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Some birational maps of Fano 3-folds

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Section 0. Introduction

Iskovskih [2] investigated the double projections of Fano 3-folds from lines on them assuming the existence of lines, and classified the Fano 3-folds. (Later Shokurov [11] showed that this assumption always holds.) Mori [6] treated these projections with the numerical method using the extremal ray theory.

In this paper, we study the projections from general points or conics (instead of the ones from lines in Iskovskih's works [2]; see also [11], where the projections are studied by different methods) on Fano 3-folds using the extremal ray theory. We can give an alternate simpler proof of Iskovskih's result without using the existence of lines:

(0.1) THEOREM. *Let $V = V_{2g-2}$ in $\mathbb{P}^{g+1}$ be a Fano 3-fold satisfying* Pic $V = \mathbb{Z}(-K_V)$. *Then $g = 12$ or $g \leqslant 10$.* (*Here the number g is called the genus of V* (*see* (1.0)).)

Thus this approach simplifies the coarse classification of Fano 3-folds. We should mention that the existence of lines needed in the classification of Fano 3-folds with the Picard number $\rho \geqslant 2$ [8] can be replaced with the above coarse classification.

The existence of lines mentioned above was one of the key results needed in Iskovskih's approach, which was proved by elaborate geometric arguments. We can also give another simpler proof of the result of Shokurov [11][(1)]:

(0.2) THEOREM. (*The existence of lines and smooth conics*). *Let V be a Fano 3-fold as in* (1.0) *and assume $g \geqslant 8$. Then there exist a line* (*i.e. a rational curve C such that $C \cdot (-K_V) = 1$*) *and a smooth conic on V.*

The extremal ray theory enables us to prove Theorem 0.2 with only numerical calculation. Our arguments thus considerably simplify the proof of the classification of Fano 3-folds.

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(1) Though this simplifies Shokurov's proof, it still needs his result (1.8).

Our approach also gives birational maps between Fano 3-folds systematically, and in particular the following remarkable birational map.

(0.3) THEOREM. *Let* $V = V_{14}$ *in* $\mathbb{P}^9$ *be a Fano 3-fold with* $g = 8$ *satisfying* $\operatorname{Pic} V = \mathbb{Z}(-K_V)$. *There is a birational map of a cubic 3-fold* B_3 *in* $\mathbb{P}^4$ *to* V, *whose indeterminacy is the union of a quartic rational curve and* 16 *lines in* B_3.

Fano and Iskovskih [4] constructed a different birational map of B_3 to V_{14}. In their case, the indeterminacy in B_3 is the union of a quintic elliptic curve and 25 lines, but the union of rational curves in our case.

In Section 1, we recall the well-known results in [2] for lines and conics on Fano 3-fold V, and reproduce the work of Reid [10] on the anti-canonical morphism of the blow-up of V at a point or along a conic.

The main part of this paper is in Section 2. We construct (bi)rational maps of Fano 3-folds as the projections from points or conics, and prove Theorems 0.1 and 0.2.

We discuss in Section 3 the new birational map between a cubic 3-fold B_3 and a Fano 3-fold V_{14} (Theorem 0.3).(2)

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The following notations are used in this paper.

For a subset Y of the projective space, $\langle Y \rangle$ means a projective subspace spanned by Y. Let X be an irreducible reduced subscheme of a scheme Y. We denoted by $\operatorname{mult}_X Y$ the multiplicity of Y at the generic point of X.

Let V be a projective 3-fold. Let X be a point or a conic in V and L a divisor of V. For a positive integer n, $|L - nX|$ means the linear subsystem of $|L|$ consisting of all divisors $D \in |L|$ such that $\operatorname{mult}_X D \geqslant n$.

Section 1. Preliminary

(1.0) In this paper, all varieties are considered over the field $\mathbb{C}$ of complex numbers. Let $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ be a *Fano 3-fold of the principal series*, i.e., a smooth complete irreducible algebraic variety of dimension three over $\mathbb{C}$ whose anti-canonical Cartier divisor $-K_V$ is very ample. Hence V is embedded by the anti-canonical system $|-K_V|$, and the *genus* g of V is the integer defined by $\dim |-K_V| = g + 1$, which satisfies $\deg V = (-K_V)^3 = 2g - 2$.

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(2)S.L. Tregub obtained the same result assuming the existence of lines. (cf. S.L. Tregub: Construction of a birational isomorphism of a cubic threefold and Fano variety of the first kind with $g = 8$, associated with a normal rational curve of degree 4, Vestnik Moskov. Univ. Mat., Vol. 40, No. 6, 99–101, 1985; English transl. in Moscow Univ. Math. Bull.) The author is grateful to Prof. V.A. Iskovskih for this information.

We recall the results (1.1)–(1.5) of Iskovskih and the result (1.8) of Reid.

(1.1) PROPOSITION (Iskovskih [2, Proposition 1.7], [3, Ch. 4, Proposition 1.3]). *A Fano 3-fold $V_{2g-2} \subset \mathbb{P}^{g+1}$ of the principal series is an intersection of quadrics if $g \geqslant 5$ and if its any smooth canonical curve section is non-trigonal.*

(1.2) Let Γ (resp. Δ) be the scheme parametrizing the lines (resp. conics) on V, and S(resp. T) the universal family of lines over Γ (resp. of conics over Δ). (Γ(resp. Δ) is a closed subscheme of the Hilbert scheme of closed subschemes of $\mathbb{P}^{g+1}$ with Hilbert polynomial $n+1$ (resp. $2n+1$).) Let

$$\begin{array}{ccc} S \xrightarrow{p} V & & T \xrightarrow{q} V \\ \pi \Big\downarrow & \text{resp.} & \rho \Big\downarrow \\ \Gamma & & \Delta \end{array}$$

be the diagram of natural morphisms, and $R = p(S)$ (resp. $Q = q(T)$) the image of family S(resp. T) on V. We denote by Z(resp. C) the fiber of π (resp. ρ) and we use the same letter Z(resp. C) instead of $p(Z)$(resp. $q(C)$) to represent its image on V. (Here a conic C on V may be reducible or nonreduced.)

(1.3) PROPOSITION (Iskovskih [2, Lemma 3.2, Proposition 3.3], [3, Ch. 3, Proposition 2.1]). *Under the above notation, suppose that there exists a line $Z \subset V$. Let Γ° be the irreducible component of the scheme Γ and S° the corresponding family of lines on Γ. Then*

(i) *for any normal sheaf $\mathcal{N}_{Z/V}$ there are only two possibilities:*
   (a) $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$,
   (b) $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$;

(ii) $\dim \Gamma^\circ = 1$ *or* 2;

(iii) $\dim \Gamma^\circ = 2$ *if and only if* $p(S^\circ) = \mathbb{P}^2 \subset V$, *and in this case* $\Gamma^\circ \simeq \mathbb{P}^2$.

(1.4) PROPOSITION (Iskovskih [2, Lemma 4.2], [3, Ch. 3, Lemma 3.2]). *Suppose V is an intersection of quadrics in $\mathbb{P}^{g+1}$ and there exists a smooth conic C on V. Then there are only the following four possibilities for the normal sheaf $\mathcal{N}_{C/V}$:*

(a) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$,
(b) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$,
(c) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2)$,
(d) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-4) \oplus \mathcal{O}_C(4)$.

(Here $\mathcal{O}_C(d)$ denotes the invertible sheaf on C of degree d.)

(1.5) PROPOSITION (Iskovskih [2, Proposition 4.3], [3, Ch.3, Proposition

3.3]). *Suppose that T is not empty. Let T° be an irreducible component of the scheme T, $\Delta^\circ = \rho(T^\circ)$ and $Q^\circ = q(T^\circ)$ the corresponding component of Δ and of Q respectively.*

(i) *If $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$ for the general conic $C \subset Q^\circ$, then Δ° is nonsingular at its general point,* $\dim \Delta^\circ = 2$ *and $Q^\circ = V$, i.e., the morphism $q\colon T^\circ \to V$ is generically finite.*

(ii) *If $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$ for the general conic $C \subset Q^\circ$, then Δ° is nonsingular at its general point,* $\dim \Delta^\circ = 2$, $\dim Q^\circ = 2$ *and Q° is either a Veronese surface in $\mathbb{P}^5$ or one of its projections into a lower space with the exception of a plane $\mathbb{P}^2$ and a quadric in $\mathbb{P}^3$.*

(iii) *If $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2)$ for the general conic $C \subset Q^\circ$, then Δ° is nonsingular at its general point,* $\dim \Delta^\circ = 3$ *and Q° is 2-dimensional quadric on V.*

(iv) *If $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-4) \oplus \mathcal{O}_C(4)$ for the general conic $C \subset Q^\circ$, then Q° is the plane $\mathbb{P}^2$ on V.*

(1.6) REMARK. Let $V \subset \mathbb{P}^{g+1}$ be of the first species and index 1, i.e., $\operatorname{Pic} V = \mathbb{Z}H$ (where $H \sim -K_V$ is a hyperplane section), and of genus $g \geqslant 5$. Then V is not trigonal whence V is an intersection of quadrics. Furthermore the cases (c) and (d) in Proposition 1.4 are impossible, because V cannot contain a projective plane $F = \mathbb{P}^2$ or a quadric surface $F = \mathbb{P}^1 \times \mathbb{P}^1$. In fact, $K_F = (K_V + F)|_F$ by the adjunction formula, and $F \sim dH$ for some integer d by the hypothesis, hence $(K_F^2)_F = d(d-1)^2 \cdot \deg V \neq 9$ or 8 for any integer d.

(1.7) In the rest of this section, we assume that the Fano 3-fold $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ is of the first species and index 1 and has genus $g \geqslant 5$. Moreover, fix the following assumptions and notations:

(A) Let P be a point in V, not lying on any line contained in V. (There is such P by Proposition 1.3.) Let $\sigma\colon W \to V$ be the blow-up of V at P and $S = \sigma^{-1}(P)$. Then $S \simeq \mathbb{P}^2$, $-K_W \sim \sigma^*(-K_V) - 2S$, $(-K_W)^3 = 2g' - 2$, where $g' = g - 4$. (We treat the cases $g \geqslant 6$.)

(B) Let C be a conic in V (assuming one exists). Such C has normal sheaf $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$ or $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$ by (1.6). Let $\sigma\colon W \to V$ be the blow-up of V along C and $S = \sigma^{-1}(C)$. Then S is a rational scroll (Hirzeburch surface) $\mathbb{F}_0$ or $\mathbb{F}_2$, and $-K_W \sim \sigma^*(-K_V) - S$, $(-K_W)^3 = 2g' - 2$, where $g' = g - 3$. (We treat the cases $g \geqslant 5$.)

(1.8) THEOREM (Reid [10, §3]). (*i*) *Under these assumptions and notations, the linear system $|-K_W|$ is free and defines a generically finite morphism $\varphi_{-K_W}\colon W \to \varphi_{-K_W}(W) \subset \mathbb{P}^{g'+1}$.*

(ii) *Let $\pi \circ \varphi = \varphi_{-K_W}$ be the Stein factorization of φ_{-K_W}:*

$$\begin{tikzcd} & W \arrow[dl, "\varphi"'] \arrow[d, "\varphi_{-K_W}"] \\ \overline{W} \arrow[r] & \varphi_{-K_W}(W) \subset \mathbb{P}^{g'+1}. \end{tikzcd}$$

Then φ is a small birational morphism (i.e., W has no divisor contracted by φ), $\bar{W}$ is Gorenstein with ample anti-canonical sheaf $-K_{\bar{W}}$, and $\varphi^*(-K_{\bar{W}}) = -K_W$. As for the cohomology, we have $H^i(\mathcal{O}_W) = 0$ for $i = 1, 2$ and 3.

(iii) *Suppose in addition that the linear system $|-K_V|$ has no reducible member. Then for a curve $X \subset W$ such that $X \not\subset S$ and $X \cdot (-K_W) = 0$, we have*

(A) *if $g \geqslant 8$ then $Y = \sigma(X)$ is a conic through P;*
(B) *if $g \geqslant 7$ then $Y = \sigma(X)$ is a line meeting C.*

In cases $g \leqslant 7$, we may also have a finite number of the following types of curves $X \subset W$:

(A) $g \leqslant 7$: $X \simeq \mathbb{P}^1$ *with* $\deg Y = 4$,
$\quad\quad \mathrm{mult}_P Y = 2$, $\langle Y \rangle = \mathbb{P}^3$, $p_a(Y) = 1$.
$\quad g = 6$: $X \simeq \mathbb{P}^1$ *with* $\deg Y = 6$,
$\quad\quad \mathrm{mult}_P Y = 3$, $\langle Y \rangle = \mathbb{P}^4$, $p_a(Y) = 2$.
(B) $g \leqslant 6$: *Y is a conic such that* $\langle Y \cup C \rangle = \mathbb{P}^3$,
$\quad\quad p_a(Y \cup C) = 1$.

(1.9) REMARK. Theorem 1.8 implies that S is φ-ample. Indeed σ is a small morphism and $(S \cdot C) > 0$ for any curve C on W contracted by φ.

Section 2. Projections of V from a point or a conic

(2.0) In this section, let $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ be a Fano 3-fold of the first species and index 1 and has genus $g \geqslant 8$. We note that V is an intersection of quadrics.

We prove the following main theorem in this section:

(2.1) THEOREM. *Let $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ be a Fano 3-fold of the first species of the principal series with index* 1 *and genus $g \geqslant 8$. Then the following assertions hold.*

(i) $g \leqslant 12$.
(ii) $g \neq 11$.
(iii) *There are lines and smooth conics on V.*
(iv) *For each* $8 \leqslant g \leqslant 12$ ($g \neq 11$), *there are* (bi)*rational maps from V on the list* (2.13).

Theorem 0.1 in Section 0 corresponds to the above (i) and (ii), Theorem 0.2 to (iii). For the proof of the assertions (i), (ii), (iii) and (iv), the reader see (2.10), (2.11), (2.9) and (2.12) respectively.

(2.2) To prove the above theorem, we fix again the assumptions and notations as in (1.7); let $\sigma: W \to V$ be the blow-up of V at a general point P (resp. along a general conic C) and S its exceptional divisor on W.

It is important to understand that the conic blow-up (case (B)) is based on the hypothesis that a smooth conic exists on V.

Therefore only the computations for the point blow-up (case (A)) are justified right now, and the ones for the conic blow-up (case (B)) are justified only at (2.9) based on the results from case (A). This however does not cause any trouble because no results from case (B) are used before (2.9).

(2.2.1) If P is a general point, then there are only finitely many conics through P. In fact, under the notation in (1.2), it follows from Proposition 1.5 that the parametrizing space Δ of conics is a purely 2-dimensional scheme of finite type. The natural morphism $q: T \to V$ is therefore generically finite, which shows the finiteness. It is easy to see that there are only finitely many lines meeting a general conic by a similar argument.

(2.2.2) Thus, under the notation of (1.8), $\varphi: W \to \bar{W}$ is a resolution of $\bar{W}$, isomorphic in codimension 1, such that $K_W = \varphi^* K_{\bar{W}}$ and S is φ-ample. Then we can apply $(-S)$-flop (Kollár [5]) to φ to get a nonsingular projective 3-fold W' and a birational morphism $\varphi': W' \to \bar{W}$ which is isomorphic in codimension 1. The idea is as follows: At each fundamental point q of $\bar{W}$(q such that $\dim \varphi^{-1}(q) > 0$), there is a neighbourhood U of q in $\bar{W}$ such that (U, q) has an involution $\iota: (U, q) \to (U, q)$ and that the flop of $\varphi_U: \varphi^{-1}(U) \to U$ is $\iota \circ \varphi_U: \varphi^{-1}(U) \to U$. These $\iota \circ \varphi_U$ are patched with $W - \varphi^{-1}$(fundamental points) to get W' in the obvious way. It is shown that the proper transform S' of S by $W \cdots \to W'$ is φ'-negative, i.e. $-S'$ is φ'-ample. Thus the analytic space W' constructed this way is actually a nonsingular projective 3-fold.

Of course, we have to consider the case where φ is an isomorphism so that there are no fundamental points. In such a case, we simply take $W' = W$ and $\varphi' = \varphi$.

Because of this involution construction, it is obvious that

(i) $K_{W'} = \varphi'^* K_{\bar{W}}$,

(ii) $\mathrm{Eu}(W) = \mathrm{Eu}(W')$,

where $\mathrm{Eu}(X)$ denotes the topological Euler number of a space X.

(2.2.3) We define a nontrivial morphism $\alpha: W' \to V'$ as follows.

If $W = W'$, then W' is a Fano 3-fold with $\rho(W') = 2$. Hence W' has exactly two extremal rays, one of which is associated to φ. We take as α the contraction of the other extremal ray R.

If $W \neq W'$, then $-K_{W'}$ is base point free and thus defines φ'. The cone of curves of W' has an edge (say R_1) which is generated by φ'-exceptional curves. We note that R_1 is not an extremal ray because

$$(\varphi'\text{-exceptional curve}) \cdot K_{W'} = 0.$$

Since $K_{W'}$ is not nef, W' has an extremal ray R and the cone of curves of W' is generated by R_1 and R, because $\rho(W') = 2$. Thus R is the one and only one extremal ray of W'. We take as α the contraction of the extremal ray R. Thus we have the following diagram.

$$\begin{tikzcd} W \arrow[rr, dotted, leftrightarrow, "\text{flop}"'] \arrow[d, "\sigma"'] & & W' \arrow[d, "\alpha = \mathrm{cont}_R"] \\ V & & V' \end{tikzcd} \tag{2.2.4}$$

(2.2.5) Let S' be the strict transform of S by $W \cdots \rightarrow W'$. Then $-K_{W'}$ and S' generate Pic W' by the following.

(2.2.6) REMARK. Since $W \cdots \rightarrow W'$ is an isomorphism in codimension 1, there is a commutative diagram of isomorphisms induced by $W \cdots \rightarrow W'$:

$$\begin{tikzcd} \mathrm{Div}\, W \arrow[r, "\cong"] \arrow[d] & \mathrm{Div}\, W' \arrow[d] \\ \mathrm{Pic}\, W \arrow[r, "\cong"] & \mathrm{Pic}\, W' \end{tikzcd}$$

Indeed, W and W' have the same prime divisors and the same function field (Reid [9, Proposition (6.2)]).

To know cont_R and V', it is enough to determine the divisor L whose linear system defines cont_R. We may assume that L is one of generators of Pic W' and is linearly equivalent to $x(-K_{W'}) - yS'$ for some integers x and y. We first reduce the problem to the numerical conditions for x and y (2.3)–(2.6), next solve them (2.7)–(2.8) and last judge the realizability in the geometrical view (2.10)–(2.12).

Now Mori's theory says there are three types in the extremal rays: C-type (with conic bundle structure), D-type (with del Pezzo fibering) and E-type (having an exceptional divisor D). We will treat these types separately.

C, D-types: In these cases, (2.3)

$V' = \mathbb{P}^2$ *if R is of C-type.*

$V' = \mathbb{P}^1$ *if R is of D-type.* (2.3.1)

Indeed $q(V') = 0$, since $q(W') = q(V) = 0$ and cont_R is surjective. If R is of D-type, then $V' = \mathbb{P}^1$. We may assume that R is of C-type. From the general theory of conic bundles, $-4K_{V'}$ is numerically equivalent to $\alpha_*(-K_{W'})^2 + \Delta$, where Δ is the discriminant locus of V', and hence all the plurigenera $P_m(V')$ vanish. Therefore V' is rational and we have $V' = \mathbb{P}^2$ by $\rho(V') = 1$.

(2.3.2) $L = \alpha^*\mathcal{O}_{V'}(1) \sim x(-K_{W'}) - yS'$ *for some positive coprime integers x and y.*

We remark that the positivity of x and y is not effected by replacing L with its multiples. Suppose x were not positive, then the linear system $|L| = |x(-K_{W'}) - yS'| \subset \| y|\cdot S'|$ would have dimension $\leqslant 0$. Thus x is positive. If y were not positive, the linear system $|L| \supset |x(-K_{W'})| + |-yS'|$ would contain $|x(-K_{W'})|$, which contradicts the situation that the image of the morphism defind by $|-K_{W'}|$ is of dimension three (cf. Theorem 1.8.i). We may assume that x and y are coprime, taking L as one of generators of Pic W'.

$$y = 1 \text{ or } 2, \qquad \text{if } R \text{ is of } C\text{-type.}$$
$$y = 1, 2 \text{ or } 3, \quad \text{if } R \text{ is of } D\text{-type.} \tag{2.3.3}$$

Let W'_v be a general fiber of α. Then W'_v is a conic if R is of C-type and W'_v is a del Pezzo surface if R is of D-type. Restricting Pic W' to a fiber, we have Pic $W'/\mathbb{Z}L \subset$ Pic W'_v and under this inclusion $-K_{W'}$ corresponds to $-K_{W'_v}$. Thus we have $\mathbb{Z}/y\mathbb{Z} \simeq$ Pic $W'/(\mathbb{Z}L + \mathbb{Z}(-K_{W'})) \subset$ Pic $W'_v/\mathbb{Z}(-K_{W'_v})$, and hence $-K_{W'_v}$ must be divisible by y. Therefore $y = 1$ or 2 if W'_v is a conic and $y = 1, 2$ or 3 if W'_v is a del Pezzo surface.

(2.3.4) *We have the following relations in each case.*

C-*type*:
$$L^3 = 0,$$
$$L^2\cdot(-K_{W'}) = 2, \quad \textit{and}$$
$$L\cdot(-K_{W'})^2 = 12 - \deg \Delta,$$

where Δ is a discriminant locus.

D-type:
$$L^2\cdot(-K_{W'}) = 0,$$
$$L^2\cdot S' = 0, \quad \textit{and}$$
$$L\cdot(-K_{W'})^2 = (\textit{degree of a general fiber } W'_v).$$

Indeed, in the case of C-type we have

$$\begin{aligned} -\deg \Delta &= (-K_{W'/V'})^2\cdot L = (-K_{W'} + \alpha^* K_{V'})^2\cdot L = (-K_{W'} - 3L)^2\cdot L \\ &= (-K_{W'})^2\cdot L - 6(-K_{W'})\cdot L^2 = (-K_{W'})^2\cdot L - 12, \end{aligned}$$

and in the case of D-type we have

$$(-K_{W'})^2\cdot L = (-K_L)^2 = (\text{degree of a general fiber}).$$

The rest are obvious.

(2.4) E-type: In this case, we have an exceptional divisor $D \sim z(-K_{W'}) - uS'$

for some integers z and u.

$$\operatorname{Pic} W' = \mathbb{Z}(-K_{W'}) \oplus \mathbb{Z}S'. \tag{2.4.1}$$

(2.4.2) $D \sim z(-K_{W'}) - uS'$ *for some positive coprime integers* z *and* u.

If $z \leqslant 0$, then $D = S'$ because D and S' are effective and no multiple of them move. So a positive multiple of $-K_{W'} + aS'$ with $a = 2, 1, \frac{1}{2}$ defines the morphism α (cf. (2.4.3)). However these define a rational mapping to V of W, which is a contradiction. So $z > 0$. If $u \leqslant 0$, then $\dim|D| \geqslant 1$, which is a contradition. Thus z, $u > 0$.

In cases E_1–E_4, there is a rational curve $C \subset D$ contracted by α such that $(C \cdot D) = -1$ (by description of contraction). In case E_5, we have $D \cdot (-K_{W'})^2 = 1$. Thus we see that z and u are coprime.

(2.4.3) *In the cases of* E_2–E_5*-types, we have following relations:*

		E_2	E_3	E_4	E_5
$\alpha^*(-K_{V'}) + K_{W'}$	$\equiv$	$2D$	D	D	$(\frac{1}{2})D$
D^3	$=$	1	2	2	4
$D^2 \cdot (-K_{W'})$	$=$	-2	-2	-2	-2
$D \cdot (-K_{W'})^2$	$=$	4	2	2	1

(2.4.4) In the cases of E_1, E_2-types, V' is a Fano 3-fold of the first species because $\operatorname{Pic} V' \simeq \mathbb{Z}$ and $-K_{V'}$ is positive. Moreover we have $\operatorname{Pic} W' \simeq \alpha^*(\operatorname{Pic} V') \oplus \mathbb{Z}D$ and

$$\begin{aligned}\mathbb{Z}/u\mathbb{Z} &\simeq \operatorname{Pic} W'/(\mathbb{Z}D + \mathbb{Z}(-K_{W'})) \simeq \operatorname{Pic} W'/(\mathbb{Z}D + \mathbb{Z}\alpha^*(-K_{V'})) \\ &\simeq \operatorname{Pic} V'/\mathbb{Z}(-K_{V'}).\end{aligned}$$

Therefore u is equal to the index of V', and hence there are four possibilities: $u = 1$, 2, 3 or 4.

(2.4.5) In the case of E_1-type, we have $-K_{W'} \sim \alpha^*(-K_{V'}) - D$ and the following relations:

$$\begin{aligned}
(-K_{W'} + D)^3 &= (-K_{V'})^3, \\
(-K_{W'} + D)^2 \cdot D &= \alpha^*(-K_{V'})^2 \cdot D = 0 \\
(-K_{W'} + D) \cdot D^2 &= \alpha^*(-K_{V'}) \cdot D^2 = -(-K_{V'} \cdot \Gamma) = -u \cdot \deg \Gamma, \quad \text{and} \\
(-K_{W'}) \cdot D^2 &= (-K_{W'} + D) \cdot D^2 - D^3 = -(K_{V'} \cdot \Gamma) + c_1(\mathcal{N}_{\Gamma/V'}) \\
&= 2g(\Gamma) - 2,
\end{aligned}$$

where $\Gamma = \alpha(D)$ is the curve of the center of the blow-up α. We will use the following equalities instead of the first and third ones:

$$(-K_{W'}+D)^2\cdot(-K_{W'})=(-K_{V'})^3,\ (-K_{W'}+D)\cdot D\cdot(-K_{W'})=u\cdot\deg\Gamma.$$

(2.5) As already mentioned in (2.2.5), Pic W' is generated by $-K_{W'}$ and S'. Consider the intersection numbers on W' of these generators. The intersection numbers with canonical divisor $-K_{W'}$ are the same as on W. On the other hand, the self-intersection number S'^3 of S' on W' may be different from S^3 on W, and put $e = S^3 - S'^3$. Then we have the following multiplication table:

$$(-K_{W'})^3 = 2g' - 2 = 2g - 10 \text{ (resp. } = 2g - 8)$$

$$(-K_{W'})^2\cdot S' = 4$$

$$(-K_{W'})\cdot S'^2 = -2$$

$$S'^3 = 1 - e \text{ (resp. } = -e).$$

It can be shown that the number e is non-negative, and that e can be interpreted as the number (counted with multiplicity) of conics through P if $g \geqslant 8$ (resp. of lines meeting C if $g \geqslant 7$). However, we use here only the obvious fact that $e \in \mathbb{Z}$ and that if $e \neq 0$ then $W \neq W'$ so that conics exist if $g \geqslant 8$ (resp. lines exist if $g \geqslant 7$).

(2.6) From (2.3.4), (2.4.3), (2.4.5) and (2.5), we obtain a system of Diophantine equations for each case.

(2.6.1) *C-type*: From (2.3.4),

$$(2g'-2)x^3 - 12x^2y - 6xy^2 - (1-e)y^3 = 0$$

$$\text{(resp. } (2g'-2)x^3 - 12x^2y - 6xy^2 + ey^3 = 0),$$

$$(2g'-2)x^2 - 8xy - 2y^2 = 2, \quad \text{and}$$

$$(2g'-2)x - 4y = 12 - \deg\Delta.$$

(2.6.2) *D-type*: From (2.3.4),

$$(2g'-2)x^2 - 8xy - 2y^2 = 0,$$

$$4x^2 + 4xy + (1-e)y^2 = 0 \text{ (resp. } 4x^2 + 4xy - ey^2 = 0), \quad \text{and}$$

$$(2g'-2)x - 4y = \deg W'_v.$$

(2.6.3) E_2–E_5-*types*: From (2.4.3),

E_2	E_3, E_4	E_5	
1	2	4	$= (2g'-2)z^3 - 12z^2u - 6zu^2 - (1-e)u^3$
	(resp.		$= (2g'-2)z^3 - 12z^2u - 6zu^2 + eu^3),$

$$-2 \quad -2 \quad -2 = (2g'-2)z^2 - 8zu - 2u^2, \quad \text{and}$$
$$4 \quad 2 \quad 1 = (2g'-2)z - 4u.$$

(2.6.4) *E_1-type*: From (2.4.5),

$(2g'-2)(z+1)^2 - 8(z+1)u - 2u^2 = (-K_{V'})^3,$

$(2g'-2)z(z+1)^2 - 4(3z+1)(z+1)u - 2(3z+2)u^2 - (1-e)u^3 = 0$

(resp. $(2g'-2)z(z+1)^2 - 4(3z+1)(z+1)u - 2(3z+2)u^2 + eu^3 = 0$),

$(2g'-2)z(z+1) - 4(2z+1)u - 2u^2 = u \cdot \deg \Gamma$, and

$(2g'-2)z^2 - 8zu - 2u^2 = 2g(\Gamma) - 2.$

(2.7) The above system of Diophantine equations (2.6.1)–(2.6.4) with conditions (2.3.2), (2.3.3), (2.4.2) and (2.4.4) have the solutions (2.8.1) (resp. (2.8.2)) as follows.

(2.7.1) *Case of C-type.*
We have $y = 1$ or 2 by (2.3.3). If $y = 1$ then $(g'-1)x^2 - 4x - 2 = 0$ from the second equation in (2.6.1), hence $x = 1$ and $g' = 7$. If $y = 2$ then $(g'-1)x^2 - 8x - 5 = 0$, hence $x = 1$ and $g' = 14$. But the latter is impossible because no integer e satisfies the first equation in (2.6.1).

(2.7.2) *Case of D-type.*
From this case, we obtain $x = 1$, $y = 1$, $g' = 6$ and $e = 9$ (resp. $e = 8$) similarly as (2.7.1).

(2.7.3) *Cases of E_2-E_5-types.*
From the third equation in (2.6.3), E_5-type does not occur and $2u = (g'-1)z - 2$ (E_2-type) or $2u = (g'-1)z - 1$ (E_3, E_4-types). In the case of E_2-type, we have three solutions from the second equation in (2.6.3) and from the above. In the cases E_3, E_4-types there is no solution.

(2.7.4) *Case of E_1-type.*
Here we use the fact that $\mathcal{O}_S(-K_W)$ is generated by global sections and $|\mathcal{O}_S(-K_W)|$ induces an embedding of S into $\mathbb{P}^5$ as a quartic surface which is an intersection of quadrics. This is easy to see by the construction of $S(\simeq \mathbb{P}^2, \mathbb{F}_0$ or $\mathbb{F}_2)$, $(\mathcal{O}_S(-K_W)^2) = 4$, and $h^0(\mathcal{O}_S(-K_W)) = 6$. It is obvious that $\varphi(S)$ is the image of a linear projection of the quartic surface above.
Since u is the index of V' and

$$\alpha^*(-K_{V'}) = (1+z)(-K_{W'}) - uS',$$

we may set $z + 1 = u \cdot t$ for some positive integer t. Then the first equation of

(2.6.4) reduces to

$$\{(g'-1)t-4\}t=\begin{cases}3 & \text{if } u=4\\ 4 & \text{if } u=3,\\ d+1 & \text{if } u=2\end{cases}$$

where $d=(-K_{V'}/2)^3=1,\dots,5$ in case $u=2$. Thus it is easy to figure out the solutions as in the Table 2.8 if $u\geqslant 2$.

Then we assume $u=1$. Since $\varphi(S)$ spans at most $\mathbb{P}^5$ in $\mathbb{P}^{g'+1}$ as above, we have $g'\leqslant 5$. Because otherwise,

$$\dim|D|\geqslant\dim|-K_{W'}-S'|=\dim|-K_W-S|\geqslant g'-5\geqslant 1,$$

which is a contradiction. We treat two cases.

Case $g'=5$: We see similarly that $\dim|-K_{W'}-S'|\geqslant 0$. Thus $z=1$, which is in the Table 2.8.

Case $g'=4$: We claim that $z=1$. Indeed if $\varphi(S)$ does not span $\mathbb{P}^5$, then $\dim|-K_{W'}-S'|\geqslant 0$ similarly, and $z=1$ since $\dim|D|=0$. If $\varphi(S)$ spans $\mathbb{P}^5$ then $\varphi(S)\subset\mathbb{P}^5$ is the embedding of S by $|\mathcal{O}_S(-K_W)|$ and hence $\varphi(S)$ is an intersection of quadrics of $\mathbb{P}^5$ containing it. Thus $\dim|2(-K_{W'})-S'|=\dim|2(-K_W)-S|\geqslant 1$, again $z=1$ by $\dim|D|=0$. Hence we have $z=1$ anyway. This is in the Table 2.8.

(2.8) *The list of solutions of the systems of equations.*

In the following lists, the data in the row with symbol # can be realized in the geometrical way. (See (2.10)–(2.12).) We write simply by H and S instead of $-K_{W'}$ and S' in the list.

(2.8.1) *Case* (A), i.e. *starting with point-blow-up*:

g	e	L	D	The type of extremal ray			
13	5	$H-S$	$2H-3S$	E_1	$\deg\Gamma=6$,	$g(\Gamma)=0$	on Q
# 12	6	$H-S$	$3H-4S$	E_1	$\deg\Gamma=6$,	$g(\Gamma)=0$	on $\mathbb{P}^3$
11	7	$H-S$		C	$\deg\Delta=4$		
# 10	9	$H-S$		D	$\deg W'_v=6$		
# 9	12	$3H-2S$	$H-S$	E_2	$V'=V_{16}$		
	11	$2H-S$	$H-S$	E_1	$\deg\Gamma=2$,	$g(\Gamma)=0$	on V_{14}
8	15	$2H-S$	$5H-3S$	E_1	$\deg\Gamma=10$,	$g(\Gamma)=7$	on Q
#	16	$2H-S$	$3H-2S$	E_1	$\deg\Gamma=4$,	$g(\Gamma)=0$	on B_3

(2.8.2) *Case* (B), i.e. *starting with conic-blow-up*:

g	e	L	D	The type of extremal ray			
#12	4	$H - S$	$2H - 3S$	E_1	$\deg \Gamma = 6$,	$g(\Gamma) = 0$ on	Q
11	5	$H - S$	$3H - 4S$	E_1	$\deg \Gamma = 6$,	$g(\Gamma) = 0$ on	$\mathbb{P}^3$
#10	6	$H - S$		C	$\deg \Delta = 4$		
# 9	8	$H - S$		D	$\deg W'_v = 6$		
8	11	$3H - 2S$	$H - S$	E_2	$V' = V_{16}$		
#	10	$2H - S$	$H - S$	E_1	$\deg \Gamma = 2$,	$g(\Gamma) = 0$ on	V_{14}

In the above lists, Q is a quadric 3-fold in $\mathbb{P}^4$ and B_3 is a cubic 3-fold in $\mathbb{P}^4$.

(2.9) THEOREM. *There exist a line and a smooth conic in* V_{2g-2} *if* $g \geqslant 8$.

Since $e > 0$ in all the possible cases in Table 2.8.1, there exists a smooth conic if $g \geqslant 8$. Therefore we can blow up a general conic as proposed in (2.2) and we have Table 2.8.2 as the result. Since $e > 0$ in all the possible cases in (2.8.2), there exists a line if $g \geqslant 8$.

(2.10) Proof of (2.1.i). *Non-existence of* V_{2g-2} *with* $g \geqslant 13$.

From (2.8.1), there is no 3-fold V_{2g-2} with $g > 13$; in the cases $g \leqslant 13$ there is a conic C on V_{2g-2} with the normal sheaf $\mathcal{O}_C \oplus \mathcal{O}_C$ or $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$, and hence V_{2g-2} exists only in the cases $g \leqslant 12$ from (2.8.2).

(2.11) Proof of (2.1.ii). *Non-existence of* V_{20}.

By (2.8.1), W' is a conic bundle over $\mathbb{P}^2$ with discriminant locus Δ a degree 4 curve. Therefore $\mathrm{Eu}(W')$ can be computed as

$$\begin{aligned}\mathrm{Eu}(W') &= \mathrm{Eu}(\mathbb{P}^1)\cdot\mathrm{Eu}(\mathbb{P}^2 - \Delta) + \mathrm{Eu}(F)\cdot\mathrm{Eu}(\Delta - \Sigma) + \mathrm{Eu}(\mathbb{P}^1)\cdot \#|\Sigma| \\ &= \mathrm{Eu}(\mathbb{P}^1)\cdot\mathrm{Eu}(\mathbb{P}^2) + \{\mathrm{Eu}(F) - \mathrm{Eu}(\mathbb{P}^1)\}\cdot\{\mathrm{Eu}(\Delta) - \#|\Sigma|\} \\ &= 2\cdot 3 + 1\cdot(2 - 2p_a(\Delta)) = 2,\end{aligned}$$

where F denotes the reducible reduced conic and Σ denotes the singular locus of Δ. Thus

$$\mathrm{Eu}(V) = \mathrm{Eu}(W) - \mathrm{Eu}(S) + 1 = \mathrm{Eu}(W) - 2 = \mathrm{Eu}(W') - 2 = 0,$$

by (2.2.2). On the other hand, by conic-blow-up we get a different W' in (2.8.2) which is a blow up of $\mathbb{P}^3$ along a rational curve Γ.

$$\begin{aligned}\mathrm{Eu}(W') &= \mathrm{Eu}(\mathbb{P}^3 - \Gamma) + \mathrm{Eu}(\mathbb{P}^1)\cdot\mathrm{Eu}(\Gamma) \\ &= \mathrm{Eu}(\mathbb{P}^3) + \{\mathrm{Eu}(\mathbb{P}^1) - 1\}\cdot\mathrm{Eu}(\mathbb{P}^1) \\ &= 4 + 1\cdot 2 = 6.\end{aligned}$$

Thus

$$\mathrm{Eu}(V) = \mathrm{Eu}(W) - \mathrm{Eu}(S) + \mathrm{Eu}(\mathbb{P}^1) = \mathrm{Eu}(W) - 2 = \mathrm{Eu}(W') - 2 = 4.$$

We thus get different values of $\mathrm{Eu}(V)$, which is a contradiction and the case $g = 11$ does not occur.

(2.12) *Judgment of the geometrical realization of solutions* (2.8.1)–(2.8.2).

Cases $g \geqslant 10$ in case (A) and cases $g \geqslant 9$ in case (B) are all settled.

To settle the remaining cases, we use the computed value of $\mathrm{Eu}(V_{16})$ and $\mathrm{Eu}(V_{14})$: since we have settled the existence of lines, we can apply line-blow-up and do similar computations which are done by Iskovskih [2] (or [6]). Thus one obtains

$$\mathrm{Eu}(V_{14}) = -6, \; \mathrm{Eu}(V_{16}) = -2.$$

From these it is easy to decide which case occurs. We put # in front of the cases actually occurring.

(2.13) *Illustration of (bi)rational maps.*

(2.13.1) *Cases of point-blow-up* (A).

$g = 12$

$$\begin{tikzcd}
S \arrow[d, "\sigma"'] & \subset & W \arrow[dll] \arrow[rr, dotted, leftrightarrow, "\text{flop}"] & & W' \arrow[d, "\alpha"] & \supset & D \sim 3(-K_{W'}) - 4S' \arrow[d, "|-K_{W'} - S'|"] \\
P \in V_{22} \arrow[rrrr, dotted, "|-K_V - 3P|"] & & & & \mathbb{P}^3 & \supset & \Gamma
\end{tikzcd}$$

$\deg \Gamma = 6$, $g(\Gamma) = 0$

$g = 10$

$$\begin{tikzcd}
S \arrow[d, "\sigma"'] & \subset & W \arrow[dll] \arrow[rr, dotted, leftrightarrow, "\text{flop}"] & & W' \arrow[d, "\alpha"] \\
P \in V_{18} \arrow[rrrr, dotted, "|-K_V - 3P|"] & & & & \mathbb{P}^1
\end{tikzcd}$$

$|-K_{W'} - S'|$

del Pezzo fibering $\deg W'_v = 6$

$g = 9$

$$\begin{tikzcd}
S \arrow[d, "\sigma"'] & \subset & W \arrow[dll] \arrow[rr, dotted, leftrightarrow, "\text{flop}"] & & W' \arrow[d, "\alpha"] & \supset & D \sim (-K_{W'}) - S' \arrow[d, "|3(-K_{W'}) - 2S'|"] \\
P \in V_{16} \arrow[rrrr, dotted, "|3(-K_V) - 8P|"] & & & & V_{16} & \ni & P
\end{tikzcd}$$

$g = 8$

$$\begin{tikzcd}
S \subset W \arrow[d, "\sigma"] \arrow[rr, dashed, leftrightarrow, "\text{flop}"] & & W' \supset D \sim 3(-K_{W'}) - 2S' \arrow[d, "\alpha"', "{|2(-K_{W'}) - S'|}"] \\
P \in V_{14} \arrow[rr, dashed, "{|2(-K_V) - 5P|}"] & & B_3 \supset \Gamma \qquad \deg \Gamma = 4,\ g(\Gamma) = 0
\end{tikzcd}$$

(2.13.2) *Cases of conic-blow-up* (B).

$g = 12$

$$\begin{tikzcd}
S \subset W \arrow[d, "\sigma"] \arrow[rr, dashed, leftrightarrow, "\text{flop}"] & & W' \supset D \sim 2(-K_{W'}) - 3S' \arrow[d, "\alpha"', "{|-K_{W'} - S'|}"] \\
C \subset V_{22} \arrow[rr, dashed, "{|-K_V - 2C|}"] & & Q \supset \Gamma \qquad \deg \Gamma = 6,\ g(\Gamma) = 0
\end{tikzcd}$$

$g = 10$

$$\begin{tikzcd}
S \subset W \arrow[d, "\sigma"] \arrow[rr, dashed, leftrightarrow, "\text{flop}"] & & W' \arrow[d, "\alpha"', "{|-K_{W'} - S'|}"] \\
C \subset V_{18} \arrow[rr, dashed, "{|-K_V - 2C|}"] & & \mathbb{P}^2 \supset \Delta
\end{tikzcd}$$

conic bundle
$\deg \Delta = 4$

$g = 9$

$$\begin{tikzcd}
S \subset W \arrow[d, "\sigma"] \arrow[rr, dashed, leftrightarrow, "\text{flop}"] & & W' \arrow[d, "\alpha"', "{|-K_{W'} - S'|}"] \\
C \subset V_{16} \arrow[rr, dashed, "{|-K_V - 2C|}"] & & \mathbb{P}^1
\end{tikzcd}$$

del Pezzo
fibering
$\deg W'_v = 6$

$g = 8$

$$\begin{tikzcd}
S \subset W \arrow[d, "\sigma"] \arrow[rr, dashed, leftrightarrow, "\text{flop}"] & & W' \supset D \sim (-K_{W'}) - S' \arrow[d, "\alpha"', "{|2(-K_{W'}) - S'|}"] \\
C \subset V_{14} \arrow[rr, dashed, "{|2(-K_V) - 3C|}"] & & V_{14} \supset \Gamma \qquad \deg \Gamma = 2,\ g(\Gamma) = 0
\end{tikzcd}$$

Section 3. Birational map of a cubic 3-fold to a Fano 3-fold of g = 8

(3.0) In this section, we prove the following (cf. Theorem 0.3 in Section 0):

(3.1) THEOREM. *For every Fano 3-fold* $V = V_{14} \subset \mathbb{P}^9$ *of genus* 8, *there is a bi-*

rational map from a cubic 3-fold $V' = B_3 \subset \mathbb{P}^4$ *to* V:

$$\begin{array}{ccccc} D \subset W' & \leftarrow \cdots \rightarrow & W \supset S \\ \swarrow\ \alpha \swarrow & & \searrow\ \sigma \searrow \\ \Gamma \subset V' & \cdots \xrightarrow{g} & V \ni P \end{array}$$

Here Γ *is a rational normal quartic curve on* V' *and the rational map* g *is given by the linear system* $|8L' - 5\Gamma|$ *for a hyperplane section* L' *of* V'. *The exceptional divisor* S *of* σ *is the strict transform of the unique member of the linear system* $|3L' - 2\Gamma|$ *on* V'. (*This unique member is an intersection of* V' *and* M *which is a cubic hypersurface with singularities along* Γ *swept by chords of* Γ.)

(3.2) REMARK. Fano and Iskovskih [4] have another birational map between $V' = B_3$ and $V = V_{14}$:

$$\begin{array}{ccccc} R' \subset W' & \leftarrow \cdots \rightarrow & W \supset R \\ \swarrow\ \alpha \swarrow & & \searrow\ \sigma \searrow \\ B' \subset V' & \cdots \xrightarrow{\chi} & V \supset B \end{array}$$

In this diagram, B and B' are quintic elliptic curves on V and V' respectively, the map χ is defined by the linear system $|7L' - 4B'|$ on V', and R is the strict transform of the unique member of $|5L' - 3B'|$.

(3.3) First we will recall the result [0], [12] for Fano surface of a cubic 3-fold.

Let $V' = B_3$ be a cubic 3-fold in $\mathbb{P}^4$, and $\Phi = \Phi(V')$ the Fano surface parametrizing lines in V'. Then Φ is a nonsingular irreducible surface and there is a diagram of natural morphisms:

$$\begin{array}{l} \Psi \xrightarrow{\psi} V' \subset \mathbb{P}^4 \\ \pi \downarrow \\ \Phi \end{array}$$

where Ψ is a universal family of lines in V'. In this diagram,

(3.3.1) *The morphism* ψ *is a covering of degree* 6.

Let R be the ramification divisor of ψ in Ψ, then

(3.3.2) $R = \pi^{-1}(\Sigma)$ *for some curve* Σ *on* Φ *such that* $\Sigma \sim 2K_\Phi$, *where* K_Φ *is a canonical divisor of* Φ.

For each $u \in \Phi$, let $L_u = \psi(\pi^{-1}(u))$ be a line in V' corresponding to u and let

$$C_u = (x \in \Phi \mid L_x \text{ intersects the fixed line } L_u\}$$

be a curve in Φ. Then

(3.3.3) *the intersection number* $(C_u \cdot D)$ *for any divisor* D *on* Φ *is independent of* $u \in \Phi$, *and* $(C_u \cdot C_u) = 5$.

For a canonical divisor,

(3.3.4) $K_\Phi \sim C_u + V_v + C_w$ *for* $u, v, w \in \Phi$ *such that three lines* L_u, L_v, L_w *are coplanar.*

(3.4) Now let Γ be a rational normal quartic curve in $\mathbb{P}^4$. Then the chords of Γ are distinct. Indeed, if two chords of Γ coincide, this line is intersecting with Γ at least at three points, which is a contradiction. Moreover each pair of chords is not intersecting except on Γ. If two chords of Γ are intersecting, then these chords determine a plane in $\mathbb{P}^4$, but the cut of Γ by the plane is at most three geometric points. This means two chords intersect only at a point on Γ.

(3.5) LEMMA. *The rational normal quartic curve* Γ *on a cubic* 3*-fold* V' *in* $\mathbb{P}^4$ *has exactly* 16 *chords in* V'.

Proof. Let $Y = \psi^{-1}(\Gamma)$ and $X = \pi(Y)$. Then Y is a curve in Ψ and $\psi|_Y\colon Y \to \Gamma$ is a covering of degree 6. First we will consider the case where Y is nonsingular. The curve X is parametrizing lines intersecting with Γ, and its double points are corresponding to chords of Γ in V'. By the argument in (3.4), Γ has only double points as singularities. To compute the number of chords of Γ, we have only to count the double points of X, i.e., $p_a(X) - g(X)$, where $p_a(X)$ is the arithmetic genus of X and $g(X)$ is the geometric genus. Because Y is a nonsingular model of X, $g(X) = g(Y) = (\deg K_Y/2) + 1$. Using (3.3.1)–(3.3.4), we get $g(X) = 55$ and $p_a(X) = 71$, hence we obtain this lemma in this case. For the case where Y has singularities, the same reason shows that the number of the chords of Γ is equal to $p_a(X) - p_a(Y)$, and this difference is equal to 16.

(3.6) LEMMA. *Let* $\alpha\colon W' \to V'$ *be a blow-up of a cubic* 3*-fold* V' *along a rational normal quartic* Γ. *Then the anti-canonical divisor* $-K_{W'}$ *of* W' *defines a morphism* $\varphi' = \varphi_{-K_{W'}}\colon W' \to \bar{W} \subset \mathbb{P}^5$, *and* φ' *contracts only strict transform of chords of* Γ.

Proof. Let L' be a hyperplane section of V' and $D = \alpha^{-1}(\Gamma)$ an exceptional divisor for α. Then $-K_{V'} \sim 2L'$ and $-K_{W'} \sim 2\alpha^* L' - D$. Let H' be a hyperplane in $\mathbb{P}^4$, then $L' = H' \cap V'$. We have the commutative diagram

$$\begin{array}{ccc} & & W' \\ & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle\varphi'} \\ V' & \xrightarrow[f]{\cdots} & \bar{W} \\ \cap & & \cap \\ \mathbb{P}^4 & \xrightarrow[F]{\cdots} & \mathbb{P}^5 \end{array}$$

of the natural maps, where $F = \varphi_{2H'-\Gamma}: \mathbb{P}^4 \cdots\to \mathbb{P}^5$, $f = \varphi_{2L'-\Gamma}: V' \cdots\to \bar{W}, f = F|_{V'}$ and φ' is a resolution of f.

It is obvious that the linear system $|-K_{W'}|$ is free, so we will prove the second assertion.

Choosing a suitable coordinates $[X_0:X_1:X_2:X_3:X_4]$ of $\mathbb{P}^4$, we may assume the curve Γ is defined by the following system of equations:

$$\begin{aligned} &F_0 = X_0X_2 - X_1^2 = 0, && F_1 = X_0X_3 - X_1X_2 = 0,\\ &F_2 = X_0X_4 - X_1X_3 = 0, && F_3 = X_1X_3 - X_2^2 = 0,\\ &F_4 = X_1X_4 - X_2X_3 = 0 \quad \text{and} && F_5 = X_2X_4 - X_3^2 = 0. \end{aligned}$$

Then the member of $|2H' - \Gamma|$ is defined by the equation of linear combination $\Sigma_{i=0}^5 \lambda_i F_i = 0$ and the rational map F is given by $[X_0:X_1:X_2:X_3:X_4] \to [F_0(X):F_1(X):F_2(X):F_3(X):F_4(X):F_5(X)]$. The image $F(\mathbb{P}^4)$ is a hypersurface in $\mathbb{P}^5$ defined by $Y_0Y_5 - Y_1Y_4 + Y_2Y_3 = 0$, where $[Y_0:Y_1:Y_2:Y_3:Y_4:Y_5]$ are coordinates of $\mathbb{P}^5$ with respect to the basis of the linear system $|2H' - \Gamma|$. The straightforward calculation shows that the fiber of F at a point of $F(\mathbb{P}^4)$ is a point or a line, and that these lines are chords of Γ in $\mathbb{P}^4$. Therefore the fiber of $f: V' \cdots\to \bar{W}$ is only a point or a chord in V' of Γ. Q.E.D.

(3.7) Proof of (3.1). By Lemma 3.6, W' is a small resolution of $\bar{W}$, and $\bar{W}$ has another small resolution $\varphi: W \to \bar{W}$. We have $\rho(W) = \rho(W') = \rho(V') + 1$, and Pic W is generated by L and D^* or L and $-K_W$ where L and D^* are strict transforms of α^*L' and of $D = \alpha^{-1}(\Gamma)$ respectively. Thus we can determine the extremal ray of cone $\overline{\mathrm{NE}}(W)$ of W using numerical way similar as in Section 2.

Let e be the number of chords of Γ in V'. Suppose that e is unknown, then we obtain the following two possibilities.

(1) $e = 16$.

$$\begin{tikzcd} D \subset W' \arrow[d, "\alpha"'] \arrow[r, dashed, leftrightarrow, "\text{flop}"] & W \supset S \sim 2(-K_W) - L \sim 3L - 2D^* \arrow[d, "\sigma"] \\ \Gamma \subset V' & \mathbb{P}^9 \supset V \ni P \end{tikzcd}$$

In this diagram, V is a Fano 3-fold $V_{14} \subset \mathbb{P}^9$ of index 1 of genus 8, and σ is a blow-up at a point P of V.

(2) $e = 15$.

$$\begin{tikzcd} D \subset W' \arrow[d, "\alpha"'] \arrow[r, dashed, leftrightarrow, "\text{flop}"] & W \supset E \sim 2(-K_W) - L \sim 3L - 2D^* \arrow[d, "\beta"] \\ \Gamma \subset V' & \mathbb{P}^8 \supset \bar{V} \supset C \end{tikzcd}$$

In this diagram, $\bar{V}$ is a Fano 3-fold $V_{12} \subset \mathbb{P}^8$ of index 1 of genus 7, and C is a conic in $\bar{V}$.

But Lemma 3.5 shows that the case (1) is realized and the case (2) is impossible. Thus we construct the birational map $g: B_3 \cdots\to V_{14}$ which is defined by the linear system $|8L' - 5\Gamma|$ on B_3. This proves Theorem 3.1. Its inverse map h: $V_{14} \cdots\to B_3$ is defined by the linear system $|2H - 3P|$ for a hyperplane section H of V_{14} in $\mathbb{P}^9$.

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