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# Invariant analysis and contractions of symmetric spaces*

## *Part I*

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**Abstract.** For any symmetric space $S = G/H$, we define and study a function $e(X, Y)$ of two tangent vectors at the origin of $S$, obtained from the corresponding infinitesimal structure of Lie triple system. Our approach to $e$ relies on contractions of $S$ into its tangent space.

The exponential mapping carries convolution products of $H$-invariant functions on $S$ into ordinary convolutions on the tangent space, twisted by $e$; thus this function plays a significant rôle in harmonic analysis on $S$.

## Introduction

1. This paper is motivated by the following related problems.

PROBLEM 1. Can one transform an invariant differential operator on a homogeneous space into a constant coefficients differential operator on some vector space? Answering the question in the affirmative for a single operator leads to solvability results for this operator. Doing it simultaneously for all invariant operators can give informations on the algebra of all these operators, and on their joint eigendistributions; therefore it is a tool for harmonic analysis on the given homogeneous space.

Here we consider the case of a simply connected symmetric space $S = G/H$, with the algebra $\mathbb{D}(S)$ of all $G$-invariant linear differential operators on $S$; by invariant analysis, we mean the study of $H$-invariant functions (or distributions) $u$ on $S$. We look for a map $u \to u'$, where $u'$ is a function on some vector space $V$, and a map $D \to D'$ from $\mathbb{D}(S)$ into an algebra $\mathbb{D}(V)$ of constant coefficients

*This article is dedicated to François Trèves.

differential operators on $V$, such that

$$(Du)' = D'u'. \tag{1}$$

PROBLEM 2. An inflated sphere tends to a plane, its Laplace-Beltrami operator tends to the Euclidean Laplacian, Legendre polynomials (eigenfunctions of the former) tend to Bessel functions (eigenfunctions of the latter).... These well-known facts extend to more general symmetric spaces (see J.-L. Clerc [1], A.H. Dooley [2], A.H. Dooley–J.W. Rice [3]...), by means of Lie group contractions: inflating a sphere amounts to contracting its motion group SO(3) into the Euclidean motion group of the plane.

Our second problem is: can we go backwards? Can harmonic analysis on a symmetric space be deduced from harmonic analysis on its tangent space? The answer is obviously no, as the same Euclidean plane appears as the limit of a sphere, or torus, or hyperbolic disc.... But we shall see that much of the lacking information can be obtained from the corresponding infinitesimal structure of Lie triple system, through one function defined on the tangent space. This will provide a common approach to Problems 1 and 2.

The aim of this paper is to develop the formal tools required in this approach, with first applications to Problem 1. The second question will be considered in a forthcoming paper.

2. As regards problem 1, three examples are well-known.

EXAMPLE 1. $S$ is a semi-simple Lie group $G_0$ considered as a symmetric space, that is $G = G_0 \times G_0$ and $H$ is the diagonal subgroup. Then $D$ is a bi-invariant operator on $G_0$, and $u$ is a conjugacy invariant function. Equality (1) holds taking as $V$ the tangent space at the origin of $S$ (i.e. the Lie algebra of $G_0$) and $u'(X) = j(X)^{1/2}u(\exp X)$, where $j$ is the Jacobian of the exponential mapping at $X \in V$ (Harish-Chandra [8], 1965). This result was extended in different ways by M. Duflo [4] (1977), and M. Kashiwara–M. Vergne [12] (1978).

EXAMPLE 2. $G$ is a complex semi-simple Lie group, $K$ is a maximal compact subgroup, and $S = G/K$. Then (1) holds taking as $V$ the tangent space $S_0$ at the origin of $S$, and $u'(X) = J(X)^{1/2}u(\operatorname{Exp} X)$, where $J$ is the Jacobian of the exponential mapping $\operatorname{Exp}: S_0 \to S$ (S. Helgason [9], 1964).

EXAMPLE 3. This last result is no longer true if we drop the assumption $G$ complex, but (1) still holds taking as $V$ a Cartan subspace of $S_0$, and replacing the map $'$ by the Radon (or Abel) transform (S. Helgason [9], 1964).

The proofs of these results require a deep knowledge of the structure of semi-simple Lie groups, although it should be natural (for Examples 1 and 2 at least) to search for a proof only relying on general properties of the exponential mapping. This is the point of view chosen here, in the spirit of Kashiwara-Vergne [12]. The method applies to any $S$, however gives complete answers only in certain cases – up to now.

3. Let us now describe our results more precisely. Throughout $S = G/H$ is a simply connected symmetric space, exp its exponential mapping, and $J$ the Jacobian of exp. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ be the decomposition of the Lie algebra of $G$ given by the symmetry, and $\mathfrak{s}'$ an "invariant exponential subset" of $\mathfrak{s}$ (see §2.2), such that Exp is a diffeomorphism of $\mathfrak{s}'$ onto $S' = \operatorname{Exp} \mathfrak{s}'$. If $u$ is a function on $\mathfrak{s}'$ we define a function $\tilde{u}$ on $S'$ by

$$u(X) = J(X)^{1/2}\tilde{u}(\operatorname{Exp} X)$$

(thus $\tilde{}$ will be the inverse map of $'$ above). Then for $H$-invariant $u$ and $v$, considered as densities, we have (Proposition 4.1)

$$\langle \tilde{u} * \tilde{v}, \tilde{f} \rangle = \langle u(X)v(Y), e(X, Y)f(X + Y)\rangle \tag{2}$$

for any test function $f$ on $\mathfrak{s}'$. Here $*$ is the convolution product on the symmetric space $S$ (under some assumption on the supports of $u$ and $v$), the brackets mean duality of distributions and functions on $S$ and $\mathfrak{s} \times \mathfrak{s}$ respectively, and $e(X, Y)$ is a specific function of two vectors in $\mathfrak{s}$ which will be described below.

The previous paper [16] was entirely devoted to the case (now called *special*) when $e$ is identically one. Then (2) implies (Proposition 4.3)

$$\tilde{u} * \tilde{v} = (u * v)\tilde{} \quad \text{on } S' \tag{3}$$

for any $H$-invariant functions, or distributions, on the tangent space (with suitable supports); the $*$ on the right hand side of (3) is the ordinary convolution on the vector space $\mathfrak{s}$. In particular, this solves Problem 1, when taking $v$ supported at the origin, i.e. an invariant differential operator (see §4.3, and [16] §6–7 for more details on this case).

In the forthcoming Part II it will be shown that, for Riemannian symmetric spaces, the spherical functions of $S$ are (locally) entirely determined by the function $e$ of $S$, together with the structure of the flat symmetric space $\mathfrak{s}$ (with action of $H$). We shall also investigate the relations between $e$, the Radon transform, and the spherical Plancherel measure of $S$, by means of expansions with respect to some contraction parameter.

4. The function $e$ arises as follows. A *Campbell–Hausdorff formula* ("Schur's formula" should be more appropriate, according to J.J. Duistermaat) for $S$ is an expression of the vector $Z(X, Y)$ which describes the action (.) of $G$ on $S$

in exponential coordinates:

$$\operatorname{Exp} Z(X, Y) = \operatorname{Exp} X \,.\, \operatorname{Exp} Y, \; X, Y \in \mathfrak{s}.$$

Finding $Z$ is computing the third side of a certain geodesic triangle in $S$ (see §2). Locally near the origin, $Z$ can be written as (Theorem 2.2)

$$Z(X, Y) = h.X + k.Y, \tag{4}$$

where dots mean here adjoint action of two elements $h, k$ of $H$, depending on $X$ and $Y$ (more precisely, $h$ and $k$ belong to the "holonomy subgroup" with Lie algebra $[\mathfrak{s}, \mathfrak{s}]$). Besides, the map $\Phi: (h.X, k.Y) \to (X, Y)$ is (locally) an analytic diffeomorphism of $\mathfrak{s} \times \mathfrak{s}$ onto itself, transforming $Z$ into the corresponding function for the flat case: $Z_0(X, Y) = X + Y$. This diffeomorphism is obtained by solving differential equations with respect to a variable $t$; the meaning of this method (learnt from Moser, Duistermaat) is to flatten the space $S$ into its tangent space $S_0 = \mathfrak{s}$ through a family of symmetric space structures $S_t$, with $0 \leqslant t \leqslant 1$, and to follow the evolution of $Z(X, Y)$ etc. The relevant definitions on *contractions* are given in Section 1; they are expressed simply by means of Lie triple systems, the infinitesimal analogue of symmetric spaces.

Now the $e$-function can be defined in terms of Jacobians (Proposition 3.14) by

$$e(X, Y) = \left(\frac{J(X)J(Y)}{J(X + Y)}\right)^{1/2} \det D\Phi(X, Y), \tag{5}$$

assuming (for simplicity) that $S$ has a $G$-invariant measure. The proof of (2) above is then a mere change of variables in an integral, by means of $\Phi$.

The equality (4) might have independent interest. In fact, putting the Campbell–Hausdorff formula of a matrix Lie group under a form similar to (4):

$$e^X e^Y = e^{hXh^{-1} + kY k^{-1}}$$

was a problem raised in 1979 by R.C. Thompson, who solved it (globally) for unitary groups; see [17], [18], and Section 2.4 hereunder.

5. To study $e(X, Y)$, which is the main goal of this paper, it is convenient to use a more technical definition (§3.3). Without going here into details, we mention that $e$ is obtained from the trace of a specific endomorphism $E(X, Y)$ of $\mathfrak{h}$ (§3.2). For given $X$ and $Y$, $E(X, Y)$ belongs to the algebra $\mathscr{A}$ of (formal) series in the non-commuting variables $x = \operatorname{ad} X$ and $y = \operatorname{ad} Y$. The main result of Section 3 (Theorem 3.15) states that $E(X, Y)$ actually belongs to the two-sided ideal of $\mathscr{A}$ generated by $xy - yx$; the proof of this result is postponed to

Section 5. An easy corollary is that $e(X, Y) = 1$ whenever $X$ and $Y$ belong to a solvable subalgebra of $\mathfrak{g}$ (Corollary 3.16); in particular $S$ is special when $G$ is a solvable group.

We conjecture that $E(X, Y)$ belongs to the smaller subspace $[\mathscr{A}, \mathscr{A}]$ (Conjecture 3.9). This would imply that the following spaces are special (Proposition 4.5):

- $S = G_{\mathbb{C}}/G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is a real form of a complex Lie group $G_{\mathbb{C}}$;
- $S = G \times G/\text{diagonal}$, i.e. any Lie group considered as a symmetric space.

Thus Conjecture 3.9 can be considered as a variant of the Kashiwara-Vergne conjecture in [12]. I could only check it up to order 7 in $x$ and $y$, by explicit computation of the first terms in the series $E$ (Lemma 3.8).

Incidentally, expansions have been given up to order 5 or 7 for the main functions considered in the paper. For instance, let $B_{\mathfrak{g}}$ and $B_{\mathfrak{h}}$ be the Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$, and $b = B_{\mathfrak{g}} - 2B_{\mathfrak{h}}$, as a bilinear form on $\mathfrak{h}$; then (Lemma 3.12)

$$e(X, Y) = 1 - \tfrac{1}{240} b(T, T) + \tfrac{1}{1512} b(T, (x^2 + xy + y^2)T) + \cdots$$

with $T = [X, Y]$, whenever $S$ has a $G$-invariant measure. This suggests (Conjecture 3.13) that $e(X, Y) = 1 + b(T, \ldots)$; in particular $S$ should be special when $b$ vanishes identically.

Finally let us mention that $e$ is analytic on some neighborhood of the origin, even, that $e(h.X, h.Y) = e(X, Y)$ for $h \in H$ (Proposition 3.14) and, above all, that the Lie triple system structure determines $e$. It follows that the $e$-function of the symmetric space $S_*$ dual to $S$ is $e(iX, iY)$, and that $S_*$ is special if and only if $S$ is (Propositions 3.17 and 4.4).

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**Notations**

Only real manifolds are considered here. *Throughout the paper $S = G/H$ will denote a connected and simply connected symmetric coset space*; $G$ is a connected

Lie group with identity $e, \sigma$ is an involutive automorphism of $G$, and $H$ is the connected component of $e$ in the fixed point subgroup of $G$ under $\sigma$.

Let $p: G \to G/H$ be the canonical projection (i.e. $p(g) = gH$), and $o = p(e) = H$ the origin of $S$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ be the decomposition of the Lie algebra of $G$ induced by $\sigma$, as the sum of the Lie algebra of $H$ and a vector space $\mathfrak{s}$, which can be identified with the tangent space $S_0$ to $S$ at the origin. The notation $S_0$ will be used rather than $\mathfrak{s}$ when it is considered as the (flat) symmetric space $S_0 = G_0/H$, where $G_0$ is the semi-direct product of $\mathfrak{s}$ with $H$. Let exp, and $\mathrm{Exp} = p \circ \exp$, denote the exponential mappings of $G$ and $S$, defined on $\mathfrak{g}$ and $\mathfrak{s}$ respectively.

Dots will be used to denote several natural actions. For instance $g.x$ is the result of $g \in G$ acting on $x \in S$, or $h.X = \mathrm{Ad}\, h(X)$ for $h \in H$ and $X \in \mathfrak{s}$; here Ad, resp. ad, is the adjoint representation of $G$, resp. $\mathfrak{g}$. When doing formal computations in the Lie algebras, we shall often write $x$ for ad $X$ and $y$ for ad $Y$.

Let $\mathbb{D}(S)$ denote the algebra of $G$-invariant differential operators (with complex coefficients) on $S = G/H$. In particular $\mathbb{D}(S_0)$ is the algebra of $H$-invariant constant coefficients differential operators on the vector space $\mathfrak{s}$; it is canonically isomorphic to $S_H(\mathfrak{s})$, the subalgebra of $H$-invariant elements in the complexified symmetric algebra of $\mathfrak{s}$.

If $u$ is an endomorphism of a vector space, and $V$ an $u$-invariant finite dimensional subspace, we write $\mathrm{tr}_V\, u$, or $\det_V u$, for the trace, or determinant, of $u$ restricted to $V$.

If $f$ is a smooth map between manifolds, its differential at $x_0$ will be denoted by $D_{x_0} f$, or sometimes $D_{x = x_0} f$, as a linear map between tangent spaces.

## 1. Contractions of symmetric spaces

For the general theory of symmetric spaces, we refer to the classical books by Kobayashi-Nomizu [13], Loos [15] and, for the Riemannian case, Helgason [10]; see also Flensted-Jensen [5]. Let us simply recall the equivalence of categories between the category of simply connected pointed symmetric spaces $(S, o)$, and the category of finite dimensional Lie triple systems $(\mathfrak{s}, [\,,\,,\,])$. Here $\mathfrak{s}$ is the tangent space to $S$ at $o$, with trilinear structure

$$[X, Y, Z] = -R_0(X, Y)Z = [[X, Y], Z],$$

where $R_0$ is the curvature tensor at $o$, and the latter brackets are the Lie brackets of $\mathfrak{g}$; see [15] chapter II for details.

Given a Lie triple system $(\mathfrak{s}, [\,,\,,\,])$ and a real parameter $t$, we define the *deformed Lie triple system* $\mathfrak{s}_t$ as the vector space $\mathfrak{s}$ with trilinear product

$$[X, Y, Z]_t = t^2 [X, Y, Z].$$

Let $S_t$ be the corresponding simply connected pointed symmetric space (unique up to isomorphism). We shall always use the subscript $t$ for notions relative to the deformed structure; for instance, the curvature tensor of $S_t$ is $R_t = t^2 R$. For $t \neq 0$ the map $f^t: X \to tX$ is an isomorphism of the Lie triple system $\mathfrak{s}_t$ onto $\mathfrak{s}$. We still denote by $f^t$ the corresponding isomorphism of $S_t$ onto $S = S_1$:

$$f^t(\mathrm{Exp}_t X) = \mathrm{Exp}\, tX \quad \text{for } X \in \mathfrak{s}, \quad t \neq 0.$$

Of course the flat space $S_0$ is not, in general, isomorphic to other $S_t$'s; it can be identified with the tangent vector space at the origin of $S$, which gives a second reason for calling it $S_0$. We call this process *contraction of $S$ into its tangent space.*

If $\mathfrak{s}$ is given by a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$, then $\mathfrak{s}_t$ is obtained from $(\mathfrak{g}_t, \mathfrak{h}, \sigma)$, where $\mathfrak{g}_t$ is the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ with bracket

$$[A + X, B + Y]_t = ([A, B] + t^2[X, Y]) + ([A, Y] - [B, X])$$

for $A, B \in \mathfrak{h}$, $X, Y \in \mathfrak{s}$. This definition agrees with the classical "contraction of $\mathfrak{g}$ with respect to $\mathfrak{h}$" (see Dooley [2], Dooley-Rice [3]), or with the contraction of a filtered Lie algebra into its graded algebra (see Guillemin-Sternberg [6] p. 447). Again the map $f^t(A + X) = A + tX$ is, for $t \neq 0$, a Lie algebra isomorphism of $\mathfrak{g}_t$ onto $\mathfrak{g} = \mathfrak{g}_1$. Besides $\mathfrak{g}_0$ is the semi-direct product of the vector space $\mathfrak{s}$ (as an abelian Lie algebra) by $\mathfrak{h}$.

Likewise, when $S$ is given by $(G, H, \sigma)$, the space $S_0$ is $G_0/H$ where $G_0$ is the semi-direct product $\mathfrak{s} \times H$.

As a typical example, let us take $G = \mathrm{SU}(1, 1)$, $H = \mathrm{SO}(2)$ (see [11] p. 29 sq.). Then $S_t$ can be realized, for $t > 0$, as the disc $|z| < 1/t$ in $\mathbb{R}^2$ with Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - t^2|z|^2)^2}, \quad z = x + iy.$$

Here $f^t(z) = tz$, and $S_t$ has curvature $-4t^2$. The space $S_0$ is the Euclidean plane, and $G_0$ its motion group. The same $S_0$, $G_0$ arise from $G = \mathrm{SO}(3)$, $H = \mathrm{SO}(2)$ too, and the space $S_t$ can be realized then as a sphere with radius $1/t$, as a Riemannian submanifold of $\mathbb{R}^3$.

The *dual* $\mathfrak{s}_*$ of a Lie triple system $\mathfrak{s}$ is defined as the same vector space $\mathfrak{s}$ with product $[X, Y, Z]_* = -[X, Y, Z]$ (see [15] p. 150, [13] p. 253); this gives, in particular, the duality between compact and non-compact types. An obvious, but useful, remark is that $\mathfrak{s}_*$ can be considered, formally, as $\mathfrak{s}_t$ with $t = i$.

## 2. Geodesic triangles

2.1. Let $s_x$ be the symmetry of $S$ with respect to the point $x$. For $X, Y \in \mathfrak{s}$, we define $z(X, Y) \in S$ by

$$z(X, Y) = s_{\mathrm{Exp}(X/2)} s_0(\mathrm{Exp}\, Y);$$

in terms of $G$ and $H$ the definition of $z$ can also be written as

$$z(X, Y) = \exp X.\mathrm{Exp}\, Y = p(\exp X \exp Y).$$

In the present section, we summarize (with minor changes) results of [16] on the geometry of the geodesic triangle $o$, $\mathrm{Exp}\, X$, $z(X, Y)$.

2.2. Let $\mathfrak{s}'$ be the set of all $X$ in $\mathfrak{s}$ such that $|\mathrm{Im}\,\lambda| < \pi/2$, for any eigenvalue $\lambda$ of ad $X$ on $\mathfrak{g}$. Then $\mathfrak{s}'$ is an *invariant exponential set* for $S$, that is a connected open subset of $\mathfrak{s}$ such that Exp is a diffeomorphism of $\mathfrak{s}'$ onto an open subset $S' = \mathrm{Exp}\,\mathfrak{s}'$ of $S$, and $\mathfrak{s}'$ is invariant under the maps $X \to tX$ for $-1 \leqslant t \leqslant 1$, and $X \to h.X$ for $h \in H$. When $G$ is an exponential solvable group, or $S$ is a Riemannian symmetric space of the non-compact type, we may take $\mathfrak{s}' = \mathfrak{s}$, $S' = S$.

Let $\Omega$ be the set of all $(X, Y) \in \mathfrak{s}' \times \mathfrak{s}'$ such that $z(tX, tY) \in S'$ for all $t \in [0, 1]$; of course $\Omega = \mathfrak{s} \times \mathfrak{s}$ in the special cases above. Then $\Omega$ is a connected open subset of $\mathfrak{s} \times \mathfrak{s}$, which is invariant under the maps $(X, Y) \to (tX, tY)$ for $-1 \leqslant t \leqslant 1$, $(X, Y) \to (Y, X)$, and $(X, Y) \to (h.X, h.Y)$ for all $h \in H$.

We define the map $Z: \Omega \to \mathfrak{s}'$, expressing the action of $G$ on $S$ in exponential coordinates, by

$$\mathrm{Exp}\, Z(X, Y) = z(X, Y) = \exp X.\mathrm{Exp}\, Y.$$

Clearly $Z$ is analytic in $\Omega$ and $Z(-X, -Y) = -Z(X, Y), Z(h.X, h.Y) = h.Z(X, Y)$; also

$$Z(X, Y) = \tfrac{1}{2}\log(e^X e^{2Y} e^X) = \tfrac{1}{2}e^{-x}\log(e^{2X}e^{2Y}),$$

where $x = \mathrm{ad}\, X$.

For the contracted space $S_t$, with $t \neq 0$, we have $\mathrm{Exp}_t\, X = (f^t)^{-1}(\mathrm{Exp}\, tX)$ by Section 1; it follows that $\mathfrak{s}'_t = t^{-1}\mathfrak{s}'$ is an invariant exponential set for $S_t$, that $\Omega_t = t^{-1}\Omega$, and the corresponding map $Z$ is $Z_t(X, Y) = t^{-1}Z(tX, tY)$, with $t \neq 0$, $(X, Y) \in \Omega_t$. When $t = 0$, we may take $\mathfrak{s}'_0 = \mathfrak{s}$, $\Omega_0 = \mathfrak{s} \times \mathfrak{s}$, and $Z_0(X, Y) = X + Y$. In the sequel we shall always have $0 \leqslant t \leqslant 1$, and it is convenient to forget about $\mathfrak{s}'_t$ and $\Omega_t$, replacing them by the (possibly smaller) sets $\mathfrak{s}'$ and $\Omega$. The classical Campbell-Hausdorff formula easily yields the following expansion

$$Z_t(X, Y) = X + Y - \frac{t^2}{6}(x^2 + 2yx)Y + $$
$$+ t^4\left(\frac{7x^4}{360} + \frac{2yx^3}{45} + \frac{x^2yx}{30} + \frac{2y^2x^2}{15} - \frac{2xy^2x}{45} + \frac{y^3x}{45}\right)Y + \cdots \quad (1)$$

where ... have order $\geqslant 6$ with respect to $t$, and order $\geqslant 7$ with respect to $X, Y$.

2.3. To study the map $Z$, we introduce the following notations, which are motivated by Lemma 2.1 below. Let $\omega$ be the function

$$\omega(u) = D_u(u \coth u) = \coth u - u\,\mathrm{sh}^{-2} u,$$

meromorphic on $\mathbb{C}$ with poles at $\pm i\pi$, $\pm 2i\pi, \ldots$, and odd. For $(X, Y) \in \Omega$, $x = \operatorname{ad} X$, $y = \operatorname{ad} Y$, $z(t) = \operatorname{ad} Z(tX, tY)$, we set

$$A(X, Y) = \int_0^1 \frac{\mathrm{sh}\, tx}{\mathrm{sh}\, x}(\mathrm{sh}(1-t)x + \mathrm{ch}\, x.\omega(z(t)).\mathrm{ch}\, tx - $$
$$- \mathrm{sh}\, x.\omega(z(t)).\mathrm{sh}\, tx)\, dt(X + Y)$$

$$F(X, Y) = \int_0^1 \frac{\mathrm{sh}\, tx}{\mathrm{sh}\, x}(\omega(z(t)).\mathrm{ch}\, tx - \mathrm{sh}\, tx)\, dt(X + Y)$$

$$G(X, Y) = F(X, Y) + A(Y, X) - A(X, Y).$$

These definitions of $A$, $F$ and $G$ make sense, due to the properties of $\Omega$; it can be checked that they agree with the functions $F, G$ of [16] Section 2.7. For parity reasons, $A$, $F$ and $G$ are analytic maps from $\Omega$ into the "holonomy ideal" $\mathfrak{h}^* = [\mathfrak{s}, \mathfrak{s}]$ of $\mathfrak{h}$; besides $A(-X, -Y) = A(X, Y)$, $A(h.X, h.Y) = h.A(X, Y)$ for $h \in H$, and similarly for $F$ and $G$. The above expansion (1) of $Z$ yields

$$A(X, Y) = \tfrac{1}{6}xY - \tfrac{4}{45}x^3Y - \tfrac{1}{30}xyxY + \tfrac{2}{45}y^2xY + \cdots$$
$$F(X, Y) = \tfrac{1}{3}yX - \tfrac{7}{90}x^2yX - \tfrac{2}{15}xy^2X - \tfrac{2}{45}y^3X + \cdots$$
$$G(X, Y) = -\tfrac{2}{3}xY + \tfrac{11}{90}x^3Y + \tfrac{1}{5}xyxY + \tfrac{4}{45}y^2xY + \cdots$$

where ... have order $\geqslant 6$. By [16] Section 2.8, we have:

LEMMA 2.1. *For $(X, Y) \in \Omega$ and $0 \leqslant t \leqslant 1$, let $F_t(X, Y) = t^{-1}F(tX, tY)$ and $G_t(X, Y) = t^{-1}G(tX, tY)$. Then*

$$D_tZ_t = D_XZ_t.[X, F_t] + D_YZ_t.[Y, G_t] \tag{2}$$

*where all functions are taken at $(X, Y)$ and, for $V \in \mathfrak{s}$, we write $D_XZ_t.V = D_{\varepsilon=0}Z_t(X + \varepsilon V, Y)$ and similarly for $D_YZ_t.V$.*

Let $H^*$ be the connected (normal) Lie subgroup of $H$ with Lie algebra $\mathfrak{h}^*$. The following result is proved in [16] Section 4.

THEOREM 2.2. *There exist two connected open neighborhoods of $0$ in $\Omega$, say $\Omega_0$ (having the same invariance properties as $\Omega$) and $\Omega_1$, and a canonical diffeo-*

*morphism $\Phi$ of $\Omega_0$ onto $\Omega_1$ endowed with the following properties:*

(i) $\Phi(X, Y) = (a.X, b.Y)$ *where* $a = a(X, Y)$ *and* $b = b(X, Y)$ *are analytic maps from* $\Omega_0$ *into* $H^*$.

(ii) $\Phi^{-1}(X, Y) = (h.X, k.Y)$ *where* $h$ *and* $k$ *are analytic from* $\Omega_1$ *into* $H^*$ *and*

$$Z(X, Y) = h.X + k.Y \quad \text{for } (X, Y) \in \Omega_1 \tag{3}$$

*or equivalently* $Z(\Phi(X, Y)) = X + Y$ *on* $\Omega_0$.

(iii) $\Phi$ *is odd and commutes with diagonal action of* $H$.

(iv) $\Phi(X, Y) = (X, Y)$ *whenever* $(X, Y) \in \Omega_0$ *and* $[X, Y] = 0$.

In other words, the diffeomorphism $\Phi$ transforms $Z$ into the corresponding function for a flat symmetric space. For later reference, we recall that $\Phi$ comes out from the differential system

$$\begin{aligned} D_t X_t &= [F_t(X_t, Y_t), X_t] \\ D_t Y_t &= [G_t(X_t, Y_t), Y_t] \end{aligned}, \quad 0 \leqslant t \leqslant 1 \tag{4}$$

with initial conditions $(X_0, Y_0) = (X, Y) \in \Omega_0$; setting $\Phi_t(X, Y) = (X_t, Y_t)$ and $\Phi = \Phi_1$, equality (3) follows from (2); furthermore

$$\Phi_t(X, Y) = t^{-1}\Phi(tX, tY). \tag{5}$$

From (4) and our expansions of $F, G$ in Section 2.3, we find

$$X_t = X - \frac{t^2}{6}xyX + \frac{t^4}{360}(7x^3y + 12(xy)^2 + 4xy^3 - 5yx^2y)X + O(t^6)$$

$$Y_t = Y + \frac{t^2}{3}yxY + \frac{t^4}{360}(-8y^3x - 20xy^2x + 12(yx)^2 - 11yx^3)Y + O(t^6);$$

see proof of Lemma 3.6 below, for more details. To expand $h$ and $k$ up to order 4, it seems simpler, reminding the parity, to look for $h = \exp(axY + bx^3Y + cyx^2Y + dy^2xY + \cdots)$ with unknown coefficients $a, b, c, d\ldots$, similarly for $k$, and identify $h.X + k.Y$ with the expansion (1) of $Z$; however this method might not determine uniquely the coefficients of higher order terms. One finds:

$$h(tX, tY) = \exp\left(\frac{t^2}{6}xY - \frac{t^4}{360}(7x^3 + 17yx^2 + 4y^2x)Y + O(t^6)\right)$$

$$k(tX, tY) = \exp\left(\frac{t^2}{3}xY - \frac{t^4}{360}(11x^3 + 28yx^2 + 8y^2x)Y + O(t^6)\right).$$

2.4. Formula (3) turns out to have independent interest for matrix Lie groups, and I am grateful to R.C. Thompson for a stimulating correspondence on this problem.

Theorem 2.2 above deals with $Z(X, Y) = \frac{1}{2}\log(e^X e^{2Y} e^X)$, related to the symmetric space structure, but it is more natural, when working on $G$ itself, to study $\log(e^X e^Y)$. This can be done by means of the functions $F^1$, $G^1$ in [16] p. 561, replacing $F$, $G$ above; the basic equation (2) is replaced by the similar Lemma 3.2 in Kashiwara-Vergne [12] p. 255. Elements $a_t, b_t$ of $G$ can be defined by the differential equations

$$D_t a_t = F_t^1(a_t X a_t^{-1}, b_t Y b_t^{-1}) a_t$$
$$D_t b_t = G_t^1(a_t X a_t^{-1}, b_t Y b_t^{-1}) b_t$$

(with ordinary products of matrices in the right-hand sides), and $a_0 = b_0 = e$. Repeating the proof of Theorem 2.2, we find

$$e^{X+Y} = e^{aXa^{-1}} e^{bYb^{-1}}, \tag{6}$$

or equivalently

$$e^X e^Y = e^{hXh^{-1} + kYk^{-1}}, \tag{7}$$

for $X$, $Y$ in suitable neighborhoods of the origin in the Lie algebra of $G$; of course $a, b, h, k$ depend analytically on $(X, Y)$. The additional symmetry $G^1(X, Y) = F^1(-Y, -X)$ valid here implies

$$b(X, Y) = a(-Y, -X), \quad k(X, Y) = h(-Y, -X), \tag{8}$$

in view of uniqueness of solutions.

Relation (7) was conjectured by R.C. Thompson in 1979 for unitary groups $G = U(n)$; he proved it, for any $X, Y$ in the corresponding Lie algebra, by means of a delicate analysis of the eigenvalues (see [17], and also [18], for several related results). The symmetry (8) was also obtained by Thompson, considering formal series expansions.

Finally we mention the following *counterexample*, given (in a more general form) in [18]. Take $G = \mathrm{GL}(2, \mathbb{C})$, and

$$X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ with } x \neq 0, \quad Y = \begin{pmatrix} 2i\pi & 0 \\ 0 & 0 \end{pmatrix}$$

in its Lie algebra. Considering eigenvalues, it is easily shown that (6) and (7) are impossible; therefore one should not hope that, for non-compact groups, these equalities hold without assuming both $X$ and $Y$ near zero.

## 3. The $e$-function

3.1. We begin with a few lemmas in non-commutative algebra. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra, $X, Y$ two given elements of $\mathfrak{s}$, and $x = \mathrm{ad}\, X, y = \mathrm{ad}\, Y$ the corresponding endomorphisms of $\mathfrak{g}$. All the functions we are interested in, such as $Z, A, F, G$ above or $E$ below, are given by (non-commutative) power series in $x, y$. Analyticity of these functions in a neighborhood of the origin is already known from their definitions; in this section we only investigate formal properties of these series, regardless of possible relations between $x$ and $y$ arising from the structure of $\mathfrak{g}$, or the choice of $X, Y$.

Thus let $\mathscr{A}_0$ be the free associative $\mathbb{C}$-algebra on two generators $x$, $y$, naturally graded by taking $x$ and $y$ of degree one. Let $\mathscr{A}$, or $\mathscr{A}_{x,y}$ to be precise, be the corresponding completion of $\mathscr{A}_0$. An element of $\mathscr{A}$ is a formal series $a = \sum_0^\infty a_n$, where $a_n$ is a finite linear combination of non-commutative monomials of degree $n$

$$u_{\alpha\beta} = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_p} y^{\beta_p},$$

with $\alpha_i, \beta_j \in \mathbb{N}$ and $\sum(\alpha_i + \beta_i) = n$. Let $\mathscr{A}^+$, resp. $\mathscr{A}^-$, be the subalgebra, resp. subspace, of even, resp. odd, elements of $\mathscr{A}$; clearly $\mathscr{A} = \mathscr{A}^+ \oplus \mathscr{A}^-$.

Let $\mathscr{I}$, or $\mathscr{I}_{x,y}$ to be precise, be the two-sided ideal of $\mathscr{A}$ generated by $xy - yx$; again $\mathscr{I} = \mathscr{I}^+ \oplus \mathscr{I}^-$.

LEMMA 3.1. *Considering $\mathscr{A}$ as a Lie algebra in the obvious way, we have $[\mathscr{A}, \mathscr{A}] \subset \mathscr{I}$.*

In fact, the bracket of two monomials of degrees $m$ and $n$ is an element of $\mathscr{I}$ of degree $m + n$, by easy induction on $m$ and $n$.

The assignment $x \to \mathrm{ad}\, X$, $y \to \mathrm{ad}\, Y$ extends to a homomorphism $j$ of $\mathscr{A}_0$ into End $\mathfrak{g}$. By restriction, elements of $\mathscr{A}_0^+$ give also rise to endomorphisms of $\mathfrak{s}$ and $\mathfrak{h}$, and elements of $\mathscr{A}_0^-$ to linear maps of $\mathfrak{h}$ into $\mathfrak{s}$, and of $\mathfrak{s}$ into $\mathfrak{h}$ (or even into $\mathfrak{h}^* = [\mathfrak{s}, \mathfrak{s}]$). The map $j$ can be extended to those elements of $\mathscr{A}$ given by absolutely convergent series, with respect to some sub-multiplicative norm; this will be the case of all relevant series here. By abuse, and to avoid clumsiness, we shall not write $j$ any more; when saying that an endomorphism of $\mathfrak{g}$ belongs to $\mathscr{A}^+$, for instance, we mean it is the image under $j$ of some (convergent) formal series in $\mathscr{A}^+$.

The following lemma will be used many times.

LEMMA 3.2. *For $a, b \in \mathscr{A}_0^-$, one has* $\mathrm{tr}_{\mathfrak{s}}\, ab = \mathrm{tr}_{\mathfrak{h}}\, ba = \mathrm{tr}_{\mathfrak{h}^*}\, ba$.

The proof is elementary, by means of bases of $\mathfrak{s}, \mathfrak{h}^*$ and $\mathfrak{h}$.

LEMMA 3.3. *If* $a \in \mathscr{A}_0$ *has no zero order term, then* $\operatorname{ad}(aX)$ *and* $\operatorname{ad}(aY)$ *belong to* $[\mathscr{A}_0, \mathscr{A}_0]$; *this result extends to all convergent series* $a$ *with no zero order term.*

*Proof.* When $a$ is a monomial of degree $n$, it is easily seen that $\operatorname{ad}(aX)$ and $\operatorname{ad}(aY)$ are homogeneous elements of $\mathscr{A}$, of degree $n+1$. If $a_0 = 0$, then $a = xb$ (for instance), with $b \in \mathscr{A}_0$, and $\operatorname{ad}(aX) = [\operatorname{ad} X, \operatorname{ad}(bX)]$ belongs to $[\mathscr{A}_0, \mathscr{A}_0]$; the same is true for $\operatorname{ad}(aY)$, whence the lemma.

LEMMA 3.4. *Let* $A, F, G$ *be as in Section* 2.3. *Assume* $X$ *and* $Y$ *close to the origin in* $\mathfrak{s}$. *Then*

$$A(X, Y) = aY, \quad F(X, Y) = fX, \quad G(X, Y) = gY,$$

*with* $a, f, g \in \mathscr{A}^-$.

The lowest order terms of $a, f, g$ have been written in Section 2.3.

*Proof.* First the formula for $Z(X, Y)$ given in Section 2.2 implies, through the adjoint representation, that

$$e^{2z(t)} = e^{tx}e^{2ty}e^{tx};$$

taking logarithms near the identity in End $\mathfrak{g}$, it follows that $z(t) = \sum_0^\infty t^{2n+1} z_{2n+1}$, with $z_{2n+1}$ homogeneous element of degree $2n+1$ in $\mathscr{A}_0$. Then $\omega(z(t))$ has a similar expansion and, expanding $\operatorname{sh} tx/\operatorname{sh} x$, $\operatorname{ch} tx$, etc., we obtain $A(X, Y) = b(X+Y)$, with $b \in \mathscr{A}^-$. As $yX = -xY$, this can also be written as claimed. The same proof works with $F$; the result now follows for $G$ too.

LEMMA 3.5. *Assume* $X$ *and* $Y$ *close to the origin in* $\mathfrak{s}$. *Then the partial derivatives* $D_X A(X, Y)$, $D_Y A(X, Y)$ *belong to* $\mathscr{A}^-$. *The same holds for derivatives of* $F, G$.

*Proof* (cf. [16] p. 560). In view of Lemma 3.4, it is enough to prove the result with $A(X, Y)$ replaced by $uY$, where $u$ is an odd monomial in $\mathscr{A}_0$. This follows inductively from the identity

$$D_X(x^m y^n uY) = x^m y^n D_X(uY) - \sum_{0 \leqslant p < m} x^{m-p-1} \operatorname{ad}(x^p y^n uY),$$

together with Lemma 3.3 for the ad(...) terms. The proof is similar for derivatives with respect to $Y$, whence the lemma.

LEMMA 3.6. *Assume* $X$ *and* $Y$ *close to the origin in* $\mathfrak{s}$. *Then*

$$X_t = X + \sum_1^\infty t^{2n} u_{2n} X, \quad Y_t = Y + \sum_1^\infty t^{2n} v_{2n} Y,$$

*where $X_t$, $Y_t$ are defined by* (5) *Section* 2.3 *and $u_{2n}, v_{2n}$ are homogeneous elements of degree $2n$ in $\mathscr{A}_0^+$.*

The first terms $u_2, u_4, v_2, v_4$ have been given at the end of Section 2.3.

*Proof.* By Lemma 3.4, we have $F(X, Y) = \sum_1^\infty f_{2n-1}(x, y)X$, with $f_{2n+1}(x, y)$ homogeneous of degree $2n+1$ in $\mathscr{A}_{x,y}$, hence

$$[F_t(X, Y), X] = -\sum_1^\infty t^{2n-1} x f_{2n-1}(x, y)X,$$

and likewise with $G$ instead of $F$. Substituting

$$x_t = x + \sum_1^\infty t^{2n}\mathrm{ad}(u_{2n}X) \quad \text{for } x, \quad \text{and} \quad y_t = y + \sum_1^\infty t^{2n}\mathrm{ad}(v_{2n}Y) \quad \text{for } y,$$

it can be checked without difficulty that the differential system:

$$D_t\left(1 + \sum_1^\infty t^{2n}u_{2n}\right) = -\sum_1^\infty t^{2n-1}x_t f_{2n-1}(x_t, y_t)\left(1 + \sum_1^\infty t^{2p}u_{2p}\right)$$
$$D_t\left(1 + \sum_1^\infty t^{2n}v_{2n}\right) = -\sum_1^\infty t^{2n-1}y_t g_{2n-1}(x_t, y_t)\left(1 + \sum_1^\infty t^{2p}v_{2p}\right)$$

determines all $u_{2n}, v_{2n}$ inductively. When we apply these endomorphisms to $X$, resp. $Y$, it follows that the system (4) Section 2.3 has a solution $(X_t, Y_t)$ of the required form. By uniqueness of Taylor expansions with respect to $t$, this $(X_t, Y_t)$ must coincide with the solution obtained in Section 2.3. This implies the lemma.

LEMMA 3.7. *Let $(X', Y') = \Phi(X, Y)$ with $X, Y$ near* 0 *in $\mathfrak{s}$ (see Theorem* 2.2), *and $x' = \mathrm{ad}\, X'$, $y' = \mathrm{ad}\, Y'$. Then $\mathscr{A}_{x',y'}$ is contained in $\mathscr{A}_{x,y}$; similar inclusions hold for $\mathscr{A}^\pm, \mathscr{I}$, and $\mathscr{I}^\pm$.*

These inclusions are equalities in fact, but this will not be needed in the sequel.

*Proof.* Lemma 3.6 yields $X' = X_1 = X + uX$, with $u \in \mathscr{A}_{x,y}^+$ and $u_0 = 0$. By Lemmas 3.3 and 3.1, we get $x' - x \in \mathscr{I}_{x,y}^-$; likewise $y' - y \in \mathscr{I}_{x,y}^-$. This proves that $x'$ and $y'$ belong to $\mathscr{A}_{x,y}^-$, therefore $\mathscr{A}_{x',y'} \subset \mathscr{A}_{x,y}$ and this inclusion preserves parity. Furthermore $x'y' - y'x'$ belongs to $\mathscr{I}_{x,y}$, by Lemma 3.1, therefore $\mathscr{I}_{x',y'} \subset \mathscr{I}_{x,y}$ and the lemma is proved.

3.2. For $(X, Y) \in \Omega$ we define an endomorphism of $\mathfrak{g}$ by

$$E(X, Y) = D_2A(Y, X).x + D_2A(X, Y).y - \mathrm{ad}(F(X, Y) + A(Y, X)) + \\ + \tfrac{1}{2}(x \coth x + y \coth y - z \coth z - 1); \qquad (1)$$

here $x = \operatorname{ad} X$, $y = \operatorname{ad} Y$, $z = \operatorname{ad} Z(X, Y)$ and $D_2$ means derivative with respect to the second variable, to avoid confusions. The interest of $E$ in analysis will appear in Section 4, but throughout this section we shall be concerned with formal properties of $E$, first.

When $X$, $Y$ are close to the origin of $\mathfrak{s}$, we can take power series expansions. Recalling that $z \in \mathscr{A}^-_{x,y}$ (see proof of Lemma 3.4, with $t = 1$), we get $E(X, Y) \in \mathscr{A}^+_{x,y}$ in view of Lemma 3.5.

Patient computations starting from the expansions of $Z$, $A$, $F$ in Section 2 lead to

$$\begin{aligned} E(X, Y) = {} & \tfrac{1}{2}(xy - yx) + \tfrac{1}{60}(-5x^3y + 13x^2yx - 15xyx^2 + 7yx^3) + \\ & + \tfrac{1}{60}(11x^2y^2 - 6xy^2x - 20(xy)^2 + 20(yx)^2 + 4yx^2y - 9y^2x^2) + \\ & + \tfrac{1}{15}(-2xy^3 + 5yxy^2 - 3y^2xy) + \text{order} \geqslant 6. \end{aligned} \tag{2}$$

LEMMA 3.8. (i) *$E(X, Y)$ belongs to $[\mathscr{A}_{x,y}, \mathscr{A}_{x,y}]$, modulo terms of order* $\geqslant 8$. (ii) *More precisely*:

$$\begin{aligned} E(X, Y) \sim {} & \tfrac{1}{30}(x^2y^2 - xy^2x) + \tfrac{1}{126}(x^3y^2x - x^4y^2 + xy^4x - x^2y^4) + \\ & + \tfrac{1}{105}(xy^2xyx + xy^2x^2y - x^2y^2xy - x^2yxy^2) + \text{order} \geqslant 8, \end{aligned}$$

*where $\sim$ means equivalence modulo the subspace $[\mathscr{A}^+_{x,y}, \mathscr{A}^+_{x,y}] + \operatorname{ad} \mathfrak{h}^*$ of $\mathscr{A}_{x,y}$.*

*Proof.* (i) Modulo order $\geqslant 6$, property (i) is easily derived from (2); observe that the sum of coefficients in each line is zero. But looking at the 6th order terms is a very tedious job, and this will not be reproduced here. We simply make a few remarks. In view of Lemmas 3.3 and 3.4, the ad(...) term in (1) can be forgotten. When the derivatives of $A$ have been written (up to order 5 in $x$ and $y$), it is convenient to compute modulo $[\mathscr{A}, \mathscr{A}]$; for instance $(xy)^3$ can be replaced by $(yx)^3$, but not by $x^3y^3$...

(ii) Here the proof is even longer. A table of all $\operatorname{ad} U$ for $U \in \mathfrak{h}^*$, up to order 6, is helpful, so as to know which terms can be neglected in the calculations. Many remarkable cancellations occur at the end so that the above result, although obtained by hand, is very likely to be correct ... Formula (ii) obviously implies (i), and will lead to an interesting expansion of $e$ below.

Lemma 3.8 supports the following conjecture.

CONJECTURE 3.9. *For $X, Y$ near the origin in $\mathfrak{s}$, $E(X, Y)$ belongs to $[\mathscr{A}_{x,y}, \mathscr{A}_{x,y}]$.*

Unfortunately, the proof of Lemma 3.8 does not give any clear insight into the conjecture, as cancellations of terms in this lemma occur in a rather mysterious way. We shall see in Section 4.4 some consequences of this conjecture. The weaker result $E(X, Y) \in \mathscr{I}_{x,y}$ will be proved below (Theorem 3.15).

LEMMA 3.10. *For $(X, Y) \in \Omega$ we have*

$$\mathrm{tr}_{\mathfrak{h}}\, E(X, Y) = -\mathrm{tr}_{\mathfrak{s}}(xD_XF + yD_YG) + \tfrac{1}{2}\mathrm{tr}_{\mathfrak{s}}(x \coth x + y \coth y - z \coth z - 1),$$

*the derivatives of $F$, $G$ being taken at $(X, Y)$.*

As $E$ belongs to $\mathscr{A}^+$, it actually defines an endomorphism of $\mathfrak{h}$.

*Proof.* Considering (1), we first observe that $(x \coth x - 1)$ is a series of even powers $x^{2n+2}, n \geq 0$. By Lemma 3.2 with $a = x$, $b = x^{2n+1}$, it has equal traces on $\mathfrak{h}$ and $\mathfrak{s}$. The same holds for $(y \coth y - 1)$ and $(z \coth z - 1)$. For the other terms of $E$, we need an auxiliary lemma.

LEMMA 3.11. *Let $C(X, Y)$ be an $\mathfrak{h}$-valued differentiable function on $\Omega$, such that $C(h.X, h.Y) = h.C(X, Y)$ for all $h \in H$. Then the endomorphisms of $\mathfrak{g}$: $\mathrm{ad}\, C(X, Y)$ and $D_XC(X, Y)x + D_YC(X, Y)y$ have the same restriction to $\mathfrak{h}$.*

*Proof.* Take $h = \exp tU$ with $U \in \mathfrak{h}$, and compute derivatives with respect to $t$, at $t = 0$.

Applying this lemma to the function $C(X, Y) = F(X, Y) + A(Y, X) = G(X, Y) + A(X, Y)$, we see that $-(D_XF(X, Y)x + D_YG(X, Y)y)$ defines the same endomorphism of $\mathfrak{h}$ as $D_2A(Y, X)x + D_2A(X, Y)y - \mathrm{ad}(F(X, Y) + A(Y, X))$. Using Lemma 3.2 again, we obtain Lemma 3.10.

3.3. In the setting of Section 2.3, let us recall the notation $(X_t, Y_t) = \Phi_t(X, Y)$ for $(X, Y) \in \Omega_0$. Observing that $E_t(X, Y) = t^{-1}E(tX, tY)$ is analytic with respect to $(t, X, Y)$ in a neighborhood of $[0. 1] \times \Omega$, we can define an analytic real-valued function $e_t(X, Y)$, with $0 \leq t \leq 1$, $(X, Y) \in \Omega_0$, by

$$D_t \log e_t(X, Y) = \mathrm{tr}_{\mathfrak{h}}\, E_t(X_t, Y_t), \quad e_0(X, Y) = 1. \tag{3}$$

We call $e(X, Y) = e_1(X, Y)$ *the e-function of the symmetric coset space.* Its rôle in analysis will appear in Section 4 and in part II. From the expansions of $E, X_t, Y_t$, one finds:

$$\begin{aligned} e(X, Y) = 1 &+ \tfrac{1}{4}\mathrm{tr}_{\mathfrak{h}}(xy - yx) + \tfrac{1}{32}(\mathrm{tr}_{\mathfrak{h}}(xy - yx))^2 + \\ &+ \tfrac{1}{12}\mathrm{tr}_{\mathfrak{h}}(yx^3 - x^3y + y^3x - xy^3) + \tfrac{1}{24}\mathrm{tr}_{\mathfrak{h}}((yx)^2 - (xy)^2) + \\ &+ \tfrac{1}{120}\mathrm{tr}_{\mathfrak{h}}(x^2y^2 - xy^2x) + \text{order} \geq 6 \end{aligned} \tag{4}$$

where $X, Y$ are near 0 in $\mathfrak{s}$. A slightly different expression can be obtained by means of the respective Killing forms $B_{\mathfrak{g}}$ and $B_{\mathfrak{h}}$ of $\mathfrak{g}$ and $\mathfrak{h}$. Putting $T = [X, Y]$, we have $\mathrm{ad}\, T = xy - yx$ and

$$B_{\mathfrak{g}}(T, T) - 2B_{\mathfrak{h}}(T, T) = \mathrm{tr}_{\mathfrak{s}}(xy - yx)^2 - \mathrm{tr}_{\mathfrak{h}}(xy - yx)^2 = 2\mathrm{tr}_{\mathfrak{h}}(xy^2x - x^2y^2),$$

in view of Lemma 3.2. Writing down $\mathrm{ad}(x^3 Y)$, $\mathrm{ad}(xyxY)$ and $\mathrm{ad}(y^2xY)$ explicitly, it is then easy to check that

$$e(X, Y) = 1 + \tfrac{1}{4}\mathrm{tr}_{\mathfrak{h}}\,\mathrm{ad}\,T + \tfrac{1}{32}(\mathrm{tr}_{\mathfrak{h}}\,\mathrm{ad}\,T)^2 - \tfrac{1}{48}\mathrm{tr}_{\mathfrak{h}}\,\mathrm{ad}(x^2 + xy + y^2)T - \\ - \tfrac{1}{240}(B_{\mathfrak{g}}(T, T) - 2B_{\mathfrak{h}}(T, T)) + \cdots \tag{4'}$$

Thus interesting simplifications occur when

$$\mathrm{tr}_{\mathfrak{s}}\,\mathrm{ad}\,U = 0 \quad \text{for all } U \in \mathfrak{h}^* = [\mathfrak{s}, \mathfrak{s}] \tag{5}$$

(or the same with $\mathrm{tr}_{\mathfrak{h}}$, or with $\mathrm{tr}_{\mathfrak{h}^*}$, by Lemma 3.2). Since $H^*$ is connected, this is equivalent to $\det_{\mathfrak{s}} \mathrm{Ad}\, h = 1$ for all $h \in H^*$, which is true when $H^*$ is compact, or *when the space $S = G/H$ has a $G$-invariant measure.*

LEMMA 3.12. *Assume* (5). *Then*

$$e(X, Y) = 1 - \tfrac{1}{240}b(T, T) + \tfrac{1}{1512}b(T,(x^2 + xy + y^2)T) + \cdots$$

*where $b = B_{\mathfrak{g}} - 2B_{\mathfrak{h}}$, $T = [X, Y]$, and ... have order $\geqslant 8$ with respect to $(X, Y)$.*

Up to this order, $e$ is therefore *symmetric* with respect to $X$ and $Y$.

*Proof.* The second term is given by (4′). The third follows from Lemma 3.8, expansions of $X_t$, $Y_t$ in Section 2.3 and (3) above, after some calculations. The result is then compared to a table of traces, on $\mathfrak{h}$ and $\mathfrak{s}$, of all $\mathrm{ad}\, U\, \mathrm{ad}\, V$ for $U$, $V \in \mathfrak{h}^*$ up to order 6, so as to get the result of the lemma.

Lemma 3.12 supports the following conjecture.

CONJECTURE 3.13. *Assume* (5) (*for instance, assume $S$ has a $G$-invariant measure*). *Then*

$$e(X, Y) = 1 + (B_{\mathfrak{g}} - 2B_{\mathfrak{h}})([X, Y], a(x, y)[X, Y]),$$

*for $X$, $Y$ near 0 in $\mathfrak{s}$, with $a \in \mathcal{A}^+_{x,y}$ (given by a convergent series of even monomials).*

If this conjecture is true, then $e$ is identically 1 when $B_{\mathfrak{g}} = 2B_{\mathfrak{h}}$ on $\mathfrak{h}$, for instance when $\mathfrak{h}$ is a real form of a Lie algebra $\mathfrak{g}$ with complex structure; see Section 4.4 for further discussion. Also, $e$ equals 1 up to order 4 would imply $e$ equals 1 exactly, if $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{h}$; in fact $b(T, T)$ would be identically 0 for $T \in \mathfrak{h}$, and all higher order terms would vanish too. I am grateful to J.J. Duistermaat for suggesting this phenomenon; this motivated Lemma 3.12 and Conjecture 3.13.

Taking, as an example, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{h} = \mathfrak{so}(n)$, we have $B_{\mathfrak{g}}(X, Y) = 2n\, \mathrm{tr}\, XY$, $B_{\mathfrak{h}}(X, Y) = (n - 2)\, \mathrm{tr}\, XY$, where tr is the usual trace of $n \times n$ matrices, and $b(T, T) = 4\, \mathrm{tr}\, T^2 = -4\, \|T\|^2$, where $\|T\|$ denotes the Hilbert-Schmidt norm of the skew-symmetric matrix $T$. Therefore, for $SL(n, \mathbb{R})/SO(n)$ we have

$$e(X, Y) = 1 + \tfrac{1}{60}\|T\|^2 + \cdots \quad \text{with } T = [X, Y].$$

In this example, $e(X, Y)$ is 1 up to order 4 if and only if $X$ and $Y$ are commuting elements of $\mathfrak{s}$; this, in turn, implies $e(X, Y) = 1$ exactly, by Corollary 3.16 below. Thus the above phenomenon happens here. For $SL(2, \mathbb{R})/SO(2)$, Lemma 3.12 gives

$$e(X, Y) = 1 + \tfrac{1}{60}\|T\|^2 - \tfrac{1}{189}\|T\|^2 \operatorname{tr}(X^2 + XY + Y^2) + \cdots$$

Other classical semi-simple symmetric spaces can be studied in the same way: for instance

$$e(X, Y) = 1 + \frac{3-n}{240}\|T\|^2 + \cdots$$

for $SO(n+1)/SO(n)$. In a preliminary version of this paper, we gave an exact formula for the two-dimensional sphere, as an elementary exercise starting from (6) below. Since then, M. Flensted-Jensen has been able to compute $e$, by a different method, for $SO_0(n, 1)/SO(n)$; his result is

$$e(X, Y) = \left(4 \frac{u}{\operatorname{sh} u} \frac{v}{\operatorname{sh} v} \frac{w}{\operatorname{sh} w} \frac{\operatorname{ch} w - \operatorname{ch}(u-v)}{w^2 - (u-v)^2} \frac{\operatorname{ch} w - \operatorname{ch}(u+v)}{w^2 - (u+v)^2}\right)^{n-3/2}$$

where $u, v$ and $w$ are the respective norms of $X$, $Y$ and $X + Y$; thus $e(X, Y) = e(Y, X)$ in this example.

Conjecture 3.13 is not a mere consequence of Conjecture 3.9. Indeed $E(X, Y)$ is, according to 3.9, a sum of $ab - ba$ with $a, b \in \mathscr{A}$, both even or both odd. If they are even, $\operatorname{tr}_{\mathfrak{h}}(ab - ba) = 0$; if they are odd, $\operatorname{tr}_{\mathfrak{h}}(ab - ba) = (\operatorname{tr}_{\mathfrak{s}} - \operatorname{tr}_{\mathfrak{h}})(ba) = (\operatorname{tr}_{\mathfrak{g}} - 2\operatorname{tr}_{\mathfrak{h}})(ba)$ (see Lemma 3.2), but this cannot be written by means of Killing forms, giving 3.13, unless we know that $ba = \Sigma \operatorname{ad} U_j \operatorname{ad} V_j + \operatorname{ad} W$, for some $U_j, V_j, W \in \mathfrak{h}^*$.

3.4. The following properties of $e$ are consequences of Theorem 2.2. Here enters $J$, the Jacobian of Exp.

PROPOSITION 3.14. (i) *The function $e$ is analytic on $\Omega_0$, strictly positive, even, and invariant under diagonal action of $H$: $e(-X, -Y) = e(X, Y) = e(h.X, h.Y)$ for $(X, Y) \in \Omega_0$, $h \in H$. Also $e_t(X, Y) = e(tX, tY)$ for $0 \leqslant t \leqslant 1$.*

(ii) *Let* $\Phi(X, Y) = (a.X, b.Y)$ *be as in Theorem 2.2, with* $(X, Y) \in \Omega_0$. *Then*

$$e(X, Y) = \left(\frac{J(X)J(Y)}{J(X+Y)}\right)^{1/2} \det_{\mathfrak{s}\times\mathfrak{s}} D\Phi(X, Y).\det_{\mathfrak{s}} \operatorname{Ad} a^{-1}.\det_{\mathfrak{s}} \operatorname{Ad} b^{-1}, \tag{6}$$

*where* $J(X) = \det_{\mathfrak{s}}(\operatorname{sh} x/x)$.

Since $a, b$ belong to $H^*$, the latter two determinants in (6) are equal to 1 when $H^*$ is compact – or when $S$ has a $G$-invariant measure.

*Proof.* (i) By (5) Section 2.3 we have $(tX_t, tY_t) = \Phi(tX, tY)$, and invariance properties of $e$ follow from the corresponding properties of $\Phi$ (Theorem 2.2) and $E$. The definition of $e_t$ becomes

$$D_t \log e_t(X, Y) = t^{-1} \operatorname{tr}_{\mathfrak{h}} E \circ \Phi(tX, tY),$$

which gives

$$\log e_t(X, Y) = \int_0^t \operatorname{tr}_{\mathfrak{h}} E \circ \Phi(uX, uY)\, u^{-1}\, du.$$

Changing the variable $u$ into $v = t^{-1}u$ shows that $e_t(X, Y) = e_1(tX, tY)$.

(ii) Let

$$f_t(X, Y) = \left(\frac{J(tX)J(tY)}{J(tX+tY)}\right)^{1/2} \det D\Phi_t(X, Y).\det \operatorname{Ad} a_t^{-1}.\det \operatorname{Ad} b_t^{-1},$$

where $\Phi_t(X, Y) = (a_t.X, b_t.Y)$. The behaviour of $a_t, b_t, \Phi_t$ under homotheties on $X$ and $Y$ implies $f_t(X, Y) = f_1(tX, tY)$. Since $f_0(X, Y) = 1$, the proposition will be proved if we show that $\log e_t$ and $\log f_t$ have the same derivative at $t = 1$; in fact $D_s \log f_s(X, Y) = s^{-1} D_{t=1} \log f_t(sX, sY)$, and the same holds for $e_t$ by (i). This will come out from several facts. First

$$D_{t=1} \log J(tX) = \operatorname{tr}_{\mathfrak{s}}(x \coth x - 1),$$

an easy consequence of the definition of $J$ and differential of the determinant map; this trace will not change when replacing $x = \operatorname{ad} X$ by $\operatorname{ad} X_1 = \operatorname{Ad} a_1.x.\operatorname{Ad} a_1^{-1}$. When looking at $D_t \log J(tX + tY)$ in the same way, we may use the equality $X + Y = Z(X_1, Y_1)$ to get

$$D_{t=1} \log \frac{J(tX)J(tY)}{J(tX+tY)} = \operatorname{tr}_{\mathfrak{s}}(x \coth x + y \coth y - z \coth z - 1)(X_1, Y_1).$$

Secondly, $D_t \log \det D\Phi_t$ is the trace of the divergence of the vector field giving rise to $\Phi_t$ by (4) Section 2.3; it follows that (see (20) and (21) in [16] p. 570):

$$D_{t=1} \log(\det D\Phi_t . \det \operatorname{Ad} a_t^{-1} . \det \operatorname{Ad} b_t^{-1}) = -\operatorname{tr}_{\mathfrak{s}}(xD_X F + yD_Y G)(X_1, Y_1).$$

On the other hand, $D_{t=1} \log e_t(X, Y) = \operatorname{tr}_{\mathfrak{h}} E(X_1, Y_1)$, and our claim follows from Lemma 3.10. This proves the proposition.

3.5. Our main results on $e$ are the following theorem and corollary. As usual, $S = G/H$ is a simply connected symmetric coset space. The notation $\mathscr{I}_{x,y}$, introduced in Section 3.1, means the two-sided ideal generated by $xy - yx$ in the completion of the free associative algebra on $x$, $y$, and $\mathscr{I}^+_{x,y}$ is the subspace of even elements. Here we take $x = \operatorname{ad} X$, $y = \operatorname{ad} Y$; again we omit the map $j$ of Section 3.1.

THEOREM 3.15. *If $X$ and $Y$ are near the origin in $\mathfrak{s}$, then*
(i) *$E(X, Y)$ belongs to $\mathscr{I}^+_{x,y}$.*
(ii) *There exist elements $u$, $v$ of $\mathscr{I}^+_{x,y}$ such that $e(X, Y) = \exp(\operatorname{tr}_{\mathfrak{h}} u) = \exp(\operatorname{tr}_{\mathfrak{s}} v)$.*

*Proof.* (i) The proof is long and technical, and will be postponed until Section 5. Let us remind the reader Conjecture 3.9, which would give a stronger result.

(ii) Assuming (i), we have $t^{-1}E \circ \Phi(tX, tY) \in \mathscr{I}^+_{x,y}$ by Lemma 3.7, with $tX$, $tY$ instead of $X$, $Y$. But we know that, for $0 \leqslant t \leqslant 1$,

$$t^1 E \circ \Phi(tX, tY) = \sum_1^\infty t^{2n-1} u_{2n},$$

an absolutely convergent power series, where the coefficient $u_{2n}$ is homogeneous of degree $2n$ and must belong to $\mathscr{I}^+_{x,y}$. From (3) we get

$$D_t \log e_t(X, Y) = \sum_1^\infty t^{2n-1} \operatorname{tr}_{\mathfrak{h}} u_{2n},$$

and, integrating, $\log e(X, Y) = \operatorname{tr}_{\mathfrak{h}} u$, with $u = \Sigma_1^\infty (1/2n) u_{2n} \in \mathscr{I}^+_{x,y}$. To change this into traces on $\mathfrak{s}$, it is enough to observe that, by Lemma 3.2, $\operatorname{tr}_{\mathfrak{h}}(xy - yx) = \operatorname{tr}_{\mathfrak{s}}(yx - xy)$ and, for higher degree elements of $\mathscr{I}^+$ (with $a, b \in \mathscr{A}$),

$$\operatorname{tr}_{\mathfrak{h}} xa(xy - yx)b = \operatorname{tr}_{\mathfrak{s}} a(xy - yx)bx, \quad \operatorname{tr}_{\mathfrak{h}} ya(xy - yx)b = \operatorname{tr}_{\mathfrak{s}} a(xy - yx)by.$$

This completes the proof.

REMARK. A closer look at the proof of (i) and (ii) would show that the above $u$ and $v$ are series of non-commutative monomials in $x$ and $y$ with *rational* coefficients. These coefficients are the same for all symmetric spaces.

COROLLARY 3.16. *Let $(X, Y)$ be a point in $\Omega_0$ (see Theorem 2.2) such that $X$ and $Y$ generate a solvable Lie subalgebra of $\mathfrak{g}$. Then $e(X, Y) = 1$.*

A simple example is when $[X, Y] = 0$. But it does not seem that our proof can be made much shorter in this case, in spite of Theorem 2.2(iv), as derivatives of $\Phi$ at such a point are involved in $e$.

*Proof.* Let us assume $X$, $Y$ near 0 first, and let $\mathfrak{g}'$ denote the solvable subalgebra. By Lie's theorem for the adjoint representation of $\mathfrak{g}'$ on $\mathfrak{g}_{\mathbb{C}}$ (the complexification of $\mathfrak{g}$), there exists a basis of $\mathfrak{g}_{\mathbb{C}}$ in which $x = \operatorname{ad} X$ and $y = \operatorname{ad} Y$ are given by upper triangular matrices. The matrices of $xy - yx$, and $\mathscr{I}^+_{x,y}$ more generally, are strictly upper triangular then. The above $u$ (Theorem 3.15(ii)) is therefore a nilpotent endomorphism of $\mathfrak{g}_{\mathbb{C}}$, and of $\mathfrak{h}$ by restriction. The corollary follows when $X$, $Y$ are near the origin, and the general case $(X, Y) \in \Omega_0$ by analytic continuation on $t$ for the analytic function $e(tX, tY)$; the proof is complete.

PROPOSITION 3.17. *The e-function of the contracted symmetric space $S_t$ is $e_t(X, Y) = e(tX, tY)$. For the dual space $S_*$ it is $e_*(X, Y) = e(iX, iY)$.*

*Proof.* Let us assume $X$, $Y$ near the origin in $\mathfrak{s}$. By Theorem 3.15(ii), $\log e(X, Y)$ is the $\operatorname{tr}_{\mathfrak{s}}$ of a series of non-commutative even monomials in $x$ and $y$ (the full force of 3.15(i) is not needed here, where $\mathscr{A}^+_{x,y}$ would do as well as $\mathscr{I}^+_{x,y}$). Therefore

$$e(X, Y) = \operatorname{tr}_{\mathfrak{s}} f(x^2, xy, yx, y^2),$$

where $f$ is a convergent power series of four non-commuting variables, near the origin. Now it is important to observe that $f$ is built from our functions $Z$, $A$, $F$, $G$ and $\Phi$, therefore from $Z(X, Y)$ and the classical hyperbolic functions only. A glance at Section 2.2 shows that $Z$, therefore $f$, are "universal" functions, i.e. the coefficients of their power series expansions are the same for all symmetric spaces. Since $xy$ is the endomorphism $U \to [U, Y, X]$ of the Lie triple system $\mathfrak{s}$, and similarly for $x^2$, $yx$, $y^2$, we see that *$e$ can be obtained directly from the Lie triple system structure.*

When switching over from $\mathfrak{s}$ to $\mathfrak{s}_t$ (see §1), each of these endomorphisms must be multiplied by $t^2$, which gives the $e$-function $e(tX, tY)$; as proved in Proposition 3.14(i), this is coherent with the notation $e_t(X, Y)$ in (3).

When switching over from $\mathfrak{s}$ to $\mathfrak{s}_*$, a minus sign must be put in front of the endomorphisms; the result is $e(iX, iY)$, which makes sense near the origin, by analyticity of $e$. The proposition is proved.

## 4. The $e$-function and invariant analysis; special symmetric spaces

4.1. The space $\mathscr{D}$ of compactly supported $C^\infty$ functions on a manifold being equipped with the Schwartz topology, its dual $\mathscr{D}'$ is the space of distributions (densities); let $\langle\, ,\rangle$ be the duality bracket between $\mathscr{D}'$ and $\mathscr{D}$. The convolution product of two $H$-invariant distributions $\alpha, \beta$ is the $H$-invariant distribution $\alpha * \beta$ on $S$ defined by:

$$\langle \alpha * \beta, \phi \rangle = \langle \alpha(xH) \otimes \beta(yH), \phi(xyH) \rangle, \tag{1}$$

for any $\phi \in \mathscr{D}(S)$. Here $x$ and $y$ denote elements of $G$, and it should be emphasized that $H$-invariance of $\beta$ implies the definition is meaningful (independently of the choice of $x$ in $xH$), as soon as $\beta$ is compactly supported, for instance. We refer to [16] p. 557 for some examples of convolutions.

The exponential mapping, as a diffeomorphism of $\mathfrak{s}'$ onto $S'$, can be used to transfer analysis on $S$ to and from its tangent space. Let us recall the notation $J(X) = \det_{\mathfrak{s}} \operatorname{sh} x/x$, with $x = \operatorname{ad} X$, an $H$-invariant strictly positive even function on $\mathfrak{s}'$, which is the Jacobian of Exp. For $u \in \mathscr{D}'(\mathfrak{s}')$ let $\tilde{u} \in \mathscr{D}'(S')$ be the direct image of $J^{1/2}.u$ under Exp, that is

$$\langle \tilde{u}, \phi \rangle = \langle u(X), J(X)^{1/2} \phi(\operatorname{Exp} X) \rangle \tag{2}$$

for any $\phi \in \mathscr{D}(S')$. This $\tilde{}$ is a bijection of $\mathscr{D}'(\mathfrak{s}')$ onto $\mathscr{D}'(S')$, which preserves $H$-invariance.

Let $dX$ be a Lebesgue measure on $\mathfrak{s}$, and $ds = (J(X)^{1/2}\, dX)^{\sim}$ the corresponding measure on $S'$, that is

$$\int_{S'} \phi(s) ds = \int_{\mathfrak{s}'} \phi(\operatorname{Exp} X) J(X) dX. \tag{3}$$

If $dX$ is $H$-invariant on $\mathfrak{s}$, then it is classical that $ds$ is a $G$-invariant measure on $S'$ (wherever this makes sense).

If $u(X)dX$ is the distribution on $\mathfrak{s}'$ defined by a locally integrable function $u$, then its image under $\tilde{}$ is $\tilde{u}(s)ds$, where the locally integrable function $\tilde{u}$ is given by

$$u(X) = J(X)^{1/2} \tilde{u}(\operatorname{Exp} X), \quad X \in \mathfrak{s}'; \tag{4}$$

furthermore $\int_{\mathfrak{s}'} u(X) v(X) dX = \int_{S'} \tilde{u}(s) \tilde{v}(s) ds$, if the integrals converge absolutely; more generally

$$\langle \tilde{u}, \tilde{f} \rangle = \langle u, f \rangle \text{ for } u \in \mathscr{D}'(\mathfrak{s}'),\ f \in \mathscr{D}(\mathfrak{s}'). \tag{5}$$

4.2. The following propositions explain the rôle of $e$ and $E$ in analysis on $S$; the notations $\Omega_0$, $\Omega_1$ are those of Theorem 2.2.

PROPOSITION 4.1. *Let $u(X)dX$, $v(X)dX$ be $H$-invariant distributions on $\mathfrak{s}'$,* *defined by measurable functions $u$, $v$. Assume $J^{1/2}.u$ and $J^{1/2}.v$ integrable with respect to $dX$, and* $\operatorname{supp} u \times \operatorname{supp} v$ *contained in $\Omega_1$. Then, for any $f \in \mathscr{D}(\mathfrak{s}')$,*

$$\int_S (\tilde{u} * \tilde{v})(s)\tilde{f}(s)ds = \int_{\mathfrak{s}\times\mathfrak{s}} u(X)v(Y)e(X, Y)f(X + Y)dXdY.$$

Here $H$-invariance of $u(X)dX$ is equivalent to $u(h.X) = |\det_{\mathfrak{s}} \operatorname{Ad} h|^{-1} u(X)$.

*Proof.* The left-hand side is, by (1), (3) and (4),

$$\int \tilde{u}(\operatorname{Exp} X')\tilde{v}(\operatorname{Exp} Y')\tilde{f}(\operatorname{Exp} Z(X', Y'))J(X')J(Y')dX'dY' =$$

$$= \int_{\Omega_1} u(X')v(Y')f(Z(X', Y'))\left(\frac{J(X')J(Y')}{J(Z(X', Y'))}\right)^{1/2} dX'dY'.$$

Since $J^{-1/2}f$ is bounded on $\mathfrak{s}'$, our assumptions imply absolute convergence of these integrals. Changing variables by means of the diffeomorphism $\Phi$ of Theorem 2.2: $(X', Y') = \Phi(X, Y) = (a.X, b.Y)$, the integral becomes

$$\int_{\Omega_0} u(a.X)v(b.Y)f(X + Y)\left(\frac{J(a.X)J(b.Y)}{J(X + Y)}\right)^{1/2} \det D\Phi(X, Y)dXdY.$$

Using $H$-invariance of $u$, $v$ and $J$, we see $e(X, Y)$ appearing, as given in Proposition 3.14. Besides, $\operatorname{supp} u \times \operatorname{supp} v$ is contained in $\Omega_1 = \Phi(\Omega_0)$, and is $H \times H$-invariant, a fortiori $\Phi^{-1}$-invariant; therefore it is contained in $\Omega_0$ too, and we can integrate on the whole space $\mathfrak{s} \times \mathfrak{s}$ as well. The proposition is proved.

This proof does not extend in an obvious way to arbitrary distributions $u, v$. Instead we have the following result, which reformulates [16] p. 567–568.

PROPOSITION 4.2. *Let $u, v$ be $H$-invariant distributions (densities) on $\mathfrak{s}$, with suitable supports. Then, for any $f \in \mathscr{D}(\mathfrak{s}')$, $0 \leqslant t \leqslant 1$,*

$$\left\langle u(X)v(Y), (D_t - \operatorname{tr}_{\mathfrak{h}} E_t(X, Y))\left[\left(\frac{J(tX)J(tY)}{J(Z(tX, tY))}\right)^{1/2} f(Z_t(X, Y))\right]\right\rangle = 0.$$

Here $u(X)v(Y)$ is a tensor product of distributions, and we recall that $Z_t(X, Y) = t^{-1}Z(tX, tY)$, $E_t(X, Y) = t^{-1}E(tX, tY)$ and $\operatorname{tr}_{\mathfrak{h}} E_t(X_t, Y_t) = D_t \log e_t(X, Y)$; see also Lemma 3.10. We refer to [16] p. 566 for the technical assumption "suitable supports"; it holds in particular when $\operatorname{supp} u$ is arbitrary and $\operatorname{supp} v$ is the origin of $\mathfrak{s}$.

4.3 The symmetric space $S$ will be called *special* when its $e$-function is identically one (on a neighborhood of the origin in $\mathfrak{s} \times \mathfrak{s}$ or, by analytic continuation, on the whole $\Omega_0$). By (3) Section 3.3, this is equivalent to: $\mathrm{tr}_{\mathfrak{h}} E$ vanishes identically (on $\Omega$); an equivalent formulation is provided by Lemma 3.10. As noted in the proof of Proposition 3.17, being special is a property of the Lie triple system only.

PROPOSITION 4.3. *Assume $S$ is a special symmetric space, and $u, v$ are $H$-invariant distributions on $\mathfrak{s}$, with suitable supports. Then*

$$\tilde{u} * \tilde{v} = (u * v)\tilde{\ } \text{ on the open subset } S'. \tag{6}$$

In the left- (resp. right-) hand side of (6), $*$ denotes convolution on $S$ (resp. on the vector space $\mathfrak{s}$). This proposition follows easily from Proposition 4.2 (cf. [16] p. 567). In the case of functions, Proposition 4.1 gives a new, and more natural, proof:

$$\langle \tilde{u} * \tilde{v}, \tilde{f} \rangle = \langle u * v, f \rangle = \langle (u * v)\tilde{\ }, \tilde{f} \rangle,$$

in view of (5).

Applications of (6) were developed in [16] Sections 6 and 7: isomorphism of the algebras $\mathbb{D}(S_0)$ and $\mathbb{D}(S)$ of invariant differential operators, existence of an $H$-invariant fundamental solution on $S'$ for any non-zero $P \in \mathbb{D}(S)$. Also *the exponential mapping solves Problem* 1 *in the introduction* (*for special symmetric spaces*), taking as $u \to u'$ the inverse map of $\tilde{\ }$.

If $G$ is a simply connected nilpotent Lie group, Exp is a global diffeomorphism and $S' = S$; besides, G. Lion has proved $P$-convexity of $S$ (see [14]), and these facts imply global solvability: $PC^\infty(S) = C^\infty(S)$. Actually, Lion obtains (by different methods) a more general result, for any homogeneous nilmanifold.

4.4. The properties of $e$ obtained in Section 3 provide some criteria for a symmetric space to be special.

PROPOSITION 4.4. *Let $S = G/H$ be a symmetric coset space.*

(i) *$S$ is special if and only if the dual space $S_*$ is special.*

(ii) *If $S$ is special, then the contracted spaces $S_t$ are special.*

(iii) *If $\mathfrak{s}$ is contained in a solvable subalgebra of $\mathfrak{g}$ (in particular if $G$ is a solvable group), then $S$ is special.*

This is immediate by Corollary 3.16 and Proposition 3.17.

Separating orders in (4') Section 3.3, we see that $S$ special implies

$$\mathrm{tr}_{\mathfrak{h}} \operatorname{ad} T = 0 \quad \text{and} \quad B_{\mathfrak{g}}(T, T) = 2B_{\mathfrak{h}}(T, T)$$

for all $T \in \mathfrak{h}^* = [\mathfrak{s}, \mathfrak{s}]$. Thus property (5) Section 3.3 is a necessary condition for $S$ to be special. From Section 3.3 it is clear that $\mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m)$ (for any $m$), $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$ (for $n \neq 3$) are not special; but $\mathrm{SO}_0(3, 1)/\mathrm{SO}(3)$ and the dual space $\mathrm{SO}(4)/\mathrm{SO}(3)$ are special.

In [16] p. 577 we proved that, for a Riemannian symmetric space $G/H$ of the non-compact type, the convolution property (6) implies that the semi-simple group $G$ has a complex structure, and $H$ is a compact real form of $G$. Therefore, the space must be a quotient $G_{\mathbb{C}}/G_{\mathbb{R}}$. Conjecture 3.13 implies the converse, since $B_{\mathfrak{g}} = 2B_{\mathfrak{h}}$ on $\mathfrak{h}$ in such a case. (Besides, property (6) can be proved directly for these spaces, from known results of semi-simple harmonic analysis). The next proposition states a more general result.

We say that a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is *strongly symmetric* if there exists a linear isomorphism $\gamma$ of $\mathfrak{g}$ which commutes with all $\operatorname{ad} X$, for $X \in \mathfrak{g}$, and anticommutes with $\sigma$. In other words, $\gamma$ maps $\mathfrak{h}$ onto $\mathfrak{s}$ and $\mathfrak{s}$ onto $\mathfrak{h}$, and $\gamma([X, Y]) = [X, \gamma(Y)]$ for all $X, Y \in \mathfrak{g}$. The basic examples of strongly symmetric spaces are

(a) a pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{R}})$ with $\sigma =$ conjugation with respect to $\mathfrak{g}_{\mathbb{R}}$ and $\gamma =$ multiplication by $i$;

(b) a pair ($\mathfrak{g} \times \mathfrak{g}$, diagonal) with $\sigma(X', X'') = (X'', X')$ and $\gamma(X', X'') = (X', -X'')$.

These examples are dual to each other.

PROPOSITION 4.5. *Assume Conjecture* 3.9. *Then strongly symmetric spaces are special. The same follows from Conjecture* 3.13 *too, for spaces satisfying assumption* (5) *Section* 3.3.

For case (b), i.e. Lie groups considered as symmetric spaces, this is the Kashiwara-Vergne conjecture (see [12]).

*Proof.* Since $E$ is an even function, Conjecture 3.9 implies that $E(X, Y)$ belongs to $[\mathscr{A}, \mathscr{A}]^+ = [\mathscr{A}^+, \mathscr{A}^+] + [\mathscr{A}^-, \mathscr{A}^-]$. Elements of $\mathscr{A}^+$ are endomorphisms of $\mathfrak{h}$, therefore the first part gives no contribution to $\operatorname{tr}_{\mathfrak{h}} E$. In the second part we may use $\gamma$, and repeat the argument in [16] p. 573 to get $\operatorname{tr}_{\mathfrak{h}} E(X, Y) = 0$.

When starting from Conjecture 3.13 the proof is even easier, since $B_{\mathfrak{g}} = 2B_{\mathfrak{h}}$ (on $\mathfrak{h}$) for strongly symmetric spaces. This proves the proposition.

In a recent work on $G_{\mathbb{C}}/G_{\mathbb{R}}$, with semi-simple $G$, P. Harinck [7] shows that invariant eigendistributions on $S$ can be obtained by means of the map $\tilde{}$ from invariant eigendistributions on the tangent space to $S$. This supports our conjectures in this case, since the same result would follow from the above proposition.

## 5. Proof of theorem 3.15

5.1. We retain the notations of Section 3. The aim of the present section is to prove that

$$E(X, Y) = D_2 A(Y, X).x + D_2 A(X, Y).y - \operatorname{ad}(F(X, Y) + A(Y, X)) + \\ + \tfrac{1}{2}(x \coth x + y \coth y - z \coth z - 1)$$

belongs to $\mathscr{I}^+_{x,y}$. Since $E$ is even, it is enough to show that it belongs to $\mathscr{I} = \mathscr{I}_{x,y}$. In view of Lemmas 3.3 and 3.1, the term ad$(\dots)$ can be forgotten here. To study the remaining terms, we shall compute $2D_Y A(X, Y)y + x \coth x$ modulo $\mathscr{I}$, then symmetrize with respect to $X$ and $Y$, then compare to $z \coth z$.

Writing $D_Y A$ exactly would be unpractical; instead we shall use the following lemma, where $\sim$ means equal modulo $\mathscr{I}$.

LEMMA 5.1. *Let $u \in \mathscr{A} = \mathscr{A}_{x,y}$, with absolute convergence of the formal series.*
(i) *If $u$ belongs to $\mathscr{A}.x$, then $D_Y(u(X + Y)).y \sim uy$.*
(ii) *If $u$ belongs to $\mathscr{A}.y$,* then $D_Y(u(X + Y)).y \sim -ux$.

*Proof.* It suffices to assume $u$ is a monomial in $x, y$. To obtain $D_Y(u(X + Y))$, we must differentiate either the final $Y$ (which gives $u$) or every single factor $y$ in $u$: for every way of writing $u = ayb$ (with $a, b$ monomials in $x, y$), we shall have to differentiate $ayb(X + Y) = a.[Y, b(X + Y)]$ with respect to this first $Y$, which yields $-a.\mathrm{ad}(b(X + Y))$. Finally

$$D_Y(u(X + Y)) = u - \sum a_j.\mathrm{ad}(b_j(X + Y)),$$

where $\sum$ runs over all possible ways of writing $u$ as some $a_j y b_j$ (with $a_j, b_j$ monomials in $x, y$).

If $u$ ends by $x$ (case i), then each $b_j$ has degree one at least, so that ad $b_j(X + Y)$ belongs to $\mathscr{I}$ by Lemma 3.3. Therefore $D_Y(u(X + Y)) \sim u$, and the result follows.

If $u$ ends by $y$ (case ii), say $u = vy$, then $u(X + Y) = vy(X + Y) = -vx(X + Y)$ since $(x + y)(X + Y) = 0$. We are therefore reduced to case (i), whence

$$D_Y(u(X + Y))y \sim (-vx)y \sim -vyx = -ux;$$

this proves the lemma.

5.2. To use Lemma 5.1, we need to separate terms ending by $x$ or $y$ in the integral defining $A(X, Y)$ (see §2.3):

LEMMA 5.2. *For $(X, Y) \in \Omega$ we have*

$$\begin{aligned} A(X, Y) = \int_0^1 \frac{\operatorname{sh} tx}{\operatorname{sh} x}(&\operatorname{sh}(1 - t)x + \operatorname{ch} x.\omega.\coth z(t).\operatorname{sh} tx - \\ &- \operatorname{sh} x.\omega.\operatorname{sh} tx)\,dt(X + Y) + \\ &+ \coth x \int_0^1 \frac{\omega}{\operatorname{sh} z(t)} e^{-z(t)} e^{tx} e^{ty} \operatorname{sh} ty\,dt(X + Y). \end{aligned}$$

Here $z(t) = \mathrm{ad}\, Z(tX, tY)$, and $\omega = \omega(z(t))$.

*Proof.* Since $e^{2Z(tX,tY)} = e^{tX}e^{2tY}e^{tX}$ we have, by the adjoint representation,

$$e^{2z(t)} = e^{tx}e^{2ty}e^{tx},$$

therefore

$$(e^{2z(t)} - 1)\,\mathrm{ch}\,tx = (e^{2z(t)} + 1)\,\mathrm{sh}\,tx + e^{tx}(e^{2ty} - 1).$$

Multiplying by $\omega/\mathrm{sh}\,z(t)$ on the left, we obtain

$$\omega\,.\,\mathrm{ch}\,tx = \frac{\omega}{\mathrm{sh}\,z(t)}(\mathrm{ch}\,z(t)\,\mathrm{sh}\,tx + e^{-z(t)}e^{tx}e^{ty}\,\mathrm{sh}\,ty),$$

and the lemma follows from the definition of $A(X, Y)$.

Looking at the sh functions on the right, we see that the first integral in Lemma 5.2 "ends by $x$", and the second by $y$. From Lemma 5.1 it follows that

$$D_Y A(X,Y)y \sim \int_0^1 \frac{\mathrm{sh}\,tx}{\mathrm{sh}\,x}(\mathrm{sh}(1-t)x + \mathrm{ch}\,x\,.\,\omega\,.\,\mathrm{coth}\,z(t)\,.\,\mathrm{sh}\,tx -$$
$$- \mathrm{sh}\,x\,.\,\omega\,.\,\mathrm{sh}\,tx)y\,\mathrm{d}t -$$
$$- \mathrm{coth}\,x \int_0^1 \mathrm{sh}\,tx \frac{\omega}{\mathrm{sh}\,z(t)} e^{-z(t)}e^{tx}e^{ty}\mathrm{sh}\,ty\,.\,x\,\mathrm{d}t.$$

Repeating backwards the proof of Lemma 5.2, we find

$$2D_Y A(X,Y)y \sim 2\int_0^1 \frac{\mathrm{sh}\,tx}{\mathrm{sh}\,x}\mathrm{sh}(1-t)x\,.\,y\,\mathrm{d}t + 2\,\mathrm{coth}\,x\,.$$
$$.\int_0^1 \mathrm{sh}\,tx\,.\,\omega\,.\,\mathrm{coth}\,z(t)\,.\,\mathrm{sh}\,tx\,.\,(x+y)\,\mathrm{d}t -$$
$$- 2\,\mathrm{coth}\,x \int_0^1 \mathrm{sh}\,tx\,.\,\omega\,.\,\mathrm{ch}\,tx\,.\,x\,\mathrm{d}t - 2\int_0^1 \mathrm{sh}\,tx\,.\,\omega\,.\,\mathrm{sh}\,tx\,.\,y\,\mathrm{d}t,$$

the sum of four integrals $I_1, I_2, I_3$ and $I_4$ respectively.

5.3. To compute each of these integrals modulo $\mathscr{I}$, we observe that $\mathscr{A}/\mathscr{I}$ is a commutative algebra, therefore all factors can be freely reordered. Besides $e^{2z(t)} = e^{tx}e^{2ty}e^{tx}$ implies that (introducing the notation $v$):

$$z(t) \sim t(x+y) = tv.$$

From now on we replace $z(t)$ by $tv$ everywhere.

The first integral is elementary; one finds

$$I_1 \sim y\left(\coth x - \frac{1}{x}\right).$$

To evaluate the second

$$I_2 \sim \coth x \int_0^1 \mathrm{sh}^2\, tx \,.\, 2\omega(tv) \,.\, \coth tv\, \mathrm{d}(tv),$$

we integrate by parts by means of the identity $2\omega(u)\coth u\,\mathrm{d}u = 2\mathrm{d}u - \mathrm{d}\omega(u)$, which follows from the definition of $\omega$. There appears $-I_3$, so that

$$I_2 + I_3 \sim v\coth x\left(\frac{\mathrm{sh}\, 2x}{2x} - 1\right) - \tfrac{1}{2}\mathrm{sh}\, 2x \,.\, \omega(v).$$

The last integral

$$I_4 \sim y \int_0^1 (1 - \mathrm{ch}\, 2tx)\omega(tv)\,\mathrm{d}t$$

can be integrated by parts, by means of $\omega(tv)\,\mathrm{d}t = \mathrm{d}(t\coth tv)$, another consequence of the definition of $\omega$. This gives

$$I_4 \sim y(1 - \mathrm{ch}\, 2x)\coth v + 2xy \int_0^1 \mathrm{sh}\, 2tx \,.\, \coth tv \,.\, t\,\mathrm{d}t.$$

Gathering all pieces, we have proved

$$2D_Y A(X, Y)y + x\coth x - 1 \sim y\coth v + v\frac{\mathrm{sh}^2\, x}{x} - y\,\mathrm{ch}\, 2x \,.\, \coth v -$$

$$- \tfrac{1}{2}\mathrm{sh}\, 2x \,.\, \omega(v) + 2xy \int_0^1 \mathrm{sh}\, 2tx \,.\, \coth tv \,.\, t\,\mathrm{d}t.$$

5.4. For the last step of the proof, we exchange $X$ and $Y$, and add:

$$2D_2 A(Y, X)x + 2D_2 A(X, Y)y + x\coth x + y\coth y - v\coth v - 1 \sim R,$$

with

$$R = 1 + v\left(\frac{\operatorname{sh}^2 x}{x} + \frac{\operatorname{sh}^2 y}{y}\right) - (y \operatorname{ch} 2x + x \operatorname{ch} 2y)\coth v -$$

$$-\tfrac{1}{2}(\operatorname{sh} 2x + \operatorname{sh} 2y)\omega(v) + 2xy \int_0^1 (\operatorname{sh} 2tx + \operatorname{sh} 2ty)\coth tv \,.\, t\, dt.$$

In the latter two terms, we use the elementary identity

$$(\operatorname{sh} 2tx + \operatorname{sh} 2ty)\coth tv \sim \operatorname{ch} 2tx + \operatorname{ch} 2ty$$

(since $v = x + y$). The integral can then be computed by parts, and it follows easily that

$$R \sim \tfrac{1}{2}(\operatorname{ch} 2x + \operatorname{ch} 2y)(1 - \operatorname{th} v \,.\, \omega(v)) -$$

$$- (y \operatorname{ch} 2x + x \operatorname{ch} 2y) \coth v + y \operatorname{sh} 2x + x \operatorname{sh} 2y.$$

But $1 - \operatorname{th} v \,.\, \omega(v) = v/\operatorname{sh} v \operatorname{ch} v$ by the definition of $\omega$ and $v = x + y$. It is now a simple exercise to check that the factors of $x$ and $y$ in $R$ both vanish: for instance the factor of $x$ is

$$\frac{\operatorname{ch} 2x + \operatorname{ch} 2y}{\operatorname{sh} 2v} - \operatorname{ch} 2y \coth v + \operatorname{sh} 2y$$

$$= (\operatorname{sh} 2v)^{-1}(\operatorname{ch} 2x + \operatorname{ch} 2y - \operatorname{ch} 2y(1 + \operatorname{ch} 2v) + \operatorname{sh} 2y \operatorname{sh} 2v);$$

since $x = v - y$, this is zero. To write these lines we must work, of course, on an open set where $\operatorname{sh} 2v$ is invertible, and extend the result by analytic continuation. Except this, all the above calculations are valid on $\Omega$.

Thus $R \sim 0$; since $v \coth v \sim z \coth z$, the proof is complete.

A proof of Conjecture 3.9 would require restarting the calculations modulo $[\mathscr{A}, \mathscr{A}]$ instead of $\mathscr{I}$ ....

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