic differential equation" below

$$e_\phi(az) = \phi(a)(e_\phi(z)).$$

If the lattice L has rank $r > 1$, one also has interesting quasi-periodic functions associated to L, as first noticed by P. Deligne in the case $A = \mathbb{F}_q[T]$.

To get these quasi-periodic functions, we consider $\bar{k}_\infty\{\tau\}$ as A-bimodule, with right multiplication by $\phi(a)$ and left multiplication by scalars $a, a \in A$. By a biderivation from A into $\bar{k}_\infty\{\tau\}\tau$, we mean a $\mathbb{F}_q$-linear map $\delta: A \to \bar{k}_\infty\{\tau\}\tau$ satisfying

$$\delta(ab) = a\delta(b) + \delta(a)\phi(b), \quad \text{for all } a\, b \in A.$$

Given such a biderivation, and given non-constant a in A, we can always solve the unique entire $\mathbb{F}_q$-linear solution $F(z)$ of the following "algebraic differential equation"

$$F(az) - aF(z) = \delta(a)(e_\phi(z)),$$

$$F(z) \equiv 0 \bmod z^q.$$

This solution is independent of a, and is henceforth denoted by $F_\delta(z)$. It is quasi-periodic with respect to the lattice L, in the sense that the following properties always hold

(i) $F_\delta(z + \omega) = F_\delta(z) + F_\delta(\omega)$, for $z \in \bar{k}_\infty$ and $\omega \in L$,

(ii) $F_\delta(\omega)$ is A-linear in $\omega \in L$.

We shall call the values $F_\delta(\omega), \omega \in L$, the quasi-periods of F_δ, and following G. Anderson [2], we shall adopt the integral notation

$$\int_\omega \delta \stackrel{\text{def}}{=} -F_\delta(\omega).$$

We call biderivations $\delta: A \to \bar{k}_\infty\{\tau\}\tau$ differentials of second kind on the Drinfeld A-module ϕ. The set of all such biderivations will be denoted by $BD(\phi)$.

The Drinfeld A-module ϕ itself gives rise to a biderivation satisfying

$$\delta_\phi(a) = \phi(a) - a\tau^0, \quad \text{for all } a \in A.$$

The solution of the corresponding equation is $F_{\delta_\phi}(z) = e_\phi(z) - z$. Thus, one has $\int_\omega \delta_\phi = \omega$ for all $\omega \in L$. We call scalar multiples of δ_ϕ differentials of the first kind.

One can also form inner biderivations $\delta_\phi^{(P)}$ from any $P \in \bar{k}_\infty\{\tau\}\tau$, i.e.

$$\delta_\phi^{(P)}(a) = P\phi(a) - aP, \quad \text{for all } a \in A.$$

These are also called exact differentials, since $\int_\omega \delta_\phi^{(P)} = -P(e_\phi(\omega)) \equiv 0$. All quasi-periodic functions obtained from exact differentials are actually periodic.

The set of all $\delta_\phi^{(P)}, P \in \bar{k}_\infty\{\tau\}\tau$, will be denoted by $IBD(\phi)$. The vector space $BD(\phi)/IBD(\phi)$ is therefore called the de Rham cohomology of the Drinfeld A-module ϕ, and is denoted by $H^*_{DR}(\phi)$.

Just as in the classical theory, one is able to write down genuine quasi-periodic functions only if the lattice L has rank $r > 1$. In fact, as observed by P. Deligne and G. Anderson, one has

$$\dim_{\bar{k}_\infty} H^*_{DR}(\phi) = \operatorname{rank} \phi = r.$$

An illuminating way to get this dimension is through the so-called de Rham isomorphism: $H^*_{DR}(\phi) \simeq \operatorname{Hom}_A(L, \bar{k}_\infty)$ via the mapping induced by $\delta \mapsto (\omega \mapsto \int_\omega \delta)$. We refer to E.-U. Gekeler [6] for a proof of this theorem.

Since our purpose here is to derive transcendence properties of the quasi-periodic functions, we will not go into the deeper part of Anderson's theory, which culminates in an analogue of the Legendre's relation for Drinfeld A-modules.

We now restrict ourselves to Drinfeld A-module ϕ defined over $\bar{k}$. For these ϕ, it is natural to consider biderivations δ defined over $\bar{k}$, i.e. satisfying $\delta(A) \subset \bar{k}\{\tau\}\tau$. The set of all such biderivations is denoted by $BD(\phi/\bar{k})$. The set of all $\delta_\phi^{(P)}, P \in \bar{k}\{\tau\}\tau$, is denoted by $IBD(\phi, \bar{k})$. Putting $H^*_{DR}(\phi/\bar{k})$ to be the quotient of $BD(\phi/\bar{k})$ by $IBD(\phi/\bar{k})$, then one has

$$H^*_{DR}(\phi) = H^*_{DR}(\phi/\bar{k}) \otimes_{\bar{k}} \bar{k}_\infty.$$

Thus, if $\delta \in BD(\phi/\bar{k})$ and $\int_\omega \delta = 0$ for all periods ω, the de Rham isomorphism implies $\delta \in IBD(\phi/\bar{k})$.

The fundamental theorem we want to prove is

THEOREM 3.1. *Let ϕ be a Drinfeld A-module defined over $\bar{k}$, with corresponding exponential $e_\phi(z)$. Let $\delta \in BD(\phi/\bar{k}) - IBD(\phi/\bar{k})$, with corresponding quasi-periodic function $F_\delta(z)$. Let $u \in \bar{k}_\infty$ such that $u \neq 0$ and $e_\phi(u) \in \bar{k}$. Then $F_\delta(u)$ is transcendental. In particular, $\int_\omega \delta$ is transcendental for all periods ω of ϕ.*

Proof. We let A act on $\mathbb{G}_a^2$ according to the following recipe

$$\Phi(a)(X_1, X_2) = (\phi(a)(X_1), aX_2 + \delta(a)(X_1)).$$

Then $(\mathbb{G}_a^2, \Phi)$ becomes an A-module, a fortiori a t-module for any choice of non-constant T in A. The exponential map for this module is easily seen to be the following map

$$(z_1, z_2) \mapsto (e_\phi(z_1), F_\delta(z_1) + z_2).$$

Since $(\mathbb{G}_a^2, \Phi)$ is defined over $\bar{k}$, we may apply Lemma 2.2 with $V = (1, 0)$. It follows that $e_\phi(z_1)$ and $F_\delta(z_1)$ are E_q-functions with respect to some finite extension field K/k. Suppose $F_\delta(u) \in \bar{k}$. Then $F_\delta(au) \in \bar{k}$ for all $a \in A$, since $e_\phi(u) \in \bar{k}$ by assumption. Thus, applying our Theorem 1.1, we know that $e_\phi(z_1)$ and $F_\delta(z_1)$ are algebraically dependent functions. By Artin's theorem, we then have algebraic dependence relations of the form

$$\sum_{l=0}^{m_1} \alpha_l e_\phi(z_1)^{p^l} + \sum_{j=0}^{m_2} \alpha'_j F_\delta(z_1)^{p^j} \equiv 0.$$

Let L be the period lattice of ϕ, and let $\omega_1 \in L$ be a period. The dependence relation above implies that the values $F_\delta(a\omega_1)$, $a \in A$, must be among the finitely many roots of the additive polynomial $\Sigma \alpha'_j X^{p^j}$. Thus we can find $a_1 \neq 0$ in A such that $F_\delta(a_1\omega_1) = 0$. Since L is finitely generated, we conclude that $F_\delta(z_1)$ vanishes on a sublattice of L of finite index. This implies $\int_\omega \delta = 0$ for all $\omega \in L$ which contradicts the de Rham isomorphism theorem. □

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