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1. Introduction

As proved by Riemann, the function $\zeta(s)$, defined for $\operatorname{Re} s > 1$ by the Dirichlet series $\Sigma_{n \geqslant 1} n^{-s}$, has a meromorphic continuation to the whole s-plane with only a simple pole, at $s = 1$, and satisfies the functional equation

$$\zeta(s)\mathbf{L}_{\mathbb{R}}(s) = \zeta(1 - s)\mathbf{L}_{\mathbb{R}}(1 - s), \tag{1}$$

where

$$\mathbf{L}_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

On the other hand, it is well-known (see [S], [G-S], [T]) that there is a tempered distribution Δ_s on the real line, depending meromorphically on s, and given for $\operatorname{Re} s > 0$ by the measure $|t|^s \,\mathrm{d}^\times t = |t|^{s-1}\,\mathrm{d}t$:

$$\langle \Delta_s | f \rangle = \int_{\mathbb{R}} f(t)|t|^s \,\mathrm{d}^\times t.$$

It satisfies the functional equation

$$\Delta_s/\mathbf{L}_{\mathbb{R}}(s) = \hat{\Delta}_{1-s}/\mathbf{L}_{\mathbb{R}}(1 - s),$$

where ^ stands for the Fourier transform on $\mathbb{R}$ with respect to the bicharacter $(x, y) \mapsto \mathrm{e}^{2\pi i x y}$, which identifies the Pontrjagin dual of $\mathbb{R}$ with $\mathbb{R}$. The functional equation (1) is thus equivalent to the following functional equation on tempered distributions:

$$\zeta(s)\Delta_s = \zeta(1 - s)\hat{\Delta}_{1-s}. \tag{2}$$

The distribution $\zeta(s)\Delta_s$ may be regarded as the Mellin transform of the tempered distribution $d_{\mathbb{Z}}$ defined by

$$\langle d_{\mathbb{Z}} | f \rangle = \sum_{n \in \mathbb{Z}} f(n)$$

for any function f in the Schwartz space of $\mathbb{R}$. More precisely, the meromorphic function $\zeta(s)\langle \Delta_s | f \rangle$ of s is the sum of the analytic continuations of the two integrals

$$\int_0^1 \langle d_{\mathbb{Z}} | f^t \rangle t^s \, d^\times t \quad \text{and} \quad \int_1^\infty \langle d_{\mathbb{Z}} | f^t \rangle t^s \, d^\times t$$

where $f^t(x) = f(tx)$ for $t > 0$. From this point of view the functional equation (2) reflects the Poisson summation formula

$$d_{\mathbb{Z}} = \hat{d}_{\mathbb{Z}}, \tag{3}$$

which is due to the fact that $\mathbb{Z}$ is its own orthogonal in $\mathbb{R}$.

There is a similar interpretation for the series $\mathbf{L}(s, \chi)$ attached to a Dirichlet character χ: it is the twist of the Riemann zeta function by χ, and its functional equation reflects the relation between the distribution $\chi d_{\mathbb{Z}} = \Sigma \chi(n)\delta_n$ and its Fourier transform.

In the same spirit we study twisted Dirichlet series defined over a global field F. We interprete the usual functional equations relating two Dirichlet series (like (1)) as an equality between a tempered distribution arising from one Dirichlet series and the Fourier transform of the tempered distribution arising from the other Dirichlet series (like (3)). The analytic behavior of such Dirichlet series is expressed in terms of tempered distributions (like (2)). Our viewpoint gives rise to the classification of such Dirichlet series, namely, they come from periodic functions on lattices. This is the content of our Theorem 1, which is stated with emphasis on the symmetries appearing in (1)–(3). It is reformulated in Theorem 2 in more classical terms. From our results it follows immediately that the Riemann zeta function is determined, up to a constant multiple, by some analytic conditions and by the functional equation (1), a result first obtained by Hamburger [Ha] and reproved by Siegel [Si], Hecke [He], and Bochner [B]. We refer to [G-L] for more discussions on the special case $F = \mathbb{Q}$; see also [V] for this case. It should be remarked that the interpretation described above was already employed in the case $F = \mathbb{Q}$ by Kahane and Mandelbrojt in [K-M], where they studied more general Dirichlet series following [B-N], [B].

As an application of our results, we obtain a criterion for an idèle class character in terms of analytic continuations and functional equations satisfied by a certain Dirichlet series (Theorem 3), which improves an earlier result of Li [L] on the same subject. The old criterion used Eisenstein series for GL(2), while the new criterion is based entirely on GL(1) itself.

In Section 2 we state the main results of this paper. Theorem 3 is proved in Section 3. In Section 4 we study tempered distributions and prepare the proof of Theorem 1; the proof is carried out in Section 5. For the sake of self-containedness, we give a complete proof, including some classical arguments.

2. Statements of main results

2.1. Throughout this paper, we fix a global field F. Let S be a finite nonempty set of places of F, containing the archimedean ones if F is a number field. Denote by K the product of the completions F_v of F at the places v in S and embed F in K diagonally. Write R for the ring of S-integers and U for the group of S-units. Let $|t|$ denote the module of multiplication on the additive group of K by the element t of the multiplicative group $K^\times$ and denote by K^1 its kernel.

LEMMA. *With the above notations, we have*

(a) *R is a discrete subgroup of K with compact quotient;*

(b) *U is a discrete subgroup of K^1 with compact quotient.*

Proof. The ring A of adèles of F is the restricted product of the completions F_v of F with respect to their unit balls E_v. Let $\bar{R}$ be the product of the E_v's for v not in S. It is a compact group, the closure of R embedded in the subring of S-adèles; the closed subring $K + \bar{R}$ of A is a neighborhood of 0. By the strong approximation theorem, the subgroup $F + K$ of A is dense; hence we have $A = F + K + \bar{R}$. As the only element in R which lies in the product over S of the open unit balls of F_v is 0, the subring R of K is discrete. The subgroup $F + \bar{R}$ is closed, and its intersection with K is R. By the projection of A onto K, we get an isomorphism from $A/(F+\bar{R})$ onto K/R. As the former group is the image of the compact group A/F, these groups are compact. This proves (a). For (b), it is another formulation of Dirichlet S-unit theorem (see Weil [W2], Th9, Ch.IV, Section 4).

By a character of a topological group G we mean a continuous homomorphism from G into $\mathbb{C}^\times$; the group $\mathscr{A}(G)$ of characters of G endowed with the compact-open topology is a locally compact group. For a character ω of U, denote by $\mathscr{A}_\omega(K^\times)$ the set of characters of $K^\times$ which extend ω. For ω trivial, the set $\mathscr{A}_1(K^\times)$ is the group $\mathscr{A}(K^\times/U)$ of characters of $K^\times/U$, and it acts simply transitively on each $\mathscr{A}_\omega(K^\times)$.

Since the group $K^\times/K^1$ is naturally isomorphic either to $\mathbb{Z}$ (in the function field case) or to $\mathbb{R}$ (in the number field case), it results from the lemma above that $\mathscr{A}(K^\times/U)$ has a canonical structure of a one-dimensional complex Lie group, of which the identity component is the image of $\mathbb{C}$ under the map sending the complex number s to the character $t \mapsto |t|^s$, and the connected components are parametrized by the discrete group $\mathscr{A}(K^1/U)$. The group of positive characters of $K^\times/U$ is isomorphic to the additive group of $\mathbb{R}$ via the map $t \mapsto |t|^r$; this is also the image of $\mathscr{A}(K^\times/U)$ under the map $\chi \mapsto (\chi\bar{\chi})^{1/2}$, the kernel being the Pontrjagin dual $(K^\times/U)^\wedge$ of $K^\times/U$. The real number r such that $(\chi\bar{\chi})^{1/2} = |\,|^r$ is called the exponent $\mathrm{e}(\chi)$ of χ. The action of $\mathscr{A}_1(K^\times)$ on $\mathscr{A}_\omega(K^\times)$ defines on $\mathscr{A}_\omega(K^\times)$ a structure of a one-dimensional complex manifold, and we have a map from $\mathscr{A}_\omega(K^\times)$ to an oriented affine line over $\mathbb{R}$, still called the exponent map and denoted by e, such that, for χ and χ' in $\mathscr{A}_\omega(K^\times)$, we have

$$\mathrm{e}(\chi') - \mathrm{e}(\chi) = \mathrm{e}(\chi'/\chi).$$

This gives a meaning to an expression such as "for χ of exponent large enough".

2.2. We are concerned with quadruples $(X, A, \mathbf{a}, \omega)$, where X is an invertible K-module, A is an invertible R submodule of X, ω is a character of U, and $\mathbf{a}$ is a nonzero tempered complex-valued function on the set A^* of nonzero elements of A of type ω under U, that is, $\mathbf{a}(\alpha\gamma) = \mathbf{a}(\alpha)\omega(\gamma)$ for all α in A^* and γ in U. As a product of normed spaces, the algebra K has a norm, so does the invertible K-module X. A function on a subset of X is said to be tempered if outside a compact set of X, it is dominated by some power of the norm. To a quadruple $(X, A, \mathbf{a}, \omega)$ we associate a twisted Dirichlet series $D_\mathbf{a}(\chi, x)$ defined for χ in $\mathscr{A}_\omega(K^\times)$ with $\mathrm{e}(\chi)$ large and any basis x of X over K by the function homogeneous in x:

$$D_\mathbf{a}(\chi, x) = \sum_{\alpha \in A^*/U} \mathbf{a}(\alpha)\chi(x/\alpha).$$

Note that A^*/U parametrizes the rank one free R-submodules of A. There is a natural measure $\mathrm{d}^\times x$ on the big orbit $X^\times$ of $K^\times$ in X and for χ with sufficiently large exponent, we get a homogeneous tempered distribution $D_\mathbf{a}(\chi, x)\,\mathrm{d}^\times x$ on X.

Let $(Y, B, \mathbf{b}, \omega^{-1})$ be another quadruple such that X and Y are in Pontrjagin duality given by a pairing $(x, y) \mapsto (x \mid y)$. Assume that this pairing is compatible with the action of $K^\times$, that is, $(tx \mid y) = (x \mid ty)$ for all t in $K^\times$, x in X and y in Y. We write $(y \mid x)$ for the inverse of $(x \mid y)$. Both A and B are discrete subgroups with compact quotients in their respective spaces, their Pontrjagin duals $Y/A^\perp$ and $X/B^\perp$ are compact groups, where $A^\perp$ and $B^\perp$ stand for the orthogonal of A and B respectively. We choose the Haar measures $\mathrm{d}x$ on X and $\mathrm{d}y$ on Y such that they are in Pontrjagin duality and the volumes of the dual groups of A and B are the same. For a character χ of $K^\times$, write $\check{\chi}$ for $|\,|\chi^{-1}$. Define $D_\mathbf{b}(\check{\chi}, y)$ in a similar way.

THEOREM 1. *Let $(X, A, \mathbf{a}, \omega)$ and $(Y, B, \mathbf{b}, \omega^{-1})$ be as above. The following statements are equivalent:*

(S1) *The two tempered distributions $D_\mathbf{a}(\chi, x)\,\mathrm{d}^\times x$ and $D_\mathbf{b}(\check{\chi}, y)\,\mathrm{d}^\times y$ on X and*

Y respectively can be continued meromorphically to $\mathscr{A}_\omega(K^\times)$ with poles on finitely many components, they have finite order at infinity in each vertical strip of finite width on each component of $\mathscr{A}_\omega(K^\times)$ if F is a number field, and they are Fourier transform of each other.

(S2) *There exist two complex numbers* $\mathbf{a}(0)$ *and* $\mathbf{b}(0)$ *such that for any function* f *in the Schwartz-Bruhat space of* X *one has*

$$\sum_{\alpha\in A} \mathbf{a}(\alpha)f(\alpha) = \sum_{\beta\in B} \mathbf{b}(\beta)\hat{f}(\beta),$$

where

$$\hat{f}(y) = \int_X f(x)(y\,|\,x)\,\mathrm{d}x.$$

(S3) *The subgroups* $A^\perp$ *and* $B^\perp$ *are orthogonal; there exist two complex numbers* $\mathbf{a}(0)$ *and* $\mathbf{b}(0)$ *such that the function* $\alpha \mapsto \mathbf{a}(\alpha)$ *on* A *is periodic* mod $B^\perp$ *and the function* $\beta \mapsto \mathbf{b}(\beta)$ *on* B *is periodic* mod $A^\perp$; *moreover, the factor groups* $A/B^\perp$ *and* $B/A^\perp$ *are finite and in duality with respect to the pairing* $(\,|\,)$, *and*

$$\mathbf{b}(\beta) = \int_{A/B^\perp} \mathbf{a}(\alpha)(\alpha\,|\,\beta), \qquad \mathbf{a}(\alpha) = \int_{B/A^\perp} \mathbf{b}(\beta)(\beta\,|\,\alpha),$$

where the integral means summing over the elements in the underlying set then dividing the sum by the positive square root of the index $[A: B^\perp] = [B: A^\perp]$.

2.3. REMARK. Let $\Theta_\mathbf{a}$ and $\Theta_\mathbf{b}$ denote the tempered distributions $\Sigma_{\alpha\in A}\mathbf{a}(\alpha)\delta_\alpha$ and $\Sigma_{\beta\in B}\mathbf{b}(\beta)\delta_\beta$ which appear in (S2). The identity $\Theta_\mathbf{a} = \hat{\Theta}_\mathbf{b}$ implies that the distribution $\mathbf{a}(0)\delta_0 - \mathbf{b}(0)$ is of type ω under U. As it is fixed by the action of the kernel of $|\,|$ on $K^\times$, which contains U, we have $\mathbf{a}(0) = \mathbf{b}(0) = 0$ for ω nontrivial. Also, it results from (S3) that the character ω equals 1 on the units in $1 + B^\perp: A = 1 + A^\perp: B$.

COROLLARY 1. *If A and B are orthogonal, then the statement* (S1) *holds if and only if the two functions* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant with the same value. In this case, the R-submodule A is the orthogonal of B.*

2.4. Now take the special case $X = K$ and choose a unitary additive character ψ of K such that the pairing $(x, y) \mapsto \psi(xy)$ identifies K with its Pontrjagin dual Y. Denote by d_ψ the Haar measure on K self-dual with respect to ψ. For a character χ of $K^\times$, put

$$\mathbf{L}_S(\chi) = \prod_{v\in S} \mathbf{L}_v(\chi_v) \text{ and } \varepsilon_S(\chi, \psi) = \prod_{v\in S} \varepsilon_v(\chi_v, \psi_v),$$

where **L**- and ε-factors are as computed in [T]. Theorem 1 for this case can be reformulated as

THEOREM 2. *Let A and B be two invertible R-submodules in K and let ω be a character of U. Let* **a** *be a nonzero tempered function on A^* of type ω and* **b** *be a nonzero tempered function on B^* of type ω^{-1}. Define, for χ in $\mathscr{A}_\omega(K^\times)$, the Dirichlet series*

$$D_{\mathbf{a}}(\chi) = \sum_{\alpha \in A^*/U} \mathbf{a}(\alpha)\chi(\alpha)^{-1},$$

which converges absolutely if $\mathrm{e}(\chi)$ is large. Define $D_{\mathbf{b}}(\check{\chi})$ similarly. Then the following statements are equivalent:

(S′1) *The Dirichlet series $D_{\mathbf{a}}(\chi)$ has a meromorphic continuation to the whole manifold $\mathscr{A}_\omega(K^\times)$ with poles on finitely many components, it has finite order at infinity in each vertical strip of finite width on each component of $\mathscr{A}_\omega(K^\times)$ in case F is a number field, and it satisfies the functional equation*

$$D_{\mathbf{a}}(\chi)\mathbf{L}_S(\chi) = \varepsilon_S(\chi, \psi)D_{\mathbf{b}}(\check{\chi})\mathbf{L}_S(\check{\chi}).$$

(S′2) *There exist complex numbers* $\mathbf{a}(0)$ *and* $\mathbf{b}(0)$ *such that, for any function f in the Schwartz space of K, the following identity holds:*

$$\sum_{\alpha \in A} \mathbf{a}(\alpha)f(\alpha) = \sum_{\beta \in B} \mathbf{b}(\beta)\hat{f}(\beta),$$

where $\hat{f}$ is the Fourier transform of f with respect to ψ.

(S′3) $\psi(A^\perp B^\perp) = 1$, *in other words, $A^\perp \subset B$ and $B^\perp \subset A$; there exist complex numbers* $\mathbf{a}(0)$ *and* $\mathbf{b}(0)$ *such that the function $\alpha \mapsto \mathbf{a}(\alpha)$ on A is constant* mod $B^\perp$, *the function $\beta \mapsto \mathbf{b}(\beta)$ on B is constant* mod $A^\perp$, *and*

$$\mathbf{b}(\beta) = \mathrm{vol}(K/B^\perp, d_\psi) \sum_{\alpha \in A/B^\perp} \mathbf{a}(\alpha)\psi(-\alpha\beta).$$

Consequently, the character ω is trivial on the elements of U which are congruent to 1 modulo the ideal $B^\perp : A$ of R.

COROLLARY 2. *Assume that R is a principal ideal domain and self-orthogonal with respect to ψ. If, furthermore, $A = R$ and B is contained in R, then the statement* (S′1) *is equivalent to $B = R$ and the functions* **a** *and* **b** *are constant with the same value.*

2.5. Theorem 2 has an application to characterizing the idèle class character of F which we now explain. Let $\mathfrak{N}$ be an effective divisor of F supported on a finite set N of nonarchimedean places of F. Suppose that, at each place v of F outside N, we are given a character μ_v of $F_v^\times$ satisfying the following two conditions:

(1) (continuity) the μ_v's are unramified for almost all v;

(2) (moderate growth) there is a real number c such that, for almost all v, the absolute value of μ_v at a uniformizer of F_v is $O((Nv)^c)$, where Nv denotes the cardinality of the residue field of F_v.

We are interested in knowing when the character μ^N of the subgroup I^N of idèles of F trivial at the places in N, defined as the product of μ_v over the places v of F outside N, can be extended to an idèle class character μ of F such that the conductor of μ on N is $\mathfrak{N}$, as prescribed. We shall give an analytic criterion below.

2.6. Choose a finite nonempty set S of places of F, containing the archimedean ones if F is a number field, such that S is disjoint from N and μ^N is unramified outside $S \cup N$. Write ${}^S\mathfrak{O}$ for the ring of S-integers in F, ${}^SU$ for the group of S-units, and ${}^Sh$ for the class number of ${}^S\mathfrak{O}$. Regard $\mathfrak{N}$ as an integral S-ideal. Let ${}^S\mathscr{A}$ denote the group of idèle class characters of F unramified outside S. For a nonzero S-fractional ideal $\mathfrak{A}$ of F and a character χ in ${}^S\mathscr{A}$, define the Dirichlet series

$$D^S(\mathfrak{A}, \mu^N, \chi) = \sum_{\alpha \in \mathfrak{A}^*/{}^SU, (\alpha\mathfrak{A}^{-1}, \mathfrak{N}) = 1} \mu^{S\cup N}(\alpha)\chi^S(\alpha),$$

which converges absolutely for e(χ) large. Put

$$D(\mathfrak{A}, \mu^N, \chi) = D^S(\mathfrak{A}, \mu^N, \chi)\mathbf{L}_S(\mu_S\chi_S).$$

Here and thereafter we employ the notation that for a finite set T of places of F and a function f defined as a product of functions f_v over (almost) all places v of F, f^T denotes the product of f_v over the places outside T, while f_T denotes the product of f_v over the places in T whenever it makes sense.

For a nontrivial additive character ψ of the adèle group of F mod F, denote by $\mathfrak{N}^\perp$ the orthogonal of $\mathfrak{N}$ in F with respect to the restriction ψ_S of ψ to F_S, the product of F_v over v in S. Of course F is embedded in F_S diagonally. As before, write $\check{\chi}$ for $|\,|\chi^{-1}$. Our criterion is as follows.

THEOREM 3. *The character μ^N above has a unique extension to an idèle class character μ of F with the conductor of μ on N equal to $\mathfrak{N}$ if and only if one can choose a set S as above and a character ψ such that the Dirichlet series $D({}^S\mathfrak{O}, \mu^N, \chi)$ and $D(\mathfrak{N}^\perp, (\mu^N)^{-1}, \check{\chi})$ satisfy the three properties:*

(1) *they can be continued analytically to meromorphic functions on ${}^S\mathscr{A}$ with poles on finitely many components,*

(2) *they are of finite order at infinity in each vertical strip of finite width on each component of ${}^S\mathscr{A}$ if F is a number field,*

(3) *there exists a nonzero constant $c(\mu^N, \psi)$ such that the following functional*

equation holds:

$$D({}^S\mathfrak{O}, \mu^N, \chi) = c(\mu^N, \psi)\varepsilon_S(\mu_S\chi_S, \psi_S)\, D(\mathfrak{N}^\perp, (\mu^N)^{-1}, \check{\chi}).$$

2.7. Several remarks are in order. Firstly, if μ exists, then the constant

$$c(\mu^N, \psi) = \mathrm{vol}(F_S/\mathfrak{N}, d_\psi)\, g(\mu^N, \psi^S)$$

with

$$g(\mu^N, \psi^S) = \sum_{\alpha\in\mathfrak{N}^\perp \bmod {}^S\mathfrak{O}^\perp, (\alpha(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1} \mu^N(\alpha)\psi^S(\alpha).$$

Here d_ψ denotes the Haar measure on F_S self-dual with respect to ψ_S. Secondly, note that $D({}^S\mathfrak{O}, \mu^N, \chi)$ is a sub-series of $\mathbf{L}^{S\cup N}(\mu^{S\cup N}\chi^{S\cup N})$, namely, it is summing over the principal S-integral ideals relatively prime to $\mathfrak{N}$. Denote by ${}^S\mathscr{C}$ the group of characters of the S-ideal class group; each character ξ in ${}^S\mathscr{C}$ has a unique extension to an unramified idèle class character of F trivial on $F_S^\times$. Under such identifications, we have

$$D({}^S\mathfrak{O}, \mu^N, \chi) = \frac{1}{{}^Sh} \sum_{\xi\in {}^S\mathscr{C}} \mathbf{L}^N(\mu^N\xi^N\chi^N),$$

which is equal to

$$\frac{1}{{}^Sh} \sum_{\xi\in {}^S\mathscr{C}} \mathbf{L}(\mu\xi\chi),$$

in case μ exists. In our criterion, the larger the set S we choose, the more terms $D({}^S\mathfrak{O}, \mu^N, \chi)$ contains and the more functional equations are needed to prove the existence of μ. Note that $D({}^S\mathscr{D}, \mu^N, \chi)$ stabilizes as soon as ${}^Sh = 1$, that is, ${}^S\mathscr{D}$ is a principal ideal domain. Thirdly, the functional equation simplifies when ${}^Sh = 1$ and ${}^S\mathscr{D}$ is self-dual with respect to ψ_S (which is equivalent to ψ having order 0 at each place outside S). In this case, let n be a generator of the ideal $\mathfrak{N}$, then n^{-1} generates $\mathfrak{N}^\perp$, $D({}^S\mathfrak{O}, \mu^N, \chi) = \mathbf{L}^N(\mu^N\chi^N)$ and $D(\mathfrak{N}^\perp, (\mu^N)^{-1}, \check{\chi}) = \check{\chi}_S(n)\mathbf{L}^N((\mu^N)^{-1}\check{\chi}^N)$. The functional equation reads

$$\mathbf{L}^N(\mu^N\chi^N) = g(\mu^N, \psi^S)\chi^S(n)\varepsilon_S(\mu_S\chi_S, \psi_S)\mathbf{L}^N((\mu^N)^{-1}\check{\chi}^N).$$

Further, if μ exists, this becomes

$$\mathbf{L}(\mu\chi) = g(\mu^N, \psi^S)\chi^S(n)\varepsilon_S(\mu_S\chi_S, \psi_S)\mathbf{L}(\mu^{-1}\check{\chi}),$$

which is the functional equation obtained by Hecke. Fourthly, when F is a number field and μ^N is unramified, we may choose S to contain only the archimedean places of F, Theorem 3 gives a criterion using only functional equations twisted by unramified idèle class characters trivial on the ideal class group. For the case $F = \mathbb{Q}$, this result was previously obtained by M.-F. Vignéras [V].

3. Proof of Theorem 3

3.1. Throughout this section, we fix a set S of places of F: when proving necessity, it is any set as described in Section 2.6 prior to Theorem 3 such that the class number ${}^S h$ is one, and when proving sufficiency, it is the one given in Theorem 3. We shall write K for F_S, R for ${}^S\mathfrak{O}$, and U for ${}^S U$ for brevity. With μ^N given, the problem is to define a character μ_N of $F_N^\times = \Pi_{v\in N} F_v^\times$ of conductor $\mathfrak{N}$ such that the character $\mu = \mu_N \mu^N$ of the group of idèles of F is trivial on $F^\times$. Let x in $F_N^\times$ be a unit at each place of N. Choose any nonzero α in R such that α embedded in F_N is congruent to x modulo $\mathfrak{N}$, then $\mu_N(x) = \mu_N(\alpha)$ which should equal $\mu^N(\alpha)^{-1}$. In order to define $\mu_N(x)$ this way, we need μ^N on R^* periodic modulo $\mathfrak{N}$. Assuming this, we can then extend μ^N uniquely to a character μ' of idèles which are units at the places in N and μ' is trivial on the elements in F contained there. To complete our extension we have to define μ_N on a uniformizer of F_v for all v in N. Given a place v in N, let α be an element of R such that α is a uniformizer in F_v and is a unit at other places in N. Define $\mu_N(\alpha)$ to be $\mu^N(\alpha)^{-1}$. If β is another such an element in R, then α/β is an element of F which is a unit at all places in N; therefore $\mu'(\alpha/\beta) = 1$ as remarked above. This proves the well-definedness of μ_N as a character of $F_N^\times$ with conductor dividing $\mathfrak{N}$ such that the resulting character μ is trivial on $F^\times$.

3.2. We shall apply Theorem 2 to the case where A is chosen to be R, B to be $\mathfrak{N}^\perp$, ψ to be ψ_S, and ω to be μ_S restricted to U embedded in K.

Define the function $\mathbf{a}$ on R^* by

$$\begin{aligned} \mathbf{a}(\alpha) &= \mu^N(\alpha) \quad \text{if } (\alpha R, \mathfrak{N}) = 1,\\ &= 0 \qquad\quad \text{otherwise;} \end{aligned}$$

and the function $\mathbf{b}'$ on $(\mathfrak{N}^\perp)^*$ by

$$\begin{aligned} \mathbf{b}'(\beta) &= \mu^N(\beta)^{-1} \quad \text{if } (\beta(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1,\\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

As μ^N is unramified outside S, it is clear that $\mathbf{a}$ is of type ω and $\mathbf{b}'$ is of type ω^{-1}. We can write $\mathscr{A}_\omega(K^\times)$ as $\mu_S\mathscr{A}_1(K^\times)$, where $\mathscr{A}_1(K^\times)$ consists of characters of $K^\times$ trivial on U. Denote by U_v the group of units in F_v, then the group $H = F^\times K^\times \Pi_{v\notin S} U_v$ has index ${}^S h$ in the group of idèles over F. Since $F^\times \backslash H/\Pi_{v\notin S} U_v$ is isomorphic to $K^\times/U$, the restriction to H yields a surjective homomorphism from ${}^S\mathscr{A}$ to $\mathscr{A}_1(K^\times)$. In particular, we have

$$\mathscr{A}_\omega(K^\times) = \{\mu_S\chi_S \colon \chi \in {}^S\mathscr{A}\}.$$

It follows from the definitions of Dirichlet series and the function $\mathbf{a}$ that, for $\chi \in {}^S\mathscr{A}$,

$$\begin{aligned} D_{\mathbf{a}}(\mu_S\chi_S) &= \sum_{\alpha\in R^*/U} \mathbf{a}(\alpha)(\mu_S\chi_S)(\alpha)^{-1} \\ &= \sum_{\alpha\in R^*/U,(\alpha R,\mathfrak{N})=1} \mu^{S\cup N}(\alpha)\chi^S(\alpha) = D^S(R,\mu^N,\chi), \end{aligned}$$

and similarly, $D_{\mathbf{b}'}(\mu_S^{-1}\check{\chi}_S) = D^S(\mathfrak{N}^\perp, (\mu^N)^{-1}, \check{\chi})$. Then the assertions (1)–(3) in Theorem 3 with $\mathbf{b} = c(\mu^N, \psi)\mathbf{b}'$ read exactly as (S′1) in Theorem 2 when ${}^S h = 1$, and they contain the statement (S′1) when ${}^S h > 1$.

3.3. Suppose first that μ^N can be extended to an idèle class character μ with cond $\mu_N = \mathfrak{N}$. As observed above, μ^N on R^* has period exactly $\mathfrak{N}$.

LEMMA. *If μ^N on R^* has period exactly $\mathfrak{N}$, then, for $\beta \in \mathfrak{N}^\perp$,*

$$\begin{aligned} \sum_{\alpha\in R/\mathfrak{N},(\alpha R,\mathfrak{N})=1} \mu^N(\alpha)\psi^S(\alpha\beta) &= \mu^N(\beta)^{-1}g(\mu^N,\psi^S) \quad \textit{if} \quad (\beta(\mathfrak{N}^\perp)^{-1},\mathfrak{N}) = 1, \\ &= 0 \qquad\qquad \textit{otherwise}. \end{aligned}$$

In other words,

$$\sum_{\alpha\in R/\mathfrak{N}} \mathbf{a}(\alpha)\psi_S(-\alpha\beta) = g(\mu^N,\psi^S)\mathbf{b}'(\beta).$$

Proof. Observe that $\mathfrak{N}^\perp\mathfrak{N} = R^\perp$. If $\beta \in \mathfrak{N}^\perp$ satisfies $(\beta(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1$, then $\beta\alpha$'s with $\alpha \in R/\mathfrak{N}$ and $(\alpha R, \mathfrak{N}) = 1$ run through all the elements γ in $\mathfrak{N}^\perp/R^\perp$ satisfying $(\gamma(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1$, so the first equality is obvious. Now suppose $\beta \in \mathfrak{N}^\perp$ is such that $\beta R = \mathfrak{N}^\perp\mathfrak{B}$ with $\mathfrak{B}$ divisible by a prime factor $\mathfrak{P}$, say, of $\mathfrak{N}$. Write $\mathfrak{N} = \mathfrak{P}\mathfrak{N}'$ and $\mathfrak{B} = \mathfrak{P}\mathfrak{B}'$, where $\mathfrak{N}'$ and $\mathfrak{B}'$ are both ideals of R. Since

$$\beta\mathfrak{N}' = \mathfrak{N}^\perp\mathfrak{N}\mathfrak{B}' \subset \mathfrak{N}^\perp\mathfrak{N},$$

so ψ^S is trivial on $\beta\mathfrak{N}'$. Thus

$$\sum_{\alpha\in R/\mathfrak{N},(\alpha R,\mathfrak{N})=1} \mu^N(\alpha)\psi^S(\alpha\beta)$$

$$= \sum_{\alpha\in R/\mathfrak{N}',(\alpha R,\mathfrak{N})=1} \psi^S(\alpha\beta)\mu^N(\alpha) \sum_{\alpha'\in 1+\mathfrak{N}'/1+\mathfrak{N},(\alpha' R,\mathfrak{N})=1} \mu^N(\alpha') \tag{$*$}$$

and the last sum is zero because μ^N has period exactly $\mathfrak{N}$. This proves the Lemma.

To prove the necessity part of Theorem 3, we shall verify the statement (S′3) in Theorem 2 with our choice of A, B, ω, ψ, $\mathbf{a}$, and

$$\mathbf{b} = \mathrm{vol}(F_S/\mathfrak{N}, d_\psi)g(\mu^N, \psi^S)\mathbf{b}'.$$

Indeed, $A^\perp = R^\perp \subset \mathfrak{N}^\perp = B$ and $B^\perp = \mathfrak{N} \subset R = A$. As $\mathbf{a}$ is a function on R^* of period $\mathfrak{N}$, we may extend it to a function on R mod $\mathfrak{N}$ simply by defining $\mathbf{a}(0)$ to be $\mathbf{a}(\alpha)$ for any nonzero α in $\mathfrak{N}$. The Lemma above implies that $\mathbf{b}'$ and hence $\mathbf{b}$ is a function on $(\mathfrak{N}^\perp)^*$ of period $R^\perp$, so we may extend it to a function on $\mathfrak{N}^\perp$ mod $R^\perp$ by defining $\mathbf{b}(0)$ to be $\mathbf{b}(\beta)$ for any nonzero β in $\mathfrak{N}^\perp$. The relation between $\mathbf{a}$ and $\mathbf{b}$ as described in (S′3) follows from the Lemma above. This proves the necessity part of Theorem 3 with the constant $c(\mu^N, \psi)$ being $\mathrm{vol}(F_S/\mathfrak{N}, d_\psi)g(\mu^N, \psi^S)$, as remarked in Section 2. The Gauss sum $g(\mu^N, \psi^S)$ is nonzero since the Dirichlet series $D(R, \mu^N, \psi)$ is nonzero.

3.4. Conversely, assume that (1)–(3) in Theorem 3 hold. Thus the statement (S′3) in Theorem 2 is valid with $\mathbf{b} = c(\mu^N, \psi)\mathbf{b}'$. This in particular implies that μ^N on R^* has period $\mathfrak{N}$. In view of our discussion in Section 3.1 this means that μ^N has a unique extension to an idèle class character μ whose conductor on N divides $\mathfrak{N}$. It remains to show that the conductor of μ_N is exactly $\mathfrak{N}$. Suppose otherwise; let $\mathfrak{F}$ be the conductor of μ_N, and write $\mathfrak{N} = \mathfrak{F}\mathfrak{M}$ with $\mathfrak{M}$ properly contained in R. Let γ be an element of $\mathfrak{M}$ with $(\gamma R, \mathfrak{N}) = \mathfrak{M}$ and δ be an element of $\mathfrak{N}^\perp$ with $(\delta(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1$. Let $\beta = \delta\gamma$. Then β is in $\mathfrak{N}^\perp\mathfrak{M} = \mathfrak{F}^\perp \subset \mathfrak{N}^\perp$ and $\beta(\mathfrak{N}^\perp)^{-1}$ is not relatively prime to $\mathfrak{N}$. From the definition of $\mathbf{b}'$, we have $\mathbf{b}'(\beta) = 0$ and hence $\mathbf{b}(\beta) = 0$ which, by (S′3), in turn gives rise to

$$0 = \sum_{\alpha\in R/\mathfrak{N}} \mathbf{a}(\alpha)\psi_S(-\alpha\beta) = \sum_{\alpha\in R/\mathfrak{N},(\alpha R,\mathfrak{N})=1} \mu^N(\alpha)\psi^S(\alpha\beta)$$

$$= g'\mu^N(\beta)^{-1} \sum_{\alpha'\in 1+\mathfrak{F}/1+\mathfrak{N},(\alpha' R,\mathfrak{N})=1} \mu^N(\alpha'),$$

where

$$g' = \sum_{\alpha\in R/\mathfrak{F},(\alpha R,\mathfrak{N})=1} \mu^N(\alpha\beta)\psi^S(\alpha\beta),$$

as seen from (*). Since μ_N and hence μ^N is trivial on elements in $1 + \mathfrak{N}'$ relatively prime to $\mathfrak{N}$, we conclude that $g' = 0$. On the other hand, it follows from our choice of γ and δ that $(\gamma\mathfrak{M}^{-1}, \mathfrak{F}) = 1$ and $(\delta(\mathfrak{N}^\perp)^{-1}, \mathfrak{N}) = 1$, therefore $(\beta(\mathfrak{F}^\perp)^{-1}, \mathfrak{F}) = 1$. When α runs through elements in $R/\mathfrak{F}$ with $(\alpha R, \mathfrak{N}) = 1, \alpha\beta$ runs through elements τ in $\mathfrak{F}^\perp/R^\perp$ with $(\tau(\mathfrak{F}^\perp)^{-1}, \mathfrak{F}) = 1$. This shows that $g' = g(\mu^N, \psi^S)$, which, implied by the necessity part of the theorem, is a factor of $c(\mu^N, \psi)$ hence is nonzero, a contradiction. Therefore the conductor of μ_N is exactly $\mathfrak{N}$. The proof of Theorem 3 is now complete.

4. Some tempered distributions

4.1. For a locally compact commutative group G, denote by $\mathscr{S}(G)$ its Schwartz-Bruhat space; its topological dual is the space $\mathscr{S}'(G)$ of tempered distributions on G (Bruhat [Br]). Denote by $\mathscr{D}(G)$ the subspace of compactly supported functions in $\mathscr{S}(G)$, with its direct limit topology of the uniform norm on compact subsets of G; its topological dual $\mathscr{D}'(G)$ is the space of distributions on G. To a discrete closed subset D of G is associated the distribution δ_D sending f to $\Sigma_{a\in D} f(a)$. Write δ_a for $\delta_{\{a\}}$ so that $\delta_D = \Sigma_{a\in D}\delta_a$.

The Pontrjagin dual of G is denoted by G', and the action of the character $y \in G'$ on $x \in G$ is written $(y \mid x)$. We define $(x \mid y)$, for $x \in G$ and $y \in G'$ to be $(y \mid x)^{-1}$. The choice of a Haar measure dx on G defines a Fourier transformation: the function $f \in \mathscr{S}(G)$ is sent to the function $\hat{f}$ in $\mathscr{S}(G')$ by the formula

$$\hat{f}(y) = \int_G f(x)(y \mid x)\,dx,$$

and $f(x) = \int_{G'} \hat{f}(y)(x \mid y)\,dy$ for a unique Haar measure dy on G'. By transposition, we have a Fourier transform on tempered distributions. If D is a discrete subgroup of G with compact quotient, then its orthogonal $D^\perp$ in G' is a discrete subgroup with compact quotient, and the Poisson summation formula says that

$$\mathrm{vol}(G/D, dx)\hat{\delta}_D = \delta_{D^\perp}.$$

If α is an automorphism of G, and f is a function on G, we write f^α for the function $x \mapsto f(\alpha(x))$; then α acts on distributions by the contragredient action.

4.2. A local field K has a canonical absolute value $|\ |$. We denote by E the unit disc and by $E^\times$ the unit circle. We choose for Haar measure on K the one for which E has volume 1 (resp. 2, resp. 2π) for K nonarchimedean (resp. real, resp. complex). Then $d^\times x = |x|^{-1}dx$ is a Haar measure on the multiplicative group $K^\times$ of K. The group $K^\times$ has $E^\times$ for maximal compact subgroup, and its Pontrjagin dual is a one dimensional real Lie group, with connected component the image of $i\mathbb{R}$ by the map which sends s to the character $t \mapsto |t|^s$. By a multiplicative character of $K^\times$ we mean a continuous homomorphism χ from $K^\times$ to $\mathbb{C}^\times$; the preceding map gives to the group $\mathscr{A}(K^\times)$ of all multiplicative characters a one dimensional complex Lie group structure. We define the exponent $\mathrm{e}(\chi)$ of $\chi \in \mathscr{A}(K^\times)$ to be the real number such that the usual absolute value of $\chi(t)$ is $|t|^{\mathrm{e}(\chi)}$, $t \in K^\times$.

4.3. The action on K of $K^\times$ by multiplication defines a continuous action of $K^\times$ on the spaces $\mathscr{S}(K)$, $\mathscr{D}(K)$, $\mathscr{D}(K^\times)$, and also on the distribution spaces $\mathscr{S}'(K)$, $\mathscr{D}'(K)$, $\mathscr{D}'(K^\times)$: we write $t.T$ for the distribution given by the formula

$$\langle t.T | f^t \rangle = \langle T | f \rangle,$$

for any test-function f. A function or a distribution is said to be of type χ if it is multiplied by $\chi(t)$ under the action of any $t \in K^\times$. It is said to be $K^\times$-finite if its transforms by all t in $K^\times$ span a finite dimensional vector space. The structure of the group $K^\times$ shows immediately that the functions $\chi(t)(\log|t|)^n$, for χ in $\mathscr{A}(K^\times)$ and n in $\mathbb{N}$, form a basis of the space of $K^\times$-finite distributions on K. Moreover a distribution on K of type χ is tempered, and the subspace of distributions of type χ on K is one dimensional, as it results from the constructions in Schwartz ([Sc]), Gel'fand-Shilov ([G-S]), Tate ([T]). For $\mathrm{e}(\chi) > 0$, a basis is given by the distribution

$$\langle \Delta_\chi | f \rangle = \int_K f(t)\chi(t) d^\times \mathrm{t}.$$

Let $\mathbf{L}(\chi)$ be the usual $\mathbf{L}$-function attached to the characters χ of $K^\times$: it is the meromorphic function on $\mathscr{A}(K^\times)$ given as follows:

- for K nonarchimedean, $\mathbf{L}(\chi) = 1$ unless $\chi(t) = |t|^s$ where it is $(1 - q^{-s})^{-1}$, with q the cardinality of the residue field of K;
- for K real, $\mathbf{L}(\chi) = \pi^{-s/2}\Gamma(s/2)$ if $\chi(t) = |t|^s$ or $t^{-1}|t|^s$;
- for K complex, $\mathbf{L}(\chi) = 2(2\pi)^{-s-n}\Gamma(s + n)$ if χ restricted to U is read on unitary complex numbers as $u \mapsto u^n$, $n \geqslant 0$, through some isomorphism $K \cong \mathbb{C}$.

Let ∇_χ be the distribution $\Delta_\chi/\mathbf{L}(\chi)$ for $\mathrm{e}(\chi) > 0$; then ∇_χ extends holomorphically to all $\mathscr{A}(K^\times)$, and for each $\chi \in \mathscr{A}(K^\times)$ the distribution ∇_χ is a basis of the subspace of distributions of type χ in $\mathscr{D}'(K)$ and also in $\mathscr{S}'(K)$; its support is K unless χ is a pole of $\mathbf{L}(\chi)$, in which case the support is 0. Denote also by Δ_χ the meromorphic continuation of Δ_χ to $\mathscr{A}(K^\times)$.

PROPOSITION. *The $K^\times$-finite distributions on K are tempered. The image in $\mathscr{D}'(K)$ of the linear map which sends a $K^\times$-finite distribution on K to its restriction to $K^\times$ is the subspace of $K^\times$-finite measures; the kernel of this map consists of the distributions on K with support in 0. The coefficient of $s^n/n!$, $n \geqslant 0$, in the Laurent* *expansion at $s = 0$ of the distribution $\Delta_{\chi|\,|^s}$ has for image the measure $\chi(t)(\log|t|)^n d^\times t$, and is a $K^\times$-finite distribution $\Delta_\chi^{(n)}$.*

Proof. The distributions on K with restriction 0 to $K^\times$ are the distributions supported in 0; each of them is $K^\times$-finite. The image by the restriction map consists of the $K^\times$-finite distributions on $K^\times$. To prove the proposition, it is sufficient to prove the last statement, which results from the expansion $\chi|\,|^s = \Sigma_{n\geqslant 0}\chi(t)(\log|t|)^n s^n/n!$.

As a consequence, a basis for the subspace of $K^\times$-finite distributions on K consists of the distributions $\Delta_\chi^{(n)}$ for $\chi \in \mathscr{A}(K^\times)$ and $n \geqslant 0$ together with the following distributions $\Delta_\chi^{(-1)} = \mathrm{Res}_{s=0}\Delta_{\chi|\,|^s}$ when χ is a pole of $L(\chi)$. By the regular part T_r of a $K^\times$-finite distribution T on K, we mean the sum of its components on the distributions $\Delta_\chi^{(n)}$ for $n \geqslant 0$. An explicit form of ∇_χ is given by the finite part of $\chi(t)(\log|t|)^n$, in the sense of L. Schwartz ([Sc]). For example, in the case of a nonarchimedean field, the distribution $\Delta_\chi^{(0)}$ is given on the test-function f by the integral $\int_K(f(t) - f(0)1_E(t))d^\times t$, where 1_E is the characteristic function of the integers.

4.4. The choice of a nontrivial unitary additive character ψ of K identifies K with its Pontrjagin dual by $(x, y) \mapsto \psi(xy)$. Under the Fourier transformation, a distribution of type χ gives a distribution of type $\check{\chi} = \chi^{-1}|\,|$. The ε-factor is essentially the matrix in the base ∇_χ of the restriction of the Fourier transformation to the subspace spanned by the $K^\times$-eigendistributions:

$$\hat{\nabla}_{\check{\chi}} = \varepsilon(\chi, \psi)\nabla_\chi, \quad \chi \in \mathscr{A}(K^\times);$$

the ε-factor at $\chi|\,|^s$ is an exponential of a linear form in s.

4.5. From now on, assume that K is a finite product of local fields F_v. Denote by $|t|$ the module of the automorphism of the additive group K under the multiplication by $t \in K^\times$. The group $\mathscr{A}(K^\times)$ is the direct product of the groups $\mathscr{A}(F_v^\times)$. Take for Haar measure on the additive group of K the product of the Haar measures d_v on the additive groups of F_v, and for Haar measure $d^\times$ on the multiplicative group of K the product of the Haar measures $d_v^\times$ on the multiplicative groups $F_v^\times$. The distributions Δ_χ and ∇_χ for $\chi \in \mathscr{A}(K^\times)$ are defined as the tensor products of the corresponding distributions on F_v attached to the components χ_v of χ in $\mathscr{A}(F_v^\times)$. On the complex Lie group $\mathscr{A}(K^\times)$, the distribution ∇_χ is holomorphic, and is a basis of the space of distributions of type χ. For n a collection of integers $n_v \geqslant -1$ and χ in $\mathscr{A}(K^\times)$, define the distribution $\Delta_\chi^{(n)}$ to be the tensor product of the corresponding distributions on each F_v associated to the characters $\chi_v \in \mathscr{A}(F_v^\times)$ components of χ and the integers $n_v \geqslant -1$. These distributions are $K^\times$-finite and they form a basis of the subspace of distributions of type χ. Write $(\log|t|)^n$ for the product of the corresponding $(\log|t_v|_v)^{n_v}$; then the functions $\chi(t)(\log|t|)^n$ of t for $\chi \in \mathscr{A}(K^\times)$ and all $n_v \in \mathbb{N}$ form a basis of the space of

$K^\times$-finite functions on K (see for ex. Jacquet-Langlands Section 12 in [J-L]), and the measures $\chi(t)(\log|t|)^n d^\times t$ on $K^\times$ form a basis of the space of $K^\times$-finite measures on $K^\times$. From the results in the local field case, the restriction to $K^\times$ defines a linear map from the space of $K^\times$-finite distributions on K onto the space of $K^\times$-finite measures on $K^\times$; this leads to an isomorphism from the subspace of regular $K^\times$-finite distributions, which consists of those distributions $\Delta_\chi^{(n)}$ with $n_v \geqslant 0$.

LEMMA. *Let T be a $K^\times$-finite distribution on K and θ be a continuous function on an open neighborhood V of 0 in K. Assume that T is given on $V\setminus\{0\}$ by the measure $\theta(x)\,\mathrm{d}x$. Then T is the sum of a regular $K^\times$-finite distribution and of a distribution supported in* 0.

Proof. By restriction to $V \cap K^\times$, the regular part T_r of T is given there by the measure $\theta(x)\,\mathrm{d}x$. This shows that the $K^\times$-finite function defined from T_r by restriction to $K^\times$ extends continuously to V. Hence the function θ extends continuously to K as a $K^\times$-finite function; we still denote this function θ. It defines a tempered distribution $f \mapsto \int_X f(x)\theta(x)\mathrm{d}x$ on X. This distribution is $K^\times$-finite, regular, and has the same restriction to $K^\times$ as T_r has. This proves that $T_r = \theta(x)\,\mathrm{d}x$, hence that $T - T_r$ is supported in 0, and the lemma is proved.

4.6. The ε-factor for K is defined as in the case of a local field: the choice of a unitary additive character ψ of K such that each restriction ψ_v to F_v is nontrivial provides an isomorphism of K with its Pontrjagin dual by $(x\ y) \mapsto \psi(xy)$. Denote by $|\ |$ the character $t \mapsto \Pi|t_v|_v$ of $K^\times$. Then, the Fourier transformation on distributions on K exchanges type $\chi \in \mathscr{A}(K^\times)$ with type $\check{\chi} = \chi^{-1}|\ |$. The ε-factor is defined by the same formula

$$\hat{\nabla}_{\check{\chi}} = \varepsilon(\chi, \psi)\nabla_\chi, \chi \in \mathscr{A}(K^\times);$$

it is the product of the ε-factors $\varepsilon(\chi_v, \psi_v)$.

4.7. Let now A be a discrete subgroup of an invertible K-module X. Assume that A has compact quotient in X.

LEMMA. *The only $K^\times$-finite distributions on X which are fixed under every translation by elements of A are the multiples of Haar measures on X.*

Proof. Call Y the Pontrjagin dual of X, and T a distribution on X as in the statement. The assumption means that the Fourier transform $\hat{T}$ of T is fixed under the multiplications by all characters $y \mapsto (y|\alpha)$, $\alpha \in A$. The orthogonal $A^\perp$ of A in Y is also a discrete subgroup with compact quotient. Hence, the distribution $\hat{T}$ is a (possibly infinite) linear combination of distributions supported at the elements $\beta \in A^\perp$. For each $\beta \in A^\perp$, there is some $\alpha \in A$ such that $(y|\alpha) - 1$ vanishes only of order 1 at β, so $\hat{T}$ is a sum of $c(\beta)\delta_\beta$, $\beta \in A^\perp$, for some tempered function c on $A^\perp$. As the distribution T is $K^\times$-finite, so is its Fourier transform. This implies the existence of a finite set Ξ in $K^\times$ such that the subset $K^\times . \hat{T}$ of $\mathscr{S}'(Y)$ is contained in the subspace span by the $\xi . \hat{T}$for $\xi \in \Xi$. As $t . \hat{T}$is the sum of $c(\beta)\delta_{t^{-1}\beta}$, $\beta \in A^\perp$, if a coefficient $c(\beta_0)$ is not zero, then the element $t^{-1}\beta_0$ appears in the set $\xi^{-1}\beta$, $\xi \in \Xi$, $\beta \in A^\perp$. As this later set is countable, so is the set $K^\times\beta_0$; this implies that $\beta_0 = 0$. We have proved that the distribution $\hat{T}$ is a multiple of δ_0, which means that T is a multiple of some Haar measure on X.

4.8. The Schwartz-Bruhat space $\mathscr{S}(K)$ of K is the tensor product of the Bruhat spaces of the nonarchimedean simple ideals of K by the Schwartz space of the product of the archimedean simple ideals of K. By a vertical strip in $\mathscr{A}(K^\times)$, we mean a vertical strip of finite width in some connected component of $\mathscr{A}(K^\times)$, that is, a subset of the form $\{\chi| \,|^s: -c < \operatorname{Re} s < c\}$. A function g on $\mathscr{A}(K^\times)$, defined outside a compact subset, is said to have finite order in vertical strips if the function $\log|g(\chi)|$ has polynomial growth in each vertical strip of $\mathscr{A}(K^\times)$. A distribution T_χ depending on $\chi \in \mathscr{A}(K^\times)$ for χ outside a compact subset is said to have finite order in vertical strips if it is the case for each function $\langle T_\chi | f \rangle$, where f is any test-function on K.

LEMMA. *The distribution ∇_χ has finite order in vertical strips of $\mathscr{A}(K^\times)$.*

Proof. When K is a non archimedean field of module q, then, for each function f of its Bruhat space, the value at f of the distribution $\nabla_{\chi| |^s}$ is a Laurent polynomial in q^s, hence is periodic with period $2\pi i/\log q$; in this case, the assertion is proved. In general it is sufficient to prove it for K without nonarchimedean simple ideals. From the definition of the L-function in the case of $\mathbb{R}$ and $\mathbb{C}$, the Stirling formula shows that $1/\mathbf{L}(\chi)$ has finite order in vertical strips. Hence, it is sufficient to prove the lemma with ∇_χ replaced by Δ_χ. For $\chi \in \mathscr{A}(K^\times)$, we order the simple ideals of K so that $\mathrm{e}(\chi_1) \geqslant \mathrm{e}(\chi_2) \geqslant \cdots \geqslant \mathrm{e}(\chi_r)$, and we cover $\mathbb{R}$ by the open intervals $(-\infty, 1 - \mathrm{e}(\chi_1)), (-\mathrm{e}(\chi_1), -\mathrm{e}(\chi_2) + 1/2), \ldots, (-\mathrm{e}(\chi_r), \infty)$. When s lies in the ith interval, we make on f a Fourier transform with respect to the variables indexed by all $j \geqslant i$ in order to get an expression of $\langle \Delta_{\chi| |^s} | f \rangle$ as an absolutely convergent integral:

$$\int_{K^\times} f^j(t) \Big(\prod_{k<i} \chi_k(t_k) |t_k|_k^s \Big) \Big(\prod_{j \geqslant i} \chi_j(t_j)^{-1} |t_j|_j^{1-s} \, d^\times t \Big)$$

times

$$\prod_{j \geqslant i} \mathbf{L}(\chi_j| \,|_j^s) / \varepsilon_j(\chi_j| \,|_j^s, \psi_j) \mathbf{L}(\chi_j^{-1}| \,|_j^{1-s}).$$

The integral term is bounded in the vertical trips for which $\operatorname{Re} s$ lies in the ith interval, and the second term is, away from some compact in the ith interval, bounded in the ith interval. This shows that each function $\langle \Delta_{\chi|\,|^s} | f \rangle$ is bounded at infinity in vertical strips, and the lemma is proved.

PROPOSITION. *Let $h(\chi)$ be a function defined for $\chi \in \mathscr{A}(K^\times)$ outside some compact subset. Then, the distribution $h(\chi)\Delta_\chi$ has finite order in vertical strips if and only if the function $h(\chi)\mathbf{L}(\chi)$ does.*

Proof. As $h(\chi)\Delta_\chi = h(\chi)\mathbf{L}(\chi)\nabla_\chi$, the lemma shows that the distribution $h(\chi)\Delta_\chi$ has finite order in vertical strips when the function $h(\chi)\mathbf{L}(\chi)$ does. Conversely, we use the fact that for any $\chi \in \mathscr{A}(K^\times)$ there exists a test-function on which $\Delta_{\chi|\,|^s}$ takes the value $\mathbf{L}(\chi|\,|^s)$ for all complex numbers s; it is sufficient to prove it when K is a local field, and then, it is done in Tate's thesis ([T]).

5. Proof of Theorem 1

5.1. Fix a global field F, and define S, R, U, K as in Section 2. Let now X be an invertible K-module. Denote by $X^\times$ the subset of generators of X over K. Its complement in X is the union of the maximal K-submodules. There is a measure $d^\times x$ on $X^\times$ invariant under the action of $K^\times$, arising from the measure $d^\times t$ on $K^\times$ by any choice of a basis of X over K and it is independant of this choice. If A is an invertible R-submodule of X, then $KA = X$ and the set A^* of nonzero elements in A is contained in $X^\times$; moreover, the subgroup A of X is discrete with compact quotient.

5.2. Let $(X, A, \mathbf{a}, \omega)$ and $(Y, B, \mathbf{b}, \omega^{-1})$ be two quadruples as in Theorem 1. We begin to prove the equivalence between the statements (S2) and (S3) with the same choice of $\mathbf{a}(0)$ and $\mathbf{b}(0)$. Observe first that if the subgroups $A^\perp$ and $B^\perp$ are orthogonal, then A contains $B^\perp$ with finite index, and B contains $A^\perp$ with finite index. This gives a meaning to (S3). The equivalence between (S2) and (S3) is easy. We use the distributions $\Theta_\mathbf{a}$ and $\Theta_\mathbf{b}$ defined in the remark after Theorem 1. The invariance of each measure $\delta_\beta, \beta \in B$, under products by characters of Y lying in $B^\perp$ shows the invariance of $\hat{\Theta}_\mathbf{b}$ under translations from $B^\perp$. Assuming (S2), we get the invariance of $\Theta_\mathbf{a}$ under translations from $B^\perp$. As the support of $\Theta_\mathbf{a}$ is contained in the subgroup A and is not empty, this invariance implies the inclusion $B^\perp \subset A$. By orthogonality, we get $A^\perp \subset B$. Write now $\langle \Theta_\mathbf{a} | f \rangle = \Sigma_{\alpha \in A} \mathbf{a}(\alpha) f(\alpha)$ as

$$\sum_{\alpha \in A/B^\perp} \sum_{\beta' \in B^\perp} \mathbf{a}(\alpha) f(\alpha + \beta');$$

apply the Poisson summation formula to rewrite this as

$$\int_{A/B^\perp} \mathbf{a}(\alpha) \sum_{\beta \in B} (\alpha | \beta) \hat{f}(\beta) = \sum_{\beta \in B} \int_{A/B^\perp} \mathbf{a}(\alpha)(\alpha | \beta) \hat{f}(\beta),$$

which is equal to $\langle \hat{\Theta}_{\mathbf{b}} | f \rangle$, that is, to $\Sigma_{\beta \in B} \mathbf{b}(\beta) \hat{f}(\beta)$, for any function $\hat{f}$ in $\mathscr{S}(Y)$. This implies the equality

$$\mathbf{b}(\beta) = \int_{A/B^{\perp}} \mathbf{a}(\alpha)(\alpha | \beta),$$

and, by symmetry between $(X, A, \mathbf{a})$ and $(Y, B, \mathbf{b})$, or by inverse Fourier transform, also the corresponding relation for $\mathbf{a}(\alpha)$ in terms of $\mathbf{b}(\beta)$. This proves the implication from (S2) to (S3). Reversing the above steps proves the opposite implication.

5.3. Next we prove the implication from (S1) to (S2). It essentially amounts to taking the inverse Mellin transform of both sides of the functional equation in (S1). We have to check the conditions allowing it. By the assumption (S1), there is a tempered distribution $\Lambda(\chi)$ on X, depending holomorphically on $\chi \in \mathscr{A}_{\omega}(K^{\times})$, given by $D_{\mathbf{a}}(\chi, x) d^{\times} x$ for $\mathrm{e}(\chi)$ large enough, and by the Fourier transform of $D_{\mathbf{b}}(\chi, y) d^{\times} y$ for $-\mathrm{e}(\chi)$ large enough. Also, for $\mathrm{e}(\chi)$ large enough and for x_o in $X^{\times}$, the integral $\int_{X^{\times}} f(x)\chi(x/x_o) d^{\times} x$ is uniformly convergent for any f in $\mathscr{S}(X)$. Hence, the product $D_{\mathbf{a}}(\chi, x_o) \int_{X^{\times}} f(x)\chi(x/x_o) d^{\times} x$ is holomorphic for $\mathrm{e}(\chi)$ large enough. It equals $\int_{X^{\times}} f(x) D_{\mathbf{a}}(\chi, x) d^{\times} x$, that is, $\langle \Lambda(\chi) | f \rangle$. We transform this expression:

$$\begin{aligned}
\langle \Lambda(\chi) | f \rangle &= \int_{X^{\times}} \sum_{\alpha \in A^*/U} \mathbf{a}(\alpha) f(x) \chi(x/\alpha)\, d^{\times} x \\
&= \int_{X^{\times}/U} \sum_{\gamma \in U} \sum_{\alpha \in A^*/U} \mathbf{a}(\alpha) f(\gamma x) \omega(\gamma) \chi(x/\alpha) d^{\times} x \\
&= \int_{X^{\times}/U} \sum_{\alpha \in A^*/U} \sum_{\gamma \in U} \mathbf{a}(\alpha\gamma) f(\gamma x) \chi(x/\alpha) d^{\times} x \\
&= \int_{K^{\times}/U} \sum_{\alpha \in A^*/U} \sum_{\gamma \in U} \mathbf{a}(\alpha\gamma) f(\alpha\gamma t) \chi(t) d^{\times} t \\
&= \int_{K^{\times}/U} \sum_{\alpha \in A^*} \mathbf{a}(\alpha) f(\alpha t) \chi(t) d^{\times} t \\
&= \int_{K^{\times}/U} \langle \Theta_{\mathbf{a}}^* | f^t \rangle \chi(t) d^{\times} t \\
&= \int_{K^{\times}/U} \langle t . \Theta_{\mathbf{a}}^* | f \rangle \chi(t)^{-1} d^{\times} t, \\
&= \left\langle \int_{K^{\times}/U} t . \Theta_{\mathbf{a}}^* \chi(t)^{-1} d^{\times} t \,\middle|\, f \right\rangle,
\end{aligned}$$

where we have introduced the tempered distribution

$$\Theta_{\mathfrak{a}}^{*} = \sum_{A^*} \mathbf{a}(\alpha)\delta_{\alpha}.$$

This means that for $\mathrm{e}(\chi)$ large enough the distribution $\Lambda(\chi)$ is the Mellin transform on $\mathscr{A}_{\omega}(K^{\times})$ of the distribution $\Theta_{\mathfrak{a}}^{*}$. In a similar way, we have, for $-\mathrm{e}(\chi)$ large enough,

$$\langle \Lambda(\chi) | \hat{f} \rangle = \int_{K^{\times}/U} \langle \Theta_{\mathfrak{b}}^{*} | \hat{f}^{t} \rangle \check{\chi}(t) d^{\times} t = \int_{K^{\times}/U} \langle \hat{\Theta}_{\mathfrak{b}}^{*} | f^{t} \rangle \check{\chi}(t) d^{\times} t,$$

and $\Lambda(\chi)$ is there the Mellin transform on $\mathscr{A}_{\omega}(K^{\times})$ of the distribution $\hat{\Theta}_{\mathfrak{b}}^{*}$. In a given component of $\mathscr{A}_{\omega}(K^{\times})$, the function $\langle \Lambda(\chi) | f \rangle$ is bounded outside a vertical strip of finite width, due to its integral representation; the finite order assumption allows us to apply the Phrágmen-Lindelöf principle, so $\langle \Lambda(\chi) | f \rangle$ is bounded at infinity on each vertical strip of finite width in each component of $\mathscr{A}_{\omega}(K^{\times})$.

For each ϕ in the space $\mathscr{D}(K^{\times})$, the map $\chi \mapsto \langle \Delta_{\chi} | \phi \rangle$ has fast decay on vertical lines. We choose ϕ in $\mathscr{D}(K^{\times})$ such that $\langle \Delta_{\chi} | \phi \rangle = 0$ for all but finitely many connected components of $\mathscr{A}_{\omega}(K^{\times})$, a condition which is void unless there are complex places in F. The function $\phi_{\omega}(t) = \Sigma_{\gamma \in U} \phi(\gamma t)\omega(\gamma)$ on $K^{\times}$ has type ω^{-1}, and its support is compact mod U; moreover, for $\chi \in \mathscr{A}_{\omega}(K^{\times})$,

$$\langle \Delta_{\chi} | \phi \rangle = \int_{K^{\times}/U} \phi_{\omega}(t)\chi(t) d^{\times} t.$$

Now, the function $\chi \mapsto \langle \Delta_{\chi} | \phi \rangle \langle \Lambda(\chi) | f \rangle$ is integrable on any vertical line with $\mathrm{e}(\chi)$ large enough. Denote by $\int^{+}$ the integral over the union of lines $\mathrm{e}(\chi) = \sigma \gg 0$ in $\mathscr{A}_{\omega}(K^{\times})$. Then

$$\begin{aligned} \int^{+} \langle \Delta_{\chi} | \phi \rangle \langle \Lambda(\chi) | f \rangle \, \mathrm{d}\chi &= \int_{K^{\times}/U} \langle t.\Theta_{\mathfrak{a}}^{*} | f \rangle \int^{+} \langle \Delta_{\chi} | \phi \rangle \chi(t)^{-1} \, d\chi d^{\times} t \\ &= \int_{K^{\times}/U} \phi_{\omega}(t^{-1}) \langle \Theta_{\mathfrak{a}}^{*} | f^{t} \rangle \mathrm{d}^{\times} t. \end{aligned}$$

We have a similar formula for $\int^{-}$, coming from $-\mathrm{e}(\chi)$ large, with $\hat{\Theta}_{\mathfrak{b}}^{*}$ replacing $\Theta_{\mathfrak{a}}^{*}$. This leads to:

$$\left(\int^{+} - \int^{-}\right) \langle \Delta_{\chi} | \phi \rangle \langle \Lambda(\chi) | f \rangle d\chi = \int_{K^{\times}/U} \phi_{\omega}(t^{-1}) \langle \Theta_{\mathfrak{a}}^{*} - \hat{\Theta}_{\mathfrak{b}}^{*} | f \rangle d^{\times} t.$$

By the "bounded at infinity" property of $\langle\Lambda(\chi)\,|\,f\rangle$, the left hand side can be expressed as a limit of contour integrals, hence is equal to the sum of the residues of $\langle\Delta_\chi\,|\,\phi\rangle\langle\Lambda(\chi)\,|\,f\rangle d\chi$ on $\mathscr{A}_\omega(K^\times)$. Note that there are only finitely many poles due to our assumption.

Choose now for functions ϕ an approximation of unity on $K^\times$; as the function $t\mapsto\langle t.(\Theta_{\mathbf{a}}^* - \hat{\Theta}_{\mathbf{b}}^*)\,|\,f\rangle$ is continuous at 1, we get finally

$$\Theta_{\mathbf{a}}^* - \hat{\Theta}_{\mathbf{b}}^* = \Sigma \operatorname{Res} \Lambda(\chi).$$

In this equality, the distribution $\Lambda(\chi)$ is the product of the meromorphic distribution Δ_χ by a meromorphic function on $\mathscr{A}_\omega(K^\times)$, hence the residues are finite linear combinations of derivatives of $s\mapsto\Delta_{\chi|\,|^s}$, and the right hand side is a $K^\times$-finite distribution. As the distribution $\Theta_{\mathbf{b}}^*$ depends only on the restriction to B of the Fourier transform of the test-function, it is invariant under translations by the elements of the orthogonal $B^\perp$ of B. Now, for β' in $B^\perp$, the distribution $T = \Theta_{\mathbf{a}}^* - \hat{\Theta}_{\mathbf{b}}^*$ has the property that $T - T_*\delta_{\beta'} = \Theta_{\mathbf{a}}^* - \Theta_{\mathbf{a}}^*{}_*\delta_{\beta'}$ is supported in a discrete subset of X. Choose a nonzero element β' in $B^\perp$. Since T is $K^\times$-finite, the distribution $T_*\delta_{\beta'}$ is given in a neighborhood of 0 by a measure $\theta(x)\mathrm{d}x$ for some continuous function θ. As A is a discrete subset of X, there is an open neighborhood V of 0 such that $T = T_*\delta_{\beta'}$ on $V\backslash\{0\}$. By the lemma in Section 4.5, the difference between T and its regular part T_r is supported in 0, and the distribution T_r is given by integration against a continuous function. The relation $T = T_*\delta_{\beta'}$ outside a discrete subset of X for β' in $B^\perp$ implies that the distribution T_r is invariant under additive translations by $B^\perp$. This subgroup is discrete with compact quotient, hence by the lemma in Section 4.7, the distribution T_r is a multiple, say, $\mathbf{b}(0)\mathrm{d}x$ of the Haar measure $\mathrm{d}x$ on X. Since $(X, A, \mathbf{a})$ and $(Y, B, \mathbf{b})$ play symmetric roles, we obtain also that the regular part of the Fourier transform of $-T$ is a multiple $\mathbf{a}(0)\mathrm{d}y$ of the Haar measure on Y. As the Fourier transform of a distribution supported in 0 is a regular $K^\times$-finite distribution, the regular part of the Fourier transform of $-T$ is the Fourier transform of $T_r - T$, and $T_r - T = \mathbf{a}(0)\delta_0$. This proves that the distributions $\mathbf{a}(0)\delta_0 + \Theta_{\mathbf{a}}^*$ and $\mathbf{b}(0)\delta_0 + \Theta_{\mathbf{b}}^*$ are Fourier transform of each other, which is the statement (S2).

5.4. Finally, we prove that (S2) implies (S1). The method follows essentially Tate's thesis ([T]). From Section 5.2 we know that for $\chi\in\mathscr{A}_\omega(K^\times)$ with $\mathrm{e}(\chi)$ large enough, the Mellin transform of the distribution $\Theta_{\mathbf{a}}^* = \Theta_{\mathbf{a}} - \mathbf{a}(0)\delta_0$ is the distribution $D_{\mathbf{a}}(\chi, x)d^\times x$, and for $-\mathrm{e}(\chi)$ large enough the Mellin transform of the distribution $\Theta_{\mathbf{b}}^* = \Theta_{\mathbf{b}} - \mathbf{b}(0)\delta_0$ is $D_{\mathbf{b}}(\chi, y)d^\times y$. We know also that $\mathbf{a}(0) = \mathbf{b}(0) = 0$ if the character ω is not trivial, as remarked in Section 2. We decompose the Mellin transform of $\Theta_{\mathbf{a}}^*$ into a sum of three tempered distributions:

$$\int_{K^\times/U} t.\Theta_{\mathbf{a}}^*\chi(t)^{-1}d^\times t = \left(\int_{K^+/U} + \int_{K^1/U} + \int_{K^-/U}\right)t^{-1}.\Theta_{\mathbf{a}}^*\chi(t)d^\times t,$$

where K^- and K^+ are those elements of $K^\times$ with absolute value <1 and >1, respectively. The middle term does not contribute in the case of a number field. When applied to a function f in $\mathscr{S}(X)$, the first term gives $\int_{K^+/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times t$, which is an entire function of $\chi\in\mathscr{A}(K^\times)$, and is bounded on left half planes of the form $\mathrm{e}(\chi)<c$ in the number field case. The second term is also an entire function of $\chi\in\mathscr{A}_\omega(K^\times)$ since the integral is over the compact set K^1/U. The third term will be written in terms of $\Theta_{\mathfrak{b}}^*$ using the statement (S2); note that the two last terms in the following expression do not occur if ω is nontrivial:

$$\int_{K^-/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times \mathrm{t}=\int_{K^-/U}\langle\hat{\Theta}_{\mathfrak{b}}^*\,|f^t\rangle\chi(t)d^\times t+\int_{K^-/U}\mathbf{a}(0)f(0)\chi(t)d^\times t$$
$$+\int_{K^\times/U}\mathbf{b}(0)\hat{f}(0)\chi(t)^{-1}d^\times t.$$

The integral $\int_{K^-/U}\langle\hat{\Theta}_{\mathfrak{b}}^*\,|f^t\rangle\chi(t)d^\times t$ is equal to $\int_{K^+/U}\langle\Theta_{\mathfrak{b}}^*\,|\hat{f}^t\rangle\check{\chi}(t)d^\times t$, hence it is entire, and is bounded on left half-planes of the form $\mathrm{e}(\chi)<c$ for the number field case. The last two terms are easy to see: they are meromorphic functions supported on $\mathscr{A}(K^\times/U)$ with only possible poles at $\chi=1$ and $\chi=|\ |$, and the poles are simple. This proves the meromorphic continuation of the distribution $D_{\mathfrak{a}}(\chi,x)d^\times x$ to all $\mathscr{A}_\omega(K^\times)$, and the behavior in vertical strips in the number field case.

It remains to prove the functional equation that the distributions $D_{\mathfrak{a}}(\chi,x)d^\times x$ and $D_{\mathfrak{b}}(\chi,y)d^\times y$ are Fourier transforms of each other. As discussed in Section 5.4, this amounts to showing that

$$\int_{K^\times/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times t=\int_{K^\times/U}\langle\Theta_{\mathfrak{b}}^*\,|\hat{f}^t\rangle\check{\chi}(t)d^\times t$$

for all $f\in\mathscr{S}(X)$. Our computations above can be summarized as

$$\int_{K^\times/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times t=\int_{K^+/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times t+\int_{K^+/U}\langle\Theta_{\mathfrak{b}}^*\,|\hat{f}^t\rangle\check{\chi}(t)d^\times t$$
$$+\int_{K^1/U}\langle\Theta_{\mathfrak{a}}^*\,|f^t\rangle\chi(t)d^\times t-\mathbf{a}(0)f(0)\int_{K^-/U}\chi(t)d^\times t$$
$$+\mathbf{b}(0)\hat{f}(0)\int_{K^-/U}\check{\chi}(t)^{-1}d^\times t.$$

The assumption $\Theta_{\mathfrak{a}}=\hat{\Theta}_{\mathfrak{b}}$ yields

$$\langle\Theta_{\mathfrak{a}}|f^t\rangle\chi(t)=\langle\hat{\Theta}_{\mathfrak{b}}|f^t\rangle\chi(t)=\langle\Theta_{\mathfrak{b}}|\hat{f}^{t'}\rangle\check{\chi}(t'),$$

where $t' = t^{-1}$. For t in K^1 this gives rise to:

$$\int_{K^1/U} \langle \Theta_{\mathbf{a}} | f^t \rangle \chi(t) d^\times t = \int_{K^1/U} \langle \Theta_{\mathbf{b}} | \hat{f}^t \rangle \check{\chi}(t) d^\times t.$$

Thus, it suffices to check that the expression

$$-\mathbf{a}(0) f(0) \int_{(K^\times - K^+)/U} \chi(t) d^\times t + \hat{\mathbf{b}}(0) f(0) \int_{(K^\times - K^+)/U} \check{\chi}(t) d^\times t$$

is invariant when $(\mathbf{a}, f, \chi)$ is replaced by $(\mathbf{b}, \hat{f}, \check{\chi})$. Indeed, the integral is nonzero only if χ lies in the identity component of $\mathscr{A}_1(K^\times) = \mathscr{A}(K^\times/U)$, that is, for $\chi = |\ |^s$, in which case

$$\int_{(K - K^+)/U} \chi(t) d^\times t = \operatorname{vol}(K^1/U) \int_0^1 t^s d^\times t = \operatorname{vol}(K^1/U)/s,$$

and

$$\int_{(K - K^+)/U} \check{\chi}(t)^{-1} d^\times t = \operatorname{vol}(K^1/U) \int_0^1 t^{s-1} d^\times t = \operatorname{vol}(K^1/U)/(s-1).$$

This proves the desired functional equation. The proof of Theorem 1 is now completed.

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