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# On the Kodaira dimension of moduli spaces of abelian surfaces

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It is known (e.g. [İ]) that the moduli space of principally polarized abelian surfaces is rational. On the other hand it follows from general results of Mumford [M-1] that the moduli space of p.p.a.s.'s with a full level-$n$ structure is of general type for $n$ big. In this paper we prove that moduli spaces of p.p.a.s.'s with an intermediate level-$p$ structure are of general type for $p$ big. More precisely let $\mathscr{A}_2(p)$ be the moduli space of couples $(S, H)$ where $S$ is a p.p.a.s. and $H \subset S[p]$ a rank two subspace of the $p$-torsion points, non-isotropic for the Weil pairing ($p$ is a prime). Our main theorem asserts that $\mathscr{A}_2(p)$ is of general type if $p \geqslant 17$. The motivation for this work came from studying moduli spaces of $K3$ surfaces. Let $\mathscr{F}_{2d}$ be the moduli space of $K3$ surfaces with a primitive polarization of degree $2d$. For every $n, k$ there exists a finite surjective map $f_{n,k}\colon \mathscr{F}_{2n^2k} \to \mathscr{F}_{2k}$ (see the appendix). Let's fix $k$, say $k = 1$. The moduli space $\mathscr{F}_2$ is unirational hence $\kappa(\mathscr{F}_2) = -\infty$ but one is tempted to study the maps $f_{n,1}\colon \mathscr{F}_{2n^2} \to \mathscr{F}_2$ in order to determine the Kodaira dimension of $\mathscr{F}_{2n^2}$ for $n$ big. Now let $\mathscr{A}_{2,d}$ be the moduli space of abelian surfaces with a polarization with elementary divisors $\{1, d\}$; we think of $\mathscr{A}_{2,d}$ as analogous to $\mathscr{F}_{2d}$ (see the appendix). There exist maps $\tilde{g}_{n,k}\colon \mathscr{A}_{2,n^2k} \to \mathscr{A}_{2,k}$ analogous to the maps $f_{n,k}$. If we set $k = 1$ and $n$ is a prime $p$, then the defintion of $\tilde{g}_{p,1}\colon \mathscr{A}_{2,p^2} \to \mathscr{A}_{2,1}$ ($= \mathscr{A}_2$, the moduli space of p.p.a.s.'s) identifies $\mathscr{A}_{2,p^2}$ with our moduli space $\mathscr{A}_2(p)$ and the map $g_{p,1}$ is identified with the natural map from $\mathscr{A}_2(p)$ to $\mathscr{A}_2$. So the Main Theorem is equivalent to the statement that $\mathscr{A}_{2,p^2}$ is of general type for $p \geqslant 17$ (Corollary 5.1); it suggests that $\mathscr{F}_{2p^2}$ is also of general type for $p$ big.

The plan of the proof of the main theorem is the following. Let $\mathscr{A}_2$ be the moduli space of p.p.a.s.'s; we choose the (toroidal) compactification $\bar{\mathscr{A}}_2$ of $\mathscr{A}_2$ isomorphic to $\bar{\mathfrak{M}}_2$, the moduli space of stable genus two curves. In Section 1 we establish some relations between divisor classes on $\bar{\mathscr{A}}_2$. Let $\pi\colon \mathscr{A}_2(p) \to \mathscr{A}_2$ be the map obtained by associating to the couple $(S, H)$ the surface $S$, i.e. by forgetting the $p$-structure. We define $\bar{\mathscr{A}}_2(p)$ to be the natural toroidal compactification of $\mathscr{A}_2(p)$ such that $\pi$ extends to a finite surjective map $\pi\colon \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$. In Section

2 we apply Hurwitz's formula to $\pi$ in order to get an expression for the canonical class of $\bar{\mathscr{A}}_2(p)$. Not all singularities of $\bar{\mathscr{A}}_2(p)$ are canonical, i.e. some of them "impose conditions on adjoints". In Section 3 we construct a partial desingularization $\hat{\mathscr{A}}_2(p)$ of $\bar{\mathscr{A}}_2(p)$ all of whose singularities are canonical. In Section 4 we show that $h^0(nK_{\hat{\mathscr{A}}_2(p)}) \geqslant Q(p)n^3 + O(n^2)$ for $n$ sufficiently divisible, where $Q(p) > 0$ for $p \geqslant 17$. Hence tr. deg. $\oplus_{n=0}^{\infty} H^0(nK_{\hat{\mathscr{A}}_2(p)}) = 4$ for $p \geqslant 17$; since $\hat{\mathscr{A}}_2(p)$ is canonical $\mathscr{A}_2(p)$ is of general type ($p \geqslant 17$).

It's a pleasure to thank Joe Harris for crucial help during the initial stage of this work. Thanks also go to Henry Pinkham for suggesting to use Fujiki's algorithm for resolving cyclic quotient singularities.

NOTATION: Let $S$ be an abelian surface, then $S[n]$ will be the subgroup of $n$-torsion points.

Let $S$ be a p.p.a.s. (or let $C$ be a genus two curve); let $-1 \in \mathrm{Aut}(S)$ be multiplication by $-1$ (respectively let $\iota\colon C \to C$ be the hyperelliptic involution), then

$$\mathrm{Aut}'(S) = \mathrm{Aut}(S)/\{\mathrm{id.}, -1\} \text{ (respectively } \mathrm{Aut}'(C) = \mathrm{Aut}(C)/\{\mathrm{id.}, \iota\}).$$

We will refer to Aut$'$ $(S)$ (Aut$'(C)$) as the reduced group of automorphisms of $S$ (respectively $C$).

By elliptic curve we mean a curve of arithmetic genus one with at most one nodal singularity. We let $j(E)$ be the usual $j$-invariant of $E$, if $j(E) = 0$, $E \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}e^{\pi i/3}$, if $j(E) = 1728$, $E \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$, if $j(E) = \infty$, $E$ is singular.

$\langle g_1, \ldots, g_n \rangle$ will be the group generated by $g_1, \ldots, g_n$. $U(x, y, \ldots, z)$ will be the affine space with coordinates $x, y, \ldots, z$. $e_n$ denotes a primitive $n$th root of unity.

Let $M$ be the moduli space of a class of varieties, let $f\colon V \to T$ be a family of such varieties, we will denote by $m$ (sometims $m_U$ or $m_p$) the induced map from $T$ to $M$. In particular if $V$ is one such variety $m(V) \in M$ will be the moduli point of $V$.

## Section 1. Divisors on $\mathfrak{M}_2$

Let $\mathfrak{M}_2$ be the moduli space of Deligne-Mumford stable curves of arithmetic genus two.

DEFINITION 1.1. (i) Let $\Delta_0 \subset \mathfrak{M}_2$ be the divisor parametrizing curves with one (at least) nondisconnecting node.

(ii) Let $\Delta_1 \subset \mathfrak{M}_2$ be the divisor parametrizing curves with one disconnecting node.

(iii) Let $\Delta_2 \subset \mathfrak{M}_2$ be the divisor whose generic point is the moduli of a double cover of an elliptic curve.

REMARK. The generic curve whose moduli belongs to $\Delta_2$ is given by

$$y^2 = (x^2 - \alpha_1)(x^2 - \alpha_2)(x^2 - \alpha_3).$$

It has two involutions whose quotient is an elliptic curve, namely $\iota_1, \iota_2$, where

$$\iota_1^*(x, y) = (-x, y) \quad \text{and} \quad \iota_2^*(x, y) = (-x, -y).$$

DEFINITION 1.2. Let, by abuse of notation, $\Delta_0, \Delta_1, \Delta_2 \in \mathrm{Pic}(\bar{\mathfrak{M}}_2) \otimes \mathbb{Q}$ be the classes of the reduced divisors $\Delta_0, \Delta_1, \Delta_2$.

REMARK. The singularities of $\bar{\mathfrak{M}}_2$ are quotient singularities hence every Weil divisor is $\mathbb{Q}$-Cartier, so $\Delta_0, \Delta_1, \Delta_2$ are indeed elements of $\mathrm{Pic}(\bar{\mathfrak{M}}_2) \otimes \mathbb{Q}$. Our classes $\Delta_i$'s are given by the reduced $\Delta_i$'s; they differ from Munford's [M-2] classes $[\Delta_i]_Q$. In fact the relation should be

$$[\Delta_1]_Q = \tfrac{1}{2}\Delta_1, \quad [\Delta_2]_Q = \tfrac{1}{2}\Delta_2.$$

DEFINITION 1.3. Let $f: \mathscr{C} \to T$ be a family of stable genus two curves. The Hodge bundle (over T) is

$$\lambda_T = \Lambda^2(f_* \omega_{\mathscr{C}/T}).$$

The Hodge bundle can be viewed as an element of the functorial Picard group of $\bar{\mathfrak{M}}_2$. Due to curves with extra-automorphisms it does not come from a line bundle on $\bar{\mathfrak{M}}_2$. A sufficiently divisible power of the functorial Hodge bundle (a common multiple of the orders of automorphism groups of stable genus two curves will do) is the pull-back of a line bundle on $\bar{\mathfrak{M}}_2$, hence we can give

DEFINITION 1.4. Let $\lambda \in \mathrm{Pic}(\bar{\mathfrak{M}}_2) \otimes \mathbb{Q}$ be the Hodge bundle.

DEFINITION 1.5. Let $\{E_t\}$ $(t \in T \cong \mathbb{P}^1)$ be a Lefschetz pencil on a smooth cubic surface in $\mathbb{P}^3$. Let $F$ be a fixed generic elliptic curve, i.e. $j(F) \neq 0$, 1728, $\infty$. Let $C_t = E_t \cup F$ be obtained by gluing $E_t$ and $F$ along the zeroes of the group laws (notice that the pencil $\{E_t\}$ has three sections so we can choose one as the curve of zeroes of the $E_t$'s). Let $f: \mathscr{C} \to T$ be the resulting family of stable genus two curves.

LEMMA 1.1: *Let $m: T \to \bar{\mathfrak{M}}_2$ be the moduli map associated to the family $f: \mathscr{C} \to T$,*
(i) $\deg m^*(\Delta_0) = 12$,
(ii) $\deg m^*(\Delta_1) = -2$,
(iii) $\deg m^*(\Delta_2) = 24$,
(iv) $\deg m^*(\lambda) = 1$.

*Proof.* (i) There are 12 singular fibers in the pencil $\{E_t\}$, since it is a Lefschetz pencil the curve $m(T) \subset \mathfrak{M}_2$ is transverse to $\Delta_0$ at each point of intersection. Hence $\deg m^*(\Delta_0) = 12$.

(ii) Let $\varphi: \mathscr{E} \to T$, $\psi: \mathscr{F} \to T$ be two families of elliptic curves. Let $g: \mathscr{C} \to T$ be the family of curves $\{C_t = E_t \cup F_t\}$ obtained by gluing the zeroes, and let $\sigma: T \hookrightarrow \mathscr{E}$, $\tau: T \hookrightarrow \mathscr{F}$ be the sections given by the zeroes. Let $m: T \to \mathfrak{M}_2$ be the associated moduli map, by definition $m(T) \subset \Delta_1$. Following [H-M], page 51, we have that

$$m^*(\Delta_1) \cong [(\mathscr{N}_{\sigma(T)/\mathscr{E}}) \otimes (\mathscr{N}_{\tau(T)/\mathscr{F}})]^{\otimes 2}.$$

In our case $\mathscr{N}_{\sigma(T)/\mathscr{E}} \cong \mathscr{O}_T(-1)$ and $\mathscr{N}_{\tau(T)/\mathscr{F}} \cong \mathscr{O}_T$ hence $\deg m^*(\Delta_1) = -2$.

(iii) It is not difficult to check that $m(t) \in \Delta_2$ if and only if $E_t \cong F$. Since $\deg m^*(\Delta_0) = 12$ there are 12 such values of $t$. Let $t_0$ be such a value, let $U$ be the universal deformation space of $C_{t_0} = E_{t_0} \cup F$, let $\tilde{m}: T \to U$ be the map associated to the family $f: \mathscr{C} \to T$ and let $m_U: U \to \mathfrak{M}_2$ be the moduli map. Let $\Delta_1(U) \subset U$, $\Delta_2(U) \subset U$ be the divisors such that $m_U(\Delta_1(U)) = \Delta_1$, $m_U(\Delta_2(U)) = \Delta_2$; it is easy to check that they are transverse. Let $C = E \cup F$ be a curve (in the universal family over $U$) lying over $\Delta_1(U)$; let $x = j(E)$, $y = j(F)$, they are local coordinates on $\Delta_1(U)$. We have that $\Delta_2(U) \cap \Delta_1(U) = \{P \in \Delta_1(U) \mid x(P) = y(P)\}$ and $\tilde{m}(T) = \{P \in \Delta_1(U) \mid y(P) = j(F)\}$, hence $\tilde{m}(T)$ is transverse to $\Delta_2(U)$. Since $m_U: U \to \mathfrak{M}_2$ is ramified with index 2 along $\Delta_2(U)$ we get that $\deg m^*(\Delta_2) = 2 \cdot (\#\{t \in T \mid m(t) \in \Delta_2\}) = 24$.

(iv) Let $\lambda'_T$ be the Hodge bundle of the pencil $\{E_t\}$, i.e., $\lambda'_T = \varphi_* \omega_{\mathscr{E}/T}$, then $\lambda'_T \cong \lambda_T$. An easy computation gives that $\deg \lambda'_T = 1$ hence $\deg \lambda_T = 1$.

DEFINITION 1.6. Let $E$ be a fixed elliptic curve with $j(E) \neq 0, 1728, \infty$. Let $S = E \times E$ and let $\varphi: \tilde{S} \to X$ be the blow up of $S$ at $(P, P)$ where $P \in E$ is the zero of the group law. Let $\tilde{\Delta}, \tilde{\Sigma} \subset \tilde{S}$ be the strict transforms of the diagonal $\Delta$ and of $\Sigma = \{P\} \times E$ respectively. Let $\pi_2 \varphi: \tilde{S} \to E$ be the composition of $\varphi$ and projection on the second factor; $\tilde{\Delta}$ and $\tilde{\Sigma}$ are sections of the family of elliptic curves $\pi_2 \varphi$. Let $\mathscr{C}$ be obtained from $\tilde{S}$ by gluing $\tilde{\Delta}$ and $\tilde{\Sigma}$ in the obvious way so that we get a family $g: \mathscr{C} \to E$ of genus two stable curves with one nondisconnecting node each. The fiber of $g$ over $Q \neq P$ is obtained from $E$ by gluing $Q$ and $P$; the fiber over $P$ is the union of $E$ and the singular elliptic curve.

LEMMA 1.2: *Let $m: E \to \mathfrak{M}_2$ be the moduli map associated to the family $g: \mathscr{C} \to E$, then*

(i) $\deg m^*(\Delta_0) = -2$,
(ii) $\deg m^*(\Delta_1) = 2$,
(iii) $\deg m^*(\Delta_2) = 6$,
(iv) $\deg m^*(\lambda) = 0$.

*Proof.* (i) By definition $m(E) \subset \Delta_0$. We have

$$m^*(\Delta_0) \cong (\mathscr{N}_{\tilde{\Delta}/\tilde{S}}) \otimes (\mathscr{N}_{\tilde{\Sigma}/\tilde{S}}),$$

Since $\tilde{\Delta}\cdot\tilde{\Delta} = \tilde{\Sigma}\cdot\tilde{\Sigma} = -1$ we get that $\deg m^*(\Delta_0) = -2$.

(ii) Obviously $m^{-1}(\Delta_1) = \{P\}$. Let $U$ be the universal deformation space of $C_P = g^{-1}(P)$ and let $\Delta_1(U)$ be the divisor parametrizing curves with one disconnecting node. Let $\tilde{m}: E \to U$ be the map associated to the family $g: \mathscr{C} \to E$, $\tilde{m}$ is one-to-one and the image is fixed by the action of the extra automorphism of $C_P$. Hence $\tilde{m}(E)$ is transverse to the divisor fixed by this action, i.e. $\Delta_1(U)$. Since the moduli map $m_U: U \to \overline{\mathfrak{M}}_2$ is ramified with index 2 along $\Delta_1(U)$ we get that $\deg m^*(\Delta_1) = 2$.

(iii) It is easy to check that

$$m^{-1}(\Delta_2) = \{Q \in E \mid Q \neq P \quad \text{and} \quad 2Q \cong 2P\}$$

hence $\# m^{-1}(\Delta_2) = 3$. An argument similar to the previous one gives that $\deg m^*(\Delta_2) = 6$.

(iv) We have the exact sequence

$$0 \to H^0(\omega_E) \otimes \mathcal{O}_E \to \pi_*(\omega_{\mathscr{C}/E}) \xrightarrow{R} \mathcal{O}_E \to 0$$

where $R$ is the residue map. Hence

$$\deg m^*(\lambda) = c_1(\pi_*(\omega_{\mathscr{C}/E})) = 0.$$

COROLLARY 1.1. *$\{\Delta_0, \Delta_1\}$ is a basis of* $\operatorname{Pic}(\overline{\mathfrak{M}}_2) \otimes \mathbb{Q}$.

*Proof.* Igusa [I] proved that $\mathfrak{M}_2 \cong U(x, y, z)/\langle g \rangle$ where $g^*(x, y, z) = (e_5 x, e_5^2 y, e_5^3 z)$, hence $\operatorname{Pic}(\mathfrak{M}_2) \otimes \mathbb{Q} \cong \{0\}$. Since $\mathfrak{M}_2 = \overline{\mathfrak{M}}_2 \backslash (\Delta_0 \cup \Delta_1)$, $\operatorname{Pic}(\overline{\mathfrak{M}}_2) \otimes \mathbb{Q}$ is generated by $\Delta_0$ and $\Delta_1$. Lemmas 1.1 and 1.2 show that $\Delta_0$ and $\Delta_1$ are independent, hence they form a basis.

COROLLARY 1.2. $10\lambda = \Delta_0 + \Delta_1$.

*Proof.* By the previous corollary we know that $\lambda = x\Delta_0 + y\Delta_1$ for some $x, y \in \mathbb{Q}$. Using Lemmas 1.1 and 1.2 we get $x = y = \frac{1}{10}$.

COROLLARY 1.3. $\Delta_2 = 3\Delta_0 + 6\Delta_1$.

*Proof.* Same as previous corollary.

LEMMA 1.3. $K_{\overline{\mathfrak{M}}_2} = -\frac{11}{5}\Delta_0 - \frac{16}{5}\Delta_1$.

*Proof.* In [H-M] a formula is given for the canonical class of $\overline{\mathfrak{M}}_g$, the moduli space of stable genus $g$ curves, with $g \geqslant 4$. The same kind of formula holds for $K_{\overline{\mathfrak{M}}_2}$ with an extra contribution from $\Delta_2$ since the points in $\Delta_2$ represent curves with an extra automorphism. The formula one gets is

$$K_{\overline{\mathfrak{M}}_2} = 13\lambda - 2\Delta_0 - \tfrac{3}{2}\Delta_1 - \tfrac{1}{2}\Delta_2.$$

Taking into account Corollaries 1.2 and 1.3 we get that

$$K_{\overline{\mathfrak{M}}_2} = -\tfrac{11}{5}\Delta_0 - \tfrac{16}{5}\Delta_1.$$

Another way of proceeding is the following. We know that $K_{\bar{\mathfrak{M}}_2} = x\Delta_0 + y\Delta_1$ for some $x, y \in \mathbb{Q}$. Igusa's description of $\mathfrak{M}_2$ via invariants of binary sextics actually extends to a description of $\bar{\mathfrak{M}}_2 \backslash \Delta_1$. Thus one can check directly that $x = -\frac{11}{5}$. One can then obtain $y = -\frac{16}{5}$ by applying adjunction to $\Delta_0$.

Let $\mathscr{A}_2$ be the moduli space of principally polarized abelian surfaces. By associating to a genus two curve its Jacobian we get a map Jac: $\mathfrak{M}_2 \to \mathscr{A}_2$ which extends to an isomorphism Jac: $\bar{\mathfrak{M}}_2 \backslash \Delta_0 \xrightarrow{\sim} \mathscr{A}_2$.

DEFINITION 1.7. Let $\mathscr{A}_2 \subset \bar{\mathscr{A}}_2$ be the compactification of $\mathscr{A}_2$ given by $\mathrm{Jac}^{-1}$: $\mathscr{A}_2 \hookrightarrow \bar{\mathfrak{M}}_2$ (i.e. $\bar{\mathscr{A}}_2 = \bar{\mathfrak{M}}_2$).

IMPORTANT REMARK. The compactification $\bar{\mathscr{A}}_2$ is a toroidal compactification of $\mathscr{A}_2[n]$.

We will identify $\bar{\mathfrak{M}}_2$ and $\bar{\mathscr{A}}_2$ via the isomorphism Jac: $\bar{\mathfrak{M}}_2 \xrightarrow{\sim} \bar{\mathscr{A}}_2$. In particular we will denote by $\Delta_0, \Delta_1, \Delta_2$ the divisor classes $\mathrm{Jac}(\Delta_0), \mathrm{Jac}(\Delta_1), \mathrm{Jac}(\Delta_2) \in \mathrm{Pic}(\bar{\mathscr{A}}_2) \otimes \mathbb{Q}$. Notice that $\Delta_1 \subset \bar{\mathscr{A}}_2$ is the closure of the locus of moduli of p.p.a.s.'s $(S, \Theta)$ with an elliptic curve $E \subset S$ such that $E \cdot \Theta = 1$. Similarly $\Delta_2 \subset \bar{\mathscr{A}}_2$ is the closure of the locus of moduli of p.p.a.s.'s $(S, \Theta)$ with an elliptic curve $E \subset S$ such that $E \cdot \Theta = 2$.

## Section 2. The canonical divisor class on $\bar{\mathscr{A}}_2(p)$

We now come to the object of our study.

DEFINITION 2.1. Let $\mathscr{A}_2(p)$ be the coarse moduli space of couples $(S, H)$ where $S$ is a p.p.a.s. and $H \subset S[p]$ is a rank two subspace non-isotropic for the Weil pairing, where $p$ is a prime.

Let $L$ be a lattice of rank four and let $E$ be an alternating bilinear form on $L$ with elementary divisors $\{1, 1\}$. Let $E$ denote also the extension of $E$ to $L \otimes \mathbb{C}$, then $H(v, w) = \sqrt{-1}\, E(v, \bar{w})$ is a Hermitian form on $L \otimes \mathbb{C}$. Siegel's upper half space can be realized as

$$\mathbb{H}_2 \cong \{V \subset L \otimes \mathbb{C} \mid \dim V = 2, E_{|V} \equiv 0, H_{|V} > 0\}$$

i.e. as a classifying space for weight one Hodge structures. Let $p$ be a prime, let $E_p$ be the $\mathbb{F}_p$-valued alternating form that $E$ induces on $L_p = L \otimes \mathbb{F}_p$ and let $\Sigma_p \subset L_p$ be a fixed rank two subspace non-isotropic for $E_p$. Now let $S$ be a p.p.a.s. and $H \subset S[p]$ a non-isotropic subspace; the Weil pairing identifies $H_1(S, \mathbb{F}_p)$ with $H^1(S, \mathbb{F}_p)$ hence we can think of $H$ as living in $H^1(S, \mathbb{F}_p)$. Let $f: H^1(S, \mathbb{Z}) \to L$ be any isomorphism such that $f^*E$ is the polarization on $S$ and such that $f(H) = \Sigma_p$; then $f(H^{1,0}(S)) \in \mathbb{H}_2$. It is clear that by this construction $\mathscr{A}_2(p)$ can be realized as

$\Gamma_p \backslash \mathbb{H}_2$, where

$$\Gamma_p = \{g \in Sp(4, \mathbb{Z}) \mid g(\Sigma_p) = \Sigma_p\}.$$

As usual $m(S, H) \in \mathscr{A}_2(p)$ (or $m_p(S, H)$) will be the moduli point of $(S, H)$. Let $C$ be a smooth genus two curve, we will use $(C, H)$ ($H \subset \mathrm{Jac}(C)[p]$) as an alternative notation for $(\mathrm{Jac}(C), H)$.

DEFINITION 2.2. Let $\pi: \mathscr{A}_2(p) \to \mathscr{A}_2$ be defined by $\pi(m(S, H)) = m(S)$, i.e. by forgetting the $p$-structure on $S$.

REMARKS. Notice that there is an involution $\iota: \mathscr{A}_2(p) \to \mathscr{A}_2(p)$ commuting with $\pi$: let $x = m(S, H)$ then $\iota(x) = m(S, H^{\perp})$ (orthogonality is with respect to the Weil pairing).

The map $\pi$ can also be defined as the map induced from the inclusion $\Gamma_p < Sp(4, \mathbb{Z})$.

LEMMA 2.1. *Let* $\pi: \mathscr{A}_2(p) \to \mathscr{A}_2$, *then* $\deg \pi = p^4 + p^2$.

*Proof.* Let $m(S) \in \mathscr{A}_2$ be a generic point of $\mathscr{A}_2$, i.e. let the automorphism group of $S$ be generated by multiplication by $-1$. The fiber $\pi^{-1}(m(S))$ is in one-to-one correspondence with the set of isomorphism classes of couples $(S, H)$. Since multiplication by $-1$ fixes every subspace of $S[p]$ the degree of $\pi$ is equal to the number of subspaces $H \subset L_p$ such that $E_{p|H}$ is non-degenerate. The Grassmannian $\mathrm{Gr}(2, L_p)$ of planes in $L_p$ is realized by the Plucker embedding as the variety of rational points of a smooth quadric hypersurface in $\mathbb{P}(\Lambda^2 L_p)$. The isotropic subspaces correspond to points on a hyperplane section, which is smooth if $p > 2$. Hence if $p > 2$

$$\deg \pi = (1 + p + 2p^2 + p^3 + p^4) - (1 + p + p^2 + p^3) = p^4 + p^2.$$

One can check that the formula still holds if $p = 2$.

PROPOSITION 2.1. *Let* $D \subset \mathscr{A}_2$ *be an irreducible component of the branch divisor of* $\pi$, *then* $D$ *parametrizes surfaces with extra automorphisms.*

*Proof.* Let $m(S)$ be a generic point of $D$, i.e. let $\mathrm{Aut}'(S)$ be contained in the reduced automorphism group of all surfaces $T$ such that $m(T) \in D$. Let $U$ be the universal deformation space of $S$, let $m\colon U \to \mathscr{A}_2$ be the moduli map. The group $\mathrm{Aut}'(S)$ acts on $U$ and $m(U) \cong \mathrm{Aut}'(S)\backslash U$. Let $m(S, H) \in \mathscr{A}_2(p)$ be a point in the ramification divisor lying over $D$. The deformation space of $(S, H)$ is isomorphic to $U$. Let $\mathrm{Aut}'(S, H) < \mathrm{Aut}'(S)$ be the subgroup fixing $H$ (this makes sense because multplication by $-1$ fixes $H$). Let $m_p: U \to \mathscr{A}_2(p)$ be the moduli map, then

$m_p(U)) \cong \mathrm{Aut}'(S,H)\backslash U$. The map $\pi\colon m_p(U) \to m(U)$ is induced from the inclusion $\mathrm{Aut}'(S,H) < \mathrm{Aut}'(S)$. The inclusion of groups must be proper because $\pi$ is ramified hence $\mathrm{Aut}'(S)$ cannot be trivial. Q.E.D.

The preceding discussion also proves the following.

PROPOSITION 2.2. *Let $D \subset \mathscr{A}_2$ be an irreducible component of the branch divisor of $\pi$ and let $m(S)$ be a generic point of $D$. The ramification index of the component of $\pi^{-1}(D)$ through $m(S,H)$ is equal to*

$$[\mathrm{Aut}'(S)\colon \mathrm{Aut}'(S,H)].$$

It is easy to check that the divisors on $\mathscr{A}_2$ parametrizing p.p.a.s.'s with extra automorphisms are exactly $\Delta_1$ and $\Delta_2$.

DEFINITION 2.3. Let $\tilde{\Delta}_1 \subset \mathscr{A}_2(p)$ be the locus of moduli of couples $(S,H)$ with $S$ a reducible p.p.a.s. (i.e. $S \cong E \times F$) and $H = E[p]$ or $H = F[p]$.

Obviously $\pi(\tilde{\Delta}_1) = \Delta_1$ and $\tilde{\Delta}_1$ is a two sheeted cover of $\Delta_1$.

DEFINITION 2.4. Let $R_1 \subset \mathscr{A}_2(p)$ be the (reduced) divisor such that $\pi^{-1}(\Delta_1) = \tilde{\Delta}_1 \cup R_1$.

LEMMA 2.2. *If $p > 2$ the map $\pi\colon \mathscr{A}_2(p) \to \mathscr{A}_2$ is unramified along $\tilde{\Delta}_1$ and has ramification index 2 along $R_1$. If $p = 2$, $\pi$ is unramified along all of $\pi^{-1}(\Delta_1)$.*

*Proof.* Let $m(S)$ be a generic point of $\Delta_1$. Let $g \in \mathrm{Aut}'(S)$ act as multiplication by $-1$ on $E$ and as the identity on $F$, then $\mathrm{Aut}'(S) = \langle g \rangle \cong \mathbb{Z}/(2)$. If $p > 2$ $E[p]$ and $F[p]$ are the only non-isotropic subspaces fixed by $g$, hence $\mathrm{Aut}'(S, E[p]) \cong \mathrm{Aut}'(S, F[p]) \cong \mathbb{Z}/(2)$ and $\mathrm{Aut}'(S,H) \cong \langle \mathrm{id} \rangle$ if $H \neq E[p], F[p]$. Therefore by Proposition 2.2, $\pi$ is unramified along $\tilde{\Delta}_1$ and has ramification index one along $R_1$. If $p = 2$, since $g$ acts as the identity on $S[2]$, $\mathrm{Aut}'(S,H) \cong \langle g \rangle$ for all $H$, hence $\pi$ is unramified over $\Delta_1$.

COROLLARY 2.1. $\deg \pi|_{R_1} = \frac{1}{2}(p^4 + p^2) - 1, \pi^*(\Delta_1) = \tilde{\Delta}_1 + 2R_1$ $(p > 2)$.

DEFINITION 2.5. Let $m(S) \in \Delta_2$, i.e. $S$ contains an elliptic curve $E$ such that $E \cdot \Theta = 2$, where $\Theta \subset S$ is the theta divisor. Let $\alpha\colon E \hookrightarrow S$ be the inclusion, let $\alpha\colon E[p] \hookrightarrow S[p]$ be the restriction of $\alpha$ to $p$-torsion points. If $p > 2$ $\alpha(E[p])$ is non-isotropic. For $p > 2$ let $\tilde{\Delta}_2 \subset \mathscr{A}_2(p)$ be the locus of moduli of couples $(S,H)$ where $m(S) \in \Delta_2$ and $H = \alpha(E[p])$ or $H = \alpha(E[p])^{\perp}$.

DEFINITION 2.6. For $p > 2$ let $R_2 \subset \mathscr{A}_2(p)$ be the (reduced) divisor defined by $\pi^{-1}(\Delta_2) = \tilde{\Delta}_2 \cup R_2$. When $p = 2$ let $R_2 = \pi^{-1}(\Delta_2)$.

LEMMA 2.3. *If $p > 2$ the map $\pi\colon \mathscr{A}_2(p) \to \mathscr{A}_2$ is unramified along $\tilde{\Delta}_2$ and has* *ramification index* 2 *along* $R_2$. *If* $p = 2$ $\pi$ *has ramification index* 2 *along all of* $\pi^{-1}(\Delta_2) = R_2$.

*Proof.* Let $m(S) \in \Delta_2$ be generic, i.e. let $\mathrm{Aut}'(S) \cong \mathbb{Z}/(2)$. The surface $S$ is isomorphic to $E \times F/G$, where

$$G = \{(x, \varphi(x)) \mid x \in E[2] \text{ and } \varphi\colon E[2] \to F[2] \text{ is a symplectic isomorphism}\}.$$

Let $S' = E \times F$, let $\Theta' \subset S'$ be the reducible principal polarization and let $f\colon S' \to S$ be the quotient map, then $f^*(\Theta) \cong 2\Theta'$ ($\Theta$ is the principal polarization on $S$). If $p > 2$ the map $f\colon S'[p] \to S[p]$ is an isomorphism of groups. Let $W$, $W'$ be the Weil pairings on $S$, $S'$ respectively, then $W(f(x), f(y)) = 2W'(x, y)$ hence $H \subset S[p]$ is non-isotropic if and only if $f^{-1}(H) \subset S'[p]$ is non-isotropic. The reduced group $\mathrm{Aut}'(S)$ is generated by the automorphism induced from the extra automorphism of $S'$. Hence we are reduced to the case of the previous lemma and we get that $\pi$ is unramified along $\tilde{\Delta}_2$ and has ramification index 2 along $R_2$. In the case $p = 2$ one checks that there are no non-isotropic subspaces of $S[2]$ fixed by the extra automorphism hence $\pi$ is ramified with index 2 along all of $\pi^{-1}(\Delta_2)$.

COROLLARY 2.2. Let $\pi\colon \mathscr{A}_2(p) \to \mathscr{A}_2$ then

$$\deg \pi|_{R_2} = \tfrac{1}{2}(p^4 + p^2) - 1, \qquad \pi^*(\Delta_2) = \tilde{\Delta}_2 + 2R_2 \quad (p > 2).$$

Since $\pi$ is induced from the inclusion $\Gamma_p < \mathrm{Sp}(4, \mathbb{Z})$ the rational polyhedral decompositions defining the toroidal compactification $\mathscr{A}_2 \subset \bar{\mathscr{A}}_2$ also define a compactification $\mathscr{A}_2(p) \subset \bar{\mathscr{A}}_2(p)$ such that $\pi$ extends to a finite surjective map $\pi\colon \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$. We will prove that $\mathscr{A}_2(p)$ is of general type for $p \geqslant 17$ by studying $n$-canonical forms on the compactification $\bar{\mathscr{A}}_2(p)$.

The set of codimension 1 boundary components of $\mathbb{H}_2$ is in one-to-one correspondence with the set of one-dimensional subspaces of $L \otimes \mathbb{Q}$. If $\mathbb{Q}v = [v]$ is such a subspace we can think of $v$ as the vanishing cocycle. Any two such subspaces $[v]$ and $[w]$ are $\mathrm{Sp}(4, \mathbb{Z})$-equivalent (this is equivalent to $\Delta_0$ being irreducible). The smaller group $\Gamma_p$ does not act transitively on $\mathbb{P}(L \otimes \mathbb{Q})$, in fact we have

*Claim.* There are three equivalence classes for the action of $\Gamma_p$ on $\mathbb{P}(L)$:

(a) $\{[v] \in \mathbb{P}(L \otimes \mathbb{Q}) | v_p \in \Sigma_p^{\perp}\}$,

(b) $\{[v] \in \mathbb{P}(L \otimes \mathbb{Q}) | v_p \in \Sigma_p\}$,

(c) $\{[v] \in \mathbb{P}(L \otimes \mathbb{Q}) | v_p \notin \Sigma_p^{\perp}, v_p \notin \Sigma_p\}$.

where $v_p = v \otimes \bar{1}$ is the reduction of $v$ modulo $p$.

It is clear that (a), (b), (c) are not equivalent under $\Gamma_p$. It is also easy to check that $\Gamma_p$ acts transitively on each of the sets (a), (b), (c).

DEFINITION 2.7. ($\alpha$) Let $\tilde{\Delta}_0 \subset \bar{\mathscr{A}}_2(p)$ be the divisor corresponding to the eqivalence class (a).
($\beta$) Let $\tilde{\tilde{\Delta}}_0 \subset \bar{\mathscr{A}}_2(p)$ be the divisor corresponding to the equivalence class (b).
($\gamma$) Let $R_0 \subset \bar{\mathscr{A}}_2(p)$ be the divisor corresponding to the equivalence class (c).

REMARK. The involution on $\mathscr{A}_2(p)$ (p. 128) extends to an involution $\imath: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2(p)$ commuting with $\pi$. It is clear that $\imath(\tilde{\Delta}_0) = \tilde{\tilde{\Delta}}_0$ $\imath(R_0) = R_0$.

LEMMA 2.4. *The map* $\pi: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$ *is unramified along* $\tilde{\Delta}_0$ *and* $\tilde{\tilde{\Delta}}_0$, *it has ramification index p along* $R_0$.

Before proving the Lemma we give a description of the fibers of $\pi$ over $\Delta_0$. Let $C$ be a stable genus two curve with one (at least) non-disconnecting node. Let $U$ be the universal deformation space of $C$, let $\Delta_0(U) \subset U$ be the divisor parametrizing curves with one (at least) non-disconnecting node, it's a divisor with normal crossings. Let $\varphi: V \to U$ be the cover unbranched outside $\Delta_0(U)$ and with ramification of order $p$ over each component of $\Delta_0(U)$. Let $\mathscr{C}$ be the pull back to $V$ of the universal curve over $U$ and let $C'$ be a fixed smooth reference curve (in the family $\mathscr{C}$) with no extra automorphisms. The Picard-Lefschetz transformation(s) acts trivially on $p$-torsion points of Jac($C'$). The fiber $\pi^{-1}(m(C'))$ is in one-to-one correspondence with the set of isomorphism classes of couples $(C', H)$. We can associate a point of $\pi^{-1}(m(C))$ to every couple $(C, H)$ where $C$ is our singular curve and $H \subset \mathrm{Jac}(C')[p]$ is a non-isotropic subspace of the fixed smooth curve. If we choose another reference fiber $C''$ there is a well defined isomorphism between the subspaces of $\mathrm{Jac}(C')[p]$ and the subspaces of $\mathrm{Jac}(C'')[p]$ because monodromy acts trivially on $p$-torsion points of Jac($C'$). Let $m: U \to \mathfrak{M}_2$ be the moduli map, let $V^0 \subset V$ be the open set on which $m: V \to \mathfrak{M}_2$ is unramified (notice that $C'$ maps to a point of $V^0$). Let $G$ be the group of deck transformations of $m\varphi: V^0 \to m\varphi(V^0)$ and let $M$ be the group of deck transformations of $\varphi: V^0 \to \varphi(V^0)$ we have an exact sequence $1 \to M \to G \to \mathrm{Aut}'(C) \to 1$. The group $G$ acts on the set of non-isotropic subspaces of $\mathrm{Jac}(C')[p]$ because, as we have remarked, there is a well defined isomorphism between $p$-torsion points of any two smooth fibers of $\mathscr{C}$.

DEFINITION 2.8. Let $H \subset \mathrm{Jac}(C')$ be non-isotropic, we define $\mathrm{Aut}'(C, H) < G$ to be the subgroup fixing $H$.

Let $m_p: V \to \bar{\mathscr{A}}_2(p)$ be the moduli map, then

$$m_p(V) \cong \mathrm{Aut}'(C, H) \backslash V.$$

In practice in order to construct $m_p(V)$ we start with a smooth fiber $C'$ in the universal family over $U$ and we choose a non-isotropic $H \subset \mathrm{Jac}(C')[p]$. Then we let $V'$ be the cover of $U$ ramified with index $(p-1)$ only over the components of $\Delta_0(V)$ corresponding to Picard-Lefschetz transformations which do not fix $H$, and we proceed as before.

*Proof of Lemma* 2.4. Let $m(C) \in \Delta_0$ be generic, i.e. $\mathrm{Aut}'(C)$ is trivial. Let $U$ be the universal deformation space of $C$, $\Delta_0(V) \subset V$ is smooth. Let $C'$ be a fixed reference smooth curve in the universal family $\mathscr{C}$ and let $v_p \in \mathrm{Jac}(C')[p]$ be the vanishing cycle:

(i) A point in $\tilde{\Delta}_0 \cap \pi^{-1}(m(C))$ corresponds to $H \subset \mathrm{Jac}(C')[p]$ such that $H \perp v_p$, hence $H$ is fixed by the Picard-Lefschetz transformation. Using the notation we just introduced we have that $\mathrm{Aut}'(C) \cong \{1\}$, $M \cong \mathbb{Z}/(p)$ hence $G \cong \mathbb{Z}/(p)$. Since $H$ is fixed by monodromy $\mathrm{Aut}'(C,H) \cong G \cong \mathbb{Z}/(p)$. Therefore $m_p(V) = U$, $m(U) = U$ and $\pi: m_p(V) \to m(U)$ is just the identity, so $\pi$ is indeed unramified near $m(C)$).

(ii) A point in $\tilde{\tilde{\Delta}}_0 \cap \pi^{-1}(m(C))$ corresponds to $H \subset \mathrm{Jac}(C')[p]$ such that $H \ni v_p$, so again $H$ is fixed by the Picard–Lefschetz transformation. The same argument as in case (i) shows that $\pi$ is unramified along $\tilde{\tilde{\Delta}}_0$.

(iii) A point in $R_0 \cap \pi^{-1}(m(C))$ corresponds to $H \subset \mathrm{Jac}(C')[p]$ which is not orthogonal to $v_p$ and does not contain $v_p$, hence it is not fixed by the Picard–Lefschetz transformation. Therefore $\mathrm{Aut}'(C,H) \cong \{1\}$ so $m_p(V) \cong V$; since $V$ is a $p$-sheeted cover of $V$ branched over $\Delta_0(U)$ and $m(V) \cong V$ we see that $\pi$ has ramification of order $p$ along $R_0$.

COROLLARY 2.3. *Let* $\pi: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$, *then*

$$\pi^*(\Delta_0) = \tilde{\Delta}_0 + \tilde{\tilde{\Delta}}_0 + pR_0.$$

PROPOSITION 2.4. *Let* $\pi: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$, *then*

(i) $\deg \pi|_{\tilde{\Delta}_0} = \deg \pi|_{\tilde{\tilde{\Delta}}_0} = p^2$

(ii) $\deg \pi|_{R_0} = p^3 - p$.

*Proof.* Let the notation be as before, so $m(C) \in \Delta_0$ is a generic point. The fiber $\pi^{-1}(m(C)) \cap \tilde{\Delta}_0$ is in one-to-one correspondence with the set of non-isotropic subspaces $H \subset \mathrm{Jac}(C')[p]$ orthogonal to the vanishing cycle $v_p$. So we have to count the number of projective lines in $\mathbb{P}(v_p^{\perp})$ which are non-isotropic for the Weil pairing. Obviously, such a line cannot contain $[v_p]$, and since the pairing is non-degenerate this condition is sufficient for a line to be non-isotropic. Hence $\deg \pi|_{\tilde{\Delta}_0} = \#\{\text{lines in } \mathbb{P}^2(\mathbb{F}_p) \text{ not containing a fixed point}\} = p^2$. A similar count gives $\deg \pi|_{\tilde{\tilde{\Delta}}_0} = p^2$; notice that the involution $\iota$ on $\bar{\mathscr{A}}_2(p)$ commuting with $\pi$ interchanges $\tilde{\Delta}_0$ and $\tilde{\tilde{\Delta}}_0$, therefore we must have $\deg \pi|_{\tilde{\Delta}_0} = \deg \pi|_{\tilde{\tilde{\Delta}}_0}$. The degree of $\pi$ restricted to $R_0$ is readily obtained from $\deg \pi = p^4 + p^2$ and $\pi^*(\Delta_0) = \tilde{\Delta}_0 + \tilde{\tilde{\Delta}}_0 + pR_0$.

$$K_{\bar{\mathscr{A}}_2(p)} \cong \pi^*(K_{\bar{\mathscr{A}}_2}) + R_2 + R_1 + (p-1)R_0, \qquad (*)$$
$$K_{\bar{\mathscr{A}}_2(2)} \cong \pi^*(K_{\bar{\mathscr{A}}_2}) + R_2 + R_0. \qquad (**)$$

*Proof.* We apply Hurwitz's formula to the finite surjective morphism $\pi: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$. Taking into account Lemmas 2.2, 2.3, 2.4 we get formulas $(*)$ and $(**)$.

THEOREM 2.1. *Let* $\pi: \bar{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$. *If* $p > 2$ *then*

$$\pi_*(K_{\bar{\mathscr{A}}_2(p)}) = (\tfrac{3}{10}p^4 - p^3 - \tfrac{17}{10}p^2 + p - 3)\Delta_0 + (\tfrac{3}{10}p^4 + \tfrac{3}{10}p^2 - 7)\Delta_1$$

*when* $p = 2$ *we get*

$$\pi_*(K_{\bar{\mathscr{A}}_2(2)}) = -8\Delta_0 - 4\Delta_1.$$

*Proof.* From Proposition 2.5 we get that for $p > 2$

$$\pi_*(K_{\bar{\mathscr{A}}_2(p)}) = (p^4 + p^2)K_{\bar{\mathscr{A}}_2} + [\tfrac{1}{2}(p^4 + p^2) - 1](\Delta_2 + \Delta_1) + (p-1)(p^3 - p)\Delta_0.$$

Applying Corollary 1.3 and Lemma 1.3 we get the first formula. When $p = 2$ we get

$$\pi_*(K_{\bar{\mathscr{A}}_2(2)}) = 20K_{\bar{\mathscr{A}}_2} + 10\Delta_2 + 6\Delta_0$$

which together with Corollary 1.3 and Lemma 1.3 gives the second formula.

The formula for $\pi_*(K_{\bar{\mathscr{A}}_2(2)})$ agrees with the fact that $\bar{\mathscr{A}}_2(2)$ is rational. When $p = 3$ we get

$$\pi_*(K_{\bar{\mathscr{A}}_2(3)}) = -18\Delta_0 + 30\Delta_1$$

Let $C$ be an irreducible genus two curve: it can be realized as the double cover of $\mathbb{P}^1$ branched over six points (some of which might be multiple). By considering pencils of sixtuples in $\mathbb{P}^1$ we can construct a curve $T \subset \mathfrak{M}_2$ such that $m(C) \in \Gamma, \Gamma \cap \Delta_1 = 0$, $\Delta_0 \cdot \Gamma > 0$, hence through a generic point of $\mathfrak{M}_2$ there passes a curve $\Gamma$ such that $\Gamma \cdot \pi_*(K_{\bar{\mathscr{A}}_2(3)}) < 0$. It follows that the linear system $|nK_{\bar{\mathscr{A}}_2(3)}|$ is empty for all $n > 0$, i.e. the Kodaira dimension of $\bar{\mathscr{A}}_2(3)$ is $-\infty$. If $p \geqslant 5$ then $\pi_*(K_{\bar{\mathscr{A}}_2(p)})$ is a linear combination with positive coefficients of $\Delta_0$ and $\Delta_1$, therefore $h^0(n\pi_* K_{\bar{\mathscr{A}}_2(p)}) = cn^3 + O(n^2)$ for a positive $c$ ($n$ divisible enough). This suggests that $\bar{\mathscr{A}}_2(p)$ might be of general type for $p$ big, but it is not sufficient to prove it; in fact we will need to further study $\bar{\mathscr{A}}_2(p)$ to prove that it is of general type for $p \geqslant 17$.

THEOREM 2.2. *If* $p \geqslant 3$ *then*

$$K_{\bar{\mathscr{A}}_2(p)} = \pi^*\left(\left(\frac{3}{10} - \frac{1}{p}\right)\Delta_0 + \frac{3}{10}\Delta_1\right) - \tfrac{1}{2}\tilde{\Delta}_2 - \tfrac{1}{2}\tilde{\Delta}_1 - \frac{p-1}{p}\tilde{\Delta}_0 - \frac{p-1}{p}\tilde{\tilde{\Delta}}_0.$$

*Proof.* The formula is obtained from (∗) of Proposition 2.5 together with Corollary 1.3, Lemma 1.3 and the defnitions of $\tilde{\Delta}_2, \tilde{\Delta}_1, \tilde{\Delta}_0, \tilde{\tilde{\Delta}}_2$.

## Section 3. A partial desingularization of $\bar{\mathscr{A}}_2(p)$

We recall that the Kodaira dimension of a variety $X$, denoted by $\kappa(X)$, is defined as follows: let $\bar{X} \supset X$ be a compactification of $X$ and let $\tilde{X}$ be a desingularization of $\bar{X}$, then $\kappa(X) = \text{tr. deg.}(R) - 1$ where $R$ is the canonical ring $R = \oplus_{n=0}^{\infty} H^0(nK_{\tilde{X}})$. One always has that $\kappa(X) \leqslant \dim X$ (possibly $_{\kappa}(X) = -\infty$) and if $\kappa(X) = \dim X$ then $X$ is said to be of general type. The Kodaira dimenson is a birational invariant; since $\kappa(\mathbb{P}^d) = -\infty$ if $\kappa(X) \geqslant 0$ then $X$ is not rational; furthermore one sees that if $\kappa(X) \geqslant 0$ then $X$ cannot be unirational. From now on we assume $p \geqslant 5$.

We recall that a germ $(X, P)$ of a normal algebraic singularity is said to have a canonical singularity at $P$ if

(i) there exists an integer $r > 0$ such that $rK_X$ is Cartier

(ii) for a resolution $\phi: \tilde{X} \to X$ (equivalently for any resolution) with exceptional set $E = \oplus_i E_i, rK_{\tilde{X}} = \phi^*(rK_X) + \Sigma_i a_i E_i$ with $a_i \geqslant 0$ for all $i$.

If $X$ has only canonical singularities and $\tilde{X}$ is a resolution of $X$ $H^0(nK_{\tilde{X}}) \cong H^0(nK_X)$ hence we need not pass to $\tilde{X}$ in order to determine $\kappa(X)$.

We will apply (when possible) the following

Shepherd-Barron, Reid, Tai criterion [H–M]: Let a finite group $G$ act linearly on a complex vector space $V$.

Let $g \in G$ be conjugate to

$$\begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_d} \end{pmatrix}$$

where $\zeta$ is a primitive $m$th root of unity and $0 \leqslant a_i < m$. If $\Sigma_{i=1}^{d} a_i \geqslant m$ for all $g$ and $\zeta$ then $(G \backslash V, 0)$ is canonical at 0.

In our case the situation is the following. Let $m(C, H) \in \bar{\mathscr{A}}_2(p)$ be a singular point. The cotangent space to the deformation space of $C$ is canonically isomorphic to $H^0(\Omega_C^1 \otimes \omega_C)$. The group $\mathrm{Aut}'(C)$ acts on $H^0(\Omega_C^1 \otimes \omega_C)$ and a neighborhood of $m(C) \in \bar{\mathfrak{M}}_2$ is isomorphic to $\mathrm{Aut}'(C)\backslash H^0(\Omega_C^1 \otimes \omega_C)$.

If $C$ is smooth or has only one disconnecting node then a neighborhood of $m(C, H) \in \bar{\mathscr{A}}_2(p)$ is isomorphic to $\mathrm{Aut}'(C, H)\backslash H^0(\Omega_C^1 \otimes \omega_C)$. Since $\mathrm{Aut}'(C, H) < \mathrm{Aut}'(C)$ we see that $m(C, H)$ can be singular only if $\mathrm{Aut}'(C)$ is non-trivial. If $C$ has one (at least) non-disconnecting node then a neighborhood of $m(C, H)$ is isomorphic to $\mathrm{Aut}'(C, H)\backslash V$, where $V$ is a cover of $U$ (the deformation space of $C$) branched over $\Delta_0(U)$. The group $\mathrm{Aut}'(C, H)$ in this case is contained in $G$, an extension of $\mathrm{Aut}'(C)$ by the monodromy group $M$. Hence if $\mathrm{Aut}'(C)$ is trivial then $\mathrm{Aut}'(C, H) < M$. It is easy to check that if $m(C) \in \Delta_0$ and $\mathrm{Aut}'(C)$ is trivial then $C$ has exactly one non-disconnecting node so $M \cong \mathbb{Z}/(p)$, $\mathrm{Aut}'(C, H) \cong \mathbb{Z}/(p)$ or $\{1\}$ and hence $m(C, H)$ is smooth. Therefore we again conclude that $m(C, H)$ can be singular only if $\mathrm{Aut}'(C)$ is nontrivial. Igusa [I] listed all smooth genus two curves with extra-automorphisms, we can easily add a list of all the remaining stable curves with extra automorphisms.

(1) $C = E \cup F$, where $E, F$ are elliptic curves and $j(E), j(F) \neq 0, 1728$, i.e. $m(C)$ is a generic point of $\Delta_1$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(2)$, where $g|_E =$ (multiplication by $-1$), $g|_F =$ identity.

(2) $C = E \cup E, j(E) \neq 0{,}1728$ so $m(C) \in \Delta_1 \cap \Delta_2$. $\mathrm{Aut}'(C) = \langle g, h \rangle \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, $g|_E = (-1)$, $g|_F = (\mathrm{id})$, $h$ interchanges the two components

(3) $C = E \cup F, j(E) = 1728, j(F) \neq 0{,}1728$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(4)$, $g|_E =$ (multiplication by $\sqrt{-1}$), $g|_F = (\mathrm{id})$.

(4) $C = E \cup F, j(E) = 0, j(F) \neq 0, 1728$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(6)$, $g|_E =$ (multiplication by $e_6$), $g|_F = (\mathrm{id})$.

(5) $C = E \cup E, j(E) = 1728$. $\mathrm{Aut}'(C)$ acts naturally on the two components of $C$ so it fits into the exact sequence $0 \to N \to \mathrm{Aut}'(C) \to \mathbb{Z}/(2) \to 0$ and $N \cong \mathbb{Z}/(4) \oplus \mathbb{Z}/(2)$.

(6) $C = E \cup F, j(E) = 0, j(F) = 1728$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(12)$, $g|_E =$ (multiplication by $e_6$), $g|_F =$ (multiplication by $\sqrt{-1}$).

(7) $C = E \cup E, j(E) = 0$. $\mathrm{Aut}'(C)$ acts naturally on the two components of $C$, it fits into the exact sequence $0 \to N \to \mathrm{Aut}'(C) \to \mathbb{Z}/(2) \to 0$ where $N \cong \mathbb{Z}/(6) \oplus \mathbb{Z}/(3)$.

(8) $C = E/P \sim Q$, where $P - Q$ is a 2-torsion point (unless $j(E) = 1728$ and $P, Q$ are chosen so that they give case (9) below). $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(2)$; if we let $P$ be the origin then $g$ is induced from multiplication by $-1$. We have $m(C) \in \Delta_2 \cap \Delta_0$.

(9) $C = E/P \sim Q$, where $E$ is given by $y^2 = x^4 + 1$, so $j(E) = 1728$, and $P = (0, 1), Q = (0, -1)$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(4)$, $g$ is induced from $\tilde{g}^*(x, y) = (\sqrt{-1}x, y)$.

(10) $C = E/P \sim Q$, where $E$ is given by $y^2 = x^3 + 1$, so $j(E) = 0$, and $P = (0, 1), Q = (0, -1)$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(3)$, $g$ is induced from $\tilde{g}^*(x, y) = (e_3 x, y)$.

(11) $C = \mathbb{P}^1/(Q_1 \sim Q_2, Q_3 \sim Q_4)$, i.e. $C$ has two non-disconnecting nodes, and the cross ratio $(Q_1, Q_2, Q_3, Q_4)$ is not equal to $2, \frac{1}{2}$ or $-1$. $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(2)$ where $g$ is induced from a projectivity of $\mathbb{P}^1$ interchanging the couples $\{Q_1, Q_2\}$ and $\{Q_3, Q_4\}$. We have $m(C) \in \Delta_2 \cap \Delta_0$.

(12) Same as (11) but we choose $\{Q_1, Q_2, Q_3, Q_4\}$ to have cross ratio 2 (equivalently $\frac{1}{2}$ or $-1$), e.g. $\{1, -1, 0, \infty\}$. $\mathrm{Aut}'(C) = \langle g, h \rangle \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, $g$ and $h$ are induced from $\tilde{g}, \tilde{h}\colon \mathbb{P}^1 \to \mathbb{P}^1$, $\tilde{g}^*(x) = x - 1/x + 1$, $\tilde{h}^*(x) = 1/x$.

(13) $C$ has three non-disconnecting nodes, i.e. $C = \mathbb{P}^1 \cup \mathbb{P}^1$ where we join the two copies of $\mathbb{P}^1$ at three points. $\mathrm{Aut}'(C)$ is isomorphic to the group of permutations of the nodes, i.e. $\mathrm{Aut}'(C) \cong S_3$.

REMARK. Notice that it might happen that $m(C) \in \mathfrak{M}_2$ is smooth but $m(C, H) \in \mathfrak{M}_2(p)$ is singular. In fact there is only one singular point in $\mathfrak{M}_2[\mathrm{I}]$ but there are many singular points in $\mathfrak{M}_2(p)$ (i.e. $\pi^{-1}(\mathfrak{M}_2)$). For example let $C$ be given by

$$y^2 = (x - a)(x - e_3 a)(x - e_3^2 a)(x - a^{-1})(x - e_3 a^{-1})(x - e_3^2 a^{-1})$$

(case (2) in Igusa's list); one can choose $H \subset \mathrm{Jac}(C)[p]$ such that $\mathrm{Aut}'(C, H) = \langle g \rangle \cong \mathbb{Z}/(3)$, where $g^*(x, y) = (e_3 x, y)$, and the action of $g$ on $H^0(\Omega_C^1 \otimes \omega_C)$ is given by

$$g^*\left(\frac{(dx)^2}{y^2}, \frac{x(dx)^2}{y^2}, \frac{x^2(dx)^2}{y^2}\right) = \left(\frac{(dx)^2}{y^2}, e_3\frac{x(dx)^2}{y^2}, e_3^2\frac{x^2(dx)^2}{y^2}\right)$$

hence $m(C, H)$ is singular.

DEFINITION 3.1. Let $\Gamma \subset \mathfrak{M}_2$ be the locus of moduli of curves $C = E \cup F$ with $j(E) = 0$ and $F$ any elliptic curve (so $\Gamma$ is a rational curve in $\mathfrak{M}_2$). Let $\Gamma', \Gamma'' \subset \bar{\mathscr{A}}_2(p)$ be the moduli of couples $(C, H)$ where $C = E \cup F$ is as above and $H = E[p]$, respectively $H = F[p]$. Obviously $\pi(\Gamma') = \pi(\Gamma'') = \Gamma$, and $\Gamma' \cap \Gamma'' = \{m(E \cup E, E[p])\}$.

DEFINITION 3.2. Let $\Delta_{00} \subset \mathfrak{M}_2$ be the locus of moduli of curves with two (at least) non-disconnecting nodes (so $\Delta_{00}$ is a rational curve). Let $\Omega \subset \bar{\mathscr{A}}_2(p)$ be the curve $\Omega = \pi_p^{-1}(\Delta_{00}) \cap \tilde{\Delta}_2$.

REMARK. The locus $\Omega$ is not empty because as we have already noticed (page 000) $\Delta_{00} \subset \Delta_2$. Since $\pi\colon \tilde{\Delta}_2 \to \Delta_2$ is two-to-one, either $\pi\colon \Omega \to \Delta_{00}$ is two-to-one or one-to-one. If $C = \mathbb{P}^1/(Q_1 \sim Q_2, Q_3 \sim Q_4)$ is generic, i.e. the cross ratio $(Q_1, Q_2, Q_3, Q_4)$ is not $2, \frac{1}{2}$ or $-1$, then $\mathrm{Aut}'(C) \cong \mathbb{Z}/(2)$. Hence the two subspaces $H$ fixed by $\mathrm{Aut}'(C)$ give distinct points in the fiber $\pi^{-1}(m(C)) \cap \tilde{\Delta}_2$, i.e. $\pi\colon \Omega \to \Delta_{00}$

is two-to-one. The map $\pi: \Omega \to \Delta_{00}$ has two branch points, namely the moduli points of $C = \mathbb{P}^1/(1 \sim -1, 0 \sim \infty)$ and $C = E \cup E$ where $j(E) = \infty$, hence $\Omega$ is a rational curve.

PROPOSITION 3.1. *The locus of non canonical singularities of* $\bar{\mathscr{A}}_2(p)$ *is equal to* $\Omega \cup \Gamma' \cup \Gamma''$.

*Proof.* The proposition follows from an application of Shepherd-Barron, Reid, Tai's criterion to singular points of $\bar{\mathscr{A}}_2(p)\backslash(\Omega \cup \Gamma' \cup \Gamma'')$, provided we take into account the following observation. Let $C$ be a curve with extra automorphisms such that $\mathrm{Aut}'(C, H)$ contains an element $g$ acting as a reflection on $H^0(\Omega^1_C \otimes \omega_C)$ (or on $V$ if $m(C) \in \Delta_0$), then $g$ does not satisfy the conditions in Shepherd-Barron, Reid, Tai's criterion. Such a $g$ exists whenever $m(C) \in \Delta_1$ or $m(C) \in \Delta_2$ or $m(C) \in \Delta_0$ and $H$ is fixed by some monodromy.

The subgroup $B$ of $\mathrm{Aut}'(C, H)$ generated by these bad $g$'s is normal in $\mathrm{Aut}'(C, H)$. Furthermore $B\backslash U$ (or $B\backslash V$ if $m(C) \in \Delta_0$) is smooth and Shepherd-Barron, Reid, Tai's criterion does indeed apply to the action of $\mathrm{Aut}'(C, H)/B$ on $B\backslash U$ ( respectively $B\backslash V$).

We work out one example, i.e. case (10) above. A basis of $H^0(\Omega^1_C \otimes \omega_C)$ is given by $\alpha = (\mathrm{d}x)^2/y^2$, $\beta = (\mathrm{d}x)^2/xy^2$ and the torsion element $\gamma = (u/s)(\mathrm{d}s)^2$ where $u, s$ are local parameters on the two branches of the node. The action of $g$ is given by $g^*(\alpha, \beta, \gamma) = (e_3\alpha, e_3^2\beta, e_3\gamma)$. Let $U$ be the universal deformation space of $C$ and let $V$ be the p-sheeted cover totally ramified over $\Delta_0(U)$. Let $C'$ be a smooth reference fiber (with no extra automorphisms) of the pull back to $V$ of the universal family over $U$. Let $\gamma \in \mathrm{Jac}(C')[p]$ be the vanishing cycle so that $\gamma^\perp/\gamma$ is identified with $E[p]$. Let $\tilde{g} \in G$ map to $g \in \mathrm{Aut}'(C)$ in the exact sequence $1 \to M \to G \to \mathrm{Aut}'(C) \to 1$; we can decompose $\mathrm{Jac}(C')[p]$ as $\mathrm{Jac}(C')[p] = F_p\gamma \oplus W \oplus \mathbb{F}_p\lambda$ so that $\mathbb{F}_p\gamma \oplus W = \gamma^\perp$, $\tilde{g}$ fixes $W$ and acts on it as on $E[p]$, and $\tilde{g}(\lambda) = \lambda$. One can check that $W$ and $\mathbb{F}_p\gamma \oplus \mathbb{F}_p\lambda$ are the only non-isotropic $\tilde{g}$-invariant subspaces. Hence we can distinguish three possiblities for $H \subset \mathrm{Jac}(C')[p]$:

(i) $H = W$ or $H = \mathbb{F}_p\gamma \oplus \mathbb{F}_p\lambda$, hence it is fixed both by $\tilde{g}$ and the monodromy group $M$, therefore $\mathrm{Aut}'(C, H) = G$ and a neighborhood of $m(C, H) \in \bar{\mathscr{A}}_2(p)$ is isomorphic to a neighborhood of $m(C) \in \mathfrak{M}_2$. We apply S-B., R., T.'s criterion to the action of $\langle g \rangle$ on $H^0(\Omega^1_C \otimes \omega_C)$.

(ii) $H$ is fixed by $M$ but not by $\tilde{g}$. A generator of $M$ acts as a reflection on $V$ hence it does not satisfy S-B., R., T.'s criterion but $V/M \cong U$ hence $m(C, H)$ is a smooth point.

(iii) $H$ is not fixed by $M$ and also is not fixed by $\tilde{g}$. In this case a neighborhood of $m(C, H)$ is isomorphic to $V$ so $m(C, H)$ is again a smooth point.

We now proceed to partially desingularize $\bar{\mathscr{A}}_2(p)$ along $\Omega \cup \Gamma' \cup \Gamma''$. Eventually all the singular points of the resulting partial desingularization $\tilde{\mathscr{A}}_2(p)$ will be canonical.

We will follow A. Fujiki's [F] method for resolving cyclic quotient singularities.

DEFINITION 3.3. Let $R'$ (respectively $R''$) be the moduli point of the couple $(E \cup F, E[p])$ (respectively $(E \cup F, F[p])$), where $j(E) = 0$, $j(F) = 1728$.

Notice that $R' \in \Gamma'$, $R'' \in \Gamma''$. Our first step is to partially desingularize $R'$ and $R''$.

Let $C = E \cup F$, then $\mathrm{Aut}(C, E[p]) \cong \mathrm{Aut}(C, F[p]) \cong \mathrm{Aut}(C)$ hence neighborhoods of $R', R''$ are isomorphic (in fact the involution $\iota\colon \mathscr{A}_2(p) \to \mathscr{A}_2(p)$ interchanges $R'$ and $R''$). Now $\mathrm{Aut}(C) = \langle g, h \rangle$, where $g|_E =$ (multiplication by $e_6$) $g|_F =$ (identity), $h|_E =$ (identity), $h|_F =$ (multiplication by $\sqrt{-1}$). The hyperelliptic involution is given by $g^3h^2$ hence $\mathrm{Aut}'(C) = \langle gh \rangle \cong \mathbb{Z}/(12)$ and $R', R''$ are cyclic quotient singularities.

*Partial desingularization of $R'$, $R''$*

Let $\omega_E, \omega_F$ be non-zero holomorphic differentials on $E, F$ respectively, and let $x$, $y$ be local parameters at the two branches of the node of $C$. A basis of $H^0(\Omega_C^1 \otimes \omega_C)$ is given by $\alpha = (\omega_F)^{\otimes 2}$, $\beta = (x/y)(\mathrm{d}y)^{\otimes 2}$, $\gamma = (\omega_E)^{\otimes 2}$. The action of $gh$ is given by $(gh)^*(\alpha, \beta, \gamma) = (e_{12}^6\alpha, e_{12}^5\beta, e_{12}^4\gamma)$. The first step in analyzing the partial desingularization of the quotient singularity is to take the quotient of $U(\alpha, \beta, \gamma)$ by the group of reflections, i.e. $\langle g^6h^6 \rangle$. We have $(g^6h^6)^*(\alpha, \beta, \gamma) = (\alpha, -\beta, \gamma)$ hence $U(\alpha, \beta, \gamma)/\langle g^6h^6 \rangle = U(x, y, z)$ where $(x, y, z) = (\alpha, \beta^2, \gamma)$.

The action of $\langle gh \rangle$ on $U(x, y, z)$ is given by $(gh)*(x, y, z) = (e_6^3x, e_6^5y, e_6^2z)$. Notice that the action of $g^2h^2$ (and of $g^4h^4$) does not satisfy the conditions in S-B, R., T,'s criterion. Let $f_1\colon U(x_1, y_1, z_1) \to U(x, y, z)$ be the covering defined by $f_1^*(x, y, z) = (x_1^3, y_1^5, z_1^2)$. Let $H$ be the covering group of $f_1$, then $U(x, y, z)/\langle gh \rangle \cong U(x_1, y_1, z_1)/\langle gh, H \rangle$. The group $H$ is generated by $h_1, h_2, h_3$ where $h_1^*(x_1, y_1, z_1) = (e_3x_1, y_1, z_1)$, $h_2^*(x_1, y_1, z_1) = (x_2, e_5y_1, z_1)$, $h_3^*(x_1, y_1, z_1) = (x_1, y_1, -z_1)$. The action of $gh$ on $U(x_1, y_1, z_1)$ is given by $(gh)*(x_1, y_1, z_1) = (e_6x_1, e_6y_1, e_6z_1)$. Let $f_2\colon W \to U(x_1, y_1, z_1)$ be the blow up of the origin; the action of $\langle gh, H \rangle$ on $W - f_2^{-1}(0)$ extends to an action on all of $W$. The natural map $q\colon W/\langle gh, H \rangle \to U(\alpha, \beta, \gamma)/\langle gh \rangle$ is the partial desingularization of the origin, i.e. of $R'$ (or $R''$). We now examine the singularities of $W/\langle gh, H \rangle$. We consider $W$ as the union of the three standard affine pieces and we examine the action of $\langle gh, \mathrm{H} \rangle$ on each piece.

(1) Let $W_1 \subset W$ be the affine piece with coordinates $(x_1, y_1/x_1, z_1/x_1)$. The elements $h_2, h_3, gh$ act as reflections on $w_1$; we have $W_1/\langle h_2, h_3, gh \rangle \cong U(x_1^6, y_1^5/x_1^5, z_1^2/x_1^2)$. The action of $h_1$ is given by $h_1^*(x_1^6, y_1^5/x_1^5, z_1^2/x_1^2) = (x_1^6, e_3y_1^5/x_1^5, e_3z_1^2/x_1^2)$. We see that the points in $W_1/\langle gh, H \rangle$ which do not satisfy S-B., R., T.'s conditions belong to the image of the curve $\{y_1/x_1 = z_1/x_1 = 0\}$. As is easily checked this curve is just the strict transform of $\Gamma'$ (or $\Gamma''$ if we are blowing up $R''$).

(2) Let $W_2 \subset W$ be the affine piece with coordinates $(y_1, x_1/y_1, z_1/y_1)$. Elements

of $\langle h_1, h_3, gh\rangle$ act as reflections on $W_2$; we have that $W_2/\langle h_1, h_3, gh\rangle \cong U(y_1^6, x_1^3/y_1^3, z_1^2/y_1^2)$. The action of $h_2$ is given by $h_2^*(y_1^6, x_1^3/y_1^3, z_1^2/y_1^2) = (e_5\, y_1^6, e_5^2 x_1^3/y_1^3, e_5^3 z_1^2/y_1^2)$. We see that $W_2/\langle gh, H\rangle$ contains only one singular point and it satisfies S-B., R., T.'s conditions.

(3) Let $W_3 \subset W$ be the affine piece with coordinates $(z_1, x_1/z_1, y_1/z_1)$. Elements of $\langle h_1, h_2, gh\rangle$ act as reflections; we have that $W_3/\langle h_1, h_2, gh\rangle \cong U(z_1^6, x_1^3/z_1^3, y_1^5/x_1^5)$. The action of $h_3$ is given by $h_3^*(z_1^6, x_1^3/z_1^3, y_1^5/x_1^5) = (z_1^6, -x_1^3/z_1^3, -y_1^5/z_1^5)$. We see that the singular points of $W_3/\langle gh, H\rangle$ belong to the image of the curve $\{x_1/z_1 = y_1/z_1 = 0\}$. They satisfy S-B., R., T.'s criterion, in fact each such point is locally isomorphic to $\mathbb{A}^1 \times \{xy - z^2 = 0\}$. As is easily checked, this curve is just the strict transform of the curve $\{m(E \cup F, H) \mid j(E) = 1728$ and $H = E[p]$ or $H = F[p]$ depending on whether we are blowing up $R'$ or $R''\}$.

DEFINITION 3.4. Let $\varphi_1: X_1 \to \mathscr{A}_2(p)$ be the partial desingularization of $R'$ and $R''$ just defined. Let $D', D'' \subset X_1$ be the exceptional divisors lying over $R', R'$ respectively; let $\tilde{\Gamma}, \tilde{\tilde{\Gamma}}$ be the strict transforms of $\Gamma', \Gamma''$ respectively.

*Partial desingularization of* $\tilde{\Gamma}$.

The curve $\tilde{\Gamma}$ meets $D'$ in one point and doesn't intersect $D''$. As we have already remarked (p. 137) a neighborhood of $D' \cap \tilde{\Gamma}$ is isomorphic to $U(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (x, e_3 y, e_3 z)$.

*Claim.* Let $Q \in \tilde{\Gamma}$ and $Q \notin \tilde{\tilde{\Gamma}}$, i.e. $\varphi_1(Q) \neq m(E \cup E, E[p])$. A neighborhood of $Q$ is isomorphic to $U(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (x, e_3 y, e_3 z)$, and $\tilde{\Gamma} \cap U(x, y, z)/\langle g\rangle$ is exactly the singular locus.

*Proof of Claim.* Since we already know that the result holds for $Q = \tilde{\Gamma} \cap D'$ and since $X_1 \backslash (D' \cup D'')$ is isomorphic to $\mathscr{A}_2(p)\backslash(\{R', R''\})$ we just have to examine the neighborhood of a point $Q \in \Gamma' \backslash (\Gamma'' \cup \{R'\})$. Hence $Q = m(E \cup F, E[p])$ where $j(E) = 0, j(F) \neq 0, j(F) \neq 1728$. We have that $\mathrm{Aut}'(E \cup F, E[p]) = \mathrm{Aut}'(E \cup F) = \langle g\rangle \cong \mathbb{Z}/(6)$, where $g|_E =$ (multiplication by $e_6$), $g|_F =$ (identity). The action of $g$ on $\alpha = (\omega_E)^{\otimes 2}$, $\beta = (x/y)/(\mathrm{d}y)^{\otimes 2}$, $\gamma = (\omega_F)^{\otimes 2}$ is given by $g^*(\alpha, \beta, \gamma) = (e_6^2\alpha, e_6\beta, \gamma)$. A neighborhood of $Q$ is isomorphic to $U(\alpha, \beta, \gamma)/\langle g\rangle$. We first take the quotient for the action of the reflection $g^3$: $U(\alpha, \beta, \gamma)/\langle g^3\rangle = U(\alpha, \beta^2, \gamma)$. The action of $g$ on $U(\alpha, \beta^2, \gamma)$ is given by $g^*(\alpha, \beta^2, \gamma) = (e_3\alpha, e_3\beta, \gamma)$ hence the first assertion in the claim is proved. Let $C = E \cup F$ and let $v \in H^1(T_C)$ be the Kodaira-Spencer class associated to a one parameter family $C_t = E \cup F_t$, then $\alpha \cup v = \beta \cup v = 0$. Hence $\tilde{\Gamma} \cap V(\alpha, \beta^2, \gamma)/\langle g\rangle$ is the image of $\{\alpha = \beta = 0\}$, i.e. exactly the singular locus.

DEFINITION 3.5. Let $\varphi_2: X_2 \to X_1$ be the partial desingularization obtained by applying the first step in Fujiki's method for resolving cyclic quotient singularities to the singularities of $\tilde{\Gamma} \subset X_1$. Let $E_1 \subset X_2$ be the exceptional divisor of $\varphi_2$.

The claim shows that $E_1$ is smooth outside the fiber over $\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$, because over $\tilde{\Gamma} \backslash \tilde{\tilde{\Gamma}}$ $\varphi_2$ is the blow up of $\tilde{\Gamma}$ and a single blow up will resolve the singularity of $U(x, y, z)/\langle g \rangle$. Hence we proceed to examine the fiber of $E_1$ over $\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$. Since $X_1 \backslash (D' \cup D'')$ is isomorphic to $\bar{\mathscr{A}}_2(p) \backslash \{R', R''\}$ a neighborhood of $\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$ is isomorphic to a neighborhood of $\Gamma' \cap \Gamma'' = m(F_1 \cup F_2, F_1[p])$, where $j(F_1) = j(F_2) = 0$. We have that $\mathrm{Aut}(F_1 \cup F_2, F_1[p]) = \langle \varphi, \theta \rangle \cong \mathbb{Z}/(6) \oplus \mathbb{Z}/(6)$, where $\varphi|_{F_1}$ (multiplication by $e_6$), $\varphi|_{F_2} =$ (identity), $\theta|_{F_1} =$ (identity) $\theta|_{F_2} =$ (multiplication by $e_6$). The hyperelliptic involution is given by $\varphi^3\theta^3$. Let $\lambda = \varphi\theta$. Then $\mathrm{Aut}'(F_1 \cup F_2, F_1[p]) \cong \langle \lambda, \theta \rangle / \langle \lambda^3 \rangle \cong \mathbb{Z}/(3) \oplus \mathbb{Z}/(6)$. Let $\alpha = (\omega_{F_1})^{\otimes 2}$, $\beta = (x/y)(\mathrm{d}y)^2$, $\gamma = (\omega_{F_2})^{\otimes 2}$, the actions of $\lambda$, $\theta$ are given by $\lambda^*(\alpha, \beta, \gamma) = (e_6^2\alpha, e_6^2\beta, e_6^2\gamma)$, $\theta^*(\alpha, \beta, \gamma) = (\alpha, e_6\beta, e_6^2\gamma)$. A neighborhood of $m(F_1 \cup F_2, F_1[p])$ is isomorphic to $U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$, hence also a neighborhood of $\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$ is isomorphic to $U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$. The curve $\tilde{\Gamma} \cap U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$ is the image of the fixed points of $\langle \theta \rangle$, hence $\tilde{\Gamma} = \mathrm{image}\{(\alpha, 0, 0)\}$. We must examine the partial desingularization of $U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$ along $\tilde{\Gamma} \cap U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$, more specifically the fiber of the exceptional divisor $E_1$ over the image of $(0, 0, 0)$ in $U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$. The group of reflections of $\langle \lambda, \theta \rangle$ is generated by $\theta^3$. Let $x = \alpha$, $y = \beta^2$, $z = \gamma$, then $U(\alpha, \beta, \gamma)/\langle \theta^3 \rangle = U(x, y, z)$. The action of $\langle \lambda, \theta \rangle / \langle \theta^3 \rangle$ on $U(x, y, z)$ is given by $\lambda(x, y, z) = (e_3x, e_3^2y, e_3z)$, $\theta^*(x, y, z) = (x, e_3y, e_3z)$. Let $\psi\colon W \to U(x, y, z)$ be the blow up of $\{(x, 0, 0)\}$. The action of $\langle \lambda, \theta \rangle / \langle \theta^3 \rangle$ on $W \backslash \psi^{-1}(D)$ extends naturally to an action on all of $W$ and $W/\langle \lambda, \theta \rangle$ is isomorphic to the partial desingularization of $U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$ along $\tilde{\Gamma} \cap U(\alpha, \beta, \gamma)/\langle \lambda, \theta \rangle$. We consider $W$ as the union of two open pieces and examine the action of $\langle \lambda, \theta \rangle / \langle \theta^3 \rangle$ on each piece.

(1) Let $W_1 \subset W$ be the affine piece with coordinates $(x, y, z/y)$. We have $\lambda^*(x, y, z/y) = (e_3x, e_3^2y, e_3^2z/y)$ and $\theta^*(x, y, z/y) = (x, e_3y, z/y)$. Let $(x_1, y_1, z_1) = (x, y^3, z/y)$, then $W_1/\langle \theta \rangle \cong U(x_1, y_1, z_1)$. The action of $\lambda$ on $U(x_1, y_1, z_1)$ is given by $\lambda^*(x_1, y_1, z_1) = (e_3x_1, y_1, e_3^2z_1)$. So we see that in $U(x_1, y_1, z_1)$ there is a curve of singular points, namely the image of $\{(0, y_1, 0)\}$, all satisfying S-B.,R.,T.'s condition. In fact, this curve belongs to a sigular curve $\Lambda \subset X_1$ such that $\pi\varphi(\Lambda)$ is the locus of moduli of curves given by

$$y^2 = (x - a)(x - e_3a)(x - e_3^2a)(x - a^{-1})(x - e_3a^{-1})(x - e_3^2a^{-1}).$$

Notice also that a local equation for $E_1$ is $(y_1 = 0)$ hence $\Lambda$ intersects $E_1$ at one point (singular on $E_1$).

(2) Let $W_2 \subset W$ be the open affine piece with coordinates $(x, y/z, z)$, We have $\lambda^*(x, y/z, z) = (e_3x, e_3y/z, e_3z)$, $\theta^*(x, y/z, z) = (x, y/z, e_3z)$. Let $(x_2, y_2, z_2) = (x, y/z, z^3)$, then $W_2/\langle \theta \rangle \cong U(x_2, y_2, z_2)$ and $\lambda^*(x_2, y_2, z_2) = (e_3x_2, e_3y_2, z_2)$. Hence $W_2/\langle \theta, \lambda \rangle$ contains a curve of singular points not satisfying S-B.,R.,T.'s conditions. In fact it is just the strict transform of $\tilde{\tilde{\Gamma}}$. Notice that a local equation for $E_1$ is $(z_2 = 0)$.

DEFINITION 3.6. Let $\Gamma^* \subset X_2$ be the strict transform of $\tilde{\tilde{\Gamma}}$.

*Desingularization of* $\Gamma^*$

By the previous analysis of the fiber of $E_1$ over $\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$ we know that a neighborhood of $E_1 \cap \Gamma^*$ is isomorphic to $U(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (x, e_3 y, e_3 z)$. The analysis given in the proof of the claim on page 138 carries over to show that a neighborhood of any point in $\Gamma^* \backslash E_1$ is also isomorphic to $U(x, y, z)/\langle g\rangle$. So let $\varphi_3\colon X_3 \to X_2$ be the blow up of $\Gamma^*$, it will desingularize the whole of $\Gamma^*$. Our analysis of the singularities that are left after the partial desingularizations $\varphi_1, \varphi_2, \varphi_3$ proves the following:

PROPOSITION 3.2. *The locus of non canonical singularities of* $X_3$ *is equal to the pre-image* $\tilde{\Omega}$ *of* $\Omega$ *in* $X_3$

Now we have to deal with the singularities of $\tilde{\Omega} \subset X_3$. Notice first that $\Omega \cap (\Gamma' \cup \Gamma'') = \emptyset$, because $\pi(\Omega) = \Delta_{00}$ and $\Delta_{00} \cap \pi(\Gamma') = \Delta_{00} \cap \pi(\Gamma'') = \emptyset$. Hence a neighborhood of $\Omega \subset \bar{\mathscr{A}}_2(p)$ is isomorphic to a neighborhood of $\tilde{\Omega} \subset X_3$.

PROPOSITION 3.3. *Let* $P$ *be any point of* $\Omega \subset \bar{\mathscr{A}}_2(p)$. *Then a neighborhood of* $P$ *is isomorphic to* $U(x, y, z)/\langle g\rangle$, *where* $g^*(x, y, z) = (e_p x, e_p^2 y, z)$.

*Proof.* We prove the proposition for $P \in \Omega$ generic, i.e. $\pi(P) = m(C)$ where $C$ is not the union of two singular elliptic curves nor $\mathbb{P}^1/(1 \sim -1, 0 \sim \infty)$ nor the union of two copies of $\mathbb{P}^1$ joined at three points. A case by case analysis shows that the result holds also in these special cases. So let $C$ be a generic curve with exactly two non-disconnecting nodes. Let $U$ be the universal deformation space of $C$ and let $\Delta_0(U) \subset U$ be the divisor parametrizing curves with one (at least) non-disconnecting node. The divisor $\Delta_0(U)$ has two components meeting transversely along $\Delta_{00}(U)$, the locus parametrizing curves with two non-disconnecting modes. Let $\Delta_2(U) \subset U$ be the divisor parametrizing curves which are double covers of elliptic curves i.e. $m(\Delta_2(U)) \subset \Delta_2$. Let $(\alpha, \beta, \gamma)$ be coordinates on $U$, chosen so that $\Delta_{00}(U) = \{\alpha\beta = 0\}$ and $\Delta_2(U) = \{\alpha - \beta = 0\}$. Let $\varphi\colon V \to U$ be the cover which has ramification index $p$ over each of the two components of $\Delta_0(V)$. Let $(x_1, y_1, z_1)$ be coordinates on $V$ such that $\varphi^*(\alpha, \beta, \gamma) = (x_1^p, y_1^p, z_1)$. Let $P \in V$ belong to $(x_1 - y_1 = 0)$ and let us assume that $\varphi(P) \notin \Delta_{00}(U)$ (i.e. $x_1 \neq 0$) and that $\varphi(P)$ is a generic point of $\Delta_2(U)$, i.e. it represents a smooth curve $C'$ such that $\mathrm{Aut}'(C') \cong \mathbb{Z}/(2)$. As we have already remarked to every non-isotropic $H \subset \mathrm{Jac}(C')[p]$ there corresponds a point $m(C, H) \in \pi^{-1}(m(C))$. In order that $m(C, H)$ belong to $\tilde{\Delta}_2$ we must choose $H$ to be one of the two subspaces fixed by the extra automorphism of $C'$. Let $H_0 \subset \mathrm{Jac}(C')[p]$ be such a subspace. A neighborhood of $m(C, H_0)$ is isomorphic to $V/\mathrm{Aut}'\,(C, H_0)$, hence we need to determine $\mathrm{Aut}'\,(C, H_0)$. The group $G$ acting on $V$ is an extension $1 \to M \to G \to \mathrm{Aut}'\,(C)$. The monodromy group $M$ is generated by $m_1, m_2$ where $m_1^*(x_1, y_1, z_1) = (e_p x_1, y_1, z_1)$, $m_2^*(x_1, y_1, z_1) = (x_1, e_p y_1, z_1)$; also $\mathrm{Aut}'(G) \cong \mathbb{Z}(2)$.

*Claim.* Aut′ $(C, H_0) \cap M = \langle m_1 m_2 \rangle$.

*Proof of the claim.* Let $\gamma$ be a generator of $\pi_1(\Delta_2(U)\backslash\Delta_{00}(U), \varphi(P))$ the element $m_\gamma$ of $M$ corresponding to $\gamma$ acts as $m_\gamma^*(x_1, y_1, z_1) = (e_p x_1, e_p y_1, z_1)$ (or as the inverse, depending on the orientation of $\gamma$), it is clear that $m_\gamma = m_1 m_2$. The action of $m_\gamma$ on $H_0$ is obtained by deforming both $C'$ and $H_0$ over $\gamma$. Let $H' = m_\gamma(H_0)$. Since $\gamma \subset \Delta_2(U)$ $H'$ is fixed by the extra automorphism of $C'$, hence either $H' = H_0$ or $H' = H_0^\perp$. We know that $m_\gamma^p(H_0) = H_0$; since $2 \nmid p$ we get that $m_\gamma(H_0) = H_0$.Hence $\langle m_1 m_2 \rangle \subseteq$ Aut′ $(C, H_0) \cap M$. If Aut′ $(C, H_0) \neq \langle m_1 m_2 \rangle$ then Aut′$(C, H_0) = M$ which is absurd because, for example, $m_1(H_0) \neq H_0$.

Now let $h \in G$ be defined by $h^*(x_1, y_1, z_1) = (y_1, x_1, z_1)$, so that $h$ maps to the non-trivial element of Aut′ $(C)$. Our $H_0$ is fixed by $h$, hence Aut′ $(C, H_0) = \langle m_1 m_2, h \rangle$. Therefore a neighborhood of $m(C, H_0)$ is isomorphic to $V/\langle m_1 m_2, h \rangle$. Since $m_1 m_2$ and $h$ commute we first consider the quotient by $\langle h \rangle$. Let $(x, y, z) = (x_1 + y_1, x_1 y_1, z_1)$ then $V/\langle h \rangle \cong U(x, y, z)$; finally $V/\langle m_1 m_2 h \rangle \cong U(x, y, z)/\langle g \rangle$ where $g$ acts as $g^*(x, y, z) = (e_p x, e_p^2 y, z)$, q.e.d.

*Partial desingularization of* $\tilde{\Omega}$

Let $\varphi_4 : \hat{\mathscr{A}}_2(p) \to X_3$ be the partial desingularization obtained by applying the first step in Fujiki's method for resolving the singularities of $\tilde{\Omega}$. Let us examine the structure of $\hat{\mathscr{A}}_2(p)$ in a neighborhood of the exceptional divisor. So let $f_1 : U(x_1, y_1, z_1) \to U(x, y, z)$ be defined by $f_1^*(x, y, z) = (x_1, y_1^2, z_1)$, then $U(x, y, z)/\langle g \rangle \cong U(x_1, y_1, z_1)/\langle g, h \rangle$ where $h^*(x_1, y_1, z_1) = (x_1, -y_1, z_1)$. Let $f_2 : W \to U(x_1, y_1, z_1)$ be the blow up of $\{(0, 0, z_1)\}$. The group $\langle g, h \rangle$ acts naturally on $W$ and $W/\langle g, h \rangle$ is isomorphic to the partial desingularization of $U(x, y, z)/\langle g \rangle$. We decompose $W$ into the union of two open affine pieces.

(1) Let $W_1 \subset W$ be the affine piece with coordinates $(x_1, y_1/x_1, z_1)$. We have that $h^*(x_1, y_1/x_1, z) = (x_1, -y_1/x_1, z_1)$, $g^*(x_1, y_1/x_1, z_1) = (e_p x_1, y_1/x_1, z_1)$. Let $(x_2, y_2, z_2) = (x_1^p, y_1^2/x_1^2, z_1)$ then $W_1/\langle g, h \rangle \cong U(x_2, y_2, z_2)$, hence it is smooth.

(2) Let $W_2 \subset W$ be the affine piece with coordinates $(y_1, x_1/y_1, z_1)$. We have that $h^*(y_1, x_1/y_1, z_1) = (-y_1, -x_1/y_1, z_1)$ and $g^*(y_1, x_1/y_1, z_1) = (e_p y_1, x_1/y_1, z_1)$. Let $(x_3, y_3, z_3) = (y_1^p, x_1/y_1, z_1)$, then $W_2/\langle g, h \rangle \cong U(x_3, y_3, z_3)/\langle h \rangle$ where $h^*(x_3, y_3, z_3) = (-x_3, -y_3, z_3)$. Hence $W_2$ contains a curve of singular points satisfying S-B.,R.,T.'s criterion, in fact they are locally isomorphic to $\mathbb{A}^1 \times (x^2 - yz = 0)$. Notice that a local equation for the exceptional divisor of $\varphi_4$ is $(x_3^2 = 0)$, hence the singular curve is contained in the exceptional divisor.

DEFINITION 3.7. Let $\varphi : \hat{\mathscr{A}}_2(p) \to \mathscr{A}_2(p)$ be the composition

$$\varphi = \varphi_1 \varphi_2 \varphi_3 \varphi_4.$$

The conclusion of our analysis is that every singularity of $\hat{\mathscr{A}}_2(p)$ is canonical.

We have proved the following:

PROPOSITION 3.4. *Let $\psi: \tilde{\mathscr{A}}_2(p) \to \hat{\mathscr{A}}_2(p)$ be a desingularization of $\hat{\mathscr{A}}_2(p)$ and let $\omega$ be an n-canonical form on $\hat{\mathscr{A}}_2(p)$, then $\psi^*(\omega)$ is regular on all of $\tilde{\mathscr{A}}_2(p)$. In other words $\hat{\mathscr{A}}_2(p)$ has only canonical singularities.*

In view of Proposition 3.4 in order to prove that $\mathscr{A}_2(p)$ is of general type of $p \geqslant 17$ it will be enough to show that there are many $n$-canonical forms on the partial desingularization $\hat{\mathscr{A}}_2(p)$.

DEFINITION 3.8. Let $\varphi: \hat{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2(p)$; let

(i) $E' = \varphi^{-1}(R')$, $E'' = \varphi^{-1}(R'')$.
(ii) $E'_1 = \varphi^{-1}(\Gamma')$, $E''_1 = \varphi^{-1}(\Gamma'')$.
(iii) $E_2 = \varphi^{-1}(\Omega)$.
(iv) $\hat{\Delta}_2, \hat{\Delta}_1, \hat{\Delta}_0, \hat{\hat{\Delta}}_0 \subset \hat{\mathscr{A}}_2(p)$ be the strict transforms of $\tilde{\Delta}_2, \tilde{\Delta}_1, \tilde{\Delta}_0, \tilde{\tilde{\Delta}}_0 \subset \bar{\mathscr{A}}_2(p)$ respectively.

By abuse of notation we will use the same symbol for the reduced divisors $E', E'', \ldots$ and their linear equivalence classes in $\mathrm{Pic}(\hat{\mathscr{A}}_2(p)) \otimes \mathbb{Q}$.

The following is a picture of the part of $\hat{\mathscr{A}}_2(p)$ lying over $\Gamma' \cup \Gamma''$ and $\Omega$:

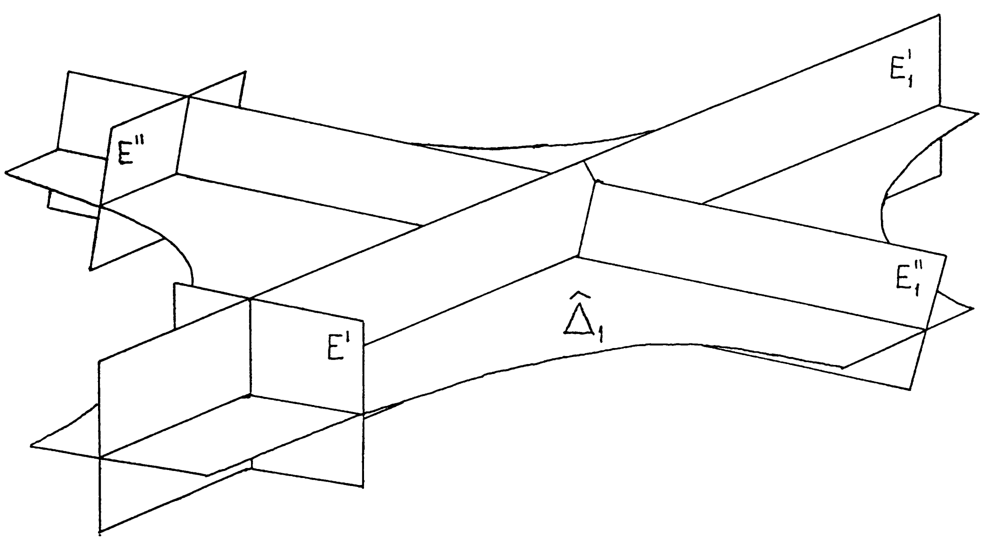



Fig. 1.

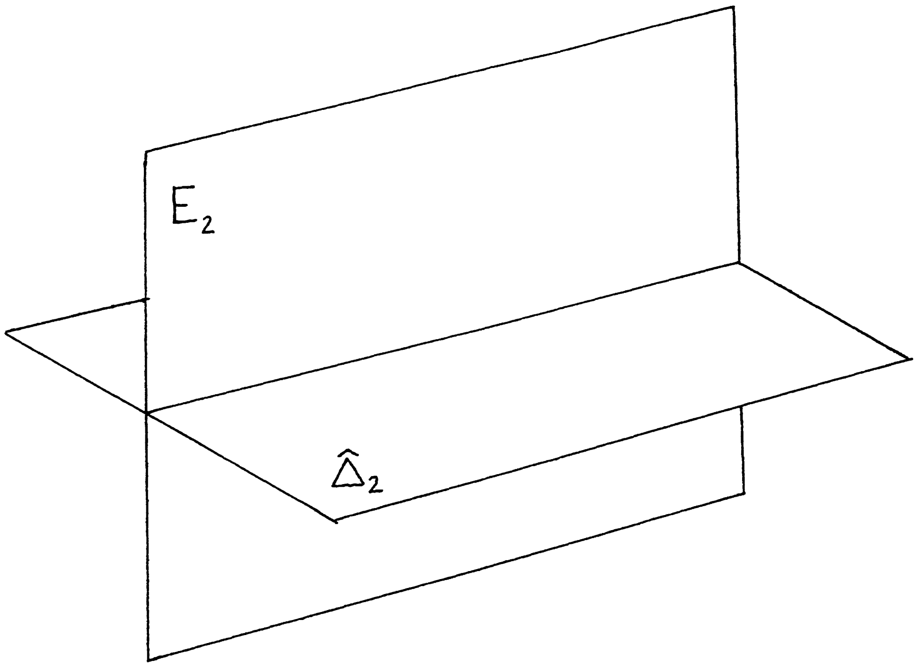



Fig. 2

PROPOSITION 3.5. *The following formula for the canonical class of $\hat{\mathcal{A}}_2(p)$ holds*:

$$K_{\hat{\mathcal{A}}_2(p)} = \varphi^*(K_{\bar{\mathcal{A}}_2(p)}) + (3/p - 1) - E_2 - \tfrac{1}{3}E'_1 - \tfrac{1}{3}E''_1 + \tfrac{2}{3}E' + \tfrac{2}{3}E''.$$

*Proof.* We know that $K_{\hat{\mathcal{A}}_2(p)} = \varphi^*(K_{\bar{\mathcal{A}}_2(p)}) + c_2E_2 + c'_1E'_1 + c''_1E''_1 + c'E' + c''E''$, we have to determine the coefficients of the exceptional divisors.

*Coefficient of $E_2$.*

$E_2$ is the exceptional divisor over $\tilde{\Omega} \subset X_3$. A neighborhood of $\tilde{\Omega}$, call it $V$, is isomorphic to $V(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (e_px, e_p^2y, z)$; the curve $\tilde{\Omega} \cap V$ is exactly the singular locus, i.e. the image of $\{(0, 0, z)\}$. A generator of $K_{\bar{\mathcal{A}}_2(p)}(V)$ is given by $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$; more precisely $\omega^p$ is invariant for the action of $\langle g\rangle$. Hence it descends to a generator of $K^{[p]}_{\bar{\mathcal{A}}_2(p)}(V)$. In the notation adopted when examining the blow up of $\tilde{\Omega}$ we have that coordinates on $W_1$ are $(x_2, y_2, z_2) = (x^p, y/x^2, z)$ and $(x_2 = 0)$ is a local equation for $E_2$. Hence $(x, y, z) = (x_2^{1/p}, x_2^{2/p}y_2, z_2)$, so $\omega = (1/p)x_2^{(3/p)-1}\,\mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}z_2$. Since $\mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}z_2$ is a local generator of $K_{\hat{\mathcal{A}}_2(p)}$ we see that the "order of vanishing" of $\varphi^*(\omega)$ along $E_2$ is $(3/p - 1)$ (since $p \geqslant 5$ this means that $\varphi^*(\omega)$ has a pole along $E_2$), and hence, in a neighborhood of $E_2$, $K_{\hat{\mathcal{A}}_2(p)} \cong \varphi^*(K_{\bar{\mathcal{A}}_2(p)}) + ((3/p) - 1)E_2$.

*Coefficients of $E_1'$, $E_1''$.*

$E_1'$ is the exceptional divisor over $\tilde{\Gamma} \subset X_1$. Let $V$ be a neighborhood of a generic point of $\tilde{\Gamma}$; we have shown that $V \cong U(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (e_3 x, e_3 y, z)$. Let $\omega = dx \wedge dy \wedge dz$. It is a generator of $K_{\bar{\mathscr{A}}_2(p)}(V)$. Let $(u, v, s) = (x^3, y/x, z)$. They are local coordinates on the partial desingularization of $V$ along $V \cap \tilde{\Gamma}$. The exceptional divisor $E_1'$ has local equasion $(u = 0)$. We have $(x, y, z) = (u^{1/3}, u^{1/3}v, s)$ hence $dx \wedge dy \wedge dz = \frac{1}{3}u^{-1/3}\, du \wedge dv \wedge ds$. Since $du \wedge dv \wedge ds$ is a local generator of $K_{\hat{\mathscr{A}}_2(p)}$ we see that $\varphi^*(\omega)$ has order of vanishing $-\frac{1}{3}$ along $E_1'$ (i.e. a pole of order $\frac{1}{3}$), hence the coefficient of $E_1'$ *is* $-\frac{1}{3}$. An analogous computation gives that the coefficient of $E_1''$ is $-\frac{1}{3}$.

*Coefficients of $E'$, $E''$*

$E'$ is the exceptional divisor of the blow up of $R'$. We have shown that a neighborhood $V$ of $R'$ is isomorphic to $U(x, y, z)/\langle gh\rangle$, where $(gh)^*(x, y, z) = (e_6^3 x, e_6^5 y, e_6^2 z)$. Adopting the notation we already used, we have that coordinates on a piece of the partial desingularization are $(u, v, s) = (x_1^6, y_1^{15} x_1^{15}, x_1^3 z_1^2/y_1^5)$. Therefore $(x, y, z) = (u^{1/2}, u^{5/6}v^{1/3}, u^{1/3}v^{1/3}s)$. A local equation for $E'$ is $(u = 0)$. Let $\omega = dx \wedge dy \wedge dz$ be the local generator of $K_{\bar{\mathscr{A}}_2(p)}(V)$, then $\varphi^*(\omega) = \frac{1}{6}u^{2/3}v^{1/3}\, du \wedge dv \wedge ds$. Hence the order of vanishing of $\varphi^*(\omega)$ along $E'$ is $\frac{2}{3}$, which justifies the coefficient of $E'$. An analogous computation holds for $E''$.

PROPOSITION 3.6. $\varphi^*(\tilde{\Delta}_1) \cong \hat{\Delta}_1 + \frac{1}{3}E_1' + \frac{1}{3}E_1'' + \frac{5}{6}E' + \frac{5}{6}E''$.

*Proof.* We know that $\varphi^*(\tilde{\Delta}_1) \cong \hat{\Delta}_1 + c_1'E_1' + c_1''E_1'' + c'E + c''E''$ for some positive coefficients $c_1', \ldots$, because $\tilde{\Delta}_1$ contains $R', R'', \Gamma', \Gamma''$.

As we have shown, a neighborhood of a generic point of $\tilde{\Gamma} \subset X_1$ is isomorphic to $U(x, y, z)/\langle g\rangle$ where $g^*(x, y, z) = (e_3 x, e_3 y, z)$. Going back to our basis $\{\alpha, \beta, \gamma\}$ of $H^0(\Omega_C^1 \otimes \omega_C)$ we see that $(\beta = 0)$ is the locus of curves in the deformation space which have a disconnecting node. Since $y = \beta^2$ and since $m\colon U(x, y, z) \to \bar{\mathscr{A}}_2(p)$ is étale outside $\{(0, 0, z)\}$ we get that $m^*(\tilde{\Delta}_1) = (y = 0)$. Now let $f\colon B \to U(x, y, z)$ be the blow up of $\{(0, 0, z)\}$ and let $q\colon B \to V$ be the qotient of $B$ by the natural action of $\langle g\rangle$. The quotient $V$ is isomorphic to $\varphi^{-1}(m(U(x, y, z)))$. We have that $mf = \varphi q$. Let $E \subset B$ be the exceptional divisor of $f$. On $V$ we have that $\varphi^*(\tilde{\Delta}_1) \cong \hat{\Delta}_1 + aE_1'$ for some $a$. The quotient map $q$ has ramification index 3 along $E$ hence $q^*(E_1') \cong 3E$. Hence $q^*\varphi^*(\tilde{\Delta}_1) = q^*(\hat{\Delta}_1 + aE_1') = q^*(\hat{\Delta}_1) + 3aE$. On the other hand we have that $f^*m^*(\tilde{\Delta}_1) = f^*\{y = 0\} = q^*(\hat{\Delta}_1) + E$. Therefore $q^*(\hat{\Delta}_1) + 3aE \cong q^*(\hat{\Delta}_1) + E$, hence $a = \frac{1}{3}$.

An analogous computation gives the coefficient of $E_1''$. A neighborhood of $R'$ is isomorphic to $U/\langle gh\rangle$ where $U = U(x, y, z)$ and $(gh)^*(x, y, z) = (e_6^3 x, e_6^5 y, e_6^2 z)$. We adopt the notation already used in analyzing the partial desingularization of $R'$. Let $m\colon U \to m(U) \subset \bar{\mathscr{A}}_2(p)$ be the moduli map. Let $f_1\colon U_1 \to U$ be the covering of $U$ and let $f_2\colon W \to U_1$ be the blow up of the origin of $U_1$. The quotient $q\colon W \to V$ of $W$ by the action of $\langle gh, H\rangle$ is isomorphic to $\varphi^{-1}(m(U))$. Hence we have that $\varphi q = mf_1 f_2$. Let $\varphi^*(\tilde{\Delta}_1) \cong \hat{\Delta}_1 + aE'$ (on $V$); let $E \subset W$ be the exceptional divisor. Since $q$ has ramification index 6 along $E$ we get that $q^*\varphi^*(\Delta_1) = q^*(\Delta_1) + 6aE'$. On the other hand $m^*(\tilde{\Delta}_1) = (y = 0)$, $f_1^*(y = 0) = (y_1^5 = 0)$ and so $f_2^* f_1^* m^*(\tilde{\Delta}_1) = q^*(\hat{\Delta}_1) + 5E$. Therefore $5 = 6a$ so $a = 5/6$.

An analogous computation gives the coefficient of $E''$.

PROPOSITION 3.7. $\varphi^*(\tilde{\Delta}_2) \cong \hat{\Delta}_2 + (2/p)E_2$.

*Proof.* We know that $\varphi^*(\tilde{\Delta}_2) \cong \hat{\Delta}_2 + aE_2$ for some positive $a$ because $\tilde{\Delta}_2$ contains $\Omega$; we need to determine $a$. A neighborhood of a point in $\tilde{\Omega}$ is isomorphic to $U/\langle g\rangle$ where $U = U(x, y, z)$ and $g^*(x, y, z) = (e_p x, e_p^2 y, z)$. Let $f_1\colon U_1 \to U$ be the covering and let $f_2\colon W \to U_1$ be the blow up of $\{(0, 0, z)\}$. Let $m\colon U \to m(U) \subset \bar{\mathscr{A}}_2(p)$ be the moduli map. The quotient $q\colon W \to V$ by the action of $\langle g, h\rangle$ is isomorphic to $\varphi^{-1}(m(U))$. Hence $mf_1 f_2 = \varphi q$. Let $E \subset W$ be the exceptional divisor, the map $q$ has ramification index $p$ along $E$. So we have $q^*\varphi^*(\tilde{\Delta}_2) = q^*(\hat{\Delta}_2 + aE_2) = q^*(\hat{\Delta}_2) + apE$. We also have that $m^*(\tilde{\Delta}_2) = (4y - x^2 = 0)$, $f_1^*(4y - x^2 = 0) = (4y_1^2 - x_1^2 = 0)$, hence $f_2^* f_1^* m^*(\tilde{\Delta}_1) = q^*(\hat{\Delta}_2) + 2E$. Therefore $2 = ap$ and $a = 2/p$.

THEOREM 3.1. *Let $\varphi\colon \hat{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2(p)$. The following formula holds.*

$$K_{\hat{\mathscr{A}}_2(p)} \cong \varphi^*\pi^*\left(\left(\frac{3}{10} - \frac{1}{p}\right)\Delta_0 + \frac{3}{10}\Delta_1\right) - \frac{1}{2}\hat{\Delta}_2 - \frac{1}{2}\hat{\Delta}_1 - \frac{p-1}{p}\hat{\Delta}_0 -$$

$$- \frac{p-1}{p}\hat{\hat{\Delta}}_0 + \left(\frac{2}{p} - 1\right)E_2 - \frac{1}{2}E_1' - \frac{1}{2}E_1'' + \frac{1}{4}E' + \frac{1}{4}E''.$$

*Proof.* Follows from Theorem 2.2, Propositions 3.5, 3.6, 3.7.

## Section 4. Proof of the main theorem

In this section we will prove that $h^0(nK_{\hat{\mathscr{A}}_2(p)}) \geqslant A(p)n^3 + O(n^2)$ for $n$ sufficiently divisible where $A(p)$ will be a sum of monomials in $p$ with positive leading coefficient. It will turn out that for $p \geqslant 17$, $A(p) > 0$.This will show that for such values of $p$ the canonical ring of $\hat{\mathscr{A}}_2(p)$ has transcendence degree equal to four hence $\hat{\mathscr{A}}_2(p)$ is of general type.

NOTATION. Let $\alpha_p = 3 - 10/p$. Since $10\lambda = \Delta_0 + \Delta_1$ (Corollary 1.2) we can write

$$\left(\frac{3}{10} - \frac{1}{p}\right)\Delta_0 + \frac{3}{10}\Delta_1 = \alpha_p\lambda + \frac{1}{p}\Delta_1.$$

Following Theorem 3.1 we have that

$$K_{\hat{\mathscr{A}}_2(p)} = \varphi^*\pi^*(\alpha_p\lambda) + \varphi^*\pi^*\left(\frac{1}{p}\Delta_1\right) - \sum_{s=1}^{7} c_s D_s + \frac{1}{4}E' + \frac{1}{4}E'',$$

where $D_s$ are the divisors appearing with negative coefficients. Our plan for estimating $h^0(nK_{\hat{\mathscr{A}}_2(p)})$ is the following. First of all $h^0(nK_{\hat{\mathscr{A}}_2(p)}) \geqslant h^0(n\varphi^*\pi^*(\alpha_p\lambda) - n\Sigma_{s=1}^7 c_s D_s)$. The Hodge bundle $\lambda$ lives on the Satake compactification of $\mathscr{A}_2$ and is ample on it, hence it is easy to estimate $h^0(n\varphi^*\pi^*(\alpha_p\lambda))$. The next thing to do is to estimate $h^0((n\varphi^*\pi^*(\alpha_p\lambda) - iD_1)|_{D_1})$ for $0 \leqslant i \leqslant c_1 n - 1$, then we estimate $h^0((n\varphi^*\pi^*(\alpha_p\lambda) - nc_1D_1 - iD_2)|_{D_2})$ for $0 \leqslant i \leqslant c_2 n - 1$ and so on up to $D_7$. Finally we subtract the number of conditions imposed by $D_1, \dots, D_7$ from the dimension of $H^0(n\varphi^*\pi^*(\alpha_p\lambda))$ and we obtain an estimate of $h^0(nK_{\hat{\mathscr{A}}_2(p)})$.

Let $\mathscr{A}_2^+ \supset \mathscr{A}_2$ be the Satake compactification of $\mathscr{A}_2$ and let $\mathscr{A}_2^+(p) \supset \mathscr{A}_2(p)$ be the Baily-Borel compactification of $\mathscr{A}_2(p)$. Let $\eta\colon \mathscr{A}_2^+(p) \to \mathscr{A}_2^+$ be the natural covering map and let $f\colon \bar{\mathscr{A}}_2 \to \mathscr{A}_2^+$, $f_p\colon \bar{\mathscr{A}}_2(p) \to \mathscr{A}_2^+(p)$ be the natural birational morphisms. We have that $f\pi = \eta f_p$. The Hodge bundle $\lambda$ on $\bar{\mathscr{A}}_2$ is the pull back of an ample bundle $\lambda^+$ on $\mathscr{A}_2^+$, i.e. $\lambda \cong f^*(\lambda^+)$. Hence $\pi^*(\lambda) \cong f_p^*\eta^*(\lambda^+)$. The bundle $\eta^*(\lambda^+)$ is ample on $\mathscr{A}_2^+(p)$. Hence $h^0(n\eta^*(\lambda^+)) = \frac{1}{6}\deg(\eta^*(\lambda^+))n^3 + O(n^2)$ for $n$ sufficiently divisible. Therefore $h^0(n\pi^*(\lambda)) = \frac{1}{6}\deg(\eta^*(\lambda^+))n^3 + O(n^2)$. Obviously $\deg(\eta^*(\lambda^+)) = (\deg\eta)\cdot\deg\lambda^+ = (p^4 + p^2)\cdot\deg\lambda$. Therefore, replacing $n$ by $\alpha_p n$, we get the following:

PROPOSITION 4.1: *Let $\pi\varphi\colon \hat{\mathscr{A}}_2(p) \to \bar{\mathscr{A}}_2$, then*

$$h^0(n\varphi^*\pi^*(\alpha_p\lambda)) = \frac{1}{6}(p^4 + p^2)(3 - 10/p)^3\lambda^3 n^3 + O(n^2)$$

*for n sufficiently divisible.*

*Computation of $\lambda^3$*

The surface $\Delta_0 \subset \mathfrak{M}_2$ is the moduli space of couples $(E, Q)$ where $E$ is an elliptic curve and $Q \in E$. In fact, let $C$ be a genus two curve with one non-disconnecting node, let $E$ be the normalization of $C$ and let $P, Q \in E$ be the points mapping to the node of $C$. Let us choose $P \in E$ to be the zero of the group law on $E$, then we can associate to $m(C) \in \Delta_0$ the moduli point of $(E, Q)$. If we choose $Q \in E$ to be the zero of the group law then we get the couple $(E, -Q)$ which is isomorphic to $(E, Q)$.

The moduli space of couples $(E, Q)$ can be described as follows. Let a semi-direct product $\Gamma = SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ act on $\mathbb{H} \times \mathbb{C}$ by

$$(\omega, z) \to \left(\frac{a\omega + b}{c\omega + d}, \frac{z + m\omega + n}{c\omega + d}\right),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $(m, n) \in \mathbb{Z}^2$.

DEFINITION 4.1: *Let S be the quotient of* $\mathbb{H} \times \mathbb{C}$ *by the action of* $\Gamma$.

The surface $S$ maps to the affine $j$-line $\mathbb{A}_j^1$ via the map assigning to $(E, Q)$ the elliptic curve $E$. We can compactify $S \subset \bar{S}$ so that the map to $\mathbb{A}_j^1$ extends to a map $\psi: \bar{S} \to \mathbb{P}_j^1$. The surface $\bar{S}$ is the moduli space of couples $(E, Q)$ where $E$ is any elliptic curve (not necessarily smooth) so $\Delta_0$ is isomorphic to $\bar{S}$. The fibers of $\psi$ are reduced projective lines except over $j = 0,1728$. In fact $\bar{S}$ has singular points on $\psi^{-1}(0)$, $\psi^{-1}(1728)$ and the fibers $\psi^*(0)$, $\psi^*(1728)$ have multiplicity 3 and 2 respectively.

PROPOSITION 4.2. $\lambda^2 \cdot \Delta_0 = 0$.

*Proof.* This follows from the fact that $\lambda = f^*(\lambda^+)$ and that $f: \bar{\mathscr{A}}_2 \to \mathscr{A}_2^+$ blows down $\Delta_0$ but we want to check it. Let $g: \mathscr{C} \to E$ be the family of singular genus two curves defined in Definition 1.2. The moduli map $m: E \to \bar{\mathfrak{M}}_2$ maps $E$ onto a fiber of $\psi: \Delta_0 \to P_j^1$. By Lemma 1.2 $m^*(\lambda) = 0$, hence $\lambda$ is trivial on fibers of $\psi$, therefore $\lambda^2 \cdot \Delta_0 = 0$.

PROPOSITION 4.3. $\lambda^2 \cdot \Delta_1 = \frac{1}{144}$.

*Proof.* The surface $\Delta_1$ is isomorphic to $\mathbb{P}^2$, hence $\mathrm{Pic}(\Delta_1)$ is generated by the hyperplane class $H$, let $\lambda|_{\Delta_1} \cong aH$. Let $f: \mathscr{C} \to T$ be the family of singular genus two curves defined in Definition 1.5 and let $m: T \to \bar{\mathfrak{M}}_2$ be the moduli map, then $m(T) \subset \Delta_1$ and $m_*(T) \cong 12H$. Therefore $\deg m^*(\lambda) = 12a$; by Lemma 1.1 we get that $a = \frac{1}{12}$. Hence $\lambda^2 \cdot \Delta_1 = (\frac{1}{12}H) \cdot (\frac{1}{12}H) = \frac{1}{144}$.

PROPOSITION 4.4. $\lambda^3 = \frac{1}{1440}$.

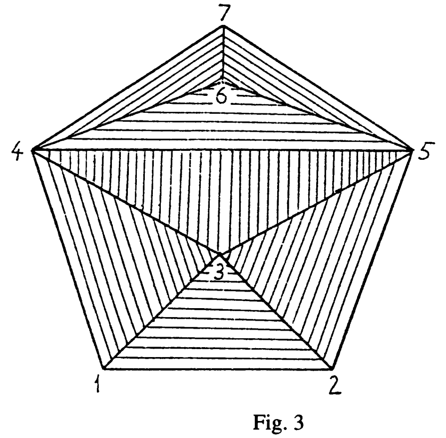
*Proof.* By Corollary 1. we get that

$$\lambda^3 = \frac{1}{10}\lambda^2(\Delta_0 + \Delta_1).$$

By Propositions 4.2 and 4.3 we get that $\lambda^3 = \frac{1}{1440}$.

COROLLARY 4.1. $h^0(n\varphi^*\pi^*(\alpha_p\lambda)) = \frac{1}{8640}(p^4 + p^2)(3 - 10/p)^3 n^3 + O(n^2)$ *for n sufficiently divisible.*

The dual graph of the configuration consisting of $E_1', E_1'', \ldots$ is the following




1-$E_1'$
2-$E_1''$
3-$\hat{\Delta}_1$
4-$\hat{\Delta}_0$
5-$\hat{\hat{\Delta}}_0$
6-$\hat{\Delta}_2$
7-$E_2$

Fig. 3

We will number the divisors in the configuration according to the key, i.e. $D_1 = E_1'$, $D_2 = E_1''$, $D_3 = \hat{\Delta}_1, \ldots$. The incidence relations between the $D_i$'s will become clear as we examine more closely $E_1', E_1'', \ldots$.

*Conditions imposed by* $E_1'$

First of all let us recall how $E_1'$ was obtained. Let $\varphi_2 \colon X_2 \to X_1$ be the blow up of $\tilde{\Gamma} \subset X_1$ and let $E_1 = \varphi_2^{-1}(\tilde{\Gamma})$. A fiber $\varphi_2^*(P)$ of the map $\varphi_2 \colon E_1 \to \tilde{\Gamma}$ is a smooth projective line if $P \neq \tilde{\Gamma} \cap \tilde{\tilde{\Gamma}}$. The fiber $\varphi_2^*(\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}})$ has multiplicity 3 and it contains two singular points one of which is $\Gamma^* \cap E_1$. Let $\varphi_3 \colon X_3 \to X_2$ be the blow up of $\Gamma^*$, then $E_1' = \varphi_3^{-1}(E_1)$. Hence $E_1'$ is the blow up of $E_1$ with center $\Gamma^* \cap E_1$. Let $\Sigma = E_1' \cap \hat{\Delta}_1$, let $G \subset E_1'$ be the proper transform of the (reduced) fiber $\varphi_2^{-1}(\tilde{\Gamma} \cap \tilde{\tilde{\Gamma}})$ and let $F \subset E_1'$ be the exceptional curve of the blow up $\varphi_3 \colon E_1' \to E_1$. Notice that if $Q = \Gamma^* \cap E_1$, then $\varphi_3^*(Q) = F + 3G$. The set $\{[\Sigma], [F], [G]\}$ is a basis of $\mathrm{Pic}(E_1') \otimes \mathbb{Q}$.

LEMMA 4.1. $E_1' \cdot \Sigma = -1$.

*Proof.* We have that $E_1' \cdot \Sigma = (\Sigma \cdot \Sigma)_{\Delta_1}$. Now $\tilde{\Delta}_1$ is isomorphic to $\mathbb{P}_j^1 \times \mathbb{P}_j^1$ and $\hat{\Delta}_1$ is the blow up of $\tilde{\Delta}_1$ with center the two points $R'$, $R''$. Let $\varphi \colon \hat{\Delta}_1 \to \tilde{\Delta}_1$, then $\Sigma \subset \hat{\Delta}_1$ is the proper transform of $\Gamma' \subset \tilde{\Delta}_1$. Since $\Gamma'$ belongs to one of the two rulings of $\tilde{\Delta}_1$ we have that $(\Gamma' \cdot \Gamma')_{\tilde{\Delta}_1} = 0$. Since $R' \in \Gamma'$ and $R'' \notin \Gamma'$ we get that $(\Sigma \cdot \Sigma)_{\tilde{\Delta}_1} = -1$, q.e.d.

LEMMA 4.2. (i) $E_1' \cdot F = 0$, (ii) $E_1' \cdot G = -1$.

*Proof.* (i) We clearly have that $F = E_1' \cap E_1''$, therefore $\deg E_1'|_F = (F^2)_{E_1''}$. The curve $F \subset E_1''$ is a fiber of the map $\varphi \colon E_1'' \to \Gamma''$, hence $(F^2)_{E_1''} = $ . Therefore $\deg E_1'|_F = 0$.

(ii) Let $\varphi^*(P) \subset E_1'$ be a generic fiber of $\varphi \colon E_1' \to \Gamma$. A local computation gives that $E_1' \cdot \varphi^*(P) = -3$. Since $\varphi^*(P) \cong F + 3G$ we get $E_1' \cdot (F + 3G) = -3$; by the previous formula we get that $E_1' \cdot G = -1$.

LEMMA 4.3. *Let* $\Sigma, F, G \in \mathrm{Pic}(E_1') \otimes \mathbb{Q}$, *then*

(i) $F \cdot G = 1$,
(ii) $\Sigma \cdot F = 1$,
(iii) $\Sigma \cdot G = 0$,
(iv) $F \cdot F = -3$,
(v) $G \cdot G = -\frac{1}{3}$,
(vi) $\Sigma \cdot \Sigma = -1$.

*Proof.* (i) Obvious.

(ii) The proper transform of $\tilde{\Delta}_1$ for the map $\varphi_1 \varphi_2 \colon X_2 \to \mathscr{A}_2(p)$ contains $\Gamma^*$. Hence $\hat{\Delta} \cdot F = 1$. Since $\Sigma = E_1' \cap \hat{\Delta}_1$ we get that $(\Sigma \cdot F)_{E_1'} = 1$.

(iii) Let $P \in \Gamma'$ be a generic point, then $\hat{\Delta}_1 \cdot \varphi^*(P) = 1$. Since $\varphi^*(P) \cong F + 3G$ we get that $\hat{\Delta}_1 \cdot (F + 3G) = 1$. By part (ii) we get that $\hat{\Delta}_1 \cdot G = 0$.

(iv), (v) Let again $P \in \Gamma'$ be a generic point, then $F \cdot \varphi^*(P) = G \cdot \varphi^*(P) = 0$. Hence $F \cdot (F + 3G) = G \cdot (F + 3G) = 0$. By formula (i) we get that $F \cdot F = -3$ and $G \cdot G = -\frac{1}{3}$.

(vi) Since $\Sigma = E_1' \cdot \hat{\Delta}_1$, $(\Sigma^2)_{E_1'} = \hat{\Delta}_1 \cdot \Sigma$. By Proposition 3.6 we have that $\varphi^*(\tilde{\Delta}_1) = \hat{\Delta}_1 + \frac{1}{3}E_1' + \frac{1}{3}E_1'' + \frac{5}{6}E' + \frac{5}{6}E''$, hence

$$\hat{\Delta}_1 \cdot \Sigma = \varphi^*(\tilde{\Delta}_1) \cdot \Sigma - \tfrac{1}{3}E_1' \cdot \Sigma - \tfrac{1}{3}E_1'' \cdot \Sigma - \tfrac{5}{6}E' \cdot \Sigma - \tfrac{5}{6}E'' \cdot \Sigma. \qquad (*)$$

The map $\varphi \colon \Sigma \to \Gamma'$ is one-to-one, hence $\varphi^*(\tilde{\Delta}_1) \cdot \Sigma = \tilde{\Delta}_1 \cdot \Gamma'$. We will prove in Lemma 4.6 that $\tilde{\Delta}_{1|\tilde{\Delta}_1} \cong \pi^*(\Delta_1)|_{\tilde{\Delta}_1}$, hence $\varphi^*(\tilde{\Delta}_1) \cdot \Sigma = \pi^*(\Delta_1) \cdot \Gamma'$. Since $\pi \colon \Gamma' \to \Gamma$ is one-to-one we have that $\pi^*(\Delta_1) \cdot \Gamma' = \Delta_1 \cdot \Gamma$. By Lemma 1.1 we get that $\Delta_1 \cdot \Gamma = -\frac{1}{6}$, hence $\varphi^*(\tilde{\Delta}_1) \cdot \Sigma = -\frac{1}{6}$. Obviously $E_1'' \cdot \Sigma = 1$, $E' \cdot \Sigma = 1$, $E'' \cdot \Sigma = 0$ and by Lemma 4.1, $E_1' \cdot \Sigma = 1$, hence $(*)$ becomes $\hat{\Delta}_1 \cdot \Sigma = -1$, q.e.d.

LEMMA 4.4. (i) $\varphi^*\pi^*(\lambda)|_{E_1'} \cong \frac{1}{12}(E + 3G)$.
(ii) $E'_{1|E_1'} \cong -3\Sigma - 4F - 9G$.

*Proof.* (i) The map $\pi \colon \Gamma' \to \Gamma$ is one-to-one, hence $\varphi^*\pi^*(\lambda) = (\deg \lambda|_\Gamma)\,(F + 3G)$. By Lemma 1.1 $\deg \lambda|_\Gamma = \frac{1}{12}$.

(ii) We know that $E'_{1|E_1'} \cong x\Sigma + yF + wG$ for some $x, y, w \in \mathbb{Q}$. Lemmas 4.1, 4.2, 4.3 determine uniquely $x, y, w$.

COROLLARY 4.1.

$$(n\varphi^*\pi^*(\alpha_p\lambda) - \mathrm{i}E_1')|_{E_1'} \cong 3i\Sigma + \left(\frac{\alpha_p}{12}n + 4i\right)F + \left(\frac{\alpha_p}{4}n + 9i\right)G.$$

In estimating the dimension of linear systems on the surfaces $E_1', \ldots$ we will often use the following:

PROPOSITION 4.5. *Let $S$ be a surface, let $F, D$ be effective divisors on $S$ and assume $F^2 = 0$, $F \cdot D \geqslant 0$, then*

$$h^0(xF + yD) \leqslant xy(F \cdot D) + x + h^0(yD)(x, y \geqslant 0).$$

*Proof.* By induction on $x$. The first step, $x = 0$, is clear. The induction from $(x - 1, y)$ to $(x, y)$ is provided by the sequence of spaces of global sections in the long exact sequence associated to

$$0 \to \mathcal{O}_S((x - 1)F + yD) \to \mathcal{O}_S(xF + yD) \to \mathcal{O}_F(xF + yD) \to 0$$

THEOREM 4.1.

$$\sum_{i=0}^{N} h^0((n\varphi^*\pi^*(\alpha_p\lambda) - iE_1')|_{E_1'}) \leqslant \left(\frac{19}{32} - \frac{5}{16p}\right)n^3 + O(n^2)(N = \tfrac{1}{2}n - 1).$$

*Proof.* Let $L \in \mathrm{Pic}(E'_1)$ be defined as $L = F + 3G$. By Corollary 4.1 we have that

$$h^0((n\varphi^*\pi^*(\alpha_p\lambda) - E'_1)|_{E'_1} = h^0\left(3i\Sigma + \left(\frac{\alpha_p}{12}n + 4i\right)L - 3iG\right)$$
$$\leqslant h^0\left(3i\Sigma + \left(\frac{\alpha_p}{12}n + 4i\right)L\right).$$

We apply Proposition 4.5 to the surface $E'_1$ with $F = L$ and $D = \Sigma$. Since $\Sigma$ is irreducible and $\Sigma\cdot\Sigma = -1$ we have that $h^0(y\Sigma) = 1$. We get

$$h^0((n\varphi^*\pi^*(\alpha_p\lambda) - iE'_1)|_{E'_1} \leqslant \frac{\alpha_p}{4}ni + 12i^2 + \frac{\alpha_p}{12}n + 4i + 1. \qquad (*)$$

Therefore the summation of the left hand side of $(*)$ for $i = 0, 1, \ldots, n/2 - 1$ is bounded above by the summation of the right hand side. The latter is a polynomial in $n$ of third degree. In order to prove the proposition we must show that the leading term (l.t.) of this polynomial is equal to $(19/32 - 5/16p)n^3$. Hence we compute

$$\text{l.t.} \sum_{i=0}^{n/2}\left(\frac{\alpha_p}{4}ni + 12i^2\right) = \left(\frac{19}{32} - \frac{5}{16p}\right)n^3.$$

*Conditions imposed by* $E''_1$

Let $\varphi_3\colon X_3 \to X_2$ be the blow up of $\Gamma^*$, then $E''_1$ is the exceptional divisor of $\varphi_3$. Let $\Sigma = E''_1 \cap \hat{\Delta}_1$, $F = \varphi^*(P)$ where $P \in \Gamma''$. A basis of $\mathrm{Pic}(E''_1)$ is given by $\{\Sigma, F\}$.

LEMMA 4.5. (i) $E''_1\cdot F = -3$.
(ii) $E''_1\cdot\Sigma = -1$.

*Proof.* (i) This is the same local computation that gives $E'_1\cdot\varphi^*(P) = -3$.
(ii) Since $\Sigma = E''_1\cdot\hat{\Delta}_1$ we have that $E''_1\cdot\Sigma = (\Sigma\cdot\Sigma)_{\hat{\Delta}_1}$. The argument that proved that $E'_1\cdot\Sigma = -1$ also show that $E''_1\cdot\Sigma = -1$.

LEMMA 4.6. (i) $(F^2)_{E''_1} = 0$, (ii) $(F\cdot\Sigma)_{E''_1} = 1$, (iii) $(\Sigma^2)_{E''_1} = -1$.

*Proof.* (i) and (ii) are clear.
(iii) We have that $(\Sigma^2)_{E''_1} = \hat{\Delta}_1\cdot\Sigma$. From Proposition 3.6 we get that

$$\hat{\Delta}_1\cdot\Sigma = \varphi^*(\tilde{\Delta}_1)\cdot\Sigma - \tfrac{5}{6}E'\cdot\Sigma - \tfrac{5}{6}E''\cdot\Sigma - \tfrac{1}{3}E'_1\cdot\Sigma - \tfrac{1}{3}E''_1\cdot\Sigma.$$

At this point we proceed in a way completely analogous to the proof of (vi) of Lemma 4.3.

LEMMA 4.7. (i) $\varphi^*\pi^*(\alpha_p\lambda)|_{E''_1} \cong \frac{1}{12}F$,

(ii) $E'_{1|E''_1} \cong F$,

(iii) $E''_{1|E''_1} \cong -3\Sigma - 4F$.

*Proof.* (i) analogous to (i) of Lemma 4.4.

(ii) clear

(iii) We know that $E''_{1|E''_1} \cong x\Sigma + yF$ for some $x, y \in \mathbb{Z}$, Lemmas 4.5, 4.6 uniquely determine the coefficients $x, y$.

COROLLARY 4.2. $\left(n\varphi^*\pi^*(\alpha_p\lambda) - \frac{n}{2}E'_1 - iE''_1\right)\Big|_{E''_1} \cong 3i\Sigma + \left(4i - \left(\frac{1}{4} + \frac{5}{6p}\right)n\right)F.$

THEOREM 4.2.

$$\sum_{i=0}^{N} h^0\left(\left(n\varphi^*\pi^*(\alpha_p\lambda) - \frac{n}{2}E'_1 - iE''_1\right)\Big|_{E''_1}\right) \leqslant \frac{3}{7}n^3 + O(n^2)(N = \tfrac{1}{2}n - 1).$$

*Proof.* By the Corollary in order that $h^0((n\varphi^*\pi^*(\alpha_p\lambda) - (n/2)E'_1 - iE''_1)|_{E''_1})$ be non-zero we must have

$$i > \left(\frac{1}{16} + \frac{5}{24p}\right)n$$

We apply Proposition 4.5 to $E''_1$ for $(1/16 + 5/24p)n \leqslant i \leqslant n/2 - 1$. We get that $\Sigma_{i=0}^{N} h^0(\ldots)$ is bounded above by a polynomial whose leading term is $\frac{3}{7}n^3$.

*Conditions imposed by* $\hat{\Delta}_1$.

As we have already pointed out $\tilde{\Delta}_1 \cong \mathbb{P}^1_j \times \mathbb{P}^1_j$. Furthermore $\Delta_1 \cong \mathbb{P}^2$. Let $(j_1, j_2) \in \tilde{\Delta}_1$, then $\pi(j_1, j_2) = (j_1 + j_2, j_1 \cdot j_2)$. The surface $\hat{\Delta}_1$ is the blow up of $\Delta_1$ at the points $R', R''$.

DEFINITION 4.2. Let $L' = E' \cap \hat{\Delta}_1$, $L'' = E'' \cap \hat{\Delta}_1$, $M' = E'_1 \cap \hat{\Delta}_1$, $M'' = E''_1 \cap \hat{\Delta}_1$.

It is clear that $L', L''$ are the exceptional divisors of $\varphi: \hat{\Delta}_1 \to \tilde{\Delta}_1$, lying above $R', R''$ respectively. The divisors $M', M''$ are the proper transforms of the divisors belonging to the two rulings of $\tilde{\Delta}_1$ and passing through $R', R''$ respectively.

LEMMA 4.8. $\pi^*(\Delta_1)|_{\tilde{\mathcal{A}}'} \cong \tilde{\Delta}_1|_{\tilde{\mathcal{A}}'}$.

*Proof.* $\pi^*(\Delta_1) = \tilde{\Delta}_1 + 2R_1$ so we must show that $R_1 \cap \tilde{\Delta}_1 = \emptyset$. Assume $R_1 \cap \tilde{\Delta}_1 \neq \emptyset$; let $D \subset \tilde{\Delta}_1 \cap R_1$ be an irreducible component, then either $\pi(D)$ is a divisor whose points parametrize curves $C$ such that $\mathrm{Aut}'(C)$ is larger than $\mathbb{Z}/(2)$ or else $\pi(D) = \Delta_1 \cap \Delta_0$. In the first case $\pi(D)$ can be either $\Delta_1 \cap \Delta_2$ or $\Gamma$ or $\Gamma_1 = \{m(E \cup F) \mid j(E) = 1728\}$. Let $m(E_1 \cup E_2)(E_1 \cong E_2)$ be a generic point of $\Delta_1 \cap \Delta_2$, then $\mathrm{Aut}'(E_1 \cup E_2) = \langle g, h\rangle$ where $g|_{E_1} = $ (mult by $-1$), $g|_{E_2} = $ (identity), $h$ interchanges $E_1$ and $E_2$. We have that $m(E_1 \cup E_2, H) \in \tilde{\Delta}_1$ if and only if $H = E_1[p]$ or $H = E_2[p]$, while $m(E_1 \cup E_2, H) \in R_1$ in all other cases. Since the automorphism $h$ interchanges $E_1[p]$ with $E_2[p]$ we see that $\pi: \tilde{\Delta}_1 \to \Delta_1$ is ramified over $\Delta_1 \cap \Delta_2$ (in fact if we identify $\Delta_1 \cong \mathbb{P}^2$, $\Delta_1 \cap \Delta_2$ gets identified with a conic) and that $\pi^{-1}(\Delta_1 \cap \Delta_2) \cap \tilde{\Delta}_1 \cap R_1 = \emptyset$. A similar analysis or $\pi: \tilde{\Delta}_1 \to \Delta_1$ over $\Gamma$ or $\Gamma_1$ will show that $\Gamma, \Gamma_1$ are not in the branch locus so $\pi^{-1}(\Gamma) \cap \tilde{\Delta}_1 \cap R_1 = \pi^{-1}(\Gamma_1) \cap \tilde{\Delta}_1 \cap R_1 = \emptyset$. For the analysis of $\pi: \tilde{\Delta}_1 \to \Delta_1$ over $\Delta_1 \cap \Delta_0$ we just have to notice the following: let $m(C)$ be a generic point of $\Delta_1 \cap \Delta_0$ and let $m(C, H) \in \tilde{\Delta}_1$, then the Picard-Lefschetz transformation fixes $H$, hence $m(C, h) \notin R_1$.

Applying Proposition 3.6 we get

COROLLARY 4.3. $\hat{\Delta}_1|_{\hat{\Delta}_1} \cong (\varphi^*\pi^*(\Delta_1) - \frac{1}{3}E'_1 - \frac{1}{3}E''_1 - \frac{5}{6}E' - \frac{5}{6}E'')|_{\hat{\Delta}_1}$.

PROPOSITION 4.6. *Let $H$ be the hyperplane class in $\Delta_1$, then* $\Delta_1|_{\Delta_1} \cong -\frac{1}{6}H$.

*Proof.* Follows from (ii) of Lemma 1.1.

LEMMA 4.9. $\hat{\Delta}_1|_{\hat{\Delta}_1} \cong -L' - L'' - \dfrac{1}{2}M' - \dfrac{1}{2}M''$.

*Proof.* Follows from Corollary 4.3 and Proposition 4.6.

COROLLARY 4.4.

$$(n\varphi^*\pi^*(\alpha_p\lambda) - \Sigma_{s=1}^2 nc_sD_s - i\hat{\Delta}_1)|_{\hat{\Delta}_1} \cong \left(\frac{\alpha_p}{12}n + 1\right)(L' + L'') + $$

$$+ \left(\frac{\alpha_p}{12}n - \frac{1}{2}n + \frac{i}{2}\right)(M' + M'').$$

*Proof.* Follows from Lemma 4.9 and the fact that $\lambda|_{\Delta_1} \cong \frac{1}{12}H$, which was proved in Proposition 4.3.

THEOREM 4.3.

$$\sum_{i=0}^{N} h^0\left(\left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{2} nc_sD_s - i\hat{\Delta}_1\right)\Big|_{\hat{\Delta}_1}\right) = O(N = \tfrac{1}{2}n - 1).$$

*Proof.* Follows from Corollary 4.4.

*Conditions imposed by* $\hat{\Delta}_0$.

The first thing we notice is that $\varphi^*(\tilde{\Delta}_0)|_{\hat{\Delta}_0} \cong \hat{\Delta}_0|_{\hat{\Delta}_0}$ because $\tilde{\Delta}_0$ does not contain any of the centers of the successive partial desingularizations by which $\hat{\mathscr{A}}_2(p)$ is obtained. The map $\varphi: \hat{\Delta}_0 \to \tilde{\Delta}_0$ is a birational morphism hence if $L \in \mathrm{Pic}(\tilde{\Delta}_0)$ then

$h^0(\varphi^*(L)) = h^0(L)$. We also have the obvious inequality

$$h^0\left(\left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{3} nc_s D_s - i\hat{\Delta}_0\right)\Big|_{\hat{\Delta}_0}\right) \leqslant h^0((n\varphi^*\pi^*(\alpha_p\lambda) - i\hat{\Delta}_0)|_{\hat{\Delta}_0}).$$

Since $\hat{\Delta}_0|_{\hat{\Delta}_0} \cong \varphi^*(\tilde{\Delta}_0)|_{\hat{\Delta}_0}$ we get that

$$h^0((n\varphi^*\pi^*(\alpha_p\lambda) - i\hat{\Delta}_0)|_{\hat{\Delta}_0}) = h^0(\varphi^*(n\pi^*(\alpha_p\lambda) - i\tilde{\Delta}_0)|_{\hat{\Delta}_0}).$$

PROPOSITION 4.7. (i) $\tilde{\Delta}_0 \cap \tilde{\tilde{\Delta}}_0$ *is non-empty and the intersection is transverse.* (ii) $\tilde{\Delta}_0 \cap R_0$ *is non-empty and the intersection is transverse.*

*Proof.* It is clear that if $m(C, H) \in \tilde{\Delta}_0 \cap \tilde{\tilde{\Delta}}_0$ or $m(C, H) \in \tilde{\Delta}_0 \cap R_0$ then $m(C) \in \Delta_{00}$. Let $U$ be the deformation space of a generic curve with two non-disconnecting nodes, call it $C$. Let $\Delta_0(v), \Delta_0(w) \subset U$ be the two components of the divisor parametrizing singular curves; $v, w$ will be the corresponding vanishing cycles. Notice that $v \perp w$. The divisors $\Delta_0(v)$ and $\Delta_0(w)$ intersect transversely along $\Delta_{00}(U)$, the curve parametrizing curves with two non-disconnecting nodes. Let $\Delta_2(U) \subset U$ be the divisor mapping to $\Delta_2$, then $\Delta_2(U) \supset \Delta_{00}(U)$ and $\Delta_2(U)$ is transverse to $\Delta_0(v)$ and $\Delta_0(w)$. Let $C'$ be *a* fixed smooth reference fiber with no extra automorphisms. *A* point of $\pi^{-1}(m(C)) \cap \tilde{\Delta}_0$ corresponds to a subspace $H \subset \mathrm{Jac}(C')[p]$ orthogonal to $v$. We distinguish two cases:

(i) $w \in H$; in this case $H$ is fixed by both the Picard-Lefschetz transformations, the extra automorphism of $C$ interchanges the two vanishing cycles, hence it does not fix $H$. Hence a neighborhood of $m(C, H)$ is isomorphic to $U$; the isomorphism takes $\Delta_0(v)$ into $\tilde{\Delta}_0$ and $\Delta_0(w)$ into $\tilde{\tilde{\Delta}}_0$, and thus we see that $\tilde{\Delta}_0 \cap \tilde{\tilde{\Delta}}_0 \neq \emptyset$ and the intersection is transverse.

(ii) $w \notin H$ and $w \not\perp H$; in this case $H$ is not fixed by the Picard-Lefschetz transformation associated associated to $w$. We see that a neighborhood of $m(C, H)$ is isomorphic to the $p$th cover of $U$ totally branched over $\Delta_0(w)$; the isomorphism takes the ramification divisor into $R_0$ and the inverse image of $\Delta_0(V)$ into $\tilde{\Delta}_0$. Hence $\tilde{\Delta}_0$ and $R_0$ intersect transversely. Notice also that $\pi: \tilde{\Delta}_0 \to \Delta_0$ has ramification index $p$ at $\tilde{\Delta}_0 \cap R_0$.

DEFINITION 4.3. (i) Let $\tilde{\Delta}_{00} = \tilde{\Delta}_0 \cap \tilde{\tilde{\Delta}}_0$. (ii) Let $\tilde{R}_{00} = \tilde{\Delta}_0 \cap R_0$.

COROLLARY 4.5. Let $\pi: \tilde{\Delta}_0 \to \Delta_0$; then

$$\pi^*(\Delta_0)|_{\tilde{\Delta}_0} = \mathcal{O}_{\tilde{\Delta}_0}(\tilde{\Delta}_0) \otimes \mathcal{O}_{\tilde{\Delta}_0}(\tilde{\Delta}_{00} + p\tilde{R}_{00}).$$

REMARK. Notice that one should expect $\tilde{\Delta}_0$ to intersect $\tilde{\tilde{\Delta}}_0$ and $R_0$ above $\Delta_{00}$ because $\Delta_0$ and $\Delta_2$ are tangent along $\Delta_{00}$.

Recall that there is a natural map $\psi: \Delta_0 \to \mathbb{P}^1_j$. By composition we get a map $\psi\pi: \tilde{\Delta} \to \mathbb{P}^1_j$.

DEFINITION 4.4. (i) Let $F \in \mathrm{Pic}(\Delta_0)$ be the class of a fiber of $\psi$.

(ii) Let $\Sigma \in \mathrm{Pic}(\Delta_0)$ be the class of the section of $\psi$ defined as $\{m(E \cup F) \mid F$ is the singular elliptic curve$\}$.

Notice that in the isomorphism between $\Delta_0$ and $\bar{S}$ the section we just defined corresponds to the zero section. Notice also that $\Sigma = \Delta_0 \cap \Delta_1$ and that the intersection is transverse.

LEMMA 4.9. $\Delta_0|_{\Delta_0} \cong \frac{5}{6}F - \Sigma$.

*Proof.* By Corollary 1.2 we have that $\Delta_0 \cong 10\lambda - \Delta_1$. We have already remarked that $\lambda|_{\Delta_0} \cong aF$; Lemma 1.1 shows that $a = \frac{1}{12}$. By definition $\Delta_1|_{\Delta_0} \cong \Sigma$, hence $\Delta_0|_{\Delta_0} \cong \frac{5}{6}F - \Sigma$.

DEFINITION 4.5. Let $\pi\colon \tilde{\Delta}_0 \to \Delta_0$. (i) Let $\tilde{F} \in \mathrm{Pic}(\tilde{\Delta}_0)$ be defined as $\tilde{F} = \pi^*(F)$. (ii) Let $\tilde{\Sigma} \in \mathrm{Pic}(\tilde{\Delta}_0)$ be defined as $\tilde{\Sigma} = \pi^*(\Sigma)$.

LEMMA 4.10. $(n\pi^*(\alpha_p\lambda) - i\tilde{\Delta}_0|_{\tilde{\Delta}_0} \cong \left(\frac{\alpha_p}{12}n + \frac{i}{6}\right)\tilde{F} + i\tilde{\Sigma}$.

*Proof.* From Corollary 4.5 we get that

$$\tilde{\Delta}_0|_{\tilde{\Delta}_0} \cong \pi^*(\Delta_0) \otimes \mathcal{O}_{\tilde{\Delta}_0}(-\tilde{\Delta}_{00} - p\tilde{R}_{00}).$$

By Lemma 4.9. $\pi^*(\Delta_0) \cong \frac{5}{6}\tilde{F} - \tilde{\Sigma}$. Notice also that $\Delta_{00} \subset \Delta_0$ is a fiber of $\psi\colon \Delta_0 \to \mathbb{P}^1_j$ and that $\pi^*\mathcal{O}_{\Delta_0}(\Delta_{00}) \cong \mathcal{O}_{\tilde{\Delta}_0}(\tilde{\Delta}_{00} + p\tilde{R}_{00})$, hence $\mathcal{O}_{\tilde{\Delta}_0}(-\tilde{\Delta}_{00} - p\tilde{R}_{00}) \cong -\tilde{F}$. Concluding $\tilde{\Delta}_0|_{\tilde{\Delta}_0} \cong -\frac{1}{6}\tilde{F} - \tilde{\Sigma}$. The formula follows by recalling once again that $\lambda|_{\Delta_0} \cong \frac{1}{12}F$.

THEOREM 4.4.

$$\sum_{i=0}^{N} h^0((n\pi^*(\alpha_p\lambda) - i\tilde{\Delta}_0)|_{\tilde{\Delta}_0}) \leqslant \frac{1}{72}\left(13 - \frac{34}{p}\right)(p-1)^2 n^2 + O(n^2) \left(N = \frac{p-1}{p}n - 1\right).$$

*Proof.* We apply Proposition 4.5 to the surface $\tilde{\Delta}_0$, with $F = \tilde{F}$, $D = \tilde{\Sigma}$. Notice that $\pi\colon \tilde{\Delta}_0 \to \Delta_0$ has degree $p^2$, hence $\tilde{F}\cdot\tilde{\Sigma} = p^2$. Notice also that by Lemma 1.1 $\Sigma\cdot\Sigma < 0$, hence $\tilde{\Sigma}\cdot\tilde{\Sigma} < 0$, therefore $h^0(y\Sigma) = 1$ for all $y \geqslant 0$.

*Conditions imposed by* $\hat{\hat{\Delta}}_0$

We proceed in a way completely analogous to the previous one and we obtain

THEOREM 4.5.

$$\sum_{i=1}^{N} h^0((n\pi^*(\alpha_p\lambda) - i\tilde{\tilde{\Delta}}_0)|_{\tilde{\tilde{\Delta}}_0}) \leqslant \frac{1}{72}\left(13 - \frac{34}{p}\right)(p-1)^2 n^3 + \mathrm{O}(n^2) \left(N = \frac{p-1}{p}n - 1\right).$$

*Conditions imposed by* $\hat{\Delta}_2$.

First of all $\varphi: \hat{\Delta}_2 \to \tilde{\Delta}_2$ is an isomorphism because $\tilde{\Delta}_2$ does not intersect $R', R'', \Gamma', \Gamma''$ and is smooth along $\Omega$ (as we will show). Therefore we start by studying $\tilde{\Delta}_2$.

*Digression on* $\tilde{\Delta}_2$

Let $S$ be the surface introduced by Definition 4.1. As is easily seen $S$ can be viewed as the moduli space of triples $(E, P, B)$ where $E$ is (smooth) elliptic, $P \in E$ is the zero of the addition law and $B \in |2P|$. The surface $\bar{S}$ is the compactification obtained by allowing $E$ to become singular. Such a triple $(E, P, B)$ uniquely determines a double cover $f: C \to E$ with branch divisor $B$. Let $f^*: E \to \mathrm{Jac}(C)$ be the pull-back map, then $f^*(E[p]) \subset \mathrm{Jac}(C)[p]$ is a non-isotropic subspace ($p > 2$) fixed by the involution on $C$ associated to $f$. Therefore $m(C, f^*(E[p])) \in \tilde{\Delta}_2$. In this way we get a map $\rho: S \to \tilde{\Delta}_2$. Notice that if $m(C, H)$ is a generic point of $\tilde{\Delta}_2$ then there exist two maps $f_1: C \to E_1, f_2: C \to E_2$ of $C$ to elliptic curves. Let $\iota_1: C \to C, \iota_2: C \to C$ be the corresponding involutions, then one of the involutions, say $\iota_1$, will act as the identity on $H$, while the other will act as multiplication by $-1$. Hence our map $\rho$ associates to $(E_1, P_1, B_1)$ the couple $(C, H)$ and it associates to $(E_2, P_2, B_2)$ the couple $(C, H^{\perp})$. Therefore we see that $\rho$ is at least generically injective; in fact it is injective. Let's define $\bar{\rho}: \bar{S} \cdots > \tilde{\Delta}_2$ to be the rational map extending $\rho$ to $\bar{S}$. Notice that $\bar{\rho}$ is a morphism outside the point corresponding to $(E_\infty, P, B)$ where $E_\infty$ is the singular elliptic curve and $B \in |2P|$ has support on the node of $E_\infty$. In fact if $E_\infty$ is the singular elliptic curve and $B \in |2P|$ is not the node then $\bar{\rho}(E_\infty, P, B) = m(C, H)$ where $C$ is a genus two curve with two non-disconnecting nodes, i.e. $m(C, H) \in \Omega$.

Now we answer the following question: when does $\pi(m(C, H))$ belong to $\Delta_0$? (with $m(C, H) \in \tilde{\Delta}_2$). Of course, one possibility is that $m(C, H) \in \Omega$, and this is the case if and only if $\pi(m(C, H)) \in \Delta_{00}$. So we must examine $(\Delta_2 \cap \Delta_0) \backslash \Delta_{00}$. Either by an explicit examination of all curves with extra automorphisms (page 00) or by the theory of admissible coverings one gets that $m(C) \in (\Delta_2 \cap \Delta_0) \backslash \Delta_{00}$ if and only if $C = \tilde{E}/P_1 \sim P_2$ where $\tilde{E}$ is a smooth elliptic curve, $2P_1 \cong 2P_2$ and $P_1 \neq P_2$. If $C$ is such a curve we can describe the two maps $f_1: C \to E_1, f_2: C \to E_2$ as follows. We let $E_1$ be the quotient of $\tilde{E}$ by the subgroup $\{0, P_1 - P_2\}$; $f_1$ is induced from the quotient map. The branch divisor of $f_1$, call it $B_1$, has become a point with multiplicity 2, i.e. $B_1 = 2Q_1$. The map $f_1$ uniquely determines $P \in E_1$ such that $2P \cong 2Q_1$ and $P \neq Q_1$. The elliptic curve $E_2$ is singular; let $\tilde{f}_2: \tilde{E} \to \mathbb{P}^1$ be the map associated to $|2P_1| = |2P_2|$, we let $E_2 = \mathbb{P}^1/\tilde{f}_2(P_1) \sim \tilde{f}_2(P_2)$ and $f_2: C \to E_2$ is the map induced from $\tilde{f}_2$. This covering corresponds to a triple $(E_\infty, P, B)$ where $E_\infty$ is the singular elliptic curve and $B$ has support on the node of $E_\infty$, so this is exactly the divisor in $\tilde{\Delta}_2$ that we don't see on $\bar{S}$.

DEFINITION 4.6. Let $\Delta_{02} \subset \mathfrak{M}_2$ be the curve whose generic point is the moduli of $C = E/P_1 \sim P_2$ with $E$ smooth elliptic and $2P_1 \cong 2P_2$.

So $\Delta_2 \cap \Delta_0 = \Delta_{00} \cup \Delta_{02}$; $\Delta_0$ and $\Delta_2$ are tangent along $\Delta_{00}$ but we have the

*Claim.* $\Delta_2$ and $\Delta_0$ are transverse along $\Delta_{02} \backslash \Delta_{00}$.

*Proof.* Since $\Delta_2 \cap \Delta_0 = \Delta_{00} \cup \Delta_{02}$ it is enough to show that $\Delta_2$ and $\Delta_0$ are transverse at a generic point $m(C) \in \Delta_{02}$. So let $C = E/P_1 \sim P_2$, with $2P_1 \cong 2P_2$, $P_1 \neq P_2$ and $\mathrm{Aut}'(C) = \langle g \rangle \cong \mathbb{Z}/(2)$, with $g: C \to C$ induced from $\tilde{g}: E \to E$ defined as $\tilde{g}(P) = P + (P_1 - P_2)$. We can write $C$ as $y^2 = x^2(x^2 - a)(x^2 - b)$, then $g$ is given by $g^*(x, y) = (-x, y)$. Let $\alpha \in H^0(\Omega_C^1 \otimes \omega_C)$ be the torsion element and let $\beta = x^2(dx)^2/y^2$, $\gamma = x(dx)^2/y^2$, then $\{\alpha, \beta, \gamma\}$ is a basis of $H^0(\Omega_C^1 \otimes \omega_C)$. As is easily checked $g^*(\alpha, \beta, \gamma) = (\alpha, \beta, -\gamma)$. We identify the deformation space of $C$, call it $U$, with $H^0(\Omega_C^1 \otimes \omega_C)$. Let $m: U \to \mathfrak{M}_2$ be the moduli map, then $m(U) \cong U/\langle g \rangle$. Coordinates on $m(U)$ are given by $(\alpha, \beta, \gamma^2)$. A local equation for $\Delta_2$ is $(\gamma^2 = 0)$ and a local equation for $\Delta_0$ is $(\alpha = 0)$, hence we see that $\Delta_0$ and $\Delta_2$ are transverse along $\Delta_{02} = \{(0, \beta, 0)\}$.

*Claim.* Let $m(C, H) \in \tilde{\Delta}_2$ be such that $m(C) \in \Delta_{02} \backslash \Delta_{00}$, then if $m(C, H)$ is generic $\pi$ is a local isomorphism at $m(C, H)$.

*Proof.* Let $U$ be the deformation space of $C$, let $\Delta_0(U) \subset U$ be the divisor parametrizing singular curves and let $\Delta_2(U) \subset U$ parametrize curves with an involution whose quotient is an elliptic curve (i.e. $m(\Delta_2(U)) \subset \Delta_2$). We have just showed that $\Delta_0(U)$ and $\Delta_2(U)$ are transverse. Let $C'$ be a smooth curve in the universal family over $U$ such that $\mathrm{Aut}'(C')$ is generated by an involution with quotient an elliptic curve (i.e. $m(C')$ is a generic point of $\Delta_2$). Let $H_0 \subset \mathrm{Jac}(C')[p]$ be one of the two non-isotropic subspaces fixed by the involution. Let $\gamma$ be a loop in $\Delta_2(U) \backslash \Delta_0(U)$ generating $\pi_1(U \backslash \Delta_0(U))$; it acts by monodromy on subspaces of $\mathrm{Jac}(C')[p]$. Since $\gamma \subset \Delta_2(U)$ we must have that $\gamma(H_0) = H_0$ or $\gamma(H_0) = H_0^{\perp}$. Since $\gamma^p(H_0) = H_0$ we get that $\gamma(H_0) = H_0$ (notice that we assume $p > 2$). Therefore we see that a neighborhood of $m(C, H_0)$ is isomorphic to $U/\langle g \rangle$, i.e. to $m(U)$ and $\pi$ is a local isomorphism. Therefore if $m(C, H) \in \tilde{\Delta}_2$ and $m(C) \in \Delta_{02} \backslash \Delta_{00}$ then either $m(C, H) \in \tilde{\Delta}_0$ or $m(C, H) \in \tilde{\tilde{\Delta}}_0$.

DEFINITION 4.7. Let $\tilde{\Delta}_{02}, \tilde{\tilde{\Delta}}_{02} \subset \tilde{\Delta}_2$ be defined as $\tilde{\Delta}_{02} = \tilde{\Delta}_0 \cap \tilde{\Delta}_2$, $\tilde{\tilde{\Delta}}_{02} = \tilde{\tilde{\Delta}}_0 \cap \tilde{\Delta}_2$ respectively.

By the preceding discussion we see that $\tilde{\Delta}_2 \cap \pi^{-1}(\Delta_0) = \Omega \cup \tilde{\Delta}_{02} \cup \tilde{\tilde{\Delta}}_{02}$. Let $X_0(2) \subset \bar{S}$ be the curve parametrizing couples $(E, Q)$ where $2P \cong 2Q, P \neq Q$ ($P$ is the zero of the group law), then $\bar{\rho}: \bar{S} \cdots > \tilde{\Delta}_2$ takes $X_0(2)$ into $\tilde{\Delta}_{02}$. The curve $\tilde{\tilde{\Delta}}_{02}$ does not appear in $\bar{S}$ and we have that $\psi(\tilde{\tilde{\Delta}}_{02}) = \infty$.

PROPOSITION 4.8 *Let* $F \subset \tilde{\Delta}_2$ *be a fiber of* $\psi: \tilde{\Delta}_2 \to \mathbb{P}_j^1$, *then* $\tilde{\Delta}_0 \cdot F = 3$.

*Proof.* Let $F = \psi^{-1}(a)$ where $a \neq 0, 1728, \infty$. The map $\rho$ identifies $F \subset \tilde{\Delta}_2$ with $\rho^{-1}(F) \subset \bar{S}$. Let $j(E) = a$ and let $P \in E$ be the zero of the addition law, $\rho^{-1}(F)$

is identified with $|2P|$. The set $\tilde{\Delta}_{02} \cap F$ is identified with the set $\{Q \in E \mid 2Q \cong 2P, Q \neq P\}$ hence $\tilde{\Delta}_{02} \cdot F = 3$. Since $\Delta_0$ and $\Delta_2$ are transverse along $\Delta_{02}$ and $\pi$ is unramified above $\Delta_{02}$ we get that $\tilde{\Delta}_0$ and $\tilde{\Delta}_2$ are transverse along $\tilde{\Delta}_{02}$, hence $\tilde{\Delta}_0 \cdot F = (\tilde{\Delta}_{02} \cdot F)_{\tilde{\Delta}_2} = 3$.

PROPOSITION 4.9. (i) $R_0 \cap F = \emptyset$.

(ii) $\tilde{\tilde{\Delta}}_0 \cap F = \emptyset$.

*Proof.* We have showed that if $F = \psi^{-1}(a)$ with $a \neq \infty, x \in F$ and $\pi(x) \in \Delta_0$ then $\pi$ is unramified at $x$, hence $x \notin R_0$. We have also shown that if $x \in \tilde{\tilde{\Delta}}_{02}$ then $\psi(x) = \infty$, hence $F \cap \tilde{\tilde{\Delta}}_0 = \emptyset$.

An easy analysis will show that, on the other hand, if $x \in \tilde{\Delta}_2$ and $\pi(x) \in \Delta_1$ then $x \in R_1$, i.e. $\tilde{\Delta}_2 \cap \tilde{\Delta}_1 = \emptyset$ while $\tilde{\Delta}_2$ and $R_1$ intersect.

DEFINITION 4.8. Let $R_{12} = \tilde{\Delta}_2 \cap R_1$.

The curve $R_{12}$ is the locus of moduli $m(C, H)$ where $C = E_1 \cup E_2, E_1 \cong E_2$ and $H = E_1[p]$. It corresponds via $\bar{\rho}$ to the locus of moduli of couples $(E, P)$ where $P$ is chosen to be equal to zero of the addition law of $E$. It is also not difficult to check that $R_1$ and $\tilde{\Delta}_2$ are transverse along $R_{12}$. Hence we have

PROPOSITION 4.9. *Let* $F \subset \tilde{\Delta}_2$ *be a fiber of* $\psi: \tilde{\Delta}_2 \to \mathbb{P}^1_j$, *then* (i) $\tilde{\Delta}_1 \cdot F = 0$. (ii) $R_1 \cdot F = 1$.

PROPOSITION 4.10. *Let* $\psi: \tilde{\Delta}_2 \to \mathbb{P}^1_j$, *then* $\psi^*(\infty) = \Omega + 2\tilde{\tilde{\Delta}}_{02}$.

*Proof.* We know that $\psi^*(\infty) = x\Omega + y\tilde{\tilde{\Delta}}_{02}$ for some coefficients $x, y$. The map $\bar{\rho}: \bar{S} \cdots > \tilde{\Delta}_2$ is an isomorphism outside one point of $\bar{S}$, call it $R$. The curve $\psi^{-1}(\infty) \backslash R \subset \bar{S}$ is identified via $\rho$ with $\Omega \backslash T$, where $T$ is a point of $\Omega$. Therefore the coefficient of $\Omega$ in the expression $\psi^*(\infty) = x\Omega + y\tilde{\tilde{\Delta}}_{02}$ is equal to the multiplicity of the fiber over $\infty$ of $\psi: \bar{S} \to \mathbb{P}^1_j$. It is easily checked that this multiplicity is one, hence the coefficient of $\Omega$ is also one.

The involution $\iota: \mathscr{A}_2(p) \to \mathscr{A}_2(p)$ leaves $\tilde{\Delta}_2$ invariant, hence it acts on it; obviously we have that $\iota(\tilde{\tilde{\Delta}}_{02}) = \tilde{\Delta}_{02}$ $\iota(\tilde{\Delta}_{02}) = \tilde{\tilde{\Delta}}_{02}$, $\iota(\Omega) = \Omega$. Let $\psi\iota: \tilde{\Delta}_2 \to \mathbb{P}^1_j$, then $\psi^*(\infty) = \Omega + 2\tilde{\tilde{\Delta}}_{02}$ is equivalent to $(\psi\iota)^*(\infty) = \Omega + 2\tilde{\Delta}_{02}$. Let $E$ be the elliptic curve with equation $y^2 = x(x-1)(x-\lambda)$, with the point at infinity $P \in E$ as zero of the addition law. Consider $x$ as a rational function on $E$, then $(x)_\infty = 2P$. Let $(E, Q)$ be any couple with $Q \in E$, then we can identify the moduli point of $(E, Q)$ in $S$ with $x(Q)$ (we assume that $j(E) \neq 0, 1728$). Hence we can identify $x$ with a local parameter on $F = \psi^{-1}(j(E)) \subset \tilde{\Delta}_2$. Let us consider the function $\psi\iota$ restricted to $F$. Let $\alpha$ be a point on the $x$-axis, let $B_\alpha = x^*(\alpha)$. The double cover of $E$ with branch divisor $B_\alpha$ is given (in affine coordinates) by:

$$C_\alpha = \{(x, y, w) \mid y^2 = x(x-1)(x-\lambda), w^2 = x - \alpha\}.$$

The involution $\iota_1^*(x, y, w) = (x, y, -w)$ has quotient $E$. The other involution

$\iota_2^*(x, y, w) = (x, -y, -w)$ has as quotient the elliptic curve

$$E_\alpha = \{(x, z) \mid z^2 = x(x-\alpha)(x-1)(x-\lambda)\}.$$

As we said $\alpha$ is a parameter on $F$; an equation for $\tilde{\Delta}_{02} \cdot F$ is given by $(\alpha(\alpha-1)(\alpha-\lambda) = 0)$. The function $\psi\iota$ restricted to $F$ is given by $\psi\iota(\alpha) = j(E_\alpha)$. We see that $j(E_\alpha) = \infty$ if and only if $\alpha = 0$, or $\alpha = 1$, or $\alpha = \lambda$, and that at each of these points $j$ as a pole of multiplicity two. Hence $\overline{\psi\iota}$ has a pole of order two along $\tilde{\Delta}_{02}$, i.e. $\psi^*(\infty) = \Omega + 2\tilde{\tilde{\Delta}}_{02}$.

PROPOSITION 4.11.

$$\Omega \cdot \tilde{\Delta}_{02} = \Omega \cdot \tilde{\tilde{\Delta}}_{02} = 1.$$

*Proof.* Let $R \in \bar{S}$ be the moduli point of $(E_\infty, P, B)$ where $E_\infty$ is the singular elliptic curve and the support of $B$ is on the node of $E_\infty$. The map $\bar{\rho}: \bar{S} \cdots > \tilde{\Delta}_2$ is not defined only at $R$. Corresponding to this we have that the fiber of $\psi: \tilde{\Delta}_2 \to \mathbb{P}_j^1$ is the union of $\Omega$ and $\tilde{\tilde{\Delta}}_{02}$, and $\tilde{\tilde{\Delta}}_{02}$ is the divisor not appearing in $\bar{S}$. The divisor $\tilde{\tilde{\Delta}}_{02}$ intersects $\Omega$ in only one point because $\bar{\rho}$ is an isomorphism outside of $R$. Hence we must determine the multiplicity of the intersection between $\tilde{\tilde{\Delta}}_{02}$ and $\Omega$. Since $\iota: \tilde{\Delta}_2 \sim \tilde{\Delta}_2$ fixes $\Omega$ we have $\Omega \cdot \tilde{\tilde{\Delta}}_{02} = \Omega \cdot \tilde{\Delta}_{02}$. We can easily find a point in $\Omega \cap \tilde{\Delta}_{02}$, namely the point corresponding via $\bar{\rho}$ to the triple $(E_\infty, P, 2Q)$ where $Q$ is the unique non-zero point of order two on $E_\infty$. It is easily checked that the multiplicity of intersection of $\Omega$ and $\tilde{\Delta}_{02}$ at this point is one. The involution $\iota$ sends this point to $\tilde{\tilde{\Delta}}_{02} \cap \Omega$, hence we see that the multiplicity of the unique point of intersection of $\tilde{\tilde{\Delta}}_{02}$ and $\Omega$ is one, i.e. $\tilde{\tilde{\Delta}}_{02} \cdot \Omega = 1$. Applying the involution $\iota$ we get also $\tilde{\Delta}_{02} \cdot \Omega = 1$.

REMARK. The way to obtain $\tilde{\Delta}_2$ from $\bar{S}$ should be the following. Let $f_1: S_1 \to \bar{S}$ be the blow up of $\bar{S}$ with center $R$, let $E_1 \subset S_1$ be the exceptional divisor and let $R_1 = E_1 \cap \tilde{\Omega}$ where $\tilde{\Omega}$ is the strict transform of $\Omega$. Let $f_2: S_2 \to S_1$ be the blow up of $S_1$ with center $R_1$, and let $\tilde{E}_1 \subset S_2$ be the strict transform of $E_1$ so $\tilde{E}_1^2 = -2$. Let $f_3: S_2 \to S_3$ be the contraction of $\tilde{E}_1$ to a point, then $S_3$ is isomorphic to $\tilde{\Delta}_2$. If $E_2 \subset S_2$ is the exceptional divisor of $f_2$, then $f_3(E_2) \subset S_3$ will correspond to $\tilde{\tilde{\Delta}}_{02} \subset \tilde{\Delta}_2$.

THEOREM 4.6.

$$\sum_{i=0}^{N} h^0\left(\left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{5} nc_sD_s - \iota\hat{\Delta}_2\right)\Big|_{\hat{\Delta}_2}\right) = 0, \quad n = \tfrac{1}{2}n - 1.$$

*Proof.* We have already noticed that $\varphi: \hat{\Delta}_2 \to \tilde{\Delta}_2$ is an isomorphism. Let $F \subset \tilde{\Delta}_2$ be a fiber of $\psi: \tilde{\Delta}_2 \to p_j^1$ not lying over $j = \infty$; let $\hat{F} \subset \hat{\Delta}_2$ be defined as

$\hat{F} = \varphi^{-1}(F)$. Since $F$ does not meet any of the centers of the partial desingularizations through which $\hat{\mathscr{A}}_2(p)$ is obtained we have that

(i) $\varphi^*\pi^*(\alpha_p\lambda)\cdot\hat{F} = \pi^*(\alpha_p\lambda)\cdot F$
(ii) $\hat{\Delta}_0\cdot\hat{F} = \tilde{\Delta}_0\cdot F$
(iii) $\hat{\hat{\Delta}}_0\cdot\hat{\hat{F}} = \tilde{\tilde{\Delta}}_0\cdot F$
(iv) $\hat{\Delta}_1\cdot\hat{F} = \tilde{\Delta}_1\cdot F$
(v) $\hat{\Delta}_2\cdot\hat{F} = \tilde{\Delta}_2\cdot F$

We also know that $E'\cdot\hat{F} = E''\cdot\hat{F} = E_1'\cdot\hat{F} = E_1''\cdot\hat{F} = 0$, hence

$$(*)\quad \left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{5} nc_sD_s - i\hat{\Delta}_2\right)\cdot\hat{F}$$

$$= (n\pi^*(\alpha_p\lambda) - nc_3\tilde{\Delta}_1 - nc_4\tilde{\Delta}_0 - nc_5\tilde{\tilde{\Delta}}_0 - i\tilde{\Delta}_2)\cdot F.$$

By Corollary 1.3 $\lambda = \frac{1}{10}(\Delta_0 + \Delta_1)$ hence

$$\pi^*(\lambda)\cdot F = \tfrac{1}{10}(\pi^*(\Delta_0)\cdot F + \pi^*(\Delta_1)\cdot F) = \tfrac{1}{2}.$$

We also have $\tilde{\Delta}_1\cdot F = 0$, $\tilde{\Delta}_0\cdot F = 3$, $\tilde{\tilde{\Delta}}_0\cdot F = 0$. By Corollary 1.3 $\Delta_2 = 3\Delta_0 + 6\Delta_1$, hence

$$\tilde{\Delta}_2\cdot F = \pi^*(\Delta_2)\cdot F = 3\pi^*(\Delta_0)\cdot F + 6\pi^*(\Delta_1)\cdot F = 21.$$

Therefore

$$(n\pi^*(\alpha_p\lambda) - nc_3\tilde{\Delta}_1 - nc_4\tilde{\Delta}_0 - nc_5\tilde{\tilde{\Delta}}_0 - i\tilde{\Delta}_2)\cdot F = (-\tfrac{3}{2} - \tfrac{2}{p})n - 21i.$$

By $(*)$ we see that

$$h^0\left(\left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{5} c_snD_s - i\hat{\Delta}_2\right)\Big|_{\hat{\Delta}_2}\right) = 0$$

for all $i \geqslant 0$, hence the theorem is proved.

*Conditions imposed by $E_2$.*

We recall that $E_2 \subset \hat{\mathscr{A}}_2(p)$ is the exceptional divisor lying over $\Omega \subset \tilde{\mathscr{A}}_2(p)$.

DEFINITION 4.9. (i) Let $F_2 \subset E_2$ be a fiber of $\varphi\colon E_2 \to \Omega$.
(ii) Let $\Omega_2 \subset E_2$ be defined as $\Omega_2 = E_2 \cap \hat{\Delta}_2$. Let, as usual, $F_2, \Omega_2$ also be the

linear equivalence classes of $F_2, \Omega_2$ in $\mathrm{Pic}(E_2)$. A basis of $\mathrm{Pic}(E_2)$ is given by $\{F_2, \Omega_2\}$.

PROPOSITION 4.12. $(\Omega_2 \cdot \Omega_2)_{\hat{\Delta}_2} = -2$.

*Proof.* The map $\varphi: \hat{\Delta}_2 \to \tilde{\Delta}_2$ is an isomorphism; obviously $\varphi(\Omega_2) = \Omega$, hence we must show that $(\Omega \cdot \Omega)_{\tilde{\Delta}_2} = -2$. Let $F$ be a fiber of $\psi: \tilde{\Delta}_2 \to \mathbb{P}^1_j$, then $(\Omega \cdot F)_{\tilde{\Delta}_2} = 0$. By Proposition 4.10, $F \cong \Omega + 2\tilde{\tilde{\Delta}}_{02}$, hence $(\Omega \cdot \Omega)_{\tilde{\Delta}_2} + 2(\Omega \cdot \tilde{\tilde{\Delta}}_{02})_{\tilde{\Delta}_2} = 0$. By Proposition 4.11, $(\Omega \cdot \tilde{\tilde{\Delta}}_{02})_{\tilde{\Delta}_2} = 1$, hence $(\Omega \cdot \Omega)_{\tilde{\Delta}_2} = -2$. q.e.d.

PROPOSITION 4.13. (i) $\varphi^* \pi^*(\lambda) \cdot \Omega_2 = 0$.

(ii) $\hat{\Delta}_0 \cdot \Omega_2 = \hat{\hat{\Delta}}_0 \cdot \Omega_2 = 1$
(iii) $E_2 \cdot \Omega_2 = -2$
(iv) $E_2 \cdot F_2 = \frac{1}{2}p$
(v) $\hat{\Delta}_2 \cdot \Omega_2 = 0$.

*Proof.* (i) Since $\varphi: \Omega_2 \to \Omega$ is an isomorphism and $\pi: \Omega \to \Delta_{00}$ is two-to-one we have $\varphi^* \pi^*(\lambda) \cdot \Omega_2 = 2\lambda \cdot \Delta_{00}$. By Lemma 1.2 we get $\lambda \cdot \Delta_{00} = 0$, hence (i).

(ii) By Proposition 4.11.

(iii) Since $\Omega_2 = E_2 \cap \hat{\Delta}_2$ (they intersect transversely) we have $E_2 \cdot \Omega_2 = (\Omega_2 \cdot \Omega_2)_{\hat{\Delta}_2}$, so (iii) follows from Proposition 4.12.

(iv) This is a local computation; it follows from the type of singularity of $\mathscr{A}_2(p)$ along $\Omega$.

(v) We apply adjunction first to $E_2$ (or $\hat{\Delta}_2$) and then to $\Omega_2 \subset E_2$.

Notice that $\hat{\mathscr{A}}_2(p)$ is singular along a curve of $E_2$ but the curve does not meet $\Omega_2$, in fact both $\hat{\mathscr{A}}_2(p)$ and $E_2$ (or $\hat{\Delta}_2$) are smooth along $\Omega_2$ so we can apply adjunction. We have

$$\deg K_{\Omega_2} = (\mathrm{K}_{\hat{\mathscr{A}}_2(p)} + E_2 + \hat{\Delta}_2) \cdot \Omega_2, \quad \text{i.e.}$$

$$-2 = \left( \varphi^* \pi^* \left( \alpha_p \lambda + \frac{1}{p} \Delta_1 \right) + \frac{1}{2} \hat{\Delta}_2 + \frac{2}{p} E_2 - \frac{p-1}{p} \hat{\Delta}_0 - \frac{p-1}{p} \hat{\hat{\Delta}}_0 \right) \cdot \Omega_2.$$

By Proposition 4.13 $\varphi^* \pi^*(\alpha_p \lambda) \cdot \Omega_2 = 0, \hat{\Delta}_0 \cdot \Omega_2 = \hat{\hat{\Delta}}_0 \cdot \Omega_2 = 1, E_2 \cdot \Omega_2 = -2$. Since $\pi\varphi: \Omega_2 \to \Delta_{00}$ is two-to-one $\varphi^* \pi^*(\Delta_1) \cdot \Omega_2 = 2\Delta_1 \cdot \Delta_{00} = 2$. Hence we get

$$-2 = -2 + \tfrac{1}{2} \hat{\Delta}_2 \cdot \Omega_2$$

i.e. $\hat{\Delta}_2 \cdot \Omega_2 = 0$.

COROLLARY 4.6. (i) $\varphi^* \pi^*(\lambda)$ *is trivial on* $E_2$.
(ii) $\hat{\Delta}_0 |_{E_2} \cong \hat{\hat{\Delta}}_0 |_{E_2} \cong F_2$.

PROPOSITION 4.14. $E_2 |_{E_2} \cong -\frac{1}{2} p \Omega_2 - 2F_2$.

*Proofs.* We know that $E_2|_{E_2} \cong x\Omega_2 + yF_2$ for some $x, y \in \mathbb{Q}$. By Proposition 4.13 $x(\Omega_2 \cdot F_2)_{E_2} + y(F_2 \cdot F_2)_{E_2} = -\frac{1}{2}p$ and $x(\Omega_2 \cdot \Omega_2)_{E_2} + y(F_2 \cdot \Omega_2)_{E_2} = -2$. But $(\Omega_2 \cdot \Omega_2)_{E_2} = \hat{\Delta}_2 \cdot \Omega_2 = 0$, hence $x = -\frac{1}{2}p$, $y = -2$.

THEOREM 4.7.

$$\sum_{i=1}^{N} \mathrm{h}^0((\mathrm{n}\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{6} nc_s D_s - iE_2)|_{E_2}) = 0, \quad (\mathrm{N} = (1 - 2/\mathrm{p})\mathrm{n} - 1).$$

*Proof.* By the preceding propositions we get that

$$\left(n\varphi^*\pi^*(\alpha_p\lambda) - \sum_{s=1}^{6} nc_s D_s - iE_2\right)\Bigg|_{E_2} \cong \left(\frac{p}{2}i - \frac{n}{2}\right)\Omega_2 + \left(2i - 2n\frac{p-1}{p}\right)F_2$$

since $0 \leqslant i \leqslant n(1 - (2/p)) - 1$ the coefficient of $F_2$ is negative hence there are no non-zero sections, q.e.d.

Finally we can prove the

Main Theorem. *Let p be a prime greater or equal to* 17, *then* $\mathscr{A}_2(p)$ *is of general type.*

*Proof.* Putting together the results in this section according to the plan described at the beginning we get that

$$h^0(nK_{\mathscr{A}_2(p)}) \geqslant Q(p)n^3 + O(n^2), \ (n \text{ sufficiently divisible}),$$

where

$$Q(p) = \frac{1}{8640}\left(3 - \frac{10}{p}\right)^3 (p^4 + p^2) - \frac{1}{36}\left(13 - \frac{34}{p}\right)\left(p - 1\right)^2 - \frac{19}{32} - \frac{3}{7} + \frac{5}{16p}.$$

It is not difficult to check that if $p \geqslant 17$ $Q(p) > 0$ (while if $p \leqslant 13$ ($Q(p) < 0$), hence for $p \geqslant 17$, $\mathscr{A}_2(p)$ is of general type.

## Appendix

Let $S$ be an abelian surface and let $D$ be an ample primitive divisor on $S$, so $D$ defines a polarization on $S$. We say that the polarization is of degree $d$ if $\varphi_D: S \to \check{S}$ has degree $d^2$, or equivalently if $D^2 = 2d$, or equivalently if the Riemann form associated to $D$ has elementary divisors $\{1, d\}$. Let $\mathscr{A}_{2,d}$ be the moduli space of polarized abelian surfaces of degree $d$ [M-3].

PROPOSITION 5.1. *Let p be a prime; the moduli space $\mathcal{A}_2(p)$ is isomorphic to $\mathcal{A}_{2,p^2}$.*

*Proof.* Let $(T, D)$ be a polarized abelian surface of degree $p^2$, then $\operatorname{Ker}\varphi_D \cong \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^2)$. Let $J \subset \operatorname{Ker}\varphi_D$ be the subgroup of $p$-torsion elements, so $J \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$. Let $S = T/J$ and let $q: T \to S$ be the quotient map. The polarization on $T$ induces a principal polarization on $S$, i.e. there is a principal polarization $\Theta$ on $S$ such that $q^*(\Theta) \cong D$ (algebraic equivalence). The image $H = q\ (\operatorname{Ker}\varphi_D) \subset S[p]$ is a rank two subspace, non-isotropic for the Weil pairing. So we have canonically associated to the degree $p^2$ abelian surface $(T, D)$ the couple $(S, H)$ where $S$ is a p.p.a.s. and $H \subset S[p]$ a rank two subspace of $p$-torsion points non-isotropic for the Weil pairing. Hence we have a map $\alpha: \mathcal{A}_{2,p^2} \to \mathcal{A}_2(p)$, which in fact is an isomorphism. Let $\check{q}: \check{S} \to \check{T}$ be the dual of $q$, then $\operatorname{Ker}\check{q} = \varphi_\Theta(H)$. So let $\beta: \mathcal{A}_2(p) \to \mathcal{A}_{2,p^2}$ be the map obtained by associating to a couple $(S, H)$ the abelian surface $V = S/H$ with the degree $p^2$ polarization induced from $\Theta$, and let $\tau: \mathcal{A}_{2,p^2} \to \mathcal{A}_{2,p^2}$ be the involution obtained by associating to a degree $p^2$ abelian surface its dual (which is again an abelian surface of degree $p^2$). The map $\tau\beta: \mathcal{A}_2(p) \to \mathcal{A}_{2,p^2}$ is the inverse of $\alpha$.

COROLLARY 5.1. *Let p be a prime and let $p \geqslant 17$, then $\mathcal{A}_{2,p^2}$ is of general type.*

The map from $\mathcal{A}_{2,p^2}$ to $\mathcal{A}_2$ that we have defined in the course of proving Proposition 5.1 generalizes to a map $\tilde{g}_{n,k}: \mathcal{A}_{2,n^2k} \to \mathcal{A}_{2,k}$ for every $n, k$. In fact let $(T, D)$ be an abelian surface of degree $n^2k$, so $\operatorname{Ker}\varphi_D = \mathbb{Z}/(n^2k) \oplus \mathbb{Z}(n^2k)$. Let $J = \operatorname{Ker}\varphi_D \cap T[n]$, let $S = T/J$ and let $q: T \to S$ be the quotient map. The surface $S$ inherits a polarization of degree $k$. Therefore we get a map $\tilde{g}_{n,k}: \mathcal{A}_{2,n^2k} \to \mathcal{A}_{2,k}$. It is easy to check that $\tilde{g}_{n,k}$ is finite surjective.

COROLLARY 5.2. *Let $p|n$, $p \geqslant 17$, then $\mathcal{A}_{2,n^2}$ is of general type.*

The isomorphism class of a polarized abelian surface is not determined by its weight two Hodge structure. In fact the $H^2$ decomposition only determines a surface up to taking the dual. Let $\mathcal{G}_{2d}$ be the period space for weight two Hodge structures of degree $d$ abelian surfaces; the natural map $\varphi_d: \mathcal{A}_{2,p} \to \mathcal{G}_{2d}$ is of degree two. One can check that the maps $\tilde{g}_{n,k}$ descend to maps $g_{n,k}: \mathcal{G}_{2n^2k} \to \mathcal{G}_{2k}$, i.e. we have $g_{n,k}\varphi_{n^2k} = \varphi_k \tilde{g}_{n,k}$.

There is an analogous picture when we look at moduli of polarized $K3$ surfaces. Let $\mathcal{F}_{2d}$ be the moduli space of $K3$ surfaces of degree $2d$, i.e. the moduli space of couples $(S, E)$ where $S$ is a $K3$ surface and $E$ a numerically effective non divisible line bundle on $S$ of degree $2d$. By the Torelli theorem for polarized $K3$ surfaces and the surjectivity of the period map $\mathcal{F}_{2d}$ is isomorphic to the period space for (polarized) weight two Hodge structures. Hence we think of $\mathcal{F}_{2d}$ as analogous to $\mathcal{G}_{2d}$; in fact we can define maps $f_{n,k}: \mathcal{F}_{2n^2k} \to \mathcal{F}_{2k}$ which are analogous to the maps $g_{n,k}$.

*Definition of* $f_{n,k}: \mathscr{F}_{2n^2k} \to \mathscr{F}_{2k}$.

Let $L = H^3 \oplus (-E_8)^2$ be the $K3$ lattice. Let $\{e, f\}$ be a standard basis of one of the copies of $H$, i.e. let $e \cup e = f \cup f = 0$ and $e \cup f = 1$. Let $\alpha_n: L \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ be the linear map defined by $\alpha_n(e) = ne, \alpha_n(f) = (1/n)f, \alpha_n(v) = v$ if $v \perp \{e, f\}$. Notice that $\alpha_n$ preserves cup product. Let $p_{2d} = e + df$, so $p_{2d} \cup p_{2d} = 2d$. The classifying space for degree $2d$ $K3$ surfaces is given by

$$D_{2d} = \{[\omega] \in \mathbb{P}(L \otimes \mathbb{C}) | \omega \cup \omega = 0, \omega \cup p_{2d} = 0, \omega \cup \bar{\omega} > 0\}.$$

Let $\alpha_n$, by abuse of notation, be the induced map on $\mathbb{P}(L \otimes \mathbb{C})$; it is easily checked that $\alpha_n$ maps $D_{2n^2k}$ onto $D_{2k}$. Let $\Gamma_{2d}$ be the group of isometries of $L$ fixing $p_{2d}$; the moduli space $\mathscr{F}_{2d}$ is given by $\Gamma_{2d} \backslash D_{2d}$. It is not difficult to check that $\alpha_n$ commutes with the actions of $\Gamma_{2n^2k}$ and $\Gamma_{2k}$, i.e. $\alpha_n \Gamma_{2n^2k} \alpha_n^{-1} < \Gamma_{2k}$. Therefore $\alpha_n$ descends to a map $f_{n,k}: \mathscr{F}_{2n^2k}\ \mathscr{F}_{2k}$ (which is finite surjective). So the Main Theorem suggests that one could analyze the maps $f_{n,k}$, e.g. $f_{p,1}$, and establish that $\mathscr{F}_{2p^2}$ is of general type for large $p$. Actually we have proved that $\mathscr{A}_{2,p^2}$ is of general type (for $p \geqslant 17$); $\mathscr{F}_{2p^2}$ should be considered analogous to $\mathscr{A}_2(p)/\iota$, or alternatively $\mathscr{A}_2(p)$ is the analogous of the double cover of $\mathscr{F}_{2p^2}$ defined by $\tilde{\mathscr{F}}_{2,p^2} = D_{2p^2}/\{\gamma \in \Gamma_{2p^2} \mid \det \gamma = 1\}$, but it is reasonable to expect $\mathscr{A}_2(p)/\iota$ to be also of general type for $p$ big.

The computations developed in this paper could also be useful if one wants to determine the Kodaira dimension of the moduli space of couples $(S, J)$ with $S$ a p.p.a.s. and $J \subset S[p]$ a rank one subspace. Let $Y(p)$ denote this moduli space, there is an obvious map $\pi: Y(p) \to \mathscr{A}_2$. Let $\bar{Y}(p)$ be the natural toroidal compactification of $Y(p)$ such that $\pi$ extends to a finite surjective $\bar{\pi}: \bar{Y}(p) \to \bar{\mathscr{A}}_2$. An easy computation shows that $\bar{\pi}_*(K_{\bar{Y}(p)})$ is asymptotic to $\frac{3}{10}p^2(\Delta_0 + \Delta_1)$.

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