COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 71, n° 2 (1989), p. 229-240
<http://www.numdam.org/item?id=CM_1989__71_2_229_0>

Compositio Mathematica **71**: 229–240, 1989.

On hypersurface singularities which are stems

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Received 12 December 1988

Section 1. Introduction

If one classifies functions of finite codimension one encounters series of functions. Well known examples in $\mathbb{C}\{x, y, z\}$ are:

$$A_k: \quad x^{k+1} + y^2 + z^2; \qquad k \geqslant 2$$
$$\mathcal{D}_k: \quad x^{k-1} + xy^2 + z^2; \qquad k \geqslant 4$$
$$T_{p,q,r}: x^p + y^q + z^r + xyz; \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

See Arnold [1].

Deleting the part which varies with the indices one gets a function one is inclined to call the stem of the series. For instance:

$$A_\infty: \qquad y^2 + z^2$$
$$D_\infty: \qquad xy^2 + z^2$$
$$T_{\infty,\infty\infty}: \quad xyz.$$

See Siersma [15].

The same phenomenon occurs if one classifies map germs $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ of finite A-codimension, see [10]. The word stem is used in [11] by Mond without giving a definition, but he suggested the following definition.

A function f is a stem if it is not finitely determined and if for some k, every function g with the same k-jet as f is either finitely determined or right-equivalent with f.

It still is a problem to define a series, see [1] page 153 or [13], but the notion of a stem seems to be a first step in understanding series in the classification of singularities, see Van Straten [16] for another approach.

The results of this paper are the following.

THEOREM 1.1. *Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function. Then f is a stem if and only if f has an irreducible curve Σ as singular locus and f has transversal A_1 singularities on $\Sigma \backslash \{0\}$.*

Following J. Montaldi we give an inductive definition of a stem of degree d.

THEOREM 1.2. *Let $f: (\mathbb{C}^{n+1}, 0)$ be a germ of an analytic function. If f is a stem of degree d then the singular locus Σ of f is a curve with at most d branches. If moreover the number of branches of Σ is equal to d then f has transversal A_1 singularities on $\Sigma \backslash \{0\}$.*

THEOREM 1.3. *Let $f: (\mathbb{C}^{n+1}, 0)$ be a germ of an analytic function. If the singular locus Σ of f is a curve with d branches and f has transversal A_1 singularities on $\Sigma \backslash \{0\}$. Then f is a stem of degree d.*

In Section 2 we collect known results, which we need in the sequel. In Section 3 we proof Theorem 1.2 and part of 1.1. In Section 4 we proof Theorem 1.3 and part of 1.1. We conclude with some questions.

We denote by $\mathcal{O}$ the local ring of germs of analytic functions $f: (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$, and m its maximal ideal. The germ in $(\mathbb{C}^{n+1}, 0)$ of the zero set of an ideal I in $\mathcal{O}$ is denoted by $V(I)$. We denote by J_f the ideal $(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)\,\mathcal{O}$.

Section 2. Finite determinacy

DEFINITION 2.1. Let $J^k: \mathcal{O} \to \mathcal{O}/m^{k+1}$ be the projection map. We call $J^k f$ the k-jet of f, for an element $f \in \mathcal{O}$. In the same way we denote by $J^k f$ the k-jet of a mapping $f \in \mathcal{O}^m$ or a matrix $f \in \mathcal{O}^{p \times q}$.

DEFINITION 2.2. We denote by $\mathcal{D}$ the group of all germs of local analytic isomorphisms $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$. Two functions f and g in $\mathcal{O}$ are called R-equivalent if $f = g \circ h$ for some $h \in \mathcal{D}$.

The function $f \in \mathcal{O}$ is called k-determined if for every $g \in \mathcal{O}$ with $J^k f = J^k g$ then f and g are R-equivalent. The function f is called *finitely determined* if it is k-determined for some k.

A function is finitely determined if and only if it has an isolated singularity, by Mather [8] and Tougeron [17] or [18].

D. Mond proposed the following definition.

DEFINITION 2.3. Let $f \in \mathcal{O}$. Suppose f is not finitely determined then f is called a k-stem if for every $g \in \mathcal{O}$ with $J^k g = J^k f$ either g is finitely determined or g is R-equivalent with f. If f is a k-stem for some $k \in \mathbb{N}$ then we call f a *stem*.

J. Montaldi suggested the following inductive definition.

DEFINITION 2.4. Let $f \in \mathcal{O}$ then f is called a k-stem of degree 0 if f is k-determined. The function f is a k-stem of degree d, if f is not a stem of degree t, for some $0 \leqslant t < d$, and if for every $g \in \mathcal{O}$ with $J^k g = J^k f$ either g is a stem of degree s, $0 \leqslant s < d$, or g is R-equivalent with f. If f is a k-stem of degree d, for some $k \in \mathbb{N}$, then we call f a *stem of degree d*.

REMARK 2.5. A stem of degree d gives rise to a series of stems of degree $d - 1$. For example

$T_{\infty,\infty,\infty}: xyz$	is a stem of degree 3,
$T_{\infty,\infty,r}: xyz + z^r$	is a stem of degree 2,
$T_{\infty,q,r}: xyz + y^q + z^r$	is a stem of degree 1,
$T_{p,q,r}: xyz + x^p + y^q + z^r$	is a stem of degree 0.

This follows from Theorem 1.3.

The finite determinacy theorem has been generalized for non-isolated singularities by Siersma [15], Izumiya and Matsuoka [4], and Pellikaan [12], [14].

DEFINITION 2.6. Let I be an ideal in $\mathcal{O}$. Define

$$\int I = \{f \in \underline{\mathcal{O}} \mid (f) + J_f \subset I\}.$$

This is called the *primitive ideal* of I and in case I is a radical ideal defining the germ $(\Sigma, 0)$ in $(\mathbb{C}^{n+1}, 0)$ then

$$\int I = \left\{ f \in m \mid \text{the singular locus of } f \text{ contains } \sum \right\}.$$

If Σ is a reduced complete intersection then $\int I = I^2$, see [12], [14].

DEFINITION 2.7. Let $\mathcal{D}_I$ be the group of all germs of local analytic isomorphisms leaving I invariant, that is to say: $\mathcal{D}_I = \{h \in \mathcal{D} \mid h^*(I) = I\}$. Two functions f and g in $\int I$ are called R-I-equivalent if $f = g \circ h$ for some $h \in \mathcal{D}_I$, that is to say f and g are in the same orbit of the action of $\mathcal{D}_I$ on $\int I$.

In case I is a radical ideal and $\dim_{\mathbb{C}}(I/J_f) < \infty$ then the *tangent space* $\tau_I(f)$ of the orbit of f under the action of $\mathcal{D}_I$, can be identified with $mJ_f \subset \int I$, see [12], [14].

DEFINITION 2.8. Let $f \in \int I$ and $\dim_{\mathbb{C}}(I/J_f) < \infty$, then we call $\dim_{\mathbb{C}}(\int I / J_f \cap \int I)$ the I-codimension of f and denote it by $c_I(f)$.

DEFINITION 2.9. If $f \in \int I$ then f is called (k, I)-determined, if for every $g \in \int I$ with the same k-jet as f one has that f and g are $R-I$-equivalent. The function f is finitely I-determined if it is (k, I)-determined for some $k \in \mathbb{N}$.

REMARK 2.10. There exists an $r \in \mathbb{N}$ such that for every $k \in \mathbb{N}$: $m^{k+r} \cap \int I \subset m^k \int I$, by Artin-Rees lemma, see [9] 11.c. Let $r(\int I)$ be the minimal number r for which the above inclusion holds.

THEOREM 2.11. *Let $f \in \int I$ and $r = r(\int I)$.*

(i) *If f is (k, I)-determined then*

$$m^{k+1} \cap \int I \subset \tau_I(f).$$

(ii) *If*

$$m^{k+1} \int I \subset m\tau_I(f) + m^{k+2} \int I$$

then f is $(k + r, I)$-determined.

Proof. See [12], [14]. □

COROLLARY 2.12. *Let $f \in \int I$ then f is finitely I-determined if and only if $c_I(f) < \infty$.*

REMARK 2.13. If I is a radical ideal defining a germ of the curve $(\Sigma, 0)$ then $c_I(f) < \infty$ if and only if $\dim_{\mathbb{C}}(I/J_f) < \infty$ if and only if f has only transversal A_1 singularities on $\Sigma \backslash \{0\}$. See [12], [14].

We also need the following finite determinacy theorem due to Hironaka:

THEOREM 2.14. *Let $(X, 0)$ be a germ of a reduced analytic space in $(\mathbb{C}^N, 0)$ with an isolated singularity. Let*

$$\mathcal{O}^q \xrightarrow{u} \mathcal{O}^p \xrightarrow{g} \mathcal{O} \to \mathcal{O}_X \to 0$$

be an exact sequence of $\mathcal{O}$-modules.

Then there exists a triple (σ, τ, ρ) of positive integers such that for all $k \geqslant \tau$ and all complexes

$$\mathcal{O}^q \xrightarrow{\bar{u}} \mathcal{O}^p \xrightarrow{\bar{g}} \mathcal{O}$$

such that $J^\sigma u = J^\sigma \bar{u}$ and $J^k g = J^k \bar{g}$, there exists a germ of a local analytic isomorphism $h: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ such that $h(\bar{X}, 0) = (X, 0)$ and $J^{k-\rho} h = id$. Where

$(\bar{X}, 0)$ is the germ of the analytic space in $(\mathbb{C}^N, 0)$ with local ring $\mathcal{O}/Im(\bar{g})$.

REMARK 2.15. This theorem is proved by Hironaka [6] Theorem 3.3, in the formal category. One uses Artin approximation [2] to get local analytic isomorphism. See also Artin [3] Theorem 3.9.

In the proof of Theorem 1.3 we need a strengthening of Artin approximation due to Wavrik [19]:

THEOREM 2.16. *Let $G = (G_1, \ldots, G_m)$ with $G_i \in \mathbb{C}\{x\}[y]$. Then for all $\alpha \in \mathbb{N}$ there exists a $\beta \in \mathbb{N}$ such that if $y(x) \in \mathbb{C}[[x]]^r$ and $J^\beta G(x, y(x)) = 0$ then there exists $\bar{y}(x) \in \mathbb{C}\{x\}^r$ such that*

$$G(x, \bar{y}(x)) = 0 \quad \text{and} \quad J^\alpha y = J^\alpha \bar{y}.$$

Section 3. The number of branches of a stem

LEMMA 3.1. *Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function which is a stem of degree d. If Σ is a curve with r branches contained in the singular locus of $f^{-1}(0)$ then $r \leqslant d$.*

Proof. By induction on r. Suppose $r = 1$ then f has not an isolated singularity at 0 and therefore f can not be a stem of degree 0, by Mather [8] and Tougeron [17], [18]. Thus $d \geqslant 1$.

Suppose f is a k-stem of degree d and the singular locus of $f^{-1}(0)$ contains the curve $\Sigma_1 \cup \cdots \cup \Sigma_{r+1}$ with $r+1$ branches. Let I be the ideal defining $\Sigma_1 \cup \cdots \cup \Sigma_r$, generated by $g_1, \ldots, g_m$. Let

$$f_\lambda = f + \sum \lambda_i g_i^{k+1}.$$

Then the singular locus of $f^{-1}(0)$ is contained in $\Sigma_1 \cup \cdots \cup \Sigma_r$, for all $\lambda \in U$, where U is a dense subset of $\mathbb{C}^m$, by Bertini's theorem. So there exists a $\lambda \in U$ such that the singular locus of $f_\lambda^{-1}(0)$ is equal to $\Sigma_1 \cup \cdots \cup \Sigma_r$. Hence f cannot be R-equivalent with f. But f_λ and f have the same k-jet and f is a k-stem of degree d. Thus f_λ must be a stem of degree $t < d$. By the induction assumption we have that $r \leqslant t$, so $r + 1 \leqslant d$. This proves the lemma.

COROLLARY 3.2. *Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function which is a stem of degree d. Then the singular locus of f is a curve with at most d branches.*

Proof. If the dimension of the singular locus of f is bigger than one, then it contains a curve with r branches, for any $r \in \mathbb{N}$. By Lemma 3.1, f cannot be a stem of finite degree.

PROPOSITION 3.3. *Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function which is a stem of degree d. If the number of branches of the singular locus Σ of f is d then f has transversal A_1 singularities on $\Sigma\setminus\{0\}$.*

Proof. Suppose f is a k-stem of degree d. Let the curve Σ be the singular locus of $f^{-1}(0)$ with branches $\Sigma_1,\ldots,\Sigma_d$. Let p_i be the prime ideal defining Σ_i. Let $I=p_1\cap\cdots\cap p_d$ then I defines Σ.
Let $z_0,z_1,\ldots,z_n$ be local coordinates of $(\mathbb{C}^{n+1},0)$ such that $\Sigma\cap V(z_0)=\{0\}$. One can choose generators $g_1,\ldots,g_m$ of I such that

$$(g_1,\ldots,g_n)\mathcal{O}_{p_i}=I\,\mathcal{O}_{p_i},\qquad \text{for all } i=1,\ldots,d.$$

Moreover for all $a\in\Sigma\setminus\{0\}$ small enough one has that $z_0-a_0,g_1,\ldots,g_n$ are local coordinates of $(\mathbb{C}^{n+1},a)$, where $a=(a_0,a_1,\ldots,a_n)$, see [12], [13].
Consider

$$f_{\lambda,\mu}=f+\mu z_0^k\left(\sum_{i=1}^{n} g_i^2\right)+\sum_{j=1}^{m}\lambda_j g_j^{k+2},$$

then by Bertini's theorem there exists a set G_1 in $\mathbb{C}^m\times\mathbb{C}$, which is the countable intersection of open dense sets, such that the singular locus of $f_{\lambda,\mu}^{-1}(0)$ is contained in $V(z_0^k(\Sigma_{i=1}^n g_i^2),g_1^{k+2},\ldots,g_m^{k+2})$ for all $(\lambda,\mu)\in G_1$. So the singular locus of $f_{\lambda,\mu}^{-1}(0)$ is equal to Σ for $(\lambda,\mu)\in G_1$. The p_i-primary components of $\int I$ and I^2 are the same, see [12], [14], hence $\dim_{\mathbb{C}}(\int I/I^2)<\infty$ and $m^l.\int I\subset I^2$ for some $l\in\mathbb{N}$. We can write $(g_1,\ldots,g_n)=I\cap K$, for some ideal K, which for every $i=1,\ldots,d$ is not contained in p_i, by the primary decomposition of the ideal $(g_1,\ldots,g_n)$. Hence m^lK^2 is not contained in $p_1\cup\cdots\cup p_d$, by [9] 1.B. So there exists an element s in $m^lK^2\setminus(p_1\cup\cdots\cup p_d)$. Thus

$$sf\in K^2m^l\int I\subset(KI)^2\subset(g_1,\ldots,g_n)^2,$$

since $f\in\int I$. Therefore we can write

$$sf=\sum_{i,j=1}^{n} h_{ij}g_ig_j.$$

Let

$$\Delta=\det(h_{ij}+s\mu z_0^k\delta_{ij}),$$

then the zeroset of Δ defines a hypersurface V in $\mathbb{C}^{n+1} \times \mathbb{C}^m \times \mathbb{C}$, which does not contain $\Sigma \times \mathbb{C}^m \times \mathbb{C}$, since Δ is a polynomial in μ and the coefficient of the highest degree term is $s^n z_0^{nk}$, which is not an element of I.

The intersection $(\Sigma \times \mathbb{C}^m \times \mathbb{C}) \cap V$ contains two sorts of components: the vertical components V_α of the form $\Sigma \times W_\alpha$, where W_α is a proper analytic subset of $\mathbb{C}^m \times \mathbb{C}$, and the horizontal components H_β, which project finitely on $\mathbb{C}^m \times \mathbb{C}$. Let $W = \cup W_\alpha$, then the complement U of W in $\mathbb{C}^m \times \mathbb{C}$, is an open dense subset. Let $G = G_1 \cap U$, then G is a countable intersection of open dense subsets, hence G is dense in $\mathbb{C}^m \times \mathbb{C}$ by Baire's category theorem.

For all $(\lambda, \mu) \in G$ the zero set $f_{\lambda,\mu}^{-1}(0)$ has singular locus Σ and for all $a \in \Sigma \backslash \{0\}$ small enough, the transversal hessian of $f_{\lambda,\mu}$ at a has determinant not equal to zero, since $\Delta(a) \neq 0$ and $s(a) \neq 0$, since $s \notin p_i$ for all $i = 1, \ldots, d$. Hence $f_{\lambda,\mu}$ has transversal A_1 singularities on $\Sigma \backslash \{0\}$, see [14], [15].

If $f_{\lambda,\mu}$ is not R-equivalent with f, then $f_{\lambda,\mu}$ is a stem of degree $t, t < d$, since $f_{\lambda,\mu}$ and f have the same k-jet and f is a k-stem of degree d. But the singular locus of $f_{\lambda,\mu}^{-1}(0)$ has d branches and this contradicts Lemma 3.1. Thus $f_{\lambda,\mu}$ and f are R-equivalent. This proves Proposition 3.3 and completes the proof of Theorem 1.2.

Section 4. Sufficiency

LEMMA 4.1. *Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function. If f has a curve Σ as singular locus and f has transversal A_1 singularities on $\Sigma \backslash \{0\}$, then for every $r \in \mathbb{N}$ there exists a $t \in \mathbb{N}$ such that for all $\phi \in m^{t+2}$: if $f + \phi$ has singular locus Σ_ϕ then there exists a local analytic isomorphism $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $h(\Sigma_\phi) \subset \Sigma$ and $J^r h = \mathrm{id}$.*

Proof. If $f + \phi$ has an isolated singularity we can take for h the identity map. So we only have to consider the case that $f + \phi$ has a non-isolated singularity.

(i) Let $z_0, z_1, \ldots, z_n$ be local coordinates of $(\mathbb{C}^{n+1}, 0)$ such that the polar curve Γ of the map

$$(f, z_0): (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$$

is reduced. Such a z_0 exists by a result of Hamm and Lê [4], in fact "almost every" z_0 will do. Let K be the vanishing ideal of Γ, then

$$V(f_1, \ldots, f_n) = \Sigma \cup \Gamma \quad \text{and}$$
$$(f_1, \ldots, f_n) = I \cap K.$$

The map

$$F = (f_1, \ldots, f_n): (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$$

defines a complete intersection curve $\Sigma \cup \Gamma$ with an isolated singularity. So F is finitely determined with respect to contact-equivalences, see Mather [8]. So there exists a $\mu \in \mathbb{N}$ such that for every map $G: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ with the same μ-jet as F, is contact-equivalent with F. In particular for every $\phi \in m^{\mu+2}$ there exists a local analytic isomorphism $H: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that

$$H(V(f_1 + \phi_1, \ldots, f_n + \phi_n)) = \Sigma \cup \Gamma,$$

where $\phi_i = \partial\phi/\partial z_i$.
So for every $\phi \in m^{\mu+2}$ such that $f + \phi$ has a non-isolated singularity, the singular locus Σ_ϕ of $f + \phi$ is isomorphic with $H(\Sigma_\phi)$, which is contained in the curve $\Sigma \cup \Gamma$ and therefore Σ_ϕ must be a curve.

(ii) By (i) we know that $H(\Sigma_\phi) \subset \Sigma \cup \Gamma$. Hence the minimal number of generators of Σ_ϕ and the minimal number of relations between the generators are bounded above by say p and q respectively.

(iii) Let $(\sigma(X), \tau(X), \rho(X))$ be the triple of integers associated to the reduced curve $(X, 0)$ as stated in Theorem 2.14 of Hironaka. Let

$$\sigma = \max\{\sigma(X) \mid X \text{ is a reduced curve and } (X, 0) \subset (\Sigma \cup \Gamma, 0)\}.$$

Then σ is finite, since there are only finitely many reduced subcurves of $(\Sigma \cup \Gamma, 0)$. In the same way one defines τ and ρ.

(iv) Let

$$G_i(x, y, z) := \begin{cases} \sum_{j=1}^{p} z_{ij} y_j, & \text{for } i = 1, .., q \\ f_{i-q-1}(x) - \sum_{j=1}^{p} z_{ij} y_j, & \text{for } i = q+1, \ldots, q+n+1 \end{cases}$$

then $G_i(x, y, z) \in \mathbb{C}\{x\}[y, z]$. Let $r \in \mathbb{N}$ and define $\alpha = \max\{\sigma, \tau, \sigma + r\}$, then there exists a β associated to α as stated in Wavrik's Theorem 2.16.

(v) Let $t = \max\{\mu, \beta\}$, then for all $\phi \in m^{t+2}$ such that $f + \phi$ has a non-isolated singularity, the vanishing ideal I_ϕ of Σ_ϕ has p generators $g_1, \ldots, g_p$ and q relations between these generators:

$$\sum_{j=1}^{p} u_{ij} g_j = 0, \qquad \text{for } i = 1, \ldots, q.$$

That is to say, the following sequence is exact

$$\mathcal{O}^q \xrightarrow{u} \mathcal{O}^p \xrightarrow{g} \mathcal{O} \to \mathcal{O}/I_\phi \to 0.$$

Moreover, there exist elements $u_{i+q+1,j} \in \mathcal{O}$ such that

$$f_i + \phi_i = \sum_{j=1}^{p} u_{i+q+1,j} g_j, \qquad \text{for } i = 0, 1, \ldots, n,$$

since $f_i + \phi_i \in I_\phi$.
Thus

$$J^\beta G(x, g(x), u(x)) = 0, \qquad \text{since } t \geqslant \beta \text{ and } \phi_i \in m^{t+1}.$$

Hence by Wavrik's theorem there exist $\bar{g} \in \mathcal{O}^p$ and $\bar{u} \in \mathcal{O}^{p(q+n+1)}$ such that $J^\alpha \bar{g} = J^\alpha g$ and $J^\alpha \bar{u} = J^\alpha u$ and $G(x, \bar{g}(x), \bar{u}(x)) = 0$, that is to say

$$\begin{cases} \sum_{j=1}^{p} \bar{u}_{ij} \bar{g}_j = 0, & \text{for } i = 1, \ldots, p \\ f_i = \sum_{j=1}^{p} \bar{u}_{i+q+1,j} \bar{g}_j, & \text{for } i = 0, \ldots, n. \end{cases}$$

Since $H(\Sigma_\phi) \subset \Sigma \cup \Gamma$ and $\alpha = \max\{\sigma, \tau, \rho + r\}$ and by (iii), we can apply Hironaka's Theorem 2.14, that is to say there exists a local analytic isomorphism $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that

$$(g_1, \ldots, g_p) = h^*(\bar{g}_1, \ldots, \bar{g}_p)$$

and

$$J^{\alpha - \rho} h = \text{id}.$$

Hence $J^r h = \text{id}$, since $\alpha \geqslant \rho + r$. Further $h(\Sigma_\phi) \subset \Sigma$, since $\Sigma_\phi = V(g_1, \ldots, g_p)$ and $\Sigma = V(J_f)$ and $J_f \subset (\bar{g}_1, \ldots, \bar{g}_p)$. This proves Lemma 4.1.

Proof of theorem 1.3. The proof is by induction on d. In case $d = 0$, that is to say f has an isolated singularity, f is a stem of degree 0. Now suppose the proposition is proved for all $t < d$. Let I be the vanishing ideal of the singular locus Σ of f, then $f \in \int I$, by 2.6. Since f has transversal A_1 singularities on $\Sigma \setminus \{0\}$ and Σ is a curve we have that f is (r, I)-determined for some $r \in \mathbb{N}$, by Theorem 2.11 and Remark 2.13. Given this r there exists a $t \in \mathbb{N}$ with the properties stated in Lemma 4.1.

Let $k = \max\{t, r\}$. Suppose $\phi \in m^{k+2}$ then there exists a local analytic isomorphism $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $h(\Sigma_\phi) \subseteq \Sigma$ and $J^r h = \mathrm{id}$. If $h(\Sigma_\phi) \neq \Sigma$ then $f + \phi$ has a singular locus Σ_ϕ with t branches, $t < d$. The ideal $(f_1 + \phi_1, \ldots, f_n + \phi_n)$ is radical, since it is equivalent with $(f_1, \ldots, f_n)$, see part (i) of the proof of Lemma 4.2. Thus for every minimal prime p lying over I_ϕ we have that

$$J_{f+\phi}\mathcal{O}_p = \begin{cases} (f_1 + \phi_1, \ldots, f_n + \phi_n)\mathcal{O}_p = p\mathcal{O}_p, & \text{if } f_0 + \phi_0 \in p \\ \mathcal{O}_p & , \quad \text{otherwise.} \end{cases}$$

Hence the p-primary components of $J_{f+\phi}$ and I_ϕ are the same for all $p \neq m$. So $\dim_{\mathbb{C}}(I_\phi / J_{f+\phi}) < \infty$ and therefore $f + \phi$ has transversal A_1 singularities on $\Sigma_\phi \backslash \{0\}$, by Remark 2.13. By the induction hypothesis $f + \phi$ is a stem of degree t. If $h(\Sigma_\phi) = \Sigma$ then $h^*(f + \phi) \in \int I$. Moreover

$$J^r(h^*(f + \phi)) = J^r f,$$

since $k \geqslant r$ and $\phi \in m^{k+2}$ and $J^r h = \mathrm{id}$. So f and $h^*(f + \phi)$ are right I-equivalent, hence f and $f + \phi$ are R-equivalent. Thus f is a k-stem of degree d.

This proves Theorem 1.3 and completes the proof of Theorem 1.1.

Section 5. Concluding remarks and questions

Stems of degree one are completely characterized by Theorem 1.1. Although Theorem 1.3 gives a sufficient condition for a function to be stem of degree d, the converse does not hold. Since it is not difficult to show that the function $f(x, y) = y^{d+1}$ is a stem of degree d, but has a line as singular locus and transversal A_d singularities.

So one may ask whether every function with a one dimension singular locus is a stem of finite degree.

In contrast with the above question one may ask whether a stem of finite degree is R-equivalent with a polynomial. Functions with a one dimensional singular locus and transversal A_1 singularities are R-equivalent with a polynomial, see [12], [14]. Whitney's example

$$f(x, y, z) = xy(x + y)(x + (z + 2)y)(x + 3e^z y)$$

is a function with a one dimensional singular locus, but it is not R-equivalent with a polynomial [20]. We do not know whether it is a stem of finite degree. Instead of R-equivalence one could as well take A- or K-equivalence and mappings instead of functions. In particular one could ask the following question. What are the stems of finite degree in the class of germs of analytic mappings $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, with respect to A-equivalence? It is in this context that the word stem is originally used [11].

Acknowledgements

I want to thank D. Mond for his ideas concerning series of singularities, the definition of a stem and his stimulating support; J. Montaldi for his suggested generalization of the notion of a stem; the University of Warwick for the hospitality I received and The Free University of Amsterdam where this paper was written.

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