COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 72, nº 1 (1989), p. 1-31
<http://www.numdam.org/item?id=CM_1989__72_1_1_0>

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Compositio Mathematica **72**: 1–31, 1989.
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The tame fundamental group of an abelian variety and integral points

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Received 16 October 1987, accepted 26 January 1989

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic and let A/k be an abelian variety. In this paper we study the Grothendieck tame fundamental group $\pi_1^D(A)$ over an effective reduced divisor D on A. Let $\pi_1(A)$ be the étale fundamental group of A; Serre and Lang showed that $\pi_1(A)$ is canonically isomorphic to the Tate module $T(A)$ of A. Let I, called the inertia subgroup, be the kernel of the natural surjection $\pi_1^D(A) \to \pi_1(A)$.

As the group I may be very complicated, we here only attempt to determine the abelianisation I^{ab} of I i.e. the quotient of the profinite group I by the closure of the commutator subgroup. We then have that I^{ab} inherits a $T(A)$-module structure under conjugation in $\pi_1^D(A)$. One of our main results (Theorem 5.3, for notation see §§2,4) implies:

THEOREM 1.1. *The group I^{ab} is a continuous finitely generated module over the complete group algebra $\mathbb{Z}^\vee[[T(A)]]$. If the irreducible components of D are geometrically unibranch (in particular if they are normal), then there is an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules*

$$I^{\mathrm{ab}} = (\mathbb{Z}^\vee)^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}^\vee[[T(E_i)]],$$

where $E_i, i = 1, \ldots, s$, are elliptic curves attached to certain components of D.

This theorem is in contrast to Abhyankar's computation [1] of tame fundamental groups (see Theorem 3.2 below) where under suitable hypotheses, principally that D has only normal crossings, the inertia subgroup I is generated by cyclic subgroups attached to the irreducible components of D. The above theorem further shows that our results are a particular geometric form of Iwasawa theory, where $\mathbb{Z}^\vee[[T(A)]]$ is now the Iwasawa algebra and I^{ab} the corresponding Iwasawa module.

The motivation for this study is the Diophantine question of determining the integral points on a smooth projective variety V defined over a finitely generated extension field of the rationals $\mathbb{Q}$, relative to an ample divisor D on V. The idea is that if the inertia subgroup of $\pi_1^D(V)$ is large enough to compensate for any negativity of the canonical line bundle K_V of V then the integral points relative to D ought to lie on a proper closed subscheme of V (see §7, Remark (2), and the proof of Theorem 7.1). In this way, Siegel's theorem for integral points on curves can be derived from a theorem of Faltings (Mordell's conjecture see [4]). We show (Theorem 7.1) that a conjecture of Lang on the finiteness of integral points on an abelian variety, relative to an ample divisor, is a consequence of a higher dimensional analogue of Mordell's conjecture due to Bombieri and Lang independently, but which unfortunately is deep and unproved at the time of writing.

The contents of this paper are the following. In Section 2, Kummer theory is applied to abelianised tame fundamental groups of projective varieties; this is refined in Section 4 for tame fundamental groups with abelianised inertia groups. In Section 3, we summarise some results of Abhyankar. The particular case of abelian varieties is studied in Sections 5 and 6. First I^{ab} is determined for divisors whose components are geometrically unibranch, then in Section 6 the case of higher singularities is considered. Finally, Section 7 contains, as we have said, the application to integral points on abelian varieties.

I am indebted to Dr. P.M.H. Wilson for some correspondence on Section 7 of this paper. I thank my colleagues at the Université de Paris-Sud, where this paper was written, for their kind hospitality.

2. Tamely ramified abelian extensions

For Sections 2–6 we assume that k is an algebraically closed field of characteristic p, possibly zero. Let μ_m be the group of mth roots of unity of k, where m is any integer prime to p (if $p = 0$, we take this to mean that m is any integer). For any discrete abelian group A, we write

$$A^\vee = \varprojlim \operatorname{Hom}(A, \mu_m),$$

for the Tate twist of A, where the limit is taken over all integers m prime to p, and where $A^\vee$ is equipped with its projective limit topology. By abuse of notation, we identify $\mathbb{Z}^\vee$ with the completion of $\mathbb{Z}$ by subgroups of index prime to p and hence $\mathbb{Z}^\vee$ is equipped with a ring structure.

For any topological group G, we write G^{ab} for the quotient $G/[G, G]^{\mathrm{cl}}$ of G by the closure of the commutator subgroup; then G^{ab} is an abelian topological group.

Let X/k be a projective smooth irreducible k-scheme. Denote by

$\mathrm{Div}(X)$ is the group of Cartier divisors on X,
$P(X)$ is the group of principal divisors on X,
$\mathrm{Pic}(X) = \mathrm{Div}(X)/P(X)$,
$\mathrm{Pic}^0(X)$ is the group of linear equivalence classes of divisors algebraically equivalent to zero,
$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the Néron–Severi group of X,
$k(X)$ the function field of X,
$h^i(X, \mathscr{E}) = \dim_k H^i(X, \mathscr{E})$, for any coherent sheaf $\mathscr{E}$ on X.

As X/k is smooth, $\mathrm{Div}(X)$ is naturally isomorphic to the group of 1-codimensional cycles on X. Let D be an effective reduced divisor on X and let Δ be the subgroup of $\mathrm{Div}(X)$ generated by the irreducible components of the support of D. Let $\tilde{\Delta}$ be the subgroup of $\mathrm{Pic}(X)$ generated by Δ.

For any integer m prime to p, put

$$P_m = P(X) \cap (\Delta + m\,\mathrm{Div}(X)),$$

and let Γ_m be the subgroup of the multiplicative group $k(X)^*$ corresponding to P_m.

PROPOSITION 2.1. *The field $k_m = k(X)(\Gamma_m^{1/m})$ is the compositum of function fields of all finite separable abelian covers of X which are tamely ramified over D and of exponent m.*

Proof. Note first that $\Gamma_m \supset k(X)^{*m}$. Let $f \in \Gamma_m$. Let X_f/k be the normalisation of X in the field $k(X)(f^{1/m})$. Then we have $k(X_f) = k(X)(f^{1/m})$. By the theorem of the purity of the branch locus (e.g. SGA2, X.3.4), the normalisation map $X_f \to X$ is ramified only above D; hence it is tamely ramified over D, finite, separable and abelian of exponent m. Furthermore, $k_m/k(X)$ is a compositum of such field extensions $k(X_f)/k(X)$ as f runs over the elements of Γ_m.

Let $k'/k(X)$ be a finite separable abelian extension of $k(X)$ tamely ramified over D and of exponent m. By Kummer theory, $k' = k(X)(L^{1/m})$ where L is a subgroup of $k(X)^*$ containing $k(X)^{*m}$. Let $f \in L$. Let X'/k be the normalisation of X in $k(X)(f^{1/m})$. Then $k(X') = k(X)(f^{1/m})$ is an intermediate field of $k'/k(X)$ hence X' is tamely ramified over D. Let $(f) = \Sigma n_i W_i$ be the Weil divisor of f, where W_i are prime divisors of X and $n_i \in \mathbb{Z}$. As $X' \to X$ is étale outside D, n_i is divisible by m for all i such that W_i is not contained in Supp D. Hence

$$(f) \in \Delta + m\,\mathrm{Div}(X),$$

whence $f \in \Gamma_m$. We conclude that $L \subset \Gamma_m$ and the result follows.

PROPOSITION 2.2. *Let Δ' be the subgroup of $NS(X)$ generated by Δ. Then we have the exact sequence of topological groups*

$$0 \to (NS(X)/\Delta')^{\vee} \to NS(X)^{\vee} \to \Delta^{\vee} \to \pi_1^D(X)^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}} \to 0. \tag{2.1}$$

Proof. Let $k_{m,\mathrm{unram}}$ be the largest subfield of k_m which is unramified over $k(X)$. As k^* is a divisible group, we have an isomorphism

$$\Gamma_m/k(X)^{*m} \cong P_m/(P_m \cap mP(X)).$$

Proposition 2.1 and Kummer theory then give canonical isomorphisms of topological groups

$$\begin{aligned} &\mathrm{Gal}(k_m/k(X)) \cong \mathrm{Hom}(P_m/(P_m \cap mP(X)), \mu_m), \\ &\mathrm{Gal}(k_{m,\mathrm{unram}}/k(X)) \cong \mathrm{Hom}(P_m \cap m\,\mathrm{Div}(X))/(P_m \cap mP(X)), \mu_m), \end{aligned} \tag{2.2}$$

where the groups on the right and side are equipped with their natural profinite topologies.

Applying the functor $\mathrm{Hom}(-, \mu_m)$ to the exact sequence of m-torsion groups

$$0 \to \frac{P_m \cap m\,\mathrm{Div}(X)}{P_m \cap mP(X)} \to \frac{P_m}{P_m \cap mP(X)} \to \frac{P_m}{P_m \cap m\,\mathrm{Div}(X)} \to 0, \tag{2.3}$$

it remains exact. Therefore (2.2) and (2.3) give the exact sequence of topological groups

$$0 \to I_m \to \mathrm{Gal}(k_m/k(X)) \to \mathrm{Gal}(k_{m,\mathrm{unram}}/k(X)) \to 0, \tag{2.4}$$

where

$$I_m = \mathrm{Hom}(P_m/(P_m \cap m\,\mathrm{Div}(X)), \mu_m).$$

Applying the functor $\mathrm{Hom}(-, \mu_m)$ to the exact sequence of finite m-torsion groups

$$0 \to \frac{P_m}{P_m \cap m\,\mathrm{Div}(X)} \to \frac{\Delta}{\Delta \cap m\,\mathrm{Div}(X)} = \Delta \otimes \mathbb{Z}/m\mathbb{Z} \to \frac{\tilde{\Delta} + m\,\mathrm{Pic}(X)}{m\,\mathrm{Pic}(X)} \to 0$$

we get the exact sequence

$$0 \to \mathrm{Hom}((\tilde{\Delta} + m\,\mathrm{Pic}(X)/m\,\mathrm{Pic}(X), \mu_m) \to \mathrm{Hom}(\Delta, \mu_m) \to I_m \to 0. \tag{2.5}$$

We now take the projective limit over all integers m prime to p of the exact sequences (2.4) and (2.5). The latter remains exact because it is a limit of exact sequences of finite groups. The former also remains exact because it is already a projective limit commuting with $\varprojlim_m$ of exact sequences of finite groups. We then obtain the exact sequence

$$0 \to K \to \Delta^\vee \to \pi_1^D(X)^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}} \to 0, \tag{2.6}$$

where

$$K = \varprojlim \operatorname{Hom}((\tilde{\Delta} + m\operatorname{Pic}(X))/m\operatorname{Pic}(X), \mu_m).$$

As k is algebraically closed, $\operatorname{Pic}^\circ(X)(k)$ is a divisible group. Hence for any integer m we have an isomorphism

$$\operatorname{Pic}(X) \otimes \mathbb{Z}/m\mathbb{Z} \cong NS(X) \otimes \mathbb{Z}/m\mathbb{Z}.$$

As $NS(X)$ is a finitely generated Abelian group, the sequence

$$0 \to (\tilde{\Delta} + m\operatorname{Pic}(X))/m\operatorname{Pic}(X) \to NS(X) \otimes \mathbb{Z}/m\mathbb{Z} \to (NS(X)/\Delta') \otimes \mathbb{Z}/m\mathbb{Z} \to 0,$$

is an exact sequence of finite m-torsion groups. Applying the functor $\varprojlim \operatorname{Hom}(-, \mu_m)$ to this sequence, where m runs over all integers prime of p, it remains exact; hence we have the exact sequence

$$0 \to (NS(X)/\Delta')^\vee \to NS(X)^\vee \to K \to 0.$$

The Proposition now follows from this and (2.6).

3. Summary of some results of Abhyankar

Let X/k be an irreducible smooth projective k-scheme. Let D be an effective divisor on X with irreducible components $D_1, \dots, D_t$.

Let $f: X' \to X$ be a finite surjective generically étale morphism of k-schemes, where X' is normal and irreducible. Let $B(f)$ be the branch locus of f. Then $B(f)$ is a closed subscheme of X and has pure codimension 1 in the case where f is tamely ramified. Suppose further than f is Galois with Galois group G. For any $x \in X$, $y \in f^{-1}(x)$, let $I(y/x) \subset G$ be the inertia group of f at y; if $f^{-1}(x)$ is irreducible, we write $I(x)$ in place of $I(y/x)$ as there is no ambiguity.

PROPOSITION 3.1. ([1], Prop. 5 and 6) *Let $f: X' \to X$ be as above and suppose* *it is tamely ramified. Let D be a normal crossings divisor on X and assume that $B(f) \subset D$. We have:*

(1) *The irreducible components of $f^{-1}(D_i)$ are disjoint, for all i,*
(2) *If further we have $h^0(X, \mathcal{O}(D_i)) > 2$ for all i, then $f^{-1}(D_i)$ is irreducible for all i.*

For the proof of this and the next theorem, we refer to Abhyankar's paper (loc. cit.). The idea is that a local analysis shows that at most one component of $f^{-1}(D_i)$ passes through a given point of X', which proves (1). The hypothesis $h^0(X, \mathcal{O}(D_i)) > 2$ means that the linear system defined by D_i is not composite with a pencil hence Bertini's theorem implies that $f^{-1}(D_i)$ is connected; therefore (1) implies that $f^{-1}(D_i)$ is irreducible.

THEOREM 3.2. ([1], [18]). *Suppose that $f: X' \to X$, as above, is Galois with Galois group G and tamely ramified with branch locus D, which is a normal crossings divisor on X. Let $D_1, \ldots, D_t$ be the irreducible components of D and assume that*

$$h(X, \mathcal{O}(D_i)) > 2, \quad \text{for } i = 1, \ldots, t.$$

Let I be the subgroup of G generated by $I(D_i)$ for all $i = 1, \ldots, t$ ($I(D_i)$ is well defined by Proposition 3.1). We have:

(1) *The group $I(D_i)$ is a cyclic normal subgroup of I for all i and of order prime to p.*
(2) *The group I is t-step nilpotent.*
(3) *If D_i and D_j have a point in common, then $I(D_i)$ and $I(D_j)$ commute. In particular, if the components of D are pairwise connected* (i.e. $D_i \cap D_j \neq \emptyset$ *for all i,j*) *then I is Abelian.*

4. Abelianised inertia groups

Let D be an effective reduced divisor on the smooth irreducible projective k-scheme X. Let I be the kernel of the natural surjective homomorphism

$$\pi_1^D(X) \to \pi_1(X).$$

Let $[I, I]^{\text{cl}}$ be the closure of the commutator subgroup of the closed normal subgroup I; $[I, I]^{\text{cl}}$ is a normal subgroup of $\pi_1^D(X)$ and we put

$$\tilde{\pi}_1^D(X) = \pi_1^D(X)/[I, I]^{\text{cl}}.$$

The group $\tilde{\pi}_1^D(X)$ is profinite and classifies the finite coverings of X which are tamely ramified over D and with abelian inertia (i.e. the subgroup of the Galois group generated by the inertia groups over all components of D is abelian). The kernel of the natural surjective homomorphism $\tilde{\pi}_1^D(X) \to \pi_1(X)$ we call *the inertia subgroup*.

For any finite surjective morphism $f\colon X' \to X$ of smooth k-schemes, let $\Delta(f^{-1}D)$ be the subgroup of $\mathrm{Div}(X')$ generated by the irreducible components of the divisor $f^{-1}D$. The restriction of the map $f^*\colon \mathrm{Div}(X) \to \mathrm{Div}(X')$ to $\Delta(D)$ induces a homomorphism

$$\Delta(D) \to \Delta(f^{-1}D)$$

given by

$$Z \mapsto \sum_{Z_i \to Z} n_i Z_i$$

for any irreducible component Z of D, where n_i is the ramification index of the prime divisor Z_i over Z.

Fix a base point $x \in X(k)$. Let $\mathscr{C}$ be the category of finite étale coverings $f_\lambda\colon X_\lambda \to X, \lambda \in \Lambda$, of X equipped with a base point $x_\lambda \in X_\lambda$ with $f_\lambda(x_\lambda) = x$. Then $f_\lambda^{-1}D$ is an effective divisor on X_λ.

Each morphism of $\mathscr{C}$

$$f_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda,$$

is finite étale and $f^*_{\lambda\lambda'}$ induces a transition homomorphism

$$\Delta(f_{\lambda'}^{-1}D)^\vee \to \Delta(f_\lambda^{-1}D)^\vee.$$

We then have a filtered inverse system of continuous homomorphisms

$$NS(X_\lambda)^\vee \to \Delta(f_\lambda^{-1}D)^\vee$$

induced by sending a divisor in Δ to its image in the Néron-Severi group.

THEOREM 4.1. *We have an exact sequence of continuous homomorphsisms*

$$\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee \to \varprojlim_{\mathscr{C}} \Delta(f_\lambda^{-1}D)^\vee \to \tilde{\pi}_1^D(X) \to \pi_1(X) \to 0.$$

Proof. Let $\mathscr{C}' = \{g_\mu\colon Y_\mu \to X, \mu \in M\}$ be the category of *pointed* separable normal irreducible Galois coverings of X which are tamely ramified over D and with abelian inertia. Note that $\mathscr{C}$ is a full subcategory of $\mathscr{C}'$. By definition, we have

$$\tilde{\pi}_1^D(X) = \varprojlim_{\mathscr{C}'} \mathrm{Aut}(Y_\mu/X),$$

where the projective limit is taken over the category $\mathscr{C}'$.

Let $G_\mu = \mathrm{Gal}(k(Y_\mu)/k(X)) = \mathrm{Aut}(Y_\mu/X)$. Let I_μ be the subgroup of G_μ generated by the inertia groups over all the components of D; evidently, I_μ is a normal subgroup of G_μ. Let $f_\mu\colon X_\mu \to X$ be the normalisation of X in the fixed field of I_μ acting on $k(Y_\mu)$. Then f_μ is a finite Galois covering which is unramified in codimension 1. By the purity of the branch locus, f_μ is therefore étale and hence X_μ is a smooth projective k-scheme. Further, the map $Y_\mu \to X_\mu$ is a finite Abelian Galois covering with Galois group I_μ and tamely ramified over $f_\mu^{-1}D$ and which does not factor through any non-trivial irreducible étale covering of X_μ. We call X_μ the *étale closure* of X in Y_μ.

The assignment of Y_μ to the étale closure X_μ of X in Y_μ defines a functor $F\colon \mathscr{C}' \to \mathscr{C}$ whose restriction to the subcategory $\mathscr{C}$ of $\mathscr{C}'$ is the identity functor.

We have the exact sequence

$$0 \to \mathrm{Aut}(Y_\mu/X_\mu) = I_\mu \to \mathrm{Aut}(Y_\mu/X) = G_\mu \to \mathrm{Aut}(X_\mu/X) \to 0.$$

Taking the projective limit of this sequence over the category $\mathscr{C}'$ gives the exact sequence

$$0 \to \varprojlim_{\mathscr{C}'} I_\mu \to \tilde{\pi}_1^D(X) \to \varprojlim_{\mathscr{C}'} \mathrm{Aut}(X_\mu/X) \to 0.$$

The functors $\mathscr{C} \to \mathscr{C}' \to \mathscr{C}$, where the first is inclusion and the second is F, give rise to two homomorphisms

$$\pi_1(X) = \varprojlim_{\mathscr{C}} \mathrm{Aut}(X_\mu/X) \to \varprojlim_{\mathscr{C}'} \mathrm{Aut}\,(X_\mu/X) \to \pi_1(X)$$

whose composite is the identity. But the first morphism is a surjection, because any compatible system of elements of $\mathrm{Aut}(X_\mu/X)$ over $\mathscr{C}'$ is induced by a compatible system of elements of $\mathrm{Aut}(X_\lambda/X)$ over $\mathscr{C}$. It follows that $\pi_1(X) \to \varprojlim_{\mathscr{C}'} \mathrm{Aut}(X_\mu/X)$ is an isomorphism.

Hence the above exact sequence becomes

$$0 \to \varprojlim_{\mathscr{C}'} I_\mu \to \tilde{\pi}_1^D(X) \to \pi_1(X) \to 0. \tag{4.1}$$

For each X_λ, let J_λ be the kernel of $\pi_1^{f_\lambda^{-1}D}(X_\lambda)^{\mathrm{ab}} \to \pi_1(X_\lambda)^{\mathrm{ab}}$. By Proposition 2.2, we have the exact sequence

$$NS(X_\lambda)^\vee \to \Delta(f_\lambda^{-1}D)^\vee \to J_\lambda \to 0.$$

Hence this gives the exact sequence

$$\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee \to \varprojlim_{\mathscr{C}} \Delta(f_\lambda^{-1}D)^\vee \to \varprojlim_{\mathscr{C}} J_\lambda = \varprojlim_{\mathscr{C}'} J_\mu \to 0. \tag{4.2}$$

We have a natural surjection $J_\mu \to I_\mu$. But J_μ is a projective limit of abelian groups $\mathrm{Aut}(Z/X_Z)$ where Z runs over the pointed separable normal irreducible abelian Galois covering of X_μ, tamely ramified over $f_\mu^{-1}D$, and where the étale closure of X_μ in Z is X_Z. It follows easily that the induced map of projective limits

$$\varprojlim_{\mathscr{C}'} J_\mu \to \varprojlim_{\mathscr{C}'} I_\mu$$

is an injection and hence an isomorphism; this combined with the exact sequences (4.1) and (4.2) proves the theorem.

COROLLARY 4.2. *Suppose that D is a normal crossings divisor on the smooth irreducible projective k-scheme X. Assume that $h^0(X, \mathcal{O}(D_i)) > 2$ for each irreducible component D_i of D. Then we have the exact sequence*

$$\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee \to \Delta(D)^\vee \to \tilde{\pi}_1^D(X) \to \pi_1(X) \to 0$$

Further, if the components of D are pairwise connected then this sequence remains exact with $\tilde{\pi}_1^D(X)$ replaced by $\pi_1^D(X)$.

Proof. For the first part, by Proposition 3.1, the components of D remain irreducible in each étale covering of X. Hence the term in the exact sequence of Theorem 4.1 given by

$$\varprojlim_{\mathscr{C}} \Delta(f_\lambda^{-1}D)^\vee.$$

is isomorphic to $\Delta(D)^\vee$. The last part follows directly from Theorem 3.2.

Theorem 4.1 expresses $\tilde{\pi}_1^D(X)$ as a group extension of

$$\pi_1(X) = \varprojlim_{\mathscr{C}} \mathrm{Aut}_X(X_\lambda)$$

by the abelian inertia group

$$I = \mathrm{coker}\,(\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee \to \varprojlim_{\mathscr{C}} \Delta(f_\lambda^{-1}D)^\vee),$$

where the projective limits are compatible. Hence $\pi_1(X)$ acts on I by conjugation. This action can be described explicitly: the group $\mathrm{Aut}_X(X_\lambda)$ permutes the components of $f_\lambda^{-1}(D)$, and therefore permutes the set of standard generators of $\Delta(f_\lambda^{-1}D)^\vee$. The action of $\pi_1(X)$ on I is obtained by taking the projective limit over all λ of these permutation actions. In particular, as I is abelian it is a continuous module over the *complete group algebra* $\mathbb{Z}^\vee[[\pi_1(X)]]$ of the profinite group $\pi_1(X)$:

$$\mathbb{Z}^\vee[[\pi_1(X)]] \underset{\text{def}}{=\!=} \varprojlim \mathbb{Z}^\vee[\pi_1(X)/U],$$

as U runs over all open normal subgroups of $\pi_1(X)$.

COROLLARY 4.3. *Under the hypotheses of Theorem* 4.1, *suppose that the divisor* D has t *irreducible components. Then the inertia subgroup of* $\tilde{\pi}_1^D(X)$ *is a finitely generated continuous* $\mathbb{Z}^\vee[[\pi_1(X)]]$*-module with t generators.*

Proof. Let $D_1, \ldots, D_t$ be the irreducible components of D. Clearly

$$\Delta(f_\lambda^{-1}D)^\vee = \bigoplus_{i=1}^{t} \Delta(f_\lambda^{-1}D_i)^\vee$$

is a $\mathbb{Z}^\vee[\mathrm{Aut}_X(X_\lambda)]$-module with t generators, as $\mathrm{Aut}_X(X_\lambda)$ acts transitively on the components of $f_\lambda^{-1}D_i$ for each i. The result follows by taking the projective limit over all λ.

Examples. (1) Let C/k be a smooth projective curve of genus g and let D be a reduced effective divisor on C consisting of $n \geqslant 0$ closed points. Let $\pi_1^D(C)^{(p)}$ be the prime-to-p part of $\pi_1^D(C)$. Then $\pi_1^D(C)^{(p)}$ is the profinite completion, with respect to subgroups of finite index prime to p, of the group with $2g+n$ generators $a_1, b_1, \ldots, a_g, b_g, \sigma_1, \ldots, \sigma_n$ and one relation [5]

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \sigma_1 \ldots \sigma_n = 1.$$

The inertia subgroup (i.e. the kernel of $\pi_1^D(C)^{(p)} \to \pi_1(C)^{(p)}$) is then the closed normal subgroup topologically generated by $\sigma_1, \ldots, \sigma_n$ and their conjugates. Therefore the inertia subgroup of $\tilde{\pi}_1^D(C)^{(p)}$ is the free $\mathbb{Z}^\vee[[\pi_1(C)^{(p)}]]$-module on n generators

$$\bigoplus_{i=1}^{n} \mathbb{Z}^\vee[[\pi_1(C)^{(p)}]]\sigma_i.$$

(2) Let D be an effective divisor on $\mathbb{P}^n$ with irreducible components of degrees

$d_1, \ldots, d_t$. Then we have

$$\tilde{\pi}_1^D(\mathbb{P}^n) = \frac{(\mathbb{Z}^\vee)^t}{(d_1, \ldots, d_t)}.$$

As $\mathbb{P}^n$ is simply connected, it has no non-trivial étale coverings. The above result therefore follows from Theorem 4.1 as the map

$$NS(\mathbb{P}^n)^\vee = \mathbb{Z}^\vee \to \Delta(D)^\vee$$

is given by $1 \to (d_1, \ldots, d_t)$.

[This result, under the additional hypotheses of Corollary 4.2, is due to Abhyankar. For the case where $n = 2$, Edmunds [3] has shown further that if the divisor D is an irreducible curve of degree d with at worst ordinary double point singularities, then the maximal solvable quotient group of $\pi_1^D(\mathbb{P}^2)$ is cyclic and hence is isomorphic to $\mathbb{Z}^\vee/d\mathbb{Z}^\vee$.]

5. Abelian varieties

Let A/k be an abelian variety. Denote by

$$[n]: A \to A,$$

the endomorphism of multiplication by the integer n on A. Let

$$T(A) = \prod_l T_l(A) = \varprojlim_n (\ker[n])(k),$$

be the Tate module of A. The Serre-Lang theorem then states [15] that there is a canonical isomorphism

$$\pi_1(A) \cong T(A).$$

In this section we study the group $\tilde{\pi}_1^D(A)$ for a divisor D on A.

Let Y be a prime divisor on A. Denote by $\mathrm{stab}_A\, Y$ the subgroup scheme of A stabilising Y. Thus $\mathrm{stab}_A\, Y$ has a finite number of connected components and the component of the identity $\mathrm{stab}_A^\circ\, Y$ is a sub-abelian variety of A. If D is an effective divisor on A, then by Corollary 4.3, the inertia subgroup of $\tilde{\pi}_1^D(A)$ is a finitely generated continuous $\mathbb{Z}^\vee[[T(A)]]$-module.

PROPOSITION 5.1. *Let D be an effective divisor, with irreducible components $D_1, \dots, D_t$ on the abelian variety A/k. Then the inertia subgroup of $\tilde{\pi}_1^D(A)$ is a $\mathbb{Z}^\vee[[T(A)]]$-module isomorphic to*

$$\bigoplus_{i=1}^{t} \varprojlim_{n} \Delta([n]^{-1} D_i)^\vee .$$

We shall require the following lemma in the proof of this proposition.

LEMMA 5.2. *Let $f: V_1 \to V_2$ be a finite surjective purely inseparable morphism of irreducible k-schemes. Then the induced map*

$$NS(f)^\vee : NS(V_1)^\vee \to NS(V_2)^\vee$$

is an isomorphism.

Proof. Let p^s be the degree of f. The sth power of the Frobenius gives a morphism of schemes

$$g: V_2 \to V_1,$$

which on Spec k induces the map $x \mapsto x^{p^s}$, $x \in k$. The composites $g \circ f$ and $f \circ g$ are endomorphisms of V_1 and V_2, respectively, given by $x \mapsto x^{p^s}$. Therefore the endomorphisms $NS(g \circ f)$, $NS(f \circ g)$ of the Néron-Severi groups are just multiplication by p^s. Hence the kernel and cokernel of $NS(f)$ are annihilated by p^s. As the functor $\varprojlim \mathrm{Hom}(-, \mu_m)$ kills p-torsion, the result follows.

Proof of Proposition 5.1. Fix a base point $x \in A(k)$ and let $\mathscr{C}$ be the category of finite étale coverings

$$f_\lambda: X_\lambda \to A, \lambda \in \Lambda,$$

of A equipped with a base point $x_\lambda \in X_\lambda$ such that $f_\lambda(x_\lambda) = x$. Then each X_λ is again an abelian variety (Theorem of Serre-Lang, see [15] p. 167). By composing f_λ with a suitable translation on X_λ, we may assume that it is an isogeny.

Let K_n° be the connected component of the identity of the kernel of the endomorphism $[n]$ on A. Then K_n° is a finite commutative group scheme over k which is connected. The endomorphism $[n]$ then factorises as

$$A \xrightarrow{h_n} A/K_n^\circ = X_n \xrightarrow{\sigma_n} A,$$

where σ_n is a finite étale covering and h_n is a purely inseparable morphism. From the determination of the fundamental group of A (see [15] pp. 167–171) we have that the pointed coverings (X_n, σ_n, x_n), where $x_n = h_n(x) \in X_n$, are cofinal in $\mathscr{C}$. Further, the transition maps between the objects of this cofinal subset are induced by the endomorphisms $[n]$: if n, m are integers, where n divides m, then there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "h_m"] \arrow[d, "{[m/n]}"] & X_m \arrow[d, "f_{mn}"] \arrow[dr, "\sigma_m"] & \\ A \arrow[r, "h_n"'] & X_n \arrow[r, "\sigma_n"'] & A \end{tikzcd} \tag{5.1}$$

The projective limit of duals of Néron-Severi groups

$$\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee$$

may then be computed over this cofinal subset (X_n, σ_n, x_n). As the morphism $[n]$ induces multiplication by n^2 on the Néron-Severi group of A, the functor $NS(-)^\vee$ applied to the diagram (5.1) gives the following commutative diagram of abelian groups:

$$\begin{tikzcd} NS(A)^\vee \arrow[r, "NS(h_m)^\vee"] \arrow[d, "m^2/n^2"] & NS(X_m)^\vee \arrow[d, "NS(f_{mn})^\vee"] \\ NS(A)^\vee \arrow[r, "NS(h_n)^\vee"'] & NS(X_n)^\vee \end{tikzcd}$$

By lemma 5.2, the homomorphisms $NS(h_m)^\vee$ are isomorphisms for all m. Therefore the image of $NS(X_m)^\vee$ under $NS(f_{mn})^\vee$ is contained in $(m^2/n^2)NS(X_n)^\vee$. But $NS(X_n)^\vee$ is a finitely generated $\mathbb{Z}^\vee$-module, for all n, and the intersection of all ideals $m\mathbb{Z}^\vee$ over all integers m is the zero ideal of $\mathbb{Z}^\vee$. Hence

$$\varprojlim_{\mathscr{C}} NS(X_\lambda)^\vee \cong \varprojlim_{n} NS(X_n)^\vee \cong 0. \tag{5.2}$$

By Theorem 4.1 we then have the exact sequence of continuous homomorphisms

$$0 \to \varprojlim_{n} \Delta(\sigma_n^{-1} D)^\vee \to \tilde{\pi}_1^D(A) \to \pi_1(A) \to 0.$$

As h_n is a finite surjective purely inseparable morphism, the pullback via h_n of an irreducible divisor of X_n is an irreducible divisor of A with multiplicity a power of p. Hence there is a canonical isomorphism

$$\Delta([n]^{-1}D)^\vee \to \Delta(\sigma_n^{-1}D)^\vee,$$

for all n, compatible with the morphisms f_{mn} and $[n]$. Therefore we have an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules

$$\varprojlim_n \Delta([n]^{-1}D)^\vee \cong \varprojlim_n \Delta(\sigma_n^{-1}D)^\vee,$$

whence the result.

THEOREM 5.3. *Let D be an effective divisor, with irreducible components $D_1, \ldots, D_t$, on the abelian variety A/k. Then the inertia subgroup of $\tilde{\pi}_1^p(A)$ is a finitely generated $\mathbb{Z}^\vee[[T(A)]]$-module isomorphic to*

$$\bigoplus_{i=1}^{t} \mathbb{Z}^\vee[[T(A/\mathrm{stab}_A^\circ D_i)/T_i]],$$

where T_i is a closed subgroup of the Tate module $T(A/\mathrm{stab}_A^\circ D_i)$ depending only on D_i and where this Tate module has the natural quotient $T(A)$-module structure.

Further:

(a) *if* $\dim \mathrm{stab}_A^\circ D_i = \dim A - 1$ *then* $T_i = 0$;
(b) *if* $\dim \mathrm{stab}_A^\circ D_i \neq \dim A - 1$ *and D_i is geometrically unibranch* (*in particular if it is normal*) *then*

$$T_i = T(A/\mathrm{stab}_A^\circ D_i).$$

Proof. By Proposition 5.1, the interia subgroup of $\tilde{\pi}_1^p(A)$ is a direct sum of $\mathbb{Z}^\vee[[T(A)]]$-modules depending only on the irreducible components of D. Thus we may reduce to the case where D is irreducible.

Let

$$f: A \to A/\mathrm{stab}_A^\circ D = B,$$

be the quotient map, which is a proper morphism with irreducible fibres. Then $f(D)$ is a closed irreducible subscheme of the abelian variety B of codimension 1. Further, $f^{-1}f(D) = D$, as a closed subscheme. Hence f induces a bijection between the irreducible components of $[n]_A^{-1}D$ on A and the irreducible components of $[n]_B^{-1}f(D)$ on B, which also commutes with the permutation action of $T(A)$ on these components. Therefore we have an isomorphism of $\mathbb{Z}^\vee$-modules

$$\Delta([n]_A^{-1}D)^\vee \to \Delta([n]_B^{-1}f(D))^\vee, \tag{5.3}$$

which is compatible with the action of $T(A)$ and the endomorphisms $[n]$. The action of $T(A)$ on $\Delta([n]_B^{-1}f(D))^\vee$ factors through the quotient $T(A) \to T(B)$ and hence we have an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules

$$\Delta([n]_B^{-1}f(D))^\vee \cong \mathbb{Z}^\vee[T(B)/T_n], \tag{5.4}$$

for some open subgroup T_n of $T(B)$. The first part of the theorem now follows by taking the projective limit over n.

For the last part, we may assume that $D = D_i$ is irreducible. If $\dim \operatorname{stab}_A^\circ D = \dim \mathrm{A} - 1$, then D is a translate of a sub-abelian variety of A and hence is geometrically unibranch. Thus we may also assume that the divisor D is geometrically unibranch in every case. Let $\sigma\colon X \to A$ be a finite étale covering of A. Let $f\colon A \to A/\operatorname{stab}_A^\circ D = B$ be the quotient map, as above. We obtain the commutative diagram of homomorphisms of abelian varieties (cf. beginning of the proof of Proposition 5.1)

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{\bar\sigma} & B \end{array}$$

where f' is a quotient morphism of X by an abelian subvariety and where $\bar\sigma$ is a finite étale surjective morphism.

As D is geometrically unibranch and σ is étale, the irreducible components of $\sigma^{-1}D$ are disjoint [6, IV_4, Cor. 18.10.3]. Therefore the irreducible components of $\bar\sigma^{-1}f(D)$ are disjoint and in $1-1$ correspondence with those of $\sigma^{-1}D$, since the fibres of f' are irreducible.

Suppose first that $\dim \operatorname{stab}_A^\circ D = \dim A - 1$. Then B is an elliptic curve and $f(D)$ is a point. Therefore $\bar\sigma^{-1}f(D)$ is a finite set of points equal in number to the degree of $\bar\sigma$.

Hence

$$\Delta([n]_B^{-1}f(D))^\vee \cong \mathbb{Z}^\vee[T(B)/nT(B)],$$

therefore we have an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules

$$\varprojlim_n \Delta([n]_A^{-1}D)^\vee \cong \mathbb{Z}^\vee[[T(B)]],$$

as required.

Suppose finally that $\dim \mathrm{stab}_A^\circ D \neq \dim A - 1$ i.e. $\dim B \geqslant 2$. As $\mathrm{stab}_B f(D)$ is a finite group scheme, $f(D)$ is an ample divisor on B (cf. [15] p. 60). Let $\mathcal{O}(f(D))$ be the line bundle on B associated to the divisor $f(D)$. We then have an exact sequence of sheaves on X'

$$0 \to \bar{\sigma}^* \mathcal{O}(-f(D)) \to \mathcal{O}_{X'} \to \mathcal{O}_V \to 0, \tag{5.5}$$

where $\mathcal{O}_V$ is the structure sheaf of the closed subscheme $V = \bar{\sigma}^{-1} f(D)$ of X'. As $\bar{\sigma}$ is étale, V is reduced; hence

$$h^0(V, \mathcal{O}_V) = \dim_k H^0(V, \mathcal{O}_V) \tag{5.6}$$

is the number of connected components of V, and is therefore equal to the number of irreducible components of $\sigma^{-1}D$. Part of the long exact sequence of cohomology of the above sequence of sheaves is

$$H^0(X', \mathcal{O}_{X'}) \to H^0(V, \mathcal{O}_V) \to H^1(X', \bar{\sigma}^* \mathcal{O}(-f(D))). \tag{5.7}$$

As $\bar{\sigma}$ is finite, $\bar{\sigma}^* \mathcal{O}(f(D))$ is ample on X'. Hence as $\dim X' = \dim B \geqslant 2$, the vanishing theorem for line bundles on abelian varieties ([15] p. 150) gives

$$H^1(X', \bar{\sigma}^* \mathcal{O}(-f(D))) = 0. \tag{5.8}$$

Therefore

$$h^0(V, \mathcal{O}_V) = h^0(X', \mathcal{O}_{X'}) = 1. \tag{5.9}$$

Hence $\sigma^{-1}D$ is irreducible. We then have an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules

$$\varprojlim_n \Delta([n]_A^{-1} D)^\vee \cong \mathbb{Z}^\vee,$$

which completes the proof.

This theorem shows in particular that if the irreducible components of D are geometrically unibranch, then the inertia subgroup of $\tilde{\pi}_1^D(A)$ is $\mathbb{Z}^\vee[[T(A)]]$-isomorphic to a direct sum of $(\mathbb{Z}^\vee)^r$ and $\bigoplus_i \mathbb{Z}^\vee[[T(E_i)]]$ for certain elliptic curves E_i (this proves Theorem 1.1).

COROLLARY 5.4. *Let D be an effective reduced divisor with t irreducible components on the abelian variety A/k. Consider the following statements:*

(a) *the group $\tilde{\pi}_1^D(A)$ is Abelian;*

(b) *the inertia subgroup of* $\tilde{\pi}_1^D(A)$ *is isomorphic to* $(\mathbb{Z}^\vee)^t$ *as a* $\mathbb{Z}^\vee[[T(A)]]$*-module*;

(b′) *the inertia subgroup of* $\tilde{\pi}_1^D(A)$ *is a finitely generated* $\mathbb{Z}^\vee$*-module*;

(c) $\dim \mathrm{stab}_A^\circ D' < \dim A - 1$, *for all irreducible components* D' *of* D.

Then we have the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (b′) $\Rightarrow$ (c). *Further, if the irreducible components of* D *are geometrically unibranch then* (a), (b), (b′), (c) *are equivalent.*

Proof. The group $\tilde{\pi}_1^D(A)$ is an extension of the Abelian group $T(A)$ by the abelian inertia group I. Therefore $\tilde{\pi}_1^D(A)$ is Abelian if and only if I has a trivial $T(A)$-action under conjugation. In the notation of Theorem 5.3, we then have that $\tilde{\pi}_1^D(A)$ is Abelian if and only if $T_i = T(A/\mathrm{stab}_A^\circ D_i)$ for all i. The implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (b′) $\Rightarrow$ (c) now follow from Theorem 5.3, as the complete group algebra of the Tate module of an elliptic curve is not a finite $\mathbb{Z}^\vee$-module.

Suppose now that the irreducible components of D are geometrically unibranch. By Theorem 5.3, (c) implies that $T(A)$ acts trivially on the inertia group I; hence (c) $\Rightarrow$ (a) and (a), (b), (b′), (c) are equivalent.

6. Abelian varieties: study of singular divisors

Let $D \neq 0$ be an effective reduced divisor on the abelian variety A/k. In Section 5, we determined the inertia subgroup I of $\tilde{\pi}_1^D(A)$ when the irreducible components of D are geometrically unibranch. Here we consider divisors with higher singularities.

By Theorem 5.3, the continuous $\mathbb{Z}^\vee[[T(A)]]$-module I is a direct sum of submodules depending only on the irreducible components of D. Hence to study the module structure of I we may reduce to the case where D is irreducible: this is assumed for the rest of this section. There is then an isomorphism of $\mathbb{Z}^\vee[[T(A)]]$-modules (cf. Theorem 5.3)

$$I \cong \mathbb{Z}^\vee[[T]],$$

where T is a quotient group of $T(A)$ by a closed subgroup. Further, by (5.3) and (5.4) there is a decreasing filtration ordered by *divisibility* $\{T_n\}_{n\geq 1}$ of open subgroups T_n of T:

$$T_m \supset T_n \quad \text{if} \quad m|n, \qquad \bigcap_{n\geq 1} T_n = \{0\},$$

such that we have a compatible isomorphisms of $\mathbb{Z}^\vee[[T(A)]]$-modules

$$\mathbb{Z}^\vee[T/T_n] \cong \Delta([n]_A^{-1}D)^\vee. \tag{6.1}$$

Note that $\{T_n\}_{n\geqslant 1}$ lies above the filtration $\{nT\}_{n\geqslant 1}$, again ordered by divisibility.
We define

$$h(n) = |T/T_n|;$$

thus $h(n)$ is a 'Hilbert function' of the profinite group T. The main aim of this section is to try to determine $h(n)$ (cf. Theorem 6.4), and thus T.
Let

$$f\colon A \to A/\mathrm{stab}_A^\circ D = B$$

be the quotient map by the abelian variety $\mathrm{stab}_A^\circ D$. Put $d = \dim B$; then $d > 0$.

DEFINITION 6.1. *The singularities of D have an embedded normalisation in B* if there is a composite of blowings up $q\colon V \to B$ of smooth subschemes lying over the singular locus $f(D)_{\mathrm{sing}}$ of $f(D)$ for which the strict transform $\tilde{D}$ of $f(D)$ by q is an irreducible geometrically unibranch closed subscheme of V.

REMARKS. (1) Note that q is proper, surjective, birational, and is an isomorphism outside a closed subscheme of B of codimension $\geqslant 2$.
(2) If char. $k = 0$, then Hironaka's embedded resolution of singularities gives such a morphism q (in a much stronger form).
(3) One may define the same notion of an embedded normalisation for any irreducible closed subscheme of B (see also Remarks (5) and (6) below).

PROPOSITION 6.2.(1) *The quantity $h(n)$ is the number of irreducible components of the divisor $[n]_A^{-1}D$ on A, disregarding multiplicity.*
(2) *If $d \geqslant 2$, then, for every integer $n \geqslant 1$, each irreducible component of the support of the divisor $[n]_A^{-1}D$ on A intersects every other. In particular, the support of $[n]_A^{-1}D$ is connected.*
(3) *There is a constant c with the following property.* Let $x \in A$ *be a closed point of A. Then for any integer $n \geqslant 1$, the number of irreducible components of the support of $[n]_A^{-1}D$ which pass through x is $\leqslant c$.*

Proof. (1) This is immediate from (6.1).
(2) Let X, Y be two irreducible components of the support of $[n]_B^{-1}f(D)$. As X, Y are translates of each other in B, they are numerically equivalent. Hence the intersection multiplicity $X\cdot Y^{\dim B-1} = Y^{\dim B}$. By the Riemann-Roch theorem for B, $Y^{\dim B} > 0$, as Y is ample. Hence $X \cap Y \neq \emptyset$, as required.
(3) In the notation of the proof of Proposition 5.1, the morphism $[n]$ factorises as $\sigma_n \circ h_n$, where σ_n is étale and h_n is purely inseparable. It then suffices to show that the number of irreducible components of the support of $\sigma_n^{-1}D$ which pass through a closed point x is bounded independently of x and n. Put $y = \sigma_n x \in A$. In a neighbourhood of y, the divisor D is cut out by the rational function f. Let $\mathcal{O}_{A,y}^{\mathrm{sh}}$ be the strict henselisation of the local ring of A at y. Then the number N of

irreducible components of $\sigma_n^{-1}D$ which pass through x is at most the number $M(y)$ of irreducible components of Spec $\mathcal{O}^{\mathrm{sh}}_{A,y}/(f)$. But $M(y)$ is easily seen to be an upper semi-continuous function of y. It follows that $M(y)$, and thus N, is bounded from above independently of y. The result follows.

REMARK 4. With respect to Proposition 6.2.(2), the argument (5.5)–(5.9) in the proof of Theorem 5.3 can also be used to show that $[n]_A^{-1}D$ is connected.

LEMMA 6.3. *Suppose that the singularities of D have an embedded normalisation $q\colon V \to B$ in B. Then for all integers n prime to $p =$ char. k, we have*

$$1 - d \leqslant h(n) - \sum_{\mathscr{L}} h^1(V, \mathscr{L} \otimes \mathcal{O}(-\tilde{D})) \leqslant 1, \tag{6.2}$$

where the sum is over all isomorphism classes of line bundles $\mathscr{L}$ on V of order dividing n.

Proof. With the notation of the proof of Proposition 5.1, for any integer $n \neq 0$, the endomorphism $[n]_B$ of B factors as

$$B \xrightarrow{h_n} X_n \xrightarrow{\sigma_n} B,$$

where X_n is an abelian variety and σ_n, h_n are étale and purely inseparable morphisms, respectively. We then obtain the Cartesian square

$$\begin{array}{ccc} X_n & \xrightarrow{\sigma_n} & B \\ \uparrow{\scriptstyle q_n} & & \uparrow{\scriptstyle q} \\ V_n & \xrightarrow[f_n]{} & V \end{array}$$

which defines the other terms in the diagram.

As f_n is étale, the closed subscheme Z_n of V_n attached to $f_n^*\tilde{D}$ is reduced. Moreover, as $\tilde{D}$ is irreducible and geometrically unibranch, the irreducible components of Z_n are disjoint [6, IV$_4$, Corollaire 18.10.3]; in fact, they are in $1-1$ correspondence with the irreducible components of $\sigma_n^* f(D)$ and thus of $[n]_B^* f(D)$ (disregarding multiplicity in the latter), as h_n is purely inseparable. Proposition 6.2.(1) then shows that

$$h(n) = h^0(Z_n, \mathcal{O}_{Z_n}). \tag{6.3}$$

As f_n is finite and étale, the long exact sequence of cohomology of the exact sequence of $\mathcal{O}_{V_n}$-modules

$$0 \to f_n^*\mathcal{O}(-\tilde{D}) \to \mathcal{O}_{V_n} \to \mathcal{O}_{Z_n} \to 0,$$

becomes

$$0 \to H^0(V_n, \mathcal{O}_{V_n}) \to H^0(Z_n, \mathcal{O}_{Z_n}) \to H^1(V_n, f_n^*\mathcal{O}(-\tilde{D})) \to H^1(V_n, \mathcal{O}_{V_n}) \to \ldots \quad (6.4)$$

Further, the morphism q is a composite of blowings up of smooth subschemes of a smooth scheme, therefore (cf. [7, Ch.V, Proposition 3.4] for the case of surfaces; the general case is similar)

$$q_*\mathcal{O}_V = \mathcal{O}_X, \quad R^i q_*\mathcal{O}_V = 0 \qquad \text{for } i > 0.$$

As taking higher direct images of sheaves by a separated morphism commutes with étale base change (cf. [7, Ch. III, Proposition 9.3]), we then have

$$q_{n*}\mathcal{O}_{V_n} = \mathcal{O}_{X_n}, \quad R^i g_{n*}\mathcal{O}_{V_n} = 0 \qquad \text{for } i > 0. \tag{6.5}$$

The Leray spectral sequence

$$H^i(X_n, R^j q_{n*}\mathcal{O}_{V_n}) \Rightarrow H^{i+j}(V_n, \mathcal{O}_{V_n})$$

thus degenerates. As X_n is an abelian variety of dimension d, this gives

$$h^i(V_n, \mathcal{O}_{V_n}) = h^i(X_n, \mathcal{O}_{X_n}) = \binom{d}{i}, \quad \text{for all } i \geqslant 0. \tag{6.6}$$

Therefore (6.3), the exact sequence (6.4) and the equations (6.6) now give

$$1 - d \leqslant h(n) - h^1(V_n, f_n^*\mathcal{O}(-\tilde{D})) \leqslant 1. \tag{6.7}$$

The morphism f_n: $V_n \to V$ is a quotient map of V_n by the finite abelian group scheme $\ker(\sigma_n)$. If n is prime to the characteristic of k, $\ker(\sigma_n)$ has order prime to p and therefore (see [15], p. 72)

$$f_{n*}\mathcal{O}_{V_n} = \bigoplus_{\mathcal{L}} \mathcal{L}, \tag{6.8}$$

where $\mathcal{L}$ runs over all elements of the kernel of f_n^*: $\mathrm{Pic}(V) \to \mathrm{Pic}(V_n)$. But V is a birational blowing up of the abelian variety B. Hence the torsion subgroup of $\mathrm{Pic}(V)$ is canonically isomorphic to the torsion subgroup of $\mathrm{Pic}(B)$ and so $\ker(f_n^*)$ consists of all line bundles of V of order dividing n. The Leray spectral sequence for the finite morphism f_n and (6.8) give

$$h^1(V_n, f_n^*\mathcal{O}(-\tilde{D})) = h^1(V, \mathcal{O}(-\tilde{D}) \otimes f_{n*}\mathcal{O}_{V_n}) = \sum_{\mathscr{L}} h^1(V, \mathscr{L} \otimes \mathcal{O}(-\tilde{D})).$$

The lemma now follows from this and (6.7).

REMARK 5. Let X be any irreducible closed subscheme of B, and $q\colon V \to B$ an embedded normalisation of X (see Remark 3 above). Let $\mathscr{I}$ be the ideal sheaf of V defining the strict transform $\tilde{X}$ = closure of $q^{-1}(X - X_{\text{sing}})$ in B. Let $f(n)$ be the number of irreducible components of $[n]_B^{-1}X$. Then the above proof of Lemma 6.3 shows, just by changing $\mathcal{O}(-\tilde{D})$ to $\mathscr{I}$ throughout, that for any integer n prime to char. k,

$$1 - d \leqslant f(n) - \sum_{\mathscr{L}} h^1(V, \mathscr{L} \otimes \mathscr{I}) \leqslant 1, \tag{6.9}$$

where the sum is over all isomorphism classes of line bundles $\mathscr{L}$ on V of order dividing n.

Using Raynaud's proof of the Manin–Mumford conjecture [17], we may describe the behaviour of $h(n)$, in the case where k has characteristic zero.

We say that a map $P\colon \mathbb{Z} \to \mathbb{Z}$ is *quasi-polynomial* if there is an integer $m > 0$ and a polynomial $q(X) \in \mathbb{Z}(X)$ so that for all integers $n > 0$,

$$P(n) = \begin{cases} q(n), & \text{if } \ m \mid n, \\ 0, & \text{if } \ m \nmid n. \end{cases}$$

We call m the associated integer and q the associated polynomial of P.

THEOREM 6.4. *Suppose that char. $k = 0$. Then there are quasi-polynomials $P_1(n), \ldots, P_r(n)$ such that*

(1) *the corresponding associated integers $m_1, \ldots, m_r$ are distinct,*
(2) *the corresponding associated polynomials $q_1(X), \ldots, q_r(X)$ are even of degree $\leqslant 2d$ and with zero constant term,*
(3) *we have as $n \to \infty$,*

$$h(n) = \sum_{i=1}^{r} P_i(n) + 0(1).$$

We shall need the following elementary lemma, whose proof is omitted.

LEMMA 6.5. *Let H be a sub-abelian variety of an abelian variety A over the algebraically closed field k of characteristic zero. Let $A(k)_{\text{tors}}$ denote the torsion subgroup of $A(k)$. Let $t \in A(k)_{\text{tors}}$. Then the number of n-torsion elements if*

$\{t + H(k)\} \cap A(k)_{\text{tors}}$ is a quasi-polynomial function of n with associated polynomial $X^{2\dim H}$ and associated integer the least integer N such that $Nt \in H(k)_{\text{tors}}$.

Proof of theorem 6.4. By Hironaka, the singularities of D have an embedded normalisation $q\colon V \to B$ in B. The dual abelian variety $\hat{B}$ is the isomorphic to the Picard variety of V; let $\mathscr{P}$ be a Poincaré line bundle on $V \times \hat{B}$. For $x \in \hat{B}(k)$, let $\mathscr{P}_x$ be the corresponding line bundle induced on V. We then have that the map

$$\begin{aligned} &\hat{B}(k) \to \mathbb{Z} \\ &x \mapsto h^1(V, \mathscr{P}_x \otimes \mathcal{O}(-\tilde{D})), \end{aligned}$$

is upper semi-continuous, for the Zariski topology cf. [15, Corollary, p. 50]. Let

$$W_0 = \hat{B} \supset W_1 \supset \ldots \supset W_i \supset \ldots$$

be the reduced closed subschemes of $\hat{B}$ defined by

$$x \in W_i(k) \Leftrightarrow h^1(V, \mathscr{P}_x \otimes \mathcal{O}(-\tilde{D})) \geqslant i. \tag{6.10}$$

As $\hat{B}$ is a noetherian scheme, the sequence $\{W_i\}_{i \geqslant 0}$ is stationary from some point on; it follows that $W_i = \emptyset$ for all sufficiently large i, from the definition of W_i, and hence that $h^1(V, \mathscr{P}_x \otimes \mathcal{O}(-\tilde{D}))$ is bounded as x varies over $\hat{B}(k)$.

From Raynaud's theorem [17], for each integer $i \geqslant 0$, there are a finite number of sub-abelian varieties A_{ij} of $\hat{B}$ and elements t_{ij} in the subgroup $\hat{B}(k)_{\text{tors}}$ of torsion elements of the group $\hat{B}(k)$ such that

$$W_i(k) \cap \hat{B}(k)_{\text{tors}} = \bigcup_{j \geqslant 0} (t_{ij} + A_{ij}(k)_{\text{tors}}). \tag{6.11}$$

Observe that the intersection of two sets of the form $t_i + A_i(k)_{\text{tors}}$, $i = 1, 2$, where A_i are sub-abelian varieties of $\hat{B}$ and $t_i \in \hat{B}(k)_{\text{tors}}$, is again of the same form: $t_3 + A_3(k)_{\text{tors}}$. If $S_1, \ldots, S_m$ are subsets of a given set, we have the elementary combinatorial lemma:

$$\left|\bigcup_{i=1}^{m} S_i\right| = \sum_{i=1}^{m} |S_i| - \sum_{i \neq j} |S_i \cap S_j| + \sum_{\substack{i \neq j \neq k \\ i \neq k}} |S_i \cap S_j \cap S_k| \ldots \tag{6.12}$$

Hence from (6.11) and Lemma 6.5, we have that the number $f_i(n)$ of n-torsion elements of $W_i(k) \cap \hat{B}(k)_{\text{tors}}$ can be expressed as a finite alternating sum of quasi-polynomial functions of n. By (6.10) and Lemma 6.3, we have

$$1 - d \leqslant h(n) - \sum_{i=0}^{\infty} i(f_i(n) - f_{i+1}(n)) \leqslant 1. \tag{6.13}$$

The theorem follows as $f_i(n) \equiv 0$ for all sufficiently large i.

REMARK. 6. Let X be an irreducible closed subscheme of B and assume that char. $k = 0$. Let $f(n)$ be the number of irreducible components of $[n]_B^{-1}X$, disregarding multiplicity. Let V and $\mathscr{I}$ be as in Remark 5 above. Then (6.9) can be used in the proof of Theorem 6.4 which then shows, simply by replacing $\mathcal{O}(-\tilde{D})$ by $\mathscr{I}$ throughout, that

$$f(n) = \sum_{i=1}^{s} P_i(n) + 0(1), \quad \text{as } n \to \infty,$$

where the P_i are quasi-polynomial functions with the properties of Theorem 6.4.

REMARK 7. The quasi-polynomial functions $P_1, \ldots, P_r$, with the properties (1), (2) and (3) of Theorem 6.4 are *unique* up to permutation. This follows from the elementary assertion that a polynomial in $\mathbb{Z}[X]$ which is bounded on an arithmetic progression in $\mathbb{Z}$ must be constant.

REMARK 8. Assume that everything is defined over a number field K. Fix a projective embedding of $\hat{B}$ by some very ample symmetric divisor. Then the effective version of Raynaud's theorem [8] implies that the associated integers of the quasi-polynomials $P_i(n)$ of Theorem 6.4 are bounded by a constant c depending only on B, K, the degrees of the subvariaties W_j, and the degrees of a set of defining equations of the W_j.

Using this, one can then give bounds for the absolute values of the coefficients of the associated polynomials of the $P_i(n)$ of Theorem 6.4 in terms of c, B, K and the number of abelian varieties, maximal with respect to inclusion, contained in the finitely many translates of the W_i by the torsion points of order bounded by c. The 0(1) term in Theorem 6.4 can be similarly bounded.

It would be interesting to bound all the latter quantities in terms of invariants of V and D.

THEOREM 6.6 *Assume that* char. $k = 0$. *Suppose that B is a simple abelian variety. Then the inertia subgroup I of $\tilde{\pi}_1^D(A)$ is $\mathbb{Z}^\vee[[T(A)]]$-isomorphic to one of the following:*

(a) $\mathbb{Z}^\vee[[T(A)]]$;
(b) $\mathbb{Z}^\vee[G]$, *where G is a finite quotient group of $T(A)$.*

Proof. As B is simple, so is $\hat{B}$. Hence the subvarieties W_i defined in the proof of Theorem 6.4 contain no proper sub-abelian varieties of B. It follows from (6.11) and that the W_i from a decreasing sequence, that either $W_1 = \hat{B}$ or $W_i(k) \cap \hat{B}(k)_{\text{tors}}$ is a finite set for all $i \geqslant 1$. We consider these two cases separately.

Case 1. $W_1 = \hat{B}$.
The inequalities (6.13) give

$$1 - d \leqslant h(n) - \sum_{i=1}^{\infty} f_i(n) \leqslant 1.$$

As $f_1(n)$ is the number of n-torsion points of W_1, this then gives $f_1(n) = n^{2d}$ for all $n \in \mathbb{Z}$. But, $h(n) \leqslant n^{2d}$ for all n, hence $h(n) = n^{2d} + 0(1)$, for all $n \in \mathbb{Z}$. On the other hand, $h(n) = |T/T_n|$, where T_n is a subgroup of T containing nT. Hence $h(n)$ divides n^{2d} for all $n \in \mathbb{Z}$. It follows that $h(n) = n^{2d}$ for all $n \gg 0$. Hence that $T/T_n = T(A)/nT(A)$ for all $n \gg 0$. We conclude that $T = T(A)$, which gives case (a) of the theorem.

Case 2. $W_i \cap \hat{B}(k)_{\text{tors}}$ is finite for all i.
We then have $f_i(n) = 0(1)$, for all $i \geqslant 1$. The inequalities (6.13) then give $h(n) = |T/T_n| = 0(1)$, as $n \to \infty$. Hence T/T_n is stationary for all $n \gg 0$. Therefore $I = \mathbb{Z}^\vee[T/T_n]$, where n is any sufficiently large integer, which gives case (b) of the theorem and finishes the proof.

THEOREM 6.7. *Suppose that* char.$k = 0$ *and* $d \geqslant 2$. *Let* $q\colon V \to B$ *be an embedded normalisation of* D *in* B. *Suppose that* $\tilde{D}^2 \cdot H^{d-2} > 0$, *for some ample divisor* H *on* V *and* $\tilde{D} \cdot C \geqslant 0$ *for all curves* C *on* V. *Then the inertia subgroup* I *of* $\tilde{\pi}_1^D(A)$ *is* $\mathbb{Z}^\vee[[T(A)]]$-*isomorphic to* $\mathbb{Z}^\vee$.

REMARK 9. The hypotheses $\tilde{D}^2 \cdot H^{d-2} > 0$, for some ample divisor H and $\tilde{D} \cdot C \geqslant 0$, for all curves C, are implied by the conditions: $\kappa(\tilde{D}, V) \geqslant 2$ and the sheaf $\mathcal{O}(n\tilde{D})$ is generated by its global sections for all $n \gg 0$.

Proof. The Viehweg-Kawamata vanishing theorem [21] under the given hypotheses implies

$$h^1(V, \mathscr{L} \otimes \mathcal{O}(-\tilde{D})) = 0,$$

for all torsion line bundles $\mathscr{L}$ on V. Hence $h(n) = 1$ for all n, by Lemma 6.3, whence the result.

Take the particular case where $d = 2$. We have $q^*f(D) = \tilde{D} + E$, where $E \geqslant 0$ consists only of exceptional divisors. The integer E^2 then depends only on the configuration and attached multiplicities of the exceptional divisors in the embedded normalisation of D in B.

COROLLARY 6.8. *Suppose that* char. $k = 0$, $d = 2$, *and* $h^0(A, \mathcal{O}(D)) > -E^2/2$. *Then the inertia subgroup* I *of* $\tilde{\pi}_1^D(A)$ *is* $\mathbb{Z}^\vee[[T(A)]]$*-isomorphic to* $\mathbb{Z}^\vee$.

Proof. If C is any irreducible curve of V, $C \neq \tilde{D}$, then $C \cdot \tilde{D} \geqslant 0$. Further,

$$\tilde{D}^2 = (q^*f(D) - E)^2 = f(D)^2 + E^2 = 2h^0(B, \mathcal{O}(f(D)) + E^2,$$

where we use the Riemann-Roch theorem for the abelian variety B in the last equality. As f is proper with connected fibres $f_*\mathcal{O}_A = \mathcal{O}_B$, by Stein factorisation. Hence

$$\begin{aligned} h^0(A, \mathcal{O}(D)) &= h^0(A, f^*\mathcal{O}(f(D))) = h^0(B, \mathcal{O}(f(D)) \otimes f_*\mathcal{O}_A) \\ &= h^0(B, \mathcal{O}(f(D))). \end{aligned}$$

Therefore

$$\tilde{D}^2 = 2(h^0(A, \mathcal{O}(D)) + E^2/2) > 0.$$

The result follows from Theorem 6.7.

Examples.

(a) Assume that char. $k = 0$, $d = 2$, and that $f(D)$ is an irreducible curve whose only singularities are ordinary cusps and m ordinary multiple points. If $h^0(A, \mathcal{O}(D)) > m/2$, then the inertia subgroup I of $\tilde{\pi}_1^D(A)$ is $\mathbb{Z}^\vee[[T(A)]]$-isomorphic to $\mathbb{Z}^\vee$ (for one may take $q\colon V \to B$ to be the blowing up at the m multiple points and apply Corollary 6.8).

(b) Suppose there is an isogeny $s\colon X \to B$ of abelian varieties with kernel the finite abelian group G, and a smooth irreducible divisor C on X for which $\mathrm{stab}_G\, C = \{1\}$ and such that $f(D) = s(C)$. It follows that $s^{-1}(f(D))$ consists of the $|G|$ distinct translates of C by G. Using Theorem 5.3, it is then easy to check that the inertia subgroup I of $\tilde{\pi}_1^D(A)$ is $\mathbb{Z}^\vee[[T(A)]]$-isomorphic to $\mathbb{Z}^\vee[G]$.

7. Application to integral points

Let k be a finitely generated field extension of the rationals $\mathbb{Q}$. Let R be a domain which is a finitely generated $\mathbb{Z}$-algebra with fraction field k.

Suppose that D is an ample divisor on a projective integral k-scheme X. Select an integer $n > 0$ so large that nD is very ample. Take a basis $x_0 = 1, x_1, \dots, x_m$ of the linear system $\Gamma(X, \mathcal{O}(nD))$. One says that a set of k-rational points $S \subset X(k)$ is *D-integral with respect to R* if there is $d \in R$, $d \neq 0$, with $x_i(P) \in (1/d)R$, for all $i = 1, \dots, n$, and all $P \in S$. The D-integrality of S is indepenent of the choice of

coordinates x_i and the integer n (cf. [22, Cor. 1.4.2] for the case of a number field).
Lang [11] conjectured:

CONJECTURE A. *Let D be an ample divisor on an abelian variety A/k. Then every D-integral subset, with respect to R, of the group of k-rational points $A(k)$ is finite.*

In dimension 1 (elliptic curves), this conjecture is a theorem of Siegel and Mahler and can be proved using either the Thue-Siegel-Roth theorem [11], or tanscendence theory [14], or Faltings' proof of Mordell's conjecture [4, Chapter 5, §5]. Silverman [20] has shown that an abelian variety A/k always has an ample divisor with the property of Conjecture A. Vojta [22, Ch. 4, §2] proved, in the case where k is a number field, that Vojta's conjecture implies Conjecture A; he also proved that if the divisor D has sufficiently many irreducible components then a D-integral set of points of $A(k)$ lies in a proper closed subscheme of A [22, Ch. 2, §4].

Bombieri [16] and Lang independently made the following conjecture (indeed, Lang made a more precise conjecture which allows the ground field k to vary cf. [13]):

CONJECTURE B. *Let V/k be a projective variety of general type. Then there is a non-empty open subscheme U of V with $U(k) = \emptyset$.*

In dimension 1, this is a theorem of Faltings (Mordell's conjecture), as curves of general type are precisely those of genus $\geqslant 2$. Noguchi [16] partially proved the function field analogue of this conjecture. In the case of a number field, Vojta [22, Ch. 5, §1] proved that Vojta's conjecture implies Conjecture B; indeed, Vojta proved implications between various Diophantine conjectures (see especially, [22, Ch. 4 and Ch. 5, Appendix]). The next theorem is another example of the relation between two Diophantine conjectures and is an application of our results on tame fundamental groups of abelian varieties.

THEOREM 7.1. *Conjecture* B *implies Conjecture* A.

Before proving this theorem, we require three preliminary lemmas. The first is a logarithmic ramification formula:

LEMMA 7.2. *Let $f\colon X \to Y$ be a surjective and generically finite morphism of smooth proper irreducible $\mathbb{C}$-schemes. Let D be a reduced effective divisor on Y with only normal crossings. Assume that $f^{-1}D$ is a normal crossings divisor on X and that f is étale outside $f^{-1}D$. Then the canonical divisors satisfy*

$$K_X = f^*K_Y + f^*D - (f^*D)_{\mathrm{red}} + R,$$

where R is an effective divisor on Y whose components are exceptional divisors for

f (i.e. $f_*R = 0$). Further,

$$f^*D - (f^*D)_{\mathrm{red}} + R$$

is supported on every exceptional divisor for f.

For the proof of this lemma, the first part is given by [10, Lemma 1.6] and [9, §4]. The last part appears to be well-known (for the special case where f is a composite of blowings up of smooth subschemes, it follows from [7, Ch. II Ex. 8.5(b)]).

In the next lemma, we use the notation of $\mathbb{Q}$-divisors, that is divisors which are linear combinations of prime divisors with rational coefficients [10].

LEMMA 7.3. *Let $f: X \to Y$ be a finite Galois covering of complete integral varieties over $\mathbb{C}$. Assume that Y is smooth and X is normal. Let D be a reduced effective divisor on Y and assume that f is étale outside $f^{-1}(D)$. Let e_i be the ramification index of f over the irreducible component D_i of D. Then the Kodaira dimension $\kappa(X)$ of X satisfies*

$$\kappa(X) \geqslant \kappa(mK_Y + \sum_i (1 - 1/e_i)D_i,\ Y)$$

for all sufficiently large integers $m > 0$.

Proof. We construct the following commutative diagram of schemes:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X' \arrow[u] \arrow[r, "f'"] & Y' \arrow[u, "u"'] \\
& X'' \arrow[ul] \arrow[u, "f''"'] \arrow[uu, bend right]
\end{tikzcd} \qquad g = u \circ f''$$

Let u be the morphism obtained by blowing up smooth closed subschemes of Y contained in D so that the total inverse image $u^{-1}(D)$ is a normal crossings divisor on Y'. The morphism f' is then obtained by base change of f over u. The morphism $X'' \to X'$ is obtained by blowing up subschemes of X' so that X'' is smooth and $g^{-1}(D)$ is a normal crossings divisor on X''.

Lemma 7.2 applied to the morphism u gives

$$K_{Y'} \sim u^*K_Y + R, \tag{7.2}$$

where $u_*R = 0$ and $R \geqslant 0$ is supported on every exceptional divisor of u. Let D_i' be the strict transform of D_i in Y'. Then

$$u^*D_i \sim D_i' + R_i, \tag{7.3}$$

where $u_*R_i = 0$ and $R_i \geqslant 0$. We have

$$f''^*D_i' \sim e_iD_i'' + E_i, \tag{7.4}$$

where $D_i'' > 0$, $E_i \geqslant 0$ and $f''_*E_i = 0$. Hence (7.3) and (7.4) give

$$g^*D_i \sim e_iD_i'' + F_i, \tag{7.5}$$

where $F_i \geqslant 0$ has $g_*F_i = 0$.

Lemma 7.2 applied to f'' and (7.4) give

$$K_{X''} \sim f''^*K_{Y'} + \sum_i (e_i - 1)D_i'' + S_1, \tag{7.6}$$

where $f''_*S_1 = 0$ and $S_1 \geqslant 0$ is supported along every exceptional divisor of f''. From (7.2) we then have

$$K_{X''} \sim g^*K_Y + \sum_i (e_i - 1)D_i'' + S_2, \tag{7.7}$$

where $S_2 \geqslant 0$ is supported along every exceptional divisor of g; in particular,

$$mS_2 \geqslant \sum_i F_i \tag{7.8}$$

for all sufficiently large integers $m \gg 0$. Therefore (7.5), (7.7), and (7.8) give that for all sufficiently large integers m we have

$$mK_{X''} \sim g^*\left(mK_Y + \sum_i \left(1 - \frac{1}{e_i}\right)D_i\right) + S_3, \tag{7.9}$$

for some $S_3 \geqslant 0$, depending on m. It follows that (see e.g. [9, §5, Lemma 1])

$$\kappa(X'') \geqslant \kappa\left(mK_Y + \sum_i \left(1 - \frac{1}{e_i}\right)D_i, Y\right).$$

As X'' is a smooth model of X, we have the result.

REMARK 1. Under the hypotheses of Lemma 7.3, assume further that the divisor D is *smooth*. Then the Kodaira dimension of X satisfies

$$\kappa(X) \geqslant \kappa\left(K_Y + \sum_i \left(1 - \frac{1}{e_i}\right)D_i, Y\right). \tag{7.10}$$

For then X is smooth and in the proof of this lemma, one can take u to be the identity map on Y and $X = X' = X''$, $f = f' = f''$. We then have in the notation of the proof, $R = R_i = F_i = S_1 = 0$ for all i. Hence S_2 and S_3 can be taken to be zero and one then as in place of (7.9)

$$K_{X''} \sim g^*\left(K_Y + \sum_i \left(1 - \frac{1}{e_i}\right)D_i\right);$$

the inequality (7.10) now follows as in the conclusion of Lemma 7.3.

The following lemma says that Conjecture B implies a conjecture of Lang about rational points on abelian varieties. [We point out an inaccuracy on p. 392, line 7, of [2] which should read: "As there is a finite extension field k' of k such that $f(A'(k')) \supset A(k)$, we may enlarge k and Γ such that $\Gamma = f(A'(k'))$ and find finitely generated subgroups . . . "].

LEMMA 7.4. ([2]). *Assume that Conjecture* B *is true. Let A/k be an abelian variety, where k is a finitely generated field extension of $\mathbb{Q}$. Let V be a closed subscheme of A. Then there are a finite number $A_1, \ldots, A_r$ of translates of sub-abelian varieties of A which are contained in V and so that*

$$V(k) = A_1(k) \cup \ldots . \cup A_r(k),$$

except possibly for a finite set of points.

We now come to:

Proof of Theorem 7.1. Assume that Conjecture B is true. We shall prove that Conjecture A is true by induction on the dimension of the abelian variety A/k. Note that we are free to make finite extensions k' of the ground field and take the integral closure of R in k' for the domain of integrality; also, by inverting a finite number of elements we may assume that R is a smooth $\mathbb{Z}$-algebra.

Let A/k be an abelian variety, where k is the fraction field of the smooth $\mathbb{Z}$-algebra R. Let D be an ample divisor on A. Let $S \subset A(k)$ be a D-integral set of points with respect to R.

By Theorem 5.3, possibly by making a finite extension of the ground field, there is a finite surjective Galois covering $f: X \to A$ which is étale outside $f^{-1}(D)$ and which is ramified over every irreducible component D_i of D, with ramification index $e_i > 1$, say. For some sufficiently large integer n, we have

$$\sum_i \left(1 - \frac{1}{e_i}\right)D_i \geqslant \frac{1}{n}D.$$

Hence by Lemma 7.3 and that $K_A = 0$, we have

$$\kappa(X) \geqslant \kappa\left(\sum_i\left(1 - \frac{1}{e_1}\right)D_i, A\right) \geqslant \kappa(D, A) = \dim A = \dim X.$$

Therefore X is of general type.

We claim that there is a finite extension field k_1 of k so that $S \subset f(X(k_1))$. For this one adapts the argument of [22, Th. 1.4.11], given for number fields, to the present case of finitely generated extensions of $\mathbb{Q}$: the generalisation of the Hermite-Minkowski lemma [22, Th. 1.4.9] is [4, p. 209] and the Chevalley-Weil theorem [22, Th. 1.4.10] can be generalised to arbitrary fields cf. [19].

Conjecture B now implies that $X(k_1)$ lies in a proper closed subscheme W/k_1 of $X \times_k k_1$. As f is a proper morphism, we have that S is contained in the proper closed subscheme $f(W)/k_1$ of $A \times_k k_1$. Let V/k be the proper closed subscheme of A/k is given by the intersection of the Galois conjugates of $f(W)$ over k. Then $S \subset V(k)$.

By Lemma 7.4, there are a finite number $A_1, \ldots, A_r$ of translates of subabelian varieties of A so that

$$V(k) = A_1(k) \cup \ldots \cup A_r(k),$$

apart from a finite set of points of $V(k)$. But $S \cap A_i(k)$ is a set of points on A_i which is integral with respect to the ample divisor induced on A_i by D. Hence $S \cap A_i(k)$ is finite, by the induction hypothesis, for all i. Therefore S is finite.

REMARK 2. Let V/k be a projective irreducible smooth k-scheme. Select any complex embedding $k \to \mathbb{C}$. Let D be a smooth divisor on V with irreducible components $D_1, \ldots, D_t$. Suppose the inertia subgroup of $\pi_1^D(V \otimes_k \mathbb{C})$ has a finite quotient G for which the ramification indices over D_i are e_i for all $i = 1, \ldots t$ and such that

$$\kappa(K_V + \Sigma(1 - 1/e_i)D_i, V) = \dim V.$$

Then Conjecture B implies that every set of D-integral points of $V(k)$ lies in a proper closed subscheme of V.

(For the proof, at the cost of a finite extension of the ground field, there is a finite Galois ramified covering of k-schemes $X \to V$ determined by G. Then by Remark 1 above, we have

$$\kappa(X) \geqslant \kappa(K_V + \Sigma(1 - 1/e_i)D_i, V) = \dim V,$$

hence X is of general type. The argument of the proof of Theorem 7.1 then applies.)

In particular, let D be a smooth irreducible divisor of degree $\geqslant d+3$ on $\mathbb{P}^d$. We have $\pi_1^D(\mathbb{P}^d \times_k \mathbb{C}) = \mathbb{Z}^\vee/(\deg D)\mathbb{Z}^\vee$ (cf. Corollary 4.2 and Example (2) of §4) hence we may take $e = \deg D$. Then $\kappa(K_{\mathbb{P}^d} + (1 - 1/e)D, \mathbb{P}^d) = d$. As above, Conjecture B then implies that every set of D-integral points of $\mathbb{P}^d(k)$ lies in a proper closed subscheme.

For the case of curves, one checks using Section 4, Example (1), that this also recovers Siegel's theorem on integral points assuming Faltings' theorem (Mordell Conjecture).

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