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Torsion groups of elliptic surfaces

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0. Introduction

The purpose of this article is to study restrictions on the torsion groups of elliptic surfaces over an arbitrary base curve. In fact our approach is dual to this: we study the restrictions on the base curve, the rank of the Mordell–Weil group of sections, and the Euler characteristic of the surface given a certain torsion group of sections.

This problem was solved in the case of rational base by David Cox and Walter Parry [C-P] using the theory of elliptic modular surfaces.

We intend to attack the problem in a more elementary way using an Euler number argument. A similar technique was employed by Nikulin [N] to study the possible finite abelian automorphism groups acting algebraically on a $K3$ surface, and the reader who is familiar with this work will not be surprised at the method's power.

In the case of a rational base curve we recapture their classification of the 19 possible torsion groups in what we think is a conceptually simpler way. We also classify the torsion groups which can occur for an elliptic base, and the proof actually shows that those are the only ones possible for any base curve, provided one only looks at surfaces with sufficiently high Euler characteristics. We call this the asymptotic case.

It is clear from our method that one may in principle determine all the possible torsion groups for any genus base curve, although the computations are quite messy. The reader should keep in mind that through the theory of elliptic modular surfaces any finite abelian group of length at most 2 (i.e., a subgroup of $\mathbb{Z}/N \times \mathbb{Z}/N$ for some N) may occur. The restrictions on the base curve corresponding to a given torsion group are a priori imposed by the genus of the corresponding modular curve. Similar results are also implicit in our method, and we will include a few token examples.

We also show that the methods of our approach can be applied to the finer question of determining the possible Mordell–Weil groups. We illustrate this by recapturing the list for rational and $K3$ surfaces first compiled by David Cox [C].

The point of our paper is to emphasize the purely combinatorial and topological aspects of the results modulo some basic facts about elliptic surfaces. We are not so much concerned about constructing examples, where we admit that the approach through modular surfaces may be canonical, especially for a non-rational base, when the curve has to be chosen with some care.

We will assume that the reader is familiar with the rudiments of the Kodaira theory of elliptic surfaces, in particular the classification of the singular fibers and their group structures. In section one we are going to review the basic facts we will need, arriving at a lower bound for the number of singular fibers. In section two we analyze the action of a torsion subgroup T of the Mordell–Weil group on the elliptic surface: this is the heart of the theory. We pause here to analyze the simplest case of the technique, i.e., the case when there is a p-torsion section for some prime p. We want to impress the reader with the conceptual simplicity of our method, which may be obscured by the rather involved combinatorics of the general case, which we discuss in Section 3. In Section 4 we will collect our forces and show how the lists of Cox-Parry and Cox can be recovered from our method, and also extend the results to base curves of genus one, and to genus $\geqslant 2$ in the asymptotic case. We finally sharpen a result of Hindry and Silverman as an illustration of the asymptotic approach.

1. Preliminaries: a lower bound for s

The basic facts we need about elliptic surfaces are given below; these are quite well-known, and we present them without proof. We denote by MW the Mordell–Weil group of sections and by T a torsion subgroup of MW. MW is a finitely generated abelian group and T is finite. Recall that the *length* of a finitely generated abelian group is the minimum number of generators.

LEMMA 1.1. *Let $\pi: X \to C$ be a smooth minimal elliptic surface.*
(a) *Any two torsion sections of π are disjoint.*
(b) *The restriction of T to any fiber is injective.*
(c) *The length of T is at most 2.*
(d) *If π has a non-semistable fiber (i.e., one not of type I_n for some n) then $|T| \leqslant 4$.*

REMARKS. Part (a) is not as well-known as it deserves to be, but the heuristic reason is clear: you cannot "continuously" approach zero by just stepping on torsion points. Part (b) follows immediately from (a), then (c) from (b), by inspecting the group of torsion points on any fiber of π; this inspection also proves (d), since for each non-semistable fiber the group of torsion points has order at most 4. We should note that singular fibers of type I_n are often called multiplicative; the group of smooth points is isomorphic to a semidirect product of $\mathbb{Z}/n\mathbb{Z}$ with $\mathbb{C}^*$.

Part (d) above shows that we can without any significant loss of generality assume that all the fibers are semi-stable i.e. of type I_n. To be honest, in the finer applications when we consider the full Mordell–Weil group we need to consider arbitrary fibers; this will present no real difficulties, so in the interest of not burdening the presentation with irrelevant detail we will concentrate on the case of semistable elliptic surfaces, i.e., all fibers are of type I_n.

We need to bring three tools to bear on the problem at hand, one topological (the Euler number formula), one algebraic (the Shioda–Tate formula) and one analytic (Hodge theory).

We will employ the following notation now and for the rest of the article:

$e = e(X) =$ the topological Euler number of the surface X,
$\chi = \chi(\mathcal{O}_X) =$ the holomorphic Euler characteristic of X,
$\rho = \rho(X) =$ the Picard number of X,
$R =$ the rank of the Mordell–Weil group MW,
$s =$ the number of singular fibers of π,
$g =$ the genus of the base curve C,
$q = h^1(\mathcal{O}_X)$ and $p_g = h^2(\mathcal{O}_X)$,

The topological input is given by the formula for the Euler characteristics of an elliptic surface. Assume that π has singular fibers of type $I_{n_1}, \ldots, I_{n_s}$. Then

$$e(X) = 12\chi = \Sigma\, n_i \tag{1.2}$$

(see for example [AS, IV.4]).

Assume that X is a Jacobian surface, i.e., that π has a section. For a singular fiber X_i let r_i denote the rank of X_i, i.e. $r_i =$ the number of components minus one. (If X_i is of type I_n, then $r_i = n - 1$.) The formula of Shioda and Tate relating these numbers is:

$$\rho = 2 + R + \Sigma\, r_i \tag{1.3}$$

(see [S]).

In our case of only semistable fibers we have that

$$\Sigma\, r_i = e(X) - s = 12\chi - s. \tag{1.4}$$

Assuming our elliptic surface X is not a product, then its irregularity q is the genus g of the base curve C. Hence the Hodge diamond of X is

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ p_g & & h^{1,1} & & p_g \\ & g & & g & \\ & & 1 & & \end{array}$$

and therefore $12\chi = e = 2 + 2p_g + h^{1,1} - 4g = 2\chi + h^{1,1} - 2g$, so that $h^{1,1} = 10\chi + 2g$. This of course forces

$$\rho \leqslant 10\chi + 2g \tag{1.5}$$

since $\rho \leqslant h^{1,1}$.

Combining everything we get the following lower estimate on the number of singular fibres.

PROPOSITON 1.6. *Let X be a smooth minimal semistable elliptic surface (i.e., with only I_n fibres) which is not a product. Then*

$$s \geqslant 2\chi + 2 - 2g + R$$

with equality if and only if X has maximal Picard number $\rho = h^{1,1}$.

Proof. We have

$s = 12\chi - \Sigma r_i$ (from (1.4))
$= 12\chi - (\rho - 2 - R) = 12\chi + 2 + R - \rho$ (from (1.3))
$\geqslant 12\chi + 2 + R - (10\chi + 2g)$ (from (1.5))
$= 2\chi + 2 - 2g + R$ as claimed. The inequality comes from (1.5), proving the last statement. □

2. Translation by torsion sections and an upper bound for s.

The next ingredient is a combinatorial fixed point analysis of actions given by translations by torsion sections. We have exploited this in our paper [M-P 1] to which we will refer for the simple proof. Much to our surprise we have not seen this used before in the literature.

Let T be a torsion subgroup of MW, let S_0 be the zero section of T, and let S be any other torsion section. Denote by τ_S the automorphism of X given by translation (in the group law of the fibers) by S.

LEMMA 2.1. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface. Assume that the order of S in T in k.*

(a) *The fixed points of τ_S are among the nodes of the I_n fibers of π.*
(b) *If S meets a different component of an I_n fiber than S_0 does, then none of the nodes of that fiber are fixed by τ_S. If this happens, then* $\gcd(n, k) \neq 1$.
(c) *If S meets the same component of an I_n fiber as does S_0, then each of the n nodes of the I_n fiber is fixed by τ_S. If x is one of these nodes, then the linearization of the action of τ_S at x is given by the 2×2 matrix $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, where ζ is a primitive kth root of unity.*

Proof. See [M-P 1]. □

The torsion group T acts on X via the translations τ_S, for $S \in T$. We need some information about the isotropy of this action.

LEMMA 2.2. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface with a torsion group T.*

(a) *The only points of X with non-trivial isotropy under the action of T are the nodes of the I_n fibers.*
(b) *The isotropy subgroup of a node x of an I_n fiber is the set of sections S of T which meet the same component of that fiber as does S_0.*
(c) *The nodes of any one I_n fiber all have the same isotropy subgroup.*
(d) *If $H \subseteq T$ occurs as a non-trivial isotropy subgroup, then both H and T/H are cyclic.*
(e) *If x is a node of an I_n fiber with isotropy subgroup $H \subseteq T$, then $|T/H|$ divides n.*

Proof. Statements (a) and (b) follow from Lemma 2.1, and (c) is a consequence of (b). To prove (d), note that H must embed into the group of smooth points of the component meeting S_0; this is the connected component of the identity of the group of smooth points of the I_n fiber, and is isomorphic to $\mathbb{C}^*$. Hence H embeds into $\mathbb{C}^*$, so H is cyclic. In fact H is the kernel of the map sending T to the group of components of the I_n fiber, which is a cyclic group of order n; therefore T/H enbeds into $\mathbb{Z}/n\mathbb{Z}$. This finishes the proof of (d), and also proves (e). □

Property (c) above justifies the following terminology: we say that the isotropy of the singular fiber I_n is H if H is the isotropy subgroup of the nodes of that fiber.

Finally we must understand the quotient surface X/T.

LEMMA 2.3. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface with a torsion group T.*

(a) *The elliptic fibration π induces an elliptic fibration $\pi_T: X/T \to C$.*
(b) *π_T has exactly s singular fibers also, one under each singular fiber of π.*
(c) *The image of an I_n fiber with isotropy H, under the quotient map, is a cycle of $n/|T/H|$ smooth rational curves meeting at singular points which are rational double points of type $A_{|H|-1}$. These double points are the only singularities of X/T. The image of a smooth fiber is a smooth fiber.*
(d) *Let Y_T be the smooth surface obtained by minimally resolving the singularities of X/T, and let $\pi'_T: Y_T \to C$ be the induced map to C; it is a smooth minimal semistable elliptic surface also. The singular fiber of π'_T corresponding to an I_n fiber of π with isotropy H is an $I_{n|H|/|T/H|}$ fiber.*
(e) *The Euler number e and the Euler characteristic χ for Y_T is the same as that for X.*
(f) *Let the singular fiber of π be $I_{n_1}, \dots, I_{n_n}$, with isotropy subgroups $H_1, \dots, H_s$ respectively. Then $e|T| = \Sigma n_i |H_i|^2$.*

Proof. Since T acts on the fibers, it preserves the map π; hence the map π_T

exists, and it is an elliptic fibration, since the quotient of a smooth elliptic curve by a finite subgroup is again smooth elliptic. This proves (a), and the last statement of (c). Statement (b) follows from (c), and to prove the first part of (c), assume that an I_n fiber F has isotropy H. Then T/H acts faithfully on the set of components of F; indeed, since T/H embeds as a subgroup of the group of components, there will be $n/|T/H|$ components in the quotient, and they will meet in a cycle. The local analysis showing that they meet at double points of type $A_{|H|-1}$ is Lemma 2.1(c).

Upon resolving these $n/|T/H|$ singularities, one introduces $(n/|T/H|)(|H|-1)$ new curves in the cycle (an additional $(|H|-1)$ at each node), giving a total of $n|H|/|T/H|$ in the cycle of the resolution Y_T. This proves (d).

As elliptic curves over the function field $k(C)$ of C, both X and Y_T are isogenous; therefore they cover each other (birationally) and so must have the same p_g. This implies that they have the same Euler characteristic χ, and therefore the same Euler number e. This proves (e), and now (f) follows by applying (1.2) to Y_T. □

We will say that a cyclic subgroup H of T such that T/H is cyclic is *cyclic and cocyclic* in T, and we will abbreviate this as "c&c". For each nontrivial cyclic and cocyclic subgroup H of T, let $\imath_{T,H}$ be the number of nodes of singular fibers of X with isotropy group H under the action of T. Note that, for example $\imath_{\{0\},\{0\}} = e$.

PROPOSITION 2.2. *Let $\pi\colon X \to C$ be a smooth minimal semistable elliptic surface, and let T be a group of torsion sections of X. Then*

(a) $$\sum_{\substack{H\subseteq T\\ H\text{c\&c}}} |H|^2 \imath_{T,H} = e|T|.$$

(b) *For each $H \subseteq T$ cyclic and cocyclic,*

$$\sum_{\substack{G\text{c\&c}\\ H\subseteq G\subseteq T}} \imath_{T,G} = \imath_{H,H}$$

Proof. Statement (a) is simply Lemma 2.3 (f), the sum being organized over the subgroups instead of the singular fibers. The second statement is obtained by noticing that a node of a singular fiber of X is fixed by H under the induced H-action if and only if it is fixed by a subgroup of T containing H, under the full T-action. □

We are finally in a position to illustrate the technique of this article; at this point we can analyze the case when π has a section of order p, where p is a prime. Let T be a subgroup of MW of order p. The only relevant isotropy numbers in this case are $\imath_{T,T}$ and $\imath_{T,\{0\}}$, which by the previous Proposition satisfy

$$\imath_{T,\{0\}} + p^2 \imath_{T,T} = 12\chi p \quad \text{and} \quad \imath_{T,\{0\}} + \imath_{T,T} = \imath_{\{0\},\{0\}} = 12\chi.$$

Solving this gives $\imath_{T,\{0\}} = 12\chi p/(p+1)$ and $\imath_{T,T} = 12\chi/(p+1)$. We immediately see a divisibility condition: $p+1$ must divide 12χ. However, we can say much more. Every singular I_n fiber has isotropy either $\{0\}$ of T; those with trivial isotropy $\{0\}$ must have n divisible by p, by Lemma 2.2(e). For fixed χ, one maximizes the number of singular fibers s by having $\imath_{T,T}$ fibers of type I_1 with isotropy T and $\imath_{T,\{0\}}/p$ fibers of type I_p with isotropy $\{0\}$. Hence the maximum number of singular fibers is $\imath_{T,T} + \imath_{T,\{0\}}/p = 24\chi/(p+1)$: $s \leqslant 2e/(p+1)$.

Combining this upper bound with the lower bound of Proposition 1.6. we see that

$$2\chi + 2 - 2g + R \leqslant 24\chi/(p+1) \tag{2.5}$$

if there is p-torsion in the Mordell–Weil group.

Statement 2.5 is the prototype of the conditions obtained by assuming a certain torsion subgroup T of MW. Without stopping to make a complete analysis at this time, let us merely draw the easy conclusions possible from (2.5).

COROLLARY 2.6. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface, and assume that there is p-torsion in MW. Then $p + 1 | 12\chi$.*
Moreover:

(a) *If $g = 0$ then $p \leqslant 7$. If $g = 0$ and χ is odd then $p \leqslant 5$.*
(b) *If $g = 0$ and $p = 2$ then $R \leqslant 6\chi - 2$; if $g = 0$ and $p = 3$ then $R \leqslant 4\chi - 2$; if $g = 0$ and $p = 5$ then $R \leqslant 2\chi - 2$; if $g = 0$ and $p = 7$ and $R \leqslant \chi - 2$.*
(c) *If $g = 1$ then $p \leqslant 11$. If $g = 1$ and χ is odd then $p \neq 7$.*
(d) *If $g \geqslant 1$ and $\chi \geqslant 7g - 6$, then $p \leqslant 11$.*

Note that if $g = 0$ *and* $\chi = 1$, if there is 5-torsion in MW, we must have 2 I_1 fibers and 2 I_5 fibers; this is a rational extremal fibration (see [M P]) and the surface is determined uniquely.

As the reader can no doubt imagine, one can play many games with the inequality (2.5); we have just given several consequences in (2.6). For example, if a rational elliptic surface ($g = 0$ and $\chi = 1$) has an infinite Mordell–Weil group ($R \geqslant 1$) then $p = 2$ or 3 is all that is allowed. If a $K3$ elliptic surface ($g = 0$ and $\chi = 2$) has $R \geqslant 1$ *or* the Picard number ρ is not maximal ($\rho = 20$) then $p = 2, 3$, or 5. (In fact the existence of 7-torsion in MW for a $K3$ elliptic surface determines the surface.)

The simpleminded discussion above illustrates the basis ideas of the paper and gives we think the essential flavor of the same. We fear however that the simplicity is somewhat obscured by the rather formidable technicalities of computing the $\imath_{T,H}$'s in the next section.

Observe though that the expression $2/(p+1)$ denotes some kind of "size" for the group $\mathbb{Z}/p\mathbb{Z}$. (In an inverse way, the "bigger" the group the smaller the "size"). The basic effort of this paper is to find the right definition of "size" for a group, given that we need only to compare this with the estimate of Proposition 1.6 to draw the conclusions we seek.

Essentially, we want the "size" of such a group T to be the upper bound on the number of singular fibers which a smooth minimal semistable elliptic surface may have, if it admits T as a group of torsion sections. Since this upper bound depends linearly on the Euler number e, we will in fact divide this upper bound by e to get our definition of "size". In anticipation of this, let us define, for each cyclic and cocyclic subgroup H of T, the number $\gamma_{T,H} = \imath_{T,H}/e$; this is the fraction of the nodes which have isotropy H under the $T-$action, and a priori depends not only on T and H but also on the representation of T as a group of torsion sections of X.

Let then T be a finite abelian group of length at most 2. Assume that T occurs as a group of torsion sections of a smooth minimal semistable elliptic surface with Euler number e. Let $\imath_{T,H}$ be the number of nodes of fibers with isotropy a given cyclic and cocyclic subgroup H of T, as above. A fiber of type I_n with isotropy H must have n divisible by $|T/H|$ by Lemma 2.2(e), so that for fixed e, the maximum number of singular fibers would be achieved when every singular fiber with isotropy H is of type $I_{|T/H|}$. This leads to a maximum of $\Sigma_H \imath_{T,H}/|T/H|$ singular fibers.

Re-expressing this in terms of the γ's leads us to the following:

DEFINITION 2.7. Let T be a finite abelian group of length at most 2. Define the *size* of $T(=\text{Size}(T))$ by

$$\text{Size}(T) = \frac{1}{|T|} \sum_{\substack{H \subseteq T \\ H \text{ c\&c}}} |H|\, \gamma_{T,H}.$$

From the analysis above and the definition of the γ's, we have the following upper bound for the number of singular fibers s:

PROPOSITION 2.2. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface with s singular fibers and Euler number e. Assume that X has a torsion group of sections T. Then*

$$s \leqslant e \cdot \text{Size}(T).$$

3. The computation of Size(T)

It is our job in this section to prove that Size (T) is well-defined (i.e., that the $\gamma_{T,H}$'s are determined by T alone, not by T's representation as a group of torsion sections) and to calculate Size (T). Note that the analysis of the previous section shows that Size$(\mathbb{Z}/p\mathbb{Z}) = 2/(p+1)$ for a prime p. Also, we have Size$(\{0\}) = 1$.

If G is any group, we denote by G_p its Sylow-p-subgroup.

If T is a cyclic group, then of course every subgroup of T is cyclic and cocyclic. We need to understand the length 2 situation. Since H is cyclic and cocyclic in T if and only if H_p is in T_p for each prime p, we need only analyze the p-group case.

LEMMA 3.1. *Let $T \cong \mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$, with $1 \leqslant m \leqslant n$. Then:*

(a) *Every cyclic and cocyclic subgroup H of T has order p^k, with $m \leqslant k \leqslant n$.*

(b) *Assume that $m < n$. Then the number of cyclic and cocyclic subgroups of T of order p^k is p^m if $k = m$* or *$k = n$, and $p^{m-1}(p-1)$ if $m < k < n$.*

(c) *Assume that $m = n$. Then the number of cyclic and cocyclic subgroups of T of order p^m is $p^{m-1}(p+1)$.*

(d) *There are no inclusions among the cyclic and cocyclic subgroups of T.*

Proof. Statement (a) is clear: since p^n is the largest order for an element of T, it is the largest order of both an element of H (forcing $|H| \leqslant p^n$) and an element of T/H (forcing $|H| \geqslant p^m$). Let H be cyclic and cocyclic, generated by $(p^r a, p^s b)$, with a and b prime to p. Then T/H can be identified with the cokernel of the map from $\mathbb{Z}^3$ to $\mathbb{Z}^2$ given by the matrix $\begin{pmatrix} p^m & 0 & p^r a \\ 0 & p^n & p^s b \end{pmatrix}$. The g.c.d. of the entries is $p^{\min(r,s)}$, and the g.c.d. of the determinants of the 2×2 minors is $p^{\min(m+s,n+r)}$, so that $T/H \cong \mathbb{Z}/p^{\min(r,s)} \times \mathbb{Z}/p^{\min(m+s,n+r)}$. Hence H is cocyclic if and only if $\min(r, s) = 0$, i.e., either $r = 0$ or $s = 0$.

Assume then that $m \neq n$. If $|H| = p^m$, then $r = 0$, and H will contain a unique element of the form $(1, \beta)$, for some $\beta \in \mathbb{Z}/p^n$; such H's are classified by this β, which can be any multiple of p^{n-m}; there are p^m of these. If $|H| = p^n$, then $s = 0$, then H contains a unique element of the form $(\alpha, 1)$; such H's are classified by this α, which may be any element of $\mathbb{Z}/p^m$. Finally assume that $|H| = p^k$ with $m <$ k $<$ n; this forces $r = 0$ again, and note that H contains a unique element of the form (a, p^{n-k}); any a prime to p will do here. There are $p^{m-1}(p-1)$ such a's, proving (b).

Assume that $m = n$; then H must have order p^m, and can be generated by a unique element of the form $(\alpha, 1)$ or one of the form $(1, \beta)$ or both. There are p^m possible elements of the form $(\alpha, 1)$ to use, all giving different H's. Those generated by an element of the form $(1, \beta)$ can also be generated by an$(\alpha, 1)$ if and only if β is a unit mod p; therefore there are p^{m-1} new subgroups of order p^m generated by the $(1, \beta)$'s with $p \mid \beta$ not yet counted. This gives a total of $p^m + p^{m-1} = p^{m-1}(p+1)$ as claimed.

The last statement is obvious now if $m = n$, so assume $m < n$. If $H_1 \subset H_2$, then H_1 must be generated by an element of the form $(1, \beta)$, by the above argument; such a generator cannot be the multiple (by p) of any other element of T, hence it cannot be a multiple of the supposed generator of H_2. Therefore H_1 cannot be properly contained in H_2. $\square$

Let us now turn to the problem of showing that the $\gamma_{T,H}$'s are well-defined. Rewriting the equations of Proposition 2.4 in terms of the γ's, we have

$$\sum_{\substack{H \subseteq T \\ H \text{ c\&c}}} |H|^2 \gamma_{T,H} = |T| \tag{3.2}$$

and

$$\sum_{\substack{G \text{ c\&c} \\ H \subseteq G \subseteq T}} \gamma_{T,G} = \gamma_{H,H} \quad \text{for every c\&c } H \text{ in } T. \tag{3.3}$$

Note that e has disappeared from these equations, and the following suffices to prove that the γ's are well-defined.

PROPOSITION 3.4. The above system of equations is of full rank, and so determines the numbers $\gamma_{T,H}$.

Proof. We work by induction on $|T|$. We have already seen that these equations determine the γ's in case T is trivial or T is cyclic of prime order. Let $\sigma(T)$ be the number of cyclic and cocyclic subgroups of T; then we have in (3.2) and (3.3) $1 + \sigma(T)$ equations in $\sigma(T)$ unknowns.

Assume first that T is not cyclic. Then in fact the equations of (3.3) are of full rank, and determine the γ's by induction. To prove this, we work by descending induction through the lattice of cyclic and cocyclic subgroups of T; note that none of these is T by assumption. If H is a maximal such subgroup, one of the equations of (3.3) is simply $\gamma_{T,H} = \gamma_{H,H}$, and so $\gamma_{T,H}$ is determined by the induction on $|T|$. Assume now that H is any cyclic and cocyclic subgroup, not maximal, and that $\gamma_{T,G}$ is known for all cyclic and cocyclic subgroups of T larger than H. Then all the terms of the equation $\gamma_{H,H} = \Sigma_{H \subseteq G \subseteq T} \gamma_{T,G}$ are known except $\gamma_{T,H}$, and we are done by induction in this case.

Assume now that T is cyclic of order N, and write γ_M for $\gamma_{T,\mathbb{Z}/M\mathbb{Z}}$ when M divides N. One of the equations of (3.3) is just $\gamma_N = \gamma_N$, and so is useless; we must really use (3.2) in this case. However the argument of the noncyclic case applies here, to the extent that all the γ_M's can be solved for in terms of γ_N, using the equations of (3.3) are of full rank, and so to finish it suffices to show that (3.2) is independent of them.

For this we may consider the associated homogeneous system of equations

$$\sum_{M|N} M^2 g_M = 0 \tag{3.5}$$

and

$$\sum_{M|L|N} g_L = 0 \quad \text{for every c\&c } H \neq T \text{ in } T, \tag{3.6}$$

(introducing new variables g_M to avoid confusion with the γ's). We have shown that the equations of (3.6) are of full rank, and we need to show that (3.5) is independent of them. For this we need only demonstrate one solution of (3.6) which is not a solution of (3.5). It is a standard fact from elementary number theory that the numbers

$$g_M = \mu(N/M)$$

satisfy (3.6), where μ is the Möbius function:

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1 \\ (-1)^r & \text{if } k \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise (i.e., if } k \text{ is not square-free)} \end{cases} .$$

Let $p_1, \ldots, p_r$ be the distinct prime divisors of N, and let $D = \Pi p_i$. Then plugging these g_M's into (3.5) gives

$$\sum_{M|N} M^2\mu(N/M) = N^2 \sum_{M|N} (M/N)^2\mu(N/M) = N^2 \sum_{d|D} d^{-2}\mu(d) = N^2 \prod_{i=1}^{r} (1 - p_i^{-2})$$

which is clearly not zero. This proves that the original system is of full rank, and completes the proof of the Proposition. □

Our next task is to compute the numbers $\gamma_{T,H}$ explicitly. We are fortunate that these numbers turn out to be multiplicative in the following sense:

LEMMA 3.7.

$$\gamma_{T,H} = \prod_p \gamma_{T_p,H_p}.$$

Proof. Simply note that the equations (3.2) and (3.3) which determine the γ's are themselves multiplicative, i.e., if we make the formal substitution of $\Pi_p \gamma_{T_p,H_p}$ for $\gamma_{T,H}$ in these equations, both sides factor completely into the corresponding equations for γ_{T_p,H_p}. Therefore, if the γ_{T_p,H_p} satisfy their defining equations, then the above definition of $\gamma_{T,H}$ will satisfy the (3.2) and (3.3). By the previous Proposition, the system has a unique solution, so this must be it. □

The argument would be somewhat cleaner if we could prove (3.7) from first principles, and proceed from there; we have not been able to do that. If we interpret $\gamma_{T,H}$ as the probability of a node having isotropy group H, then (3.7) says that if two subgroups have relatively prime order, then having them as isotropy groups are independent events.

In any case to compute the $\gamma_{T,H}$'s, we need only compute for $T \cong \mathbb{Z}/p^n\mathbb{Z}$ and $T \cong \mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$, for $1 \leqslant m \leqslant n$. This is now a straightforward computation:

LEMMA 3.8. *Suppose that* $1 \leqslant m \leqslant n$.

(a) *Assume* $T \cong \mathbb{Z}/p^n\mathbb{Z}$. *Then* $\gamma_{T,\mathbb{Z}/p^k\mathbb{Z}} = \begin{cases} 1/p^k(1+1/p) & \text{if } k = 0 \text{ or } n \\ (1-1/p)/p^k(1+1/p) & \text{if } 0 < k < n \end{cases}.$

(b) *Assume* $T \cong \mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$. Then $\gamma_{T,\mathbb{Z}/p^k\mathbb{Z}} = 1/p^k(1+1/p)$.

Proof. We need to show that the above γ's satisfy (3.2) and (3.3). Assume first that $T \cong \mathbb{Z}/p^n\mathbb{Z}$. First evaluate (3.2):

$$\begin{aligned}\sum_{k=0}^{n} p^{2k}\gamma_{T,\mathbb{Z}/p}k &= 1/(1+1/p) + \sum_{k=1}^{n-1} p^{2k}(1-1/p)/p^k(1+1/p) + p^{2n}/p^n(1+1/p)\\ &= \frac{1}{p+1}\left[p + \sum_{k=1}^{n-1} p^k(p-1) + p^{n+1}\right]\\ &= \frac{1}{p+1}\left[p + (p^n - p) + p^{n+1}\right] = p^n = |T|\end{aligned}$$

as required.

To verify (3.3), fix l between 0 and n. If $l = 0$, we have

$$\begin{aligned}\sum_{k=l}^{n} \gamma_{T,\mathbb{Z}/p}k_{\mathbb{Z}} &= 1/(1+1/p) + \sum_{k=1}^{n-1} (1-1/p)/p^k(1+1/p) + 1/p^n(1+1/p)\\ &= \frac{1}{p+1}\left[p + \sum_{k=1}^{n-1} p^{-k}(p-1) + p^{-n+1}\right]\\ &= \frac{1}{p+1}\left[p + (1 - p^{-n+1}) + p^{-n+1}\right] = 1 = \gamma_{\{0\},\{0\}}.\end{aligned}$$

If $l > 0$, then

$$\begin{aligned}\sum_{k=l}^{n} \gamma_{T,\mathbb{Z}/p}k_{\mathbb{Z}} &= \sum_{k=l}^{n-1} (1-1/p)/p^k(1+1/p) + 1/p^n(1+1/p)\\ &= \frac{1}{p+1}\left[\sum_{k=l}^{n-1} p^{-k}(p-1) + p^{-n+1}\right]\\ &= \frac{1}{p+1}\left[(p^{-l+1} - p^{-n+1}) + p^{-n+1}\right]\\ &= 1/p^l(1+1/p) = \gamma_{\mathbb{Z}/p}l_{\mathbb{Z},\mathbb{Z}/p}l_{\mathbb{Z}}.\end{aligned}$$

This proves that the formulas of (a) are correct.

Since there are no inclusions among any cyclic and cocyclic subgroups in case (b) by Lemma 3.1(d), we have $\gamma_{T,\mathbb{Z}/p^k\mathbb{Z}} = \gamma_{\mathbb{Z}/p^k\mathbb{Z},\mathbb{Z}/p^k\mathbb{Z}} = 1/p^k(1+1/p)$ by part (a). □

We are finally ready to compute the size of a group T.

THEOREM 3.9.

$$\textit{Size}\ (\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}) = \frac{1}{N}\prod_p \left[1 + v_p(N/M)\left(\frac{p-1}{p+1}\right)\right] \textit{ if } M \mid N.$$

(Here $v_p(k)$ is the p-order of k, i.e., the largest power of p dividing k.)

Proof. Since the γ's are multiplicative, we see by inspecting the Definition 2.7 of Size(T) that Size is multiplicative also, i.e., Size(T) $= \Pi_p$ Size(T_p). Since the above formula is multiplicative too, it suffices to verify the formula above for the p-groups.

First assume $T \cong \mathbb{Z}/p^n\mathbb{Z}$. Then

$$\begin{aligned}\text{Size}\ (T) &= p^{-n}\sum_{k=0}^{n} p^k \gamma_{T,\mathbb{Z}/p^k\mathbb{Z}}\\ &= p^{-n}\left[1/(1+1/p) + \sum_{k=1}^{n-1} p^k(1-1/p)/p^k(1+1/p) + p^n/p^n(1+1/p)\right]\\ &= \frac{p^{-n}}{p+1}\left[p + (n-1)(p-1) + p\right] = \frac{p^{-n}}{p+1}\left[p + n(p-1) + 1\right]\\ &= p^{-n}\left(1 + n\left(\frac{p-1}{p+1}\right)\right)\end{aligned}$$

as the formula states.

Next assume that $T \cong \mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ with $1 \leqslant m < n$. Then

$$\begin{aligned}\text{Size}(T) &= p^{-m-n}\sum_{k=m}^{n} p^k(\#\text{ of subgroups of order } p^k)/p^k(1+1/p)\\ &= \frac{p^{-m-n+1}}{p+1}\left[p^m + \sum_{k=m+1}^{n-1} p^{m-1}(p-1) + p^m\right] \quad \text{(using Lemma 3.1)}\\ &= \frac{p^{-m-n+1}}{p+1}\left[2p^m + (n-m-1)p^{m-1}(p-1)\right]\\ &= \frac{p^{-n}}{p+1}\left[2p + (n-m)(p-1) - p + 1\right]\\ &= p^{-n}\left[1 + (n-m)\left(\frac{p-1}{p+1}\right)\right],\end{aligned}$$

which is the above.

Finally assume that $T \cong (\mathbb{Z}/p^n)^2$. Then

$$\begin{aligned}\mathrm{Size}(T) &= p^{-2n}(p^n)\,(\#\text{ of subgroups of order } p^n)/p^n(1+1/p)\\ &= p^{-2n}(p^{n-1})(p+1)/(1+1/p) = p^{-n},\end{aligned}$$

again agreeing with the formula. □

Note that this agrees with our previous calculation for $\mathbb{Z}/p\mathbb{Z}$. Relating this to the work of Cox and Parry [C-P], note that

$$\mathrm{Size}(\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}) = \nu_{00}/\mu, \tag{3.10}$$

where ν_{00} is the number of cusps and μ is the index in $PSL(2, \mathbb{Z})$ of the group $\Gamma_M(N) = \{\gamma \in SL(2, \mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix}1 & b\\ 0 & 0\end{smallmatrix}\right) \bmod N \text{ and } b \equiv 0 \bmod M\}$.

We have the following special case for the computation of Size(T) in the case of p-groups:

(a) $\mathrm{Size}(\mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}) = [p(n-m+1) - (n-m-1)]p^n(p+1)$
if $0 \leqslant m \leqslant n$, and $n \geqslant 1$. (3.11)

4. Applications

By comparing the lower bound given in (1.6) with the upper bound of (2.8) for the number of singular fibers s, we obtain the following inequality on which all of the rest of our calculations are based:

THEOREM 4.1. *Let $\pi: X \to C$ be a smooth minimal semistable elliptic surface, and assume that π admits T as a group of torsion sections. Then*

$$2\chi + 2 - 2g + R \leqslant 12\chi\cdot\mathrm{Size}(T)$$

where g is the genus of C, χ is the Euler characteristic of X, and R is the rank of the Mordell–Weil group of sections.

The formula (3.10) for Size shows that the above theorem is equivalent to the inequality given in Proposition 1.1 of [C]. However, we feel that the above formulation is easier to work with than that version.

There is one more ingredient to bring to bear on the numerology of this problem, and that is a divisibility condition, which we believe is a new result. Recall that the integers $\imath_{T,H} = 12\chi\gamma_{T,H}$ are the numbers of nodes with isotropy group H, and that they occur in singular fibers whose sizes are multiples of $|T/H|$, by Lemma 2.2(e). Therefore the numbers $\imath_{T,H}/|T/H| = 12\chi\cdot\gamma_{T,H}/|T/H|$ are integers for each cyclic and cocyclic subgroup H of T. We can express this more efficiently by defining numbers λ_{T_p} for a finite abelian p-group T_p of length at most 2 as

$$\lambda_{T_p} = \begin{cases} |T_p|(p+1)/p & \text{if } T_p \cong \mathbb{Z}/p\mathbb{Z} \text{ or } T_p \text{ is not cyclic} \\ |T_p|(p+1) & \text{if } T_p \cong \mathbb{Z}/p^n\mathbb{Z} \text{ with } n \geqslant 2 \end{cases}.$$

If T is not a p-group, define $\lambda_T = \Pi_p \lambda_{T_p}$.

PROPOSITION 4.2. *Assume an elliptic surface with Euler characteristic χ admits T as a torsion group of sections. Then λ_T divides 12χ.*

Proof. We've remarked above that $12\chi \cdot \gamma_{T,H}/|T/H|$ is an integer for every cyclic and cocyclic H in T; note that this product splits as $12\chi \cdot \Pi_p \gamma_{T_p,H_p}|H_p|/|T_p|$. Using Lemma 3.8 we see that

$$\gamma_{T_p,H_p}|H_p|/|T_p| = \begin{cases} p/|T_p|(p+1) & \text{if } T_p \text{ is cyclic and } H_p = \{0\} \text{ or } T_p, \textit{ or} \\ & \text{if } T_p \text{ is not cyclic;} \\ (p-1)/|T_p|(p+1) & \text{if } T_p \text{ is cyclic and } 0 \neq H_p \neq T_p. \end{cases}$$

If T_p is not cyclic or if $T_p \cong \mathbb{Z}/p\mathbb{Z}$, this is exactly $\lambda_{T_p}^{-1}$ for every H. If $T_p \cong \mathbb{Z}/p^n\mathbb{Z}$ with $n \geqslant 2$, then both factors above appear in the product (for different H's) and so the difference, which is $\lambda_{T_p}^{-1}$, can also be used in the product; using the difference is stronger than using either one, so we conclude that $12\chi\Pi\lambda_{T_p}^{-1} = 12\chi/\lambda_T \in \mathbb{Z}$. □

Propositions 4.1 and 4.2 exhaust the theoretical content of this article. We would like to conclude by applying these conditions to classifying the possible Mordell–Weil groups of semistable elliptic surfaces in several situations. We will first assume that the base curve has genus $g = 0$, i.e., $C \cong \mathbb{CP}^1$. In this case (4.1) can be written as

$$R \leqslant (12 \cdot \mathrm{Size}(T) - 2)\chi - 2, \tag{4.3}$$

which implies that

$$\mathrm{Size}(T) > \tfrac{1}{6} \tag{4.4}$$

since $R \geqslant 0$ and $\chi \geqslant 1$. There are a finite number of T's such that (4.4) is satisfied for it and all of its subgroups, and for each of those T's, one has a divisibility condition on χ given by (4.2), and a bound on R given by (4.3); note that (4.3) also gives a lower bound on χ. A table of this information is given below, followed by the proofs.

(4.5) Table of possible Mordell–Weill groups when genus(C) = 0.

$T = Tors(MW)$	$Size(T)$	λ_T	χ	R
$\{0\}$	1	1	arb.	$R \leqslant 10\chi - 2$
$\mathbb{Z}/2\mathbb{Z}$	$\frac{2}{3}$	3	arb.	$R \leqslant 6\chi - 2$
$\mathbb{Z}/3\mathbb{Z}$	$\frac{1}{2}$	4	arb.	$R \leqslant 4\chi - 2$
$\mathbb{Z}/4\mathbb{Z}$	$\frac{5}{12}$	12	arb.	$R \leqslant 3\chi - 2$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\frac{1}{2}$	6	arb.	$R \leqslant 4\chi - 2$
$\mathbb{Z}/5\mathbb{Z}$	$\frac{1}{3}$	6	arb.	$R \leqslant 2\chi - 2$
$\mathbb{Z}/6\mathbb{Z}$	$\frac{1}{3}$	12	arb.	$R \leqslant 2\chi - 2$
$\mathbb{Z}/7\mathbb{Z}$	$\frac{1}{4}$	8	$2 \mid \chi$	$R \leqslant \chi - 2$
$\mathbb{Z}/8\mathbb{Z}$	$\frac{1}{4}$	24	$2 \mid \chi$	$R \leqslant \chi - 2$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\frac{1}{3}$	12	arb.	$R \leqslant 2\chi - 2$
$\mathbb{Z}/9\mathbb{Z}$	$\frac{2}{9}$	36	$3 \mid \chi$	$R \leqslant (\frac{2}{3})\chi - 2$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\frac{1}{3}$	12	arb.	$R \leqslant 2\chi - 2$
$\mathbb{Z}/10\mathbb{Z}$	$\frac{2}{9}$	18	$3 \mid \chi$	$R \leqslant (\frac{2}{3})\chi - 2$
$\mathbb{Z}/12\mathbb{Z}$	$\frac{5}{24}$	48	$4 \mid \chi$	$R \leqslant (\frac{1}{2})\chi - 2$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$\frac{1}{4}$	24	$2 \mid \chi$	$R \leqslant \chi - 2$
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\frac{1}{4}$	24	$2 \mid \chi$	$R \leqslant \chi - 2$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	$\frac{5}{24}$	24	$2 \mid \chi, \chi \geqslant 4$	$R \leqslant (\frac{1}{2})\chi - 2$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$\frac{2}{9}$	36	$3 \mid \chi$	$R \leqslant (\frac{2}{3})\chi - 2$
$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	$\frac{1}{5}$	30	$5 \mid \chi$	$R \leqslant (\frac{2}{5})\chi - 2$

Note that this list of 19 torsion groups is exactly the list of Cox and Parry [C-P, Theorem 5.1]. If we restrict this list to the groups allowed when $\chi = 1$, then we recover exactly the list of Cox for Mordell–Weil groups of rational elliptic surfaces [C, Proposition 2.1]. If we restrict to those allowed when $\chi = 2$, then we obtain the list of 65 Mordell–Weil groups possible for $K3$ elliptic surfaces [C, Theorem 2.2]. In [C] and [C-P] it is shown that all these groups can occur in these cases, so the method of this paper appears to be quite sharp.

Let us briefly sketch the proof that the table above is complete. We leave to the reader the computations of Size(T) and λ_T, and the deductions re χ and R, for these 19 groups; we will only show that no other are possible.

We have already seen that T can have p-parts for $p = 2$, 3, 5 and 7 only, by Corollary 2.6. Since Size($\mathbb{Z}/p^2\mathbb{Z}$) $= (3p - 1)/p^2(p + 1)$, we see that $\mathbb{Z}/25\mathbb{Z}$ and $\mathbb{Z}/49\mathbb{Z}$ are ruled out; Size($\mathbb{Z}/27\mathbb{Z}$) $= \frac{5}{54}$ and Size($\mathbb{Z}/16\mathbb{Z}$) $= \frac{7}{48}$, so these are also ruled out, and hence no cyclic p-groups other than those of the table can occur.

Size ($\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$) $= 1/p$, so $p = 7$ is not allowed here. Since Size($\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$) $= 2/(p + p^2)$, only $p = 2$ is allowed; for $p = 2$, any non-cyclic group of order 16 is OK, but the non-cyclic groups of order 32 fail the Size test ($\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$ fails because $\mathbb{Z}/16\mathbb{Z}$ did, and Size($\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$) $= \frac{1}{6}$), so no other noncyclic p-groups other than those of the table occur.

For the non-p-groups, we must have to analyze the combinations of the allowed p-groups and this is particularly simple to do since Size is multiplicative. We leave to the reader to check that the 19 groups above exhausts the possibilities.

Let us turn our attention to the case when the base curve C has genus $g = 1$. In this case (4.1) can be written as

$$R \leqslant (12 \cdot \mathrm{Size}(T) - 2)\chi \tag{4.6}$$

and so we require $\mathrm{Size}(T) \geqslant \frac{1}{6}$, since $R \geqslant 0$ and $\chi \geqslant 1$. We see that any group which occurs for $g = 0$ also can occur for $g = 1$, and in fact the only "new" groups which occur have Size $= \frac{1}{6}$ exactly. We list these groups in Table 4.8 below.

Before doing so, note that in general (4.1) can be written as

$$R \leqslant (12 \cdot \mathrm{Size}(T) - 2)\chi - 2 + 2g \tag{4.7}$$

and so for any genus g, if $\mathrm{Size}(T) < \frac{1}{6}$, then T cannot occur for large χ. Therefore the list of groups which can occur in genus 0 and 1 (i.e., those T with $\mathrm{Size}(T) \geqslant \frac{1}{6}$) are also the only groups which can occur for any genus and arbitrarily large χ. This is the asymptotic analysis alluded to in the introduction; although a result of this type can be obtained from modular surface considerations, we believe that this approach is new.

The reader should at this point recall Corollary 2.6, noting that if $\chi > 7g - 6$, then the only primes p dividing $|T|$ are 2, 3, 5, 7, and 11; This is the first precise case of the asymptotic results.

(4.8) Table of torsion groups possible if $g = 1$, not possible if $g = 0$.
All these groups below have $\mathrm{Size}(T) = \frac{1}{6}$. If $g = 1$, then $R = 0$.

$T = Tors(MW)$	λ_T	χ
$\mathbb{Z}/11\mathbb{Z}$	12	arb.
$\mathbb{Z}/14\mathbb{Z}$	24	$2 \mid \chi$
$\mathbb{Z}/15\mathbb{Z}$	24	$2 \mid \chi$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$	36	$3 \mid \chi$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$	48	$4 \mid \chi$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$	36	$3 \mid \chi$
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	48	$4 \mid \chi$
$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	72	$6 \mid \chi$

As a final example of the asymptotic analysis, we offer the following, which is (as we shall see below) a sharpening of a result of Hindry and Silverman.

PREPOSITION 4.9. *Suppose* $\chi \geqslant 2g - 2$. *Then* $\mathrm{Size}(T) \geqslant \frac{1}{12}$.

Proof. We may assume that $S = \mathrm{Size}(T)$ is less than $\frac{1}{6}$ and that $g \geqslant 2$. In this case $12S - 2 < 0$, so $(12S - 2)\chi \leqslant (12S - 2)(2g - 2)$. Hence

$$\begin{aligned} 0 \leqslant R &\leqslant (12S - 2)\chi - 2 + 2g \quad \text{(using Theorem 4.1)} \\ &\leqslant (12S - 2)(2g - 2) - 2 + 2g = (12S - 1)(2g - 2), \end{aligned}$$

which forces $12S - 1 \geqslant 0$ since $g \geqslant 2$. □

One can readily list all the groups T with $\mathrm{Size}(T) \geqslant \frac{1}{12}$, and we present the result below; actually to save space we only list those groups T with $\frac{1}{6} > \mathrm{Size}(T) \geqslant \frac{1}{12}$, in light of the previous tables.

(4.10) Table of groups T with $\frac{1}{6} > \mathrm{Size}(T) \geqslant \frac{1}{12}$.

Notation: (N) denotes $\mathbb{Z}/N$, (M, N) denotes $\mathbb{Z}/M \times \mathbb{Z}/N$.

(13), (17), (18), (19), (20), (21), (22), (23), (24), (26), (28), (2.14),
(30), (33), (35), (36), (2, 18), (3, 12), (40), (2, 20), (2, 22), (3, 15),
(2, 24), (4, 12), (5, 10), (3, 18), (2, 28), (2, 30), (3, 21), (3, 24), (6, 12),
(5, 15), (4, 20), (4, 24), (7, 14), (5, 20), (10, 10), (6, 18), (12, 12),

The groups in Table 4.10 are listed by increasing order, and inspecting the previous tables of groups with Size at least $\frac{1}{6}$, we draw the following corollary:

$$\text{If } \chi \geqslant 2g - 2, \text{ then } |T| \leqslant 144. \tag{4.11}$$

This last statement is Hindry and Silverman's result [H-S, p. 440].

The last application of this section is a bound on the rank R of the Mordell-Weil group obtained by Cox [C, Corollary 1.3]. The proof, using our method, follows directly from (4.7).

PROPOSITON 4.12. *Suppose* $\mathrm{Size}(T) \leqslant \frac{1}{6}$ *and* $\pi\colon X \to C$ *admits* T *as a group of torsion sections. Then* $R \leqslant 2g - 2$, *where* $g = \mathrm{genus}(C)$.

We would finally like to make some brief remarks concerning the existence of the Mordell–Weil groups listed above. Recall that the Tables 4.5 and 4.8 have been previously worked out by David Cox [C] thus we find it unnecessary to duplicate the existence work here. It should be remarked however that as a spin-off from the exhaustive list of all possible elliptic fibrations on a rational surface worked out by one of the present authors [P], the list of Mordell–Weil groups come more or less for free. Also most of the torsion groups in the $K3$ case appear in the two authors forthcoming paper [M-P 2]

The more subtle things occur for torsion groups only existing over non-rational base. One should remark that it is enough to exhibit those for the elliptic base, as the other manifestations can be gotten by "generic" base changes (all genera except of course the rational can occur as coverings of elliptic curves).

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