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The classification problem in Teoplitz $\mathbf{Z}_2$-extensions

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Abstract. A large class $\mathcal{T}^*$ of regular 0–1 Toeplitz sequences is determined which enjoy the following property: every finitary isomorphism between Toeplitz $\mathbf{Z}_2$-extensions $T_\eta, T_{\eta'}, \eta, \eta' \in \mathcal{T}^*$, can be extended to a topological isomorphism. Uncountably many ergodic Toeplitz $\mathbf{Z}_2$-extensions with partly continuous spectrum are constructed such that every two are finitarily isomorphic but not topologically conjugate.

Introduction

In this paper we study three kinds of isomorphisms between dynamical systems arising from regular 0–1 Toeplitz sequences and between $\mathbf{Z}_2$-extensions of such systems: metric, finitary and topological isomorphisms. In the case of Toeplitz dynamical systems we have the following property: every metric isomorphism is a finitary isomorphism. On the other hand finitary isomorphism does not imply topological conjugacy. Each Toeplitz sequence η determines a $\mathbf{Z}_2$-extension of a Toeplitz dynamical system induced by η given by a cocycle defined by the zero coordinate. The question arises: what happens in the case of such $\mathbf{Z}_2$-extensions of Toeplitz systems?

In general metric isomorphism does not imply finitary isomorphism, nor does finitary isomorphism imply topological isomorphism. Nevertheless, we find a large class $\mathcal{T}^*$ of regular Toeplitz sequences such that every finitary isomorphism of $\mathbf{Z}_2$-extensions of systems determined by elements from the class $\mathcal{T}^*$ can be extended to a topological isomorphism. The class $\mathcal{T}^*$ includes regular Toeplitz sequences having "holes" at sufficiently large distance and such that $\mathbf{Z}_2$-extensions are ergodic. In particular, finitary isomorphism of Morse dynamical systems coincides with topological conjugacy.

We also show that for sequences from $\mathcal{T}^*$ the quantity of "holes" in n_t-skeletons is an invariant of finitary isomorphisms of $\mathbf{Z}_2$-extensions.

Lastly we produce uncountably many ergodic with partly continuous spectrum $\mathbf{Z}_2$-extensions of Toeplitz sequences, such that every two (different) are finitarily isomorphic, but not topologically conjugate.

Section 1. Preliminary definitions

Let $(X, T), (X', T')$ be strictly ergodic systems (in some compact metric space). Let $\mu(\mu')$ be the unique $T(T')$ invariant measure on $X(X')$. A metric isomorphism $\varphi: (X, T, \mu) \to (X', T', \mu')$ is said to be *finitary* if φ, φ^{-1} are essentially continuous (*e.* continuous), i.e. if there exist $X_0 \subseteq X, X'_0 \subseteq X', \mu(X_0) = \mu(X'_0) = 1$ such that $\varphi|_{X_0}, \varphi^{-1}|_{X'_0}$ are continuous.

By a *topological isomorphism* we mean a homeomorphism $\varphi: X \to X'$ such that $\varphi \circ T = T' \circ \varphi$. We say that (X, T) and (X', T') are *topologically conjugate* if there exists a topological isomorphism between (X, T) and (X', T'). If $(X, T), (X', T')$ are topologically conjugate, then $(X, T, \mu), (X', T', \mu')$ are finitarily isomorphic.

If φ is a finitary isomorphism between (X, T, μ) and (X', T', μ') then we say that φ *can be extended to* a topological isomorphism, if there exists a topological isomorphism $\bar{\varphi}$ such that $\varphi = \bar{\varphi}$ on some subset of measure one.

Section 2. Dynamical systems arising from Toeplitz regular sequences

We recall some basic definitions and results; we refer the reader to [8] for more details.

Let $\Omega = \{0, 1\}^{\mathbf{Z}}$. For $x = (x[n]), y = (y[n]) \in \Omega$ we define

$$d(x, y) = \frac{1}{1 + \min\{|i|: x[i] \neq y[i]\}}.$$

Then (Ω, d) is a compact metric space. Denote by S the left shift homeomorphism on Ω.

$\eta \in \Omega$ is called a *Toeplitz sequence* if for each $i \in \mathbf{Z}$ there exists $n \geqslant 1$ such that

$$\eta[i + k \cdot n] = \eta[i], \quad k \in \mathbf{Z}. \tag{1}$$

By *n-skeleton* of η we will mean the sequence $\eta_n \in \{0, 1, \infty\}^{\mathbf{Z}}$ such that $\eta_n[i] = \eta[i]$ for i satisfying (1) and $\eta_n[i] = \infty$ in the contrary case.

By a *period structure* of nonperiodic Toeplitz sequence η we mean increasing sequence $\{n_t\}$ of positive integers such that

(i) $n_t | n_{t+1}$,
(ii) n_t is an essential period of n_t-skeleton η_{n_t}, i.e. there is no positive integer $m < n_t$ being a period of three-symbols sequence η_{n_t}, $t \geqslant 0$.

Each non-periodic Toeplitz sequence possesses the period structure.

Let $\eta \in \Omega$ be a Toeplitz sequence with the period structure $\{n_t\}$, $n_t | n_{t+1}$, $t \geqslant 0$.

In the sequel η_t denotes the n_t-skeleton of η. Put $I_t = I_t(\eta) = \{0 \leqslant i \leqslant n_t - 1 : \eta_t[i] = \infty\}$. η is said to be *regular* if $\lim_t (1/n_t)|I_t| = 0$ (here $|A|$ denotes the cardinality of A). Now take a regular, non-periodic Toeplitz sequence η with a period structure $\{n_t\}$. Denote by $\bar{\mathcal{O}}(\eta)$ the closure of the orbit of η. Then $(\bar{\mathcal{O}}(\eta), S)$ is strictly ergodic. Denote by $\mu = \mu_\eta$ the unique S invariant measure on $\bar{\mathcal{O}}(\eta)$. Let $G = G_\eta = \varprojlim \mathbf{Z}/n_t\mathbf{Z}$ be the inverse limit group. Denote by $\hat{1}$ the element $\hat{1} = (1, 1, \ldots) \in G$ and $\hat{n} = n \cdot \hat{1}$, $n \in \mathbf{Z}$. Then $(G, \hat{1})$ is a compact monothetic group with generator $\hat{1}$ and $(G, \hat{1})$ is the maximal equicontinuous factor of $(\bar{\mathcal{O}}(\eta), S)$ ([2], [8]). Denote by $\pi = \pi_\eta : \bar{\mathcal{O}}(\eta) \to G$ the factor map, such that $\pi^{-1}(\hat{0}) = \{\eta\}$. Then π is one-to-one on the set of Toeplitz sequences in $\bar{\mathcal{O}}(\eta)$ and this set has μ-measure one. Therefore the dynamical system $(\bar{\mathcal{O}}(\eta), S, \mu)$ is metrically isomorphic to $(G, \hat{1}, \lambda)$, where λ is the normalized Haar measure on G.

THEOREM 1. *Every metric isomorphism between $(\bar{\mathcal{O}}(\eta), \mu_\eta, S)$ and $(\bar{\mathcal{O}}(\eta'), \mu_{\eta'}, S)$ (η, η' are regular Toeplitz sequences) is a finitary isomorphism.*

Proof. Let $\varphi : \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(\eta')$ be a metric isomorphism. Denote by Λ_η, $\Lambda_{\eta'}$ the eigenvalue groups of $(G_\eta, \hat{1}, \lambda)$, $(G_{\eta'}, \hat{1}, \lambda)$. Since $\Lambda_\eta = \Lambda_{\eta'}$ we may assume $G = G_\eta = G_{\eta'}$.

$$\begin{array}{ccc} \bar{\mathcal{O}}(\eta) & \xrightarrow{\ \varphi\ } & \bar{\mathcal{O}}(\eta') \\ {\scriptstyle \pi_\eta}\downarrow & & \downarrow{\scriptstyle \pi_{\eta'}} \\ G & \xrightarrow{\ \psi\ } & G \end{array}$$

The map $\psi = \pi_{\eta'} \circ \varphi \circ (\pi_\eta)^{-1}$ is a metric automorphism of $(G, \hat{1}, \lambda)$ (ψ is defined on some subset of G of measure one). Thus ψ is a translation of G, $\psi(g) = g + g_0$, $g \in G$. Since ψ, ψ^{-1} are e. continuous we have that φ, φ^{-1} are e. continuous.

REMARK 1. It follows from Lemma 10 that if η, η' are regular Toeplitz sequences having the same period structure (in this case $(\bar{\mathcal{O}}(\eta), \mu_{\eta_\eta}, S)$ and $(\bar{\mathcal{O}}(\eta'), \mu_{\eta'}, S)$ are metrically and finitarily isomorphic), $(\bar{\mathcal{O}}(\eta), \mu_\eta, S)$ and $(\bar{\mathcal{O}}(\eta'), \mu_{\eta'}, S)$ need not be topologically conjugate.

Assume now η is a regular, non-periodic Toeplitz sequence with the period structure $\{n_t\}$. Then we can consider η as the map $\eta : G \to \mathbf{Z}_2 = \{0, 1\}$ defined in the following way: $\eta(g) = \eta_t[j_t]$ (for sufficiently large t), where $g = (j_t) \in G$. The map η is correctly defined on a subset of G of measure one. Similarly, if $g_0 = (j_t) \in G$, then we can consider $\eta \circ g_0$ as a sequence $(\eta \circ g_0)[i] = \eta_t[i + j_t]$ for sufficiently large t (note that in this case $\eta \circ g_0$ need not be Toeplitz sequence – the hole may be included in it), or as a map $\eta \circ g_0 : G \to \mathbf{Z}_2$. Note that

$$(\eta \circ g_0)(g) = \eta(g + g_0).$$

The following lemma will be needed in further considerations.

LEMMA 1. *If* $\varphi: \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(n')$ *is a topological isomorphism* (η, η' *are regular, non-periodic Toeplitz sequences with the period structure* $\{n_t\}$), *then* $\varphi(\eta)$ *is a Toeplitz sequence and* $\varphi(\eta) = \eta' \circ g_0$ *for some* $g_0 \in G$.

Proof. Let $A_g = (\pi_\eta)^{-1}(g)$, $B_g = (\pi_{\eta'})^{-1}(g)$, $g \in G$. Since $\pi_{\eta'} \circ \varphi: (\bar{\mathcal{O}}(\eta), S) \to (G, \hat{1})$ is a factor and $(G, \hat{1})$ is a maximal equicontinuous factor of $(\bar{\mathcal{O}}(\eta), S)$ we can find a factor map $\psi: (G, \hat{1}) \to (G, \hat{1})$ such that

$$\psi \circ \pi_\eta = \pi_{\eta'} \circ \varphi. \tag{2}$$

Clearly $\psi(g) = g + g_0$ for some $g_0 \in G$ and all $g \in G$. Take $g \in G$. If $u \in A_g$ and $\varphi(u) \in B_h$, then by (2) $\psi(g) = h$, i.e. $\varphi(A_g) \subseteq B_{g+g_0}$. Since $\{A_g\}$, $\{B_g\}$ are partitions in $\bar{\mathcal{O}}(\eta)$, $\bar{\mathcal{O}}(\eta')$ and φ is a suriective map, we have $\varphi(A_g) = B_{g+g_0}$, $g \in G$. Since $|B_{g_0}| = 1$, we obtain that $\eta' \circ g_0$ is Toeplitz sequence and $\varphi(\eta) = \eta' \circ g_0$.

REMARK 2. It follows from the equality $B_{g_0} = \{\varphi(\eta)\}$ and from Lemma 1 that g_0 is determined uniquely.

Section 3. Finitary and topological isomorphism of Toeplitz $\mathbf{Z}_2$-extensions

Let η be a regular, non-periodic Toeplitz sequence. The dynamical system $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta, \tilde{\mu})$, where $\mathbf{Z}_2 = \{0, 1\}$, $\mu = \mu_\eta \times \vartheta$, $\vartheta(\{0\}) = \vartheta(\{1\}) = \frac{1}{2}$ and

$$T_\eta(y, i) = (Sy, i + y[0]),\ y \in \bar{\mathcal{O}}(\eta),\ i \in \mathbf{Z}_2$$

is called the *Toeplitz* $\mathbf{Z}_2$*-extension of* $(\bar{\mathcal{O}}(\eta), \mu_\eta, S)$.

Note that the cocycle $\Phi: \bar{\mathcal{O}}(\eta) \to \mathbf{Z}_2$ defined by $\Phi(y) = y[0]$ satisfies the condition

$$\Phi = \eta \circ \pi \text{ a.e.} \tag{3}$$

Indeed, define in $\bar{\mathcal{O}}(\eta)$ and G the following partitions:

$$D_t^i = \{S^m\eta : m \equiv i \operatorname{mod}(n_t)\},\quad E_i^t = \{(j_t)_0^\infty : j_t = i\},\quad i = 0, 1, \ldots, n_t - 1, t \geqslant 0.$$

We have $\pi D_i^t = E_i^t$. Now let $y \in D_i^t$ and suppose that $\eta_t[i] \neq \infty$. Then $\Phi(y) = \eta_t[i]$ and $(\eta\pi)(y) = h_t[i]$. Since π, π^{-1} are e. continuous, it follows from (3) that $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta, \tilde{\mu})$ is finitarily isomorphic to $(G \times \mathbf{Z}_2, \bar{T}_\eta, \tilde{\lambda})$, where $\tilde{\lambda} = \lambda \times \vartheta$ and

$$\bar{T}_\eta(g, i) = (g + \hat{1}, i + \eta(g)),\quad g \in G,\quad i \in \mathbf{Z}_2.$$

In the sequel we use the common notation T_η for T_η, $\bar{T}_\eta$.

Denote by $\mathscr{T}$ the set of all regular, non-periodic Toeplitz sequences with the period structure $\{n_t\}$ such that T_η is ergodic. It follows from [7] the for $\eta \in \mathscr{T}$, $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta)$ is strictly ergodic.

Suppose that $\eta, \eta' \in \mathscr{T}$ and $(G \times \mathbf{Z}_2, T_\eta, \tilde{\lambda})$, $(G \times \mathbf{Z}_2, T_{\eta'}, \tilde{\lambda})$ are finitarily isomorphic. Let W be a finitary isomorphism. It follows from [6] that W is of the form

$$W(g, i) = (g + g_0, i + p(g)) \quad \text{a.e.,} \quad g \in G, \quad i \in \mathbf{Z}_2, \tag{4}$$

where $p\colon G \to \mathbf{Z}_2$ is a measurable function satisfying the equality

$$p \circ \hat{1} + p = \eta + \eta' \circ g_0 \tag{5}$$

(we use the symbol $+$ to denote addition mod 2 in $\mathbf{Z}_2$). Since W is e. continuous we obtain that p is e. continuous. On the other hand, if W is of the form (4) and $p\colon G \to \mathbf{Z}_2$ is e. continuous function satisfying the condition (5), then W is a finitary isomorphism.

Let $\mathscr{T}^*$ be the class defined as follows: $\eta \in \mathscr{T}^*$ if $\eta \in \mathscr{T}$ and there is $\rho > 0$ such that for $t \geqslant 0$

$$\eta_t[i] = \eta_t[j] = \infty, \quad i \neq j \Rightarrow |i - j| \geqslant \rho \cdot \eta_t. \tag{6}$$

The following proposition says that the quantity of holes in η_t is invariant under finitary isomorphisms (in $\mathscr{T}^*$).

PROPOSITION 1. *If $\eta, \eta' \in \mathscr{T}^*$ and $T_\eta, T_{\eta'}$ are finitarily isomorphic, then $|I_t(\eta)| = |I_t(\eta')|$ for sufficiently large t.*

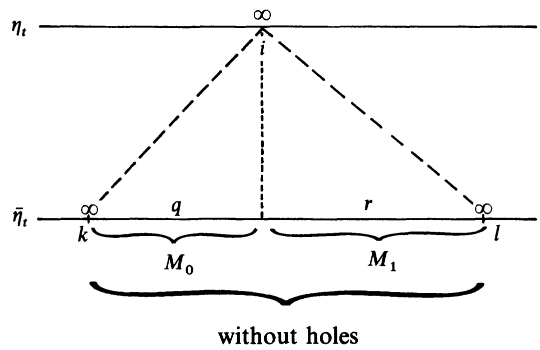
Proof. Take $\eta, \eta' \in \mathscr{T}^*$ and suppose that T_η, $T_{\eta'}$ are finitarily isomorphic. Let $p\colon G \to \mathbf{Z}_2$ be e. continuous and $g_0 \in G$ such that (5) is satisfied. Put $\bar{\eta} = \eta' \circ g_0$ and denote by $\bar{\eta}_t$ the n_t-skeleton of $\bar{\eta}$. Take $\rho > 0$ such that (6) is satisfied by η, η' (and thus by $\bar{\eta}$). Let $E_i^t = \{g = (j_s) \in G\colon j_s = i\}$, $i = 0, \ldots, n_t - 1$, $t \geqslant 0$. Define

$$J_t = \{0 \leqslant i \leqslant n_t - 1\colon p|_{E_i^t} = a \text{ a.e.} \quad \text{for some } a \in \mathbf{Z}_2\}, \quad t \geqslant 0.$$

Since p is e. continuous we obtain $1/n_t \cdot |J_t| \to 1$. Fix $t \geqslant 0$ such that

$$\frac{|J_t|}{n_t} > 1 - \tfrac{1}{4}\rho \tag{7}$$

Fix $i \in I_t = I_t(\eta)$. For $a, b \in \mathbf{Z}$ by $a \oplus b$, $a \ominus b$ we mean $(a + b)(\text{mod}\, n_t)$, $(a - b)(\text{mod}\, n_t)$. Let $q \geqslant 0$ be the smallest integer such that $k = i \ominus q \in \bar{I}_t = \bar{I}_t(\bar{\eta})$ and let $r \geqslant 1$ be the smallest integer such that $l = i \oplus r \in \bar{I}_t$.



Put

$$M_0 = \{i \ominus j : 0 \leqslant j \leqslant \min(q-1, \tfrac{1}{4}n_t \cdot \rho)\},$$
$$M_1 = \{i \ominus j : 1 \leqslant j \leqslant \min(r, \tfrac{1}{4}n_t \cdot \rho)\}$$

(if $q = 0$ then we put $M_0 = \emptyset$). It follows from the definition that for $j \in M_0 \setminus \{i\}$, we have $\eta_t[j] \neq \infty, \bar{\eta}_t[j] \neq \infty$ and for $j \in M_1 \setminus \{l\}, \eta_t[j] \neq \infty$ and $\bar{\eta}_t[j] \neq \infty$. Note that from (5) if $j \in J_t$ and $\eta_t[j] \neq \infty, \bar{\eta}_t[j] \neq \infty$, then $1 \oplus j \in J_t$. Similarly if $1 \oplus j \in J_t$ and $\eta_t[j] \neq \infty, \bar{\eta}_t[j] \neq \infty$ then $j \in J_t$. Hence for $a = 0, 1$ $M_a \cap J_t = \emptyset$ or $M_a \subseteq J_t$. Note that we can find $a \in \{0, 1\}$ with the property

$$M_a \cap J_t = \emptyset, \quad M_{1-a} \subseteq J_t. \tag{8}$$

Indeed, if $M_0, M_1 \subseteq J_t$, $M_0 \neq \emptyset$, then i, $1 \oplus i$ belong to J_t and since $\bar{\eta}_t[i] \neq \infty$ we obtain $\eta_t[i] \neq \infty$, i.e. $i \notin I_t$. Moreover $M_0 \subseteq J_t$ or $M_1 \subseteq J_t$ because in the contrary case

$$|J_t| \leqslant n_t - (|M_0| + |M_1|) \leqslant n_t - \tfrac{1}{4}\rho \cdot n_t$$

and in view of (7) we obtain a contradiction. Thus the property (8) holds. Set $\alpha_t(i) = -q$ if $a = 0$ and $\alpha_t(i) = r$ otherwise. The properties (7) and (8) give $|\alpha_t(i)| < \frac{1}{4}\rho n_t$. Therefore the map $\beta_t : I_t \to \bar{I}_t, \beta_t(i) = i + \alpha_t(i)$ is one-to-one. This implies $|I_t| \leqslant |\bar{I}_t|$. Similarly $|\bar{I}_t| \leqslant |I_t|$, so $|I_t| = |\bar{I}_t|$.

REMARK 3. For $i \in I_t$ (t sufficiently large) we put

$$K_t(i) = \begin{cases} \{i \oplus j : j = 1, \ldots, \alpha_t(i)\} & \text{if } \alpha_t(i) > 0, \\ \{i \ominus j : j = 0, 1, \ldots, \alpha_t(i) + 1\} & \text{if } \alpha_t(i) < 0, \\ \emptyset & \text{if } \alpha_t(i) = 0. \end{cases}$$

$$K_t = \bigcup_{i \in I_t} K_t(i).$$

It follows from the proof of Proposition 1 that $J_t = X_t \backslash K_t$, where $X_t = \{0, \ldots, n_t - 1\}$.

Now suppose η, η' are Toeplitz sequences. Denote by $\eta + \eta'$ the Toeplitz sequence: $(\eta + \eta')[i] = \eta[i] + \eta'[i]$. By $\check{\eta}$ we mean the following 0–1 sequence:

$$\check{\eta}[0] = 0, \check{\eta}[i] = \eta[0] + \cdots + \eta[i-1] \quad \text{for } i \geqslant 1 \text{ and}$$

$$\check{\eta}[i] = \eta[-1] + \cdots + \eta[i] \quad \text{for } i \leqslant -1.$$

Let $g \in G$ and assume that $\eta' \circ g$ is Toeplitz sequence. Then by $\theta^{(g)}$ we denote one-sided sequence defined in the following way:

$$\theta^{(g)}[i] = (\eta + \eta' \circ g)\check{}\,[i], \quad i \geqslant 1.$$

We are going to show the main theorem of this paper.

THEOREM 2. *If W is a finitary isomorphism between T_η, $T_{\eta'}$, where $\eta, \eta' \in \mathscr{T}^*$, then we can extend W to a topological isomorphism between $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta)$ and $(\bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2, T_{\eta'})$.*

The proof of this theorem consists of several lemmas. First we show

LEMMA 2. *Let $W: \bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2 \to \bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2$ be a topological isomorphism between Toeplitz $\mathbf{Z}_2$-extensions T_η and $T_{\eta'}$, where $\eta, \eta' \in \mathscr{T}$. Then there is a topological isomorphism $\varphi: \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(\eta')$ and a continuous function $p: \bar{\mathcal{O}}(\eta) \to \mathbf{Z}_2$ such that*

$$W(y, i) = (\varphi(y), i + p(y)), \quad i \in \mathbf{Z}_2, \quad y \in \bar{\mathcal{O}}(\eta), \tag{9}$$

$$p(Sy) + p(y) = y[0] + (\varphi(y))[0], \quad y \in \bar{\mathcal{O}}(\eta). \tag{10}$$

Proof. We show first that

$$W \circ \sigma = \sigma \circ W, \tag{11}$$

where $\sigma(y, i) = (y, i+1)$. We may consider W as a metric isomorphism between $\mathbf{Z}_2$-extensions $(G \times \mathbf{Z}_2, T_\eta, \tilde{\lambda})$ and $(G \times \mathbf{Z}_2, T_{\eta'}, \tilde{\lambda})$ (this isomorphism we denote by W'). Then W' is of the form (4). Since $W' \circ \sigma = \sigma \circ W'$ a.e. (here $\sigma(g, i) = (g, i+1)$) we can find $u \in \bar{\mathcal{O}}(\eta)$ such that $(W\sigma)(u, i) = (\sigma W)(u, i)$, $i \in \mathbf{Z}_2$. Put $Y = \{S^m u, m \in Z\} \times \mathbf{Z}_2$. By continuity of W, σ, σ^{-1} it suffices to show that $W = \sigma W \sigma^{-1}$ on Y. Take $(S^m u, i) \in Y$. It is obvious that $\sigma T_\eta = T_\eta \sigma$ and therefore from the equality

$$T_\eta^m(y, k) = (S^m y, k + \check{y}[m]), \quad k \in \mathbf{Z}_2 \tag{12}$$

we obtain

$$\begin{aligned}(W\sigma)(S^m u, i) &= W(S^m u, i+1) = (WT_\eta^m)(u, \check{u}[m] + i + 1)\\ &= (T_{\eta'}^m W)(u, \check{u}[m] + i + 1) = (\sigma T_{\eta'}^m W)(u, \check{u}[m] + i)\\ &= (\sigma W)(S^m u, i).\end{aligned}$$

Now, for $(y, i) \in \bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2$ we set $W(y, i) = ((W_1(y, i), W_2(y, i)))$. As a simple consequence of (11) we obtain

$$W_1(y, 1) = W_1(y, 0), \quad W_2(y, 1) = W_2(y, 0) + 1, \quad y \in \bar{\mathcal{O}}(\eta).$$

Thus, putting $\varphi(y) = W_1(y, 0), p(y) = W_2(y, 0), y \in \bar{\mathcal{O}}(\eta)$ we get (9). p is continuous, since W is continuous; φ is a topological isomorphism because W is a topological isomorphism. Lastly, the equality (10) is a consequence of $W \circ T_\eta = T_{\eta'} \circ W$.

REMARK 4. If $\varphi: \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(\eta')$ is a topological isomorphism and $p: \bar{\mathcal{O}}(\eta) \to \mathbf{Z}_2$ is a continuous function and (9), (10) hold, then it is easy to see that W is a topological isomorphism between T_η, $T_{\eta'}$.

Now, we are in a position to give another form of Theorem 2. To this end fix η, $\eta' \in \mathcal{T}$ such that T_η and $T_{\eta'}$ are topologically conjugate. Let W be a topological isomorphism between T_η and $T_{\eta'}$ and suppose that W is of the form (9). Then W determines exactly one $g_0 \in G$ which satisfies Lemma 1. Denote by $G_t \subset G$ the set of all g_0 such that there is a topological isomorphism W between T_η and $T_{\eta'}$ which determines g_0. Similarly, every finitary isomorphism W between T_η and $T_{\eta'}$ determines a unique $g_0 \in G$ such that (4) holds. Let G_f be the set of all $g_0 \in G$ such that g_0 is determined by some finitary isomorphism of T_η and $T_{\eta'}$. It is not hard to see that $g_0 \in G_f$ iff there is a measurable e. continuous function $p: G \to \mathbf{Z}_2$ such that (5) holds. Clearly $G_t \subseteq G_f$. Note that every $g_0 \in G_f$ is determined exactly by two finitary isomorphisms $W, \bar{W}$. These isomorphisms satisfy $W = \sigma \circ \bar{W}$ a.e. This is a consequence of ergodicity of $(G, \lambda, \hat{1})$ and the equality (5). Similarly every $g_0 \in G_t$ is determined exactly by two topological isomorphisms W, $\bar{W}$ and $W = \sigma\bar{W}$. Indeed, assume that

$$W(y, i) = (\varphi(y), i + p(y)), \quad \bar{W}(y, i) = (\bar{\varphi}(y), i + p(y)), \quad i \in \mathbf{Z}_2, \quad y \in \bar{\mathcal{O}}(\eta),$$

where $\varphi(A_g) = B_{g+g_0}$, $\bar{\varphi}(A_g) = B_{g+g_0}$, $g \in G$. Then $\varphi = \bar{\varphi}$ a.e. and from (10) and ergodicity of $(\bar{\mathcal{O}}(\eta), \mu_\eta, S)$ we obtain $p = p + a$ a.e. Hence there is $u \in \bar{\mathcal{O}}(\eta)$ such that $W(u, i) = \sigma^a \bar{W}(u, i)$, $i \in \mathbf{Z}_2$. This and (11) imply that $W(S^m u, i) = (\sigma^a \bar{W})(S^m u, i)$, $i \in \mathbf{Z}_2$. Since $\{(S^m u, i): m \in \mathbf{Z}, i \in \mathbf{Z}_2\}$ is a dense in $\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2$ and W, $\bar{W}, \sigma$ are continuous, we have $W = \sigma^a \bar{W}$.

By the above, for $\eta, \eta' \in \mathscr{T}^*$ Theorem 2 says that

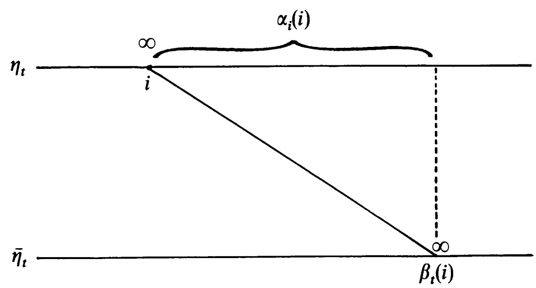
$$G_f \subseteq G_t \tag{13}$$

(and hence $G_f = G_t$).

The following three lemmas prove (13).

LEMMA 3. *Assume that* $\eta, \eta' \in \mathscr{T}^*$. *Then* $g_0 \in G_f$ *if and only if* $\eta' \circ g_0$ *is a Toeplitz sequence and* $\theta^{(g_0)}$ *is a regular Toeplitz sequence.*

Proof. Assume that $g_0 \in G_f$. Let $p\colon G \to \mathbf{Z}_2$ be e. continuous map such that (5) holds. Put $\bar\eta = \eta' \circ g_0$, $\gamma = \eta + \bar\eta$, $I_t = I_t(\eta)$, $\bar I_t = I_t(\bar\eta)$. First we show that $\bar\eta$ is a Toeplitz sequence. Fix $t \geqslant 0$ with the property (7). Fix $i \in I_t$ and consider η_t, $\bar\eta_t$ n_t-skeletons of $\eta, \bar\eta$:



There exists a constant $a_t(i)$ such that for a.e. $g \in E_i^t$

$$\eta(g) + \bar\eta(g + \hat{\alpha_t}(i)) = a_t(i) \tag{14}$$

(Recall $\hat n = n \circ \hat 1$ and $\alpha_t(i)$, $\beta_t(i)$ are defined in Proposition 1). Indeed, suppose that for example $\alpha_t(i) \geqslant 0$. Then $\gamma_t[j] \neq \infty$ for $j \in K_t(i) \backslash \beta_t(i)$. Since i, $\beta_t(i) \oplus 1 \in J_t$ we have by repeated application of (5):

$$\begin{aligned} b_t(i) &= p(g + (\alpha_t(i) + 1)\hat{}\,) + p(g) \\ &= \gamma(g) + \gamma(g + \hat 1) + \cdots + \gamma(g + \alpha_t(i)\hat{}\,) \\ &= \eta(g) + \bar\eta(g + \hat{\alpha_t}(i)) + \bar\eta_t[i] + \eta_t[\beta_t(i)] + \gamma_t[i+1] + \cdots + \gamma_t[\beta_t(i) - 1]. \end{aligned}$$

Hence (14) is obvious. In particular for $t' \geqslant t$, $k \in \mathbf{Z}$ we obtain from (14)

$$\eta_{t'}[i + k\cdot n_t] = \bar\eta_{t'}[\beta_t(i) + k\cdot n_t] + a_t(i). \tag{15}$$

The last property together with the fact that η is a Toeplitz sequence gives $\bar{\eta}$ is a Toeplitz sequence too.

Now we show that $\theta = \theta^{(g_0)}$ is a regular Toeplitz sequence. Take $j \geqslant 1$ and choose t such that the residue $l = j \pmod{n_t}$ belongs to J_t (such t exists since η, η' are Toeplitz sequences). For $g \in E_l^t$ we have

$$0 = p(g + \hat{n}_t) + p(g) = \gamma(g) + \cdots + \gamma(g + (n_t - 1)\hat{}).$$

This implies

$$\gamma[l + k \cdot n_t] + \gamma[l + k \cdot n_t + 1] + \cdots + \gamma[l + (k+1) \cdot n_t - 1] = 0, \quad k \in \mathbf{Z}. \tag{16}$$

Hence $\tilde{\gamma}[l + (k+1) \cdot n_t] = \tilde{\gamma}[l + k \cdot n_t], k \geqslant 0$, i.e. θ is a Toeplitz sequence. θ is regular since $|J_t|/n_t \to 1$.

It remains to show the sufficiency. Let $p: G \to \mathbf{Z}_2$ be defined as follows: $p(g) = \tilde{\gamma}_t[j_t]$ for sufficiently large t, $g = (j_t)_0^\infty \in G$ ($\tilde{\gamma}_t$ is the n_t-skeleton of $\tilde{\gamma}$). Then p is correctly defined for almost all $g \in G$. Moreover p is e. continuous and p satisfies the condition

$$p(g + \hat{1}) + p(g) = \tilde{\gamma}[1 + j_t] + \tilde{\gamma}[j_t] = \gamma[j_t] = \gamma(g).$$

Therefore g_0, p determines a finitary isomorphism between T_η and $T_{\eta'}$.

REMARK 5. It follows from the proof of Lemma 3 that if $\eta, \eta' \in \mathscr{T}$, $\eta' \circ g_0$ is a Toeplitz sequence and $\theta^{(g_0)}$ is a regular Toeplitz sequence, then T_η and $T_{\eta'}$ are finitarily isomorphic.

LEMMA 4. *If $\eta, \eta' \in \mathscr{T}^*$ and $g_0 \in G_f$, then*

$$\exists \delta > 0 \; d(S^m v, S^n v) < \delta \Rightarrow \tilde{\gamma}[m] = \tilde{\gamma}[n], \tag{17}$$

where $v = \eta$ or $v = \eta' \circ g_0$ and $\gamma = \eta + \eta' \circ g_0$.

Proof. Let $g_0 \in G_f$ and choose e. continuous $p: G \to \mathbf{Z}_2$ such that (5) holds. Fix t with the property (7). It is well known [3] that each Toeplitz sequence v satisfies the condition

$$\exists \delta_1 > 0 \; d(S^m v, S^n v) < \delta_1 \Rightarrow m \equiv n \pmod{n_t}. \tag{18}$$

Choose $\delta_1 > 0$ such that (18) holds for $v = \eta, \bar{\eta} = \eta' \circ g_0$. Take $v = \eta$ or $v = \bar{\eta}$. Suppose that $d(S^m v, S^n v) < \delta$, where $0 < \delta < \delta_1$ will be chosen later. We may assume that $m < n$. It follows from the definition of $\tilde{\gamma}$ that $\tilde{\gamma}[n] = \tilde{\gamma}[m] + \gamma[m] + \cdots + \gamma[n-1]$. Put $l = m \pmod{n_t}$.

A. Suppose that $l \in J_t$. Then from (18) and (16) we obtain

$$\gamma[m] + \cdots + \gamma[n-1] = 0, \text{ i.e. } \tilde{\gamma}[m] = \tilde{\gamma}[n].$$

B. $l \notin J_t$.
Let $i \in I_t$ be such that $l \in K_t(i)$ (the definition of $K_t(i)$ is in Remark 3. Let $r < l$ be the greatest integer such that $r \equiv r' \pmod{n_t}$, where $r' \in J_t$. Take

$$\delta < \min\left(\delta_1, \frac{1}{(\frac{1}{4}\cdot\rho + \delta_1)\cdot n_t + 1}\right)$$

((6) holds for $\rho > 0, \eta, \eta'$). Put $m' = m - (l - r)$, $n' = n - (l - r)$, $l' = m' \pmod{n_t}$. Since $m' \equiv n' \pmod{n_t}$ and $l' \in J_t$ from (16) we obtain $\tilde{\gamma}[m'] = \tilde{\gamma}[n']$. Moreover

$$\tilde{\gamma}[m] = \tilde{\gamma}[m'] + \gamma[m'] + \cdots + \gamma[m-1],$$

$$\tilde{\gamma}[n] = \tilde{\gamma}[n'] + \gamma[n] + \cdots + \gamma[n-1].$$

From the inequality $d(S^m v, S^n v) < \delta$ and (15) we obtain that $\gamma[m'] = \gamma[n']$ and since $\gamma_t[m' + q] = \gamma_t[n' + q] \neq \infty$ for $q = 1, 2, \ldots, m - 1 - m'$ we get $\tilde{\gamma}[m] = \tilde{\gamma}[n]$. This finishes the proof.

LEMMA 5. *Suppose that $\eta, \eta' \in \mathcal{T}$. Then $g_0 \in G_t$ iff $\eta' \circ g_0$ is Toeplitz sequence and* (16) *holds.*

Proof. Necessity.
Let $g_0 \in G_t$ and $W: \bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2 \to \bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2$ be a topological isomorphism which determines g_0. W is of the form

$$W(y, i) = (\varphi(y), p(y) + i), \quad y \in \bar{\mathcal{O}}(\eta), \quad i \in \mathbf{Z}_2,$$

where $\varphi: \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(\eta')$ is topological isomorphism, $\varphi(A_g) = B_{g+g_0}$, $g \in G$ and $p: \bar{\mathcal{O}}(\eta) \to \mathbf{Z}_2$ is a continuous map satisfying (10). Put $a = p(g)$. It follows from (10) that $p(S^r\eta) = \tilde{\gamma}[r] + a$, $r \in \mathbf{Z}$ (here $\gamma = \eta + \eta' \circ g_0$). By the continuity of p we obtain (17) for $v = \eta$. Since $\varphi^{-1}(\eta' \circ g_0) = \eta$ and φ^{-1} is continuous (17) is also true for $v = \eta' \circ g_0$.

Sufficiency.
Note first that for $\eta \in \mathcal{T}$

$$(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta) \text{ and } (\bar{\mathcal{O}}(\tilde{\eta}), S) \text{ are topologically conjugate} \tag{19}$$

In fact, put $Y = \{\tilde{y} + i, y \in \bar{\mathcal{O}}(\eta), i \in \mathbf{Z}_2\}$ and consider the map $\psi: \bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2 \to Y$, $\psi(y, i) = \tilde{y} + i$. It is not hard to see that ψ is a homeomorphism and $\psi T_\eta = S\psi$.

Thus (Y, S) is minimal and since $\check{\eta} \in Y$ we have $Y = \bar{\mathcal{O}}(\check{\eta})$. Now we show that

$$(\bar{\mathcal{O}}(\check{\eta}), S) \text{ and } (\bar{\mathcal{O}}(\check{\eta}'), S) \text{ are topologically conjugate.}$$

Put $u = \eta' \circ g_0$. Since $u \in \bar{\mathcal{O}}(\eta')$, we have $\bar{\mathcal{O}}(\check{\eta}') = \bar{\mathcal{O}}(\check{u})$. We define $\varphi\colon \bar{\mathcal{O}}(\check{\eta}) \to \bar{\mathcal{O}}(\check{u})$ as follows: if $y = \lim_t S^{r_t}\check{\eta}$, then $\varphi(y) = \lim_t S^{r_t}\check{u}$. First we show the correctness this definition. Note that from (17) for $\varepsilon > 0$ we can find $\delta' > 0$ such that

$$\begin{aligned} &d(S^m\check{\eta}, S^n\check{\eta}) < \delta' \Rightarrow d(S^m\check{u}, S^n\check{u}) < \varepsilon, \\ &d(S^m\check{u}, S^n\check{u}) < \delta' \Rightarrow d(S^m\check{\eta}, S^n\check{\eta}) < \varepsilon. \end{aligned} \tag{20}$$

Therefore if $y = \lim_t S^{r_t}\check{\eta}$, then $\{S^{r_t}\check{u}\}$ is convergent. Moreover, if $y = \lim_t S^{r_t}\check{\eta} = \lim_t S^{j_t}\check{\eta}$, then $d(S^{r_t}\check{\eta}, S^{j_t}\check{\eta}) \to 0$, so $\lim_t S^{r_t}\check{u} = \lim_t S^{j_t}\check{u}$. It follows from (20) that φ is one-to-one and onto. The continuity of φ is a consequence of the below inequality

$$d(\varphi(y), \varphi(v)) \leqslant d(\varphi(y), S^{r_t}\check{u}) + d(S^{r_t}\check{u}, S^{j_t}\check{u}) + d(S^{j_t}\check{u}, \varphi(v)),$$

where $y = \lim_t S^{r_t}\check{\eta}$, $v = \lim S^{j_t}\check{\eta} \in \bar{\mathcal{O}}(\check{\eta})$. Thus φ is a homeomorphism and since $\varphi S = S\varphi$, we obtain that φ is a topological isomorphism.

The above shows that there is a topological isomorphism $W\colon \bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2 \to \bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2$ such that $W(y, 0) = (\eta' \circ g_0, 0)$. This clearly implies $g_0 \in G_t$.

Lemma 4 and Lemma 5 give (13).

Now, consider sequences $\eta \in \{0, 1, \infty\}^{\mathbf{Z}}$ which have the following property:

$$\text{if } \eta[i] \neq \infty \text{ then there is } p \in \mathbf{N} \text{ such that } \eta[i + k \cdot p] = \eta[i],\ k \in \mathbf{Z}. \tag{21}$$

Of course, if $\eta[i] \neq \infty$ for all $i \in \mathbf{Z}$, then η is Toeplitz sequence. For such sequences we may, similarly as in the case Toeplitz sequences, define period structure, regularity, $I_t(\eta)$, $\eta \circ g$. We use $\mathcal{S}$ to denote the class of all regular sequences $\eta \in \{0, 1, \infty\}^{\mathbf{Z}}$ with period structure $\{n_t\}$, satisfying the conditions (21) and (6).

REMARK 6. If $\eta \in \mathcal{S}$, then η contains at most one ∞. Furthermore, if $\eta \in \mathcal{S}$ then there is $g \in G$ such that $\eta \circ g$ is a Toeplitz sequence. Indeed, suppose $\eta[q] = \infty$. Put

$$g = (1, 1, n_0, n_0, n_1, n_1, \dots) = (j_t)_0^\infty.$$

Suppose $(\eta \circ q)[i] = \infty$, i.e. $\eta_t[i + j_t] = \infty$, $t \geqslant 0$, where η_t is n_t-skeleton of η.

Because of $j_t \to \infty$, $j_t/n_t \to 0$ we have $(i + j_t) \not\equiv q \pmod{n_t}$ for sufficiently large t and hence in view of (6) we obtain (for sufficiently large t) $\rho \cdot n_t < i + j_t - q$. In particular

$$0 < \rho \leqslant \liminf_{t \to \infty} \frac{i + j_t - q}{n_t} = 0.$$

Therefore $(\eta \circ g)[i] \neq \infty$.

LEMMA 6. *Suppose $\eta \in \mathscr{S}$. Take $g, g' \in G$ such that $\eta \circ g, \eta \circ g'$ are Toeplitz sequences. Then $\bar{\mathcal{O}}(\eta \circ g) = \mathcal{O}(\eta \circ g')$.*

Proof. Put $h = g' - g = (j_t)_0^\infty \in G$. Since $h \circ (g + h) = (\eta \circ g) \circ h$, we have $\bar{\mathcal{O}}(\eta \circ g') = \bar{\mathcal{O}}((\eta \circ g) \circ h)$. This implies $\eta \circ g' = \lim_t S^{j_t}(\eta \circ g) \in \bar{\mathcal{O}}(\eta \circ g)$. Since $\eta \circ g$ is regular Toeplitz sequence, $(\bar{\mathcal{O}}(\eta \circ g), S)$ is minimal and $\bar{\mathcal{O}}(\eta \circ g') = \bar{\mathcal{O}}(\eta \circ g)$.

REMARK 7. If $\eta \in \mathscr{S}$ is not Toeplitz sequence, i.e. $\eta[i] = \infty$, then denote by η', η'' the sequences such that $\eta'[i] = 0$, $\eta''[i] = 1$ and for $j \neq i$ $\eta'[j] = \eta''[j] = \eta[j]$. It follows from Lemma 6 that $\eta', \eta'' \in \bar{\mathcal{O}}(\eta \circ g)$ for every $g \in G$ such that $\eta \circ g$ is Toeplitz sequence. Moreover $\bar{\mathcal{O}}(\eta') = \bar{\mathcal{O}}(\eta'') = \bar{\mathcal{O}}(\eta \circ g)$.

Now, if $\eta \in \mathscr{S}$ then by $\bar{\mathcal{O}}(\eta)$ we denote the set $\bar{\mathcal{O}}(\eta \circ g)$, where $g \in G$ is chosen in this way so that $\eta \circ g$ is a Toeplitz sequence.

Denote by $\mathscr{S}^*$ the set of all $\eta \in \mathscr{S}$ such that T_η is ergodic. From above we obtain the following version of Theorem 2.

THEOREM 2'. *Let $\eta, \eta' \in \mathscr{S}^*$. If W is a finitary isomorphism between T_η and $T_{\eta'}$, then we can extend W to a topological isomorphism between $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}, T)$ and $(\bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2, T_{\eta'})$.*

EXAMPLE 1. Let $x = b^0 \times b^1 \times \cdots$ be a Morse sequence [5] and (Ω_x, S) the dynamical system induced by x. Put $c_t = b^0 \times b^1 \times \cdots \times b^t$, $t \geqslant 0$ and let η_t be defined by

$$\eta_t[k \cdot n_t, (k+1) \cdot n_t - 1] = \hat{c}_t \infty, \quad k \in \mathbf{Z},$$

where $\hat{c}_t[i] = c_t[i] + c_t[i+1]$, $0 \leqslant i \leqslant n_t - 2$. Then $\{\eta_t\}$ determines $\eta \in \mathscr{S}^*$ (note that $I_t(\eta) = \{n_t - 1\}$ and $\eta[-1] = \infty$). Let $\omega = \eta'$ or $\omega = \eta''$, where η', η'' are defined as in Remark 7. It is not hard to see that for $i \geqslant 0$ $x[i] = \check{\omega}[i]$. Therefore $\Omega_x = \bar{\mathcal{O}}(\check{\omega})$, and since $\bar{\mathcal{O}}(\check{\omega})$ is mirror invariant (i.e. if $y \in \bar{\mathcal{O}}(\check{\omega})$, then $\tilde{y} \in \bar{\mathcal{O}}(\check{\omega})$, where $\tilde{y}[i] = 1 - y[i]$), we have by (19) $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta)$ and (Ω_x, S) are topologically conjugate. So, in the case of Morse dynamical systems, it follows from Theorem 2' that every finitary isomorphism can be extended to a topological one.

Section 4. An uncountable family of ergodic Toeplitz $\mathbf{Z}_2$-extensions with partly continuous spectrum, such that every two (different) members are finitarily isomorphic and not topologically conjugate

Here we use the following notation: if A is a block consisting of the symbols 0, 1, having l "holes" ∞ and L is a 0–1 block of length l, then by $A * L$ we mean the block arising from A by successive replacement of "holes" by elements of the block L, i.e. if $L = m_1 m_2 \ldots m_l$ then we write m_1 instead of the first ∞ in A, instead of the second ∞ in A we write m_2, etc.

Let $I \subset \Pi_0^\infty \{0, 1\}$ be an uncountable set such that if $x = (x_t), y = (y_t) \in I, x \neq y$, then $x_t \neq y_t$ for infinitely many t. Fix $r = (r_t) \in I$ and put

$$A_0 = 0\infty\infty 1,$$
$$A_{t+1} = (A_t * \underbrace{10\ldots0}_{\text{length } 2^t}\underbrace{01\ldots1}_{2^t})\, A_t A_t (A_t * (\underbrace{10\ldots0}_{2^t}\underbrace{01\ldots1}_{2^t} + r_t)),\ t \geqslant 0.$$

Here, if $A = a_1 a_2 \ldots a_k$ is a block and $l \in \mathbf{Z}_2$, by $A + 1$ we denote the block A if $l = 0$ and the block $\tilde{A} = (1 - a_1)(1 - a_2)\ldots(1 - a_k)$ in the contrary case. The sequence $\{A_t\}$ determines the Toeplitz sequence $\eta = \eta(r)$ in the following way: let η_t be the sequence such that

$$\eta_t[k \cdot n_t, (k + 1) \cdot n_t - 1] = A_t, k \in \mathbf{Z}.$$

Then η is the unique Toeplitz sequence for which the n_t-skeletons are $\eta_t, t \geqslant 0$. It is not hard to see that η is a non-periodic, regular Toeplitz sequence with the period structure $n_t = 4^{t+1}, t \geqslant 0$.

Fix $\eta = \eta(r)$.

LEMMA 7. *T_η is ergodic.*

Proof. Suppose, on contrary that T_η is not ergodic. It follows from [4] that there is a measurable function $h: \bar{\mathcal{O}} \to \{-1, 1\}$ such that $h(Sy) = (-1)^{y[0]} h(y)$ a.e. Denote by $p: \bar{\mathcal{O}}(\eta) \to \mathbf{Z}_2$ the following function: $p(y) = 1$ if $h(y) = -1$ and $p(y) = 0$ otherwise. Then $p(Sy) + p(y) = y[0]$ a.e. If we consider p as a function on G then equivalently

$$p \circ \hat{1} + p = \eta. \tag{22}$$

Let $p_t: G \to \mathbf{Z}_2$ be the function defined as follows: for $0 \leqslant i \leqslant n_t - 1$

$$p_t|_{E_i^t} = 0 \text{ or } p_t|_{E_i^t} = 1,$$

and

$$p_t|_{E_i^t} = 0 \Leftrightarrow \lambda_i^{(t)}\{g \in E_i^t : p(g) = 0\} \geqslant \tfrac{1}{2}$$

where $\lambda_i^{(t)} = \lambda(\cdot|E_i^t)$ is the conditional measure on E_i^t. Since p is measurable and the partitions $\xi_t = \{E_i^t\}$ satisfy the condition $\xi_t \nearrow \varepsilon$, where $\varepsilon = \{\{g\} : g \in G\}$ we obtain

$$p_t \to p \text{ in } G. \tag{23}$$

Let $F_t = \bigcup_{i=0}^{n_{t-1}-1} E_i^t$. Then

$$\lambda(F_t) = \tfrac{1}{4} \tag{24}$$

and from (23) $\lambda\{g \in F_t : p_t(g) \neq p(g)\} \to 0$. Fix $t > 0$ and $0 \leqslant i \leqslant n_{t-1} - 1$. Note that from the construction of η $\eta \circ \hat{\imath}|_{E_0^t}$ is constant λ a.e. Therefore

$$p \circ \hat{\imath}|_{E_0^t} = (p + \eta + \cdots + \eta \circ (i-1)\hat{\ })|_{E_0^t} = p|_{E_0^t} + a, \quad a \in \mathbf{Z}_2.$$

Hence

$$\begin{aligned}\tfrac{1}{2} \geqslant \lambda_i^{(t)}\{g \in E_i^t : p_t(g) \neq p(g)\} &= \lambda_0^{(t)}\{g \in E_0^t : p_t \circ \hat{\imath} \neq p \circ \hat{\imath}\} \\ &= \lambda_0^{(t)}\{g \in E_0^t : p_t \circ \hat{\imath} \neq p + a\}.\end{aligned}$$

Thus, it follows from the definition of p_t that

$$\lambda_i^{(t)}\{g \in E_i^t : p_t(g) \neq p(g)\} = \lambda_0^{(t)}\{g \in E_0^t : p_t \neq p\}.$$

So by (24) we have

$$\lambda\{g \in F_t : p_t(g) \neq p(g)\} = \tfrac{1}{4}\lambda_0^{(t)}\{g \in E_0^t : p_t(g) \neq p(g)\}.$$

This implies $\lambda_0^{(t)}\{g \in E_0^t : p_t(g) \neq p(g)\} \to 0$. The last condition guarantees that

$$\begin{aligned}\lambda_0^{(t)}\{g \in E_0^t : p \circ \hat{n}_t \neq p\} &\leqslant \lambda_0^{(t)}\{g : p \circ \hat{n}_t \neq p_t\} + \lambda_0^{(t)}\{g : p_t \neq p\} \\ &= 2 \cdot \lambda_0^{(t)}\{p_t \neq p\} \to 0.\end{aligned} \tag{25}$$

Let $\psi_t = \eta + \eta \circ \hat{1} + \cdots + \eta \circ (n_t - 1)\hat{\ }$. Then from (22) $p \circ \hat{n}_t = p + \psi_t$ and from

(25) $\lambda_0^{(t)}\{g \in E_0^t: \psi_t(g) = 1\} \to 0$. On the other hand we will show that for $t \geq 1$

$$\lambda_0^{(t)}\{g \in E_0^t: \psi_t = 1\} \geq \tfrac{1}{16}. \tag{26}$$

Namely, take $E_{n_t}^{t+2} \subset E_0^t$. By the construction of η, it is clear that

$$\psi_t(E_{n_t}^{t+2}) = A_{t+2}[n_t] + A_{t+2}[n_t + 1] + \cdots + A_{t+2}[2 \cdot n_t - 1].$$

Since

$$A_{t+2}[n_t, 2 \cdot n_t - 1] = A_t * \underbrace{100\ldots0}_{2^{t+1}}$$

and A_t contians an even number of one's for $t \geq 1$, we get $\psi_t(E_{n_t}^{t+2}) = 1$. The equality $\lambda_0^{(t)}(E_{n_t}^{t+2}) = \frac{1}{16}$ gives (26). This contradiction proves Lemma 7.

LEMMA 8. *T_η has partly continuous spectrum.*

Proof. Let $\mathscr{C} = \{f \in L^2(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, \tilde{\mu}): f \circ \sigma = -f\}$. Since n_{t+1}/n_t, $t \geq 0$ are even and T_η is ergodic, the same proof of Lemma 7 in [5] shows that T_η has continuous spectrum on $\mathscr{C}$.

Now, take $r = (r_t)$, $r' = (r'_t) \in I$ and put $\eta = \eta(r)$, $\eta' = \eta(r')$.

LEMMA 9. *T_η and $T_{\eta'}$ are finitarily isomorphic.*

Proof. Set $\gamma = \eta + \eta'$, $m_t = r_t + r'_t \pmod 2$, $t \geq 0$. Let us define $C_0 = 0\infty\infty0$,

$$C_{t+1} = (C_t * \underbrace{0\ldots0}_{2^{t+1}})\, C_t C_t (C_1 * (\underbrace{0\ldots0}_{2^{t+1}} + m_t)), \quad t \geq 0.$$

It is not hard to see that $\{C_t\}$ determines γ. By Remark 5 it suffices to show that $\theta = \theta^{(0)}$, $\theta[i] = \check{\gamma}[i]$, $i \geq 1$ is a regular one-sided Toeplitz sequence. Let γ_t be the n_t-skeleton of γ. Take $t \geq 1$ and choose $i \geq 1$ such that $\gamma_t[i] \neq \infty$. We show that

$$\check{\gamma}[i + k \cdot n_t] = \hat{\gamma}[i + (k + 1) \cdot n_t], \quad k \geq 0.$$

To this end we must verify the following equality:

$$\gamma[i + k \cdot n_t] + \gamma[i + k \cdot n_t + 1] + \cdots + \gamma[i + (k + 1) \cdot n_t - 1] = 0. \tag{27}$$

Note that from the construction of γ, if $j \equiv 1 \pmod 4$, then $\gamma[j + 1] + \gamma[i] = 0$ and for $j \equiv 0 \pmod 4$ or $j \equiv 3 \pmod 4$ $\gamma[j] = 0$. Thus for $i \equiv 0 \pmod 4$ or $i \equiv 1 \pmod 4$ or $i \equiv 3 \pmod 4$ (27) is clear. Now assume that $i \equiv 2 \pmod 4$. Then $i - 1 \equiv 1 \pmod 4$, and by above

$$\gamma[i - 1 + k \cdot n_t] + \cdots + \gamma[i + (k + 1) \cdot n_t - 2] = 0. \tag{28}$$

Since $\gamma_t[i-1] \neq \infty$, we obtain $\gamma[i + k\cdot n_t - 1] = \gamma[i-1] = \gamma[i + (k+1)\cdot n_t - 1]$ and hence by (28) we have (27). So θ is Toeplitz sequence. θ is regular because γ is regular.

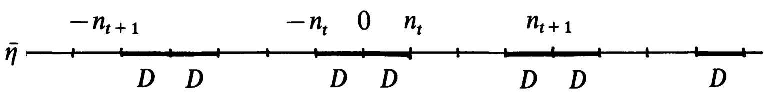
LEMMA 10. *T_η and $T_{\eta'}$ are not topologically conjugate.*

Proof. Assume that T_η, $T_{\eta'}$ are topologically conjugate. From Lemma 2 and Lemma 1 there is a topological isomorphism $\varphi: \bar{\mathcal{O}}(\eta) \to \bar{\mathcal{O}}(\eta')$ and $g_0 \in G$ such that $\varphi(\eta) = \eta' \circ g_0$ is Toeplitz sequence. From a theorem of Hedlund [see e.g. 1 page 38] $\varphi = F_\infty \circ S^k (k \in Z)$. Here F_∞ is determined by some mapping $F: \{0,1\}^l \to \{0,1\}$ $(l \geqslant 1)$ in the following way:

$$(F_\infty(y))[j] = F(y[j]y[j+1]\ldots y[j+l-1]).$$

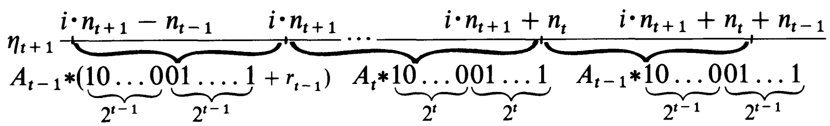
Let $J = \{t \geqslant 0: r_t = 0, r'_t = 1\}$. Interchanging η with η', if necessary, we may assume J is infinite. Fix $t \in J$ with the property: $|k| + l \leqslant n_{t-1}$. Denote by $\bar{\eta}_t$ the n_t-skeleton of $\bar{\eta} = n' \circ g_0$. We claim that

$$\bar{\eta}[i\cdot n_{t+1}, i\cdot n_{t+1} + n_t - 1] = \bar{\eta}[i\cdot n_{t+1} + 3\cdot n_t, (i+1)\cdot n_{t+1} - 1] = D, \quad i \in \mathbf{Z}. \tag{29}$$



Take $i \in \mathbf{Z}$. Then for $i\cdot n_{t+1} \leqslant j \leqslant i\cdot n_{t+1} + n_t - 1$

$$\bar{\eta}[j] = (\varphi(\eta))[j] = ((F_\infty \circ S^k)(\eta))[j] = F(\eta[j+k]\ldots\eta[j+k+l-1]) \tag{30}$$



Since $|k| + l \leqslant n_{t-1}$ and the block

$$\eta_{t+1}[i\cdot n_{t+1} - n_{t-1}, i\cdot n_{t+1} + n_t + n_{t-1} - 1]$$

does not contain ∞, we have from (30) that

$$D' = \bar{\eta}[i\cdot n_{t+1}, i\cdot n_{t+1} + n_t - 1]$$

does not depend on $i \in \mathbf{Z}$. Similarly the block

$$D'' = \bar{\eta}[i\cdot n_{t+1} + 3n_t, (i+1)\cdot n_{t+1} - 1]$$

does not depend on $i \in \mathbf{Z}$ too. We have $D'' = D'$ because

$$\begin{aligned}&\eta_{t+1}[i \cdot n_{t+1} - n_{t-1}, i \cdot n_{t+1} + n_t + n_{t-1} - 1] \\ &\quad = \eta_{t+1}[i \cdot n_{t+1} + 3n_t - n_{t-1}, (i+1) \cdot n_{t+1} + n_{t-1} - 1] \quad (r_t = 0).\end{aligned}$$

Let $q = \min\{i \geqslant 0: \eta_t[i] = \infty\}$ and $m = -j_{t+3} + q$, where $g_0 = (j_s)_0^\infty$. Put $B = \bar{\eta}[m]\bar{\eta}[m + n_t] \ldots \bar{\eta}[m + 11 + n_t]$. By construction of $\bar{\eta}$ we obtain $B = b_0 b_1 \ldots b_{11} = 110011001010$ $(r_t' = 1)$. Put $s = m \pmod{n_{t+1}}$ and write $s = z \cdot n_t + a$, where $0 \leqslant a \leqslant n_t - 1$ and $0 \leqslant z \leqslant 3$. Set

$$\mathscr{K} = \{j \in \mathbf{Z}: j \equiv a \pmod{n_{t+1}} \quad \text{or} \quad j \equiv a + 3 \cdot n_t \pmod{n_{t+1}}\}.$$

It follows from (29) that for $j \in \mathscr{K}$

$$\bar{\eta}[j] = D[a]. \tag{31}$$

Consider the following cases:

1. $z = 0$. Then m, $m + 3n_t \in \mathscr{K}$, but this in view of (30) is impossible since $\bar{\eta}[m] = b_0 = 1$, $\bar{\eta}[m + 3n_t] = b_3 = 0$.
2. $z = 1$. In this case $m + 2n_t$, $m + 10n_t \in \mathscr{K}$, but $\bar{\eta}[m + 2n_t] = b_2 = 0$, $\bar{\eta}[m + 10n_t] = b_{10} = 1$.
3. $z = 2$. Then $m + n_t$, $m + 9n_t \in \mathscr{K}$, $\bar{\eta}[m + n_t] = b_1 = 1$, $\bar{\eta}[m + 9n_t] = b_9 = 0$.
4. $z = 3$. Now $m + n_t$, $m + 9n_t \in \mathscr{K}$ and $\bar{\eta}[m + n_t] = 1$, $\bar{\eta}[m + 9n_t] = 0$.

These contradictions show that $(\bar{\mathcal{O}}(\eta), S)$ and $(\bar{\mathcal{O}}(\eta'), S)$ are not topologically conjugate so $(\bar{\mathcal{O}}(\eta) \times \mathbf{Z}_2, T_\eta)$ and $(\bar{\mathcal{O}}(\eta') \times \mathbf{Z}_2, T_{\eta'})$ are not topologically conjugate as well.

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