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Theorems of factorizations for birational morphisms

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Preface

This work is devoted to the problem of factorizing a birational morphism through blowing ups at regular centers [9].

The problem is resolved in case of surface. Zariski [1] proved around 1944 that every birational morphism between smooth surfaces over a field k is a composition of blowing ups at closed points. Later, around 1966 Shafarevich [2] proved the same theorem for regular schemes of dimension 2. This leap was fundamental for questions of number theory and the classification of algebraic surfaces. Counterexamples are known (see [3]) to the factorization theorem in general in dimension n. Nevertheless, in 1981 Danilov [4] managed to generalize the Zariski theorem. The theorem, which he proves, is that every projective and birational morphism between smooth algebraic varieties whose fibre are of dimension $\leqq 1$ is a composition of blowing ups at smooth centers of codimension 2. In this work, Danilov admits the difficulties in the regular case. In the present work we prove the theorem for regular schemes thus being valid in number theory and providing a further step in the later classification of algebraic varieties. The theorem states:

THEOREM 5.3: *Let π: $X' \to X$ be a proper and birational morphism between regular schemes whose fibres are of dimension $\leqq 1$. π then factors, locally, through a blowing up at a regular center of codimension 2. Furthermore if π is projective then π is a composition of blowing ups at regular centers.*

This theorem is obtained as a corollary of a more general theorem in codimension d which states:

THEOREM 5.2: *Let π: $X' \to X$ be a proper and birational morphism between regular schemes. Let $H_1, \ldots, H_s$ be the hypersurfaces of the exceptional* *cycle of π and let v_i be the valuations centered at H_i and whose center on X is z_i. We assume that:*

(a) *The exceptional fibres of π are equidimensional and of dimension $d - 1$.*

(b) *For every i, the factorization of the pair $(\mathcal{O}_{X,z_i}, \mathcal{O}_{v_i})$ is by local regular rings of the same dimension d.*

Then π factors, locally, through the blowing up at a regular center of codimension d. Furthermore, if π is projective then π is a composition of blowing ups at regular centers of codimension d.

We also obtain as a corollary of this theorem a necessary and sufficient condition for a birational morphism to be a composition of blowing ups at closed points. The theorem states:

THEOREM 5.4: *Let π: $X' \to X$ be a proper and birational morphism between regular schemes whose locus is a closed point x. Let $v_1, \ldots, v_n$ be the valuations centered at the hypersurfaces of the exceptional cycle of π. Then π is a composition of blowing ups at closed points if and only if the factorization of the pair $(\mathcal{O}_{X,x}, \mathcal{O}_{v_i})$ is by regular rings of same dimension.*

Another theorem obtained as a corollary of theorem 5.2 is a different version of a Moishezon theorem [5] when the exceptional fibre has a unique irreducible component.

THEOREM: *Let π: $X' \to X$ be a proper and birational morphism such that the reduced exceptional cycle H is simple. Then, $\pi(H) = Z$ is regular and π is the blowing up at Z.*

Before proving the theorem of factorization it is necessary to prove the regularity of the centers of blowing up centers and concerning this we have the following theorem.

THEOREM 4.4: *Let π: $X' \to X$ be a proper and birational morphism between regular schemes whose exceptional fibres are equidimensional. For every x belonging to the locus of π, there exists an irreducible component of the locus regular at x.*

The methodology used in this work is the systematic use of valuations and their properties (Section 1) and the duality theory for birational morphisms (Section 3). Thanks to the duality one can define, given an integral scheme X and a valuation v of its quotient field, one invariant which only depends on X and v whose existence and properties have become very useful for the study of birational morphism.

Preliminaries and notations

In this work, it is assumed known the general theory of birational morphisms which can be seen in [8]. The notations used in this paper also are in [8]. All the rings and schemes, in this paper, are noetherian and excellent. The following theorem is used no to mention

THEOREM: *If $X' \to X$ is a proper and birational morphism between noetherian schemes where X is regular, then the closed set Z where π is not an isomorphism (locus of π) has codimension $\geqq 2$ and $\pi^{-1}(Z)$ has codimension 1.*

1. Valuations

Let Σ be a field. All valuations, in this work, will be discrete of rank 1. We shall denote the valuation ring of a valuation v by $\mathcal{O}_v$ and the maximal ideal of $\mathcal{O}_v$ by p_v.

DEFINITION 1.1: Let X be a scheme having a function field Σ. We shall say that the valuation v centres on X at the point $x \in X$ (or at the irreducible subscheme $Y = \overline{\{x\}} \subset X$) if its valuation ring $\mathcal{O}_v$ dominates the local ring of X at x, $\mathcal{O}_{X,x}$.

REMARK: Let π: $X' \to X$ be a birational morphism. If v centres on X' at the point x' then v centres on X at the point $x = \pi(x')$.

DEFINITION 1.2: If $x \in X$ is a regular point, we define the normal valuation of x (or m_x-adic valuation) as the valuation v_x such that: for each $f \in \mathcal{O}_{X,x}$, $v_x(f) = n$ if and only if $f \in m_x^n$ and $f \notin m_x^{n+1}$. We denote $v_x = V_Y$ if Y is the closure set of x in X.
This valuation is the multiplicity function at x; that is $v_x(f) =$ multiplicity of $(f)_0$ at x.

REMARK: If X is normal and $x \in X$ is a point of codimension 1 then there exists a unique valuation centred at x (its normal valuation). Indeed: $\mathcal{O}_{X,x}$ is a valuation ring and there are no dominating morphisms between valuation rings.

DEFINITION 1.3: Let $\mathcal{O}$ be a local ring and let $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$ be two valuation rings containing $\mathcal{O}$. We shall say that $v_1 \leqq v_2$ (with respect to $\mathcal{O}$) if $v_1(f) \leqq v_2(f)$ for every $f \in \mathcal{O}$.

PROPOSITION 1.4: *Let $\mathcal{O}$ be a local regular ring with a closed point x.*
(a) *For every $y \in$ Spec $\mathcal{O}$ and $z \in \overline{\{y\}}$, $v_y \leqq v_z$.*
(b) *If v is a valuation with center x on Spec $\mathcal{O}$, then $v_x \leqq v$.*

Proof: (a) Is satisfied because the multiplicity is an upper semicontinuous function [6].
(b) If $\mathcal{O}_v \supset \mathcal{O}$ and $\mathfrak{p}_v \cap \mathcal{O} = \mathfrak{m}_x$ then $v(f) \geqq 1$ for every $f \in \mathfrak{m}_x$. As $v(f \cdot g) = v(f) + v(g)$ and $v(f + g) \geqq \min\{v(f), v(g)\}$ then $f \in \mathfrak{m}_x^n$ implies $v(f) \geqq n$.

PROPOSITION 1.5: *Let π: $X \to Y$ be a birational and proper morphism between regular schemes. If v_y is the normal valuation of $y \in Y$ then v_y centres on X at an irreducible component F of $\pi^{-1}(y)$ and coincides with the normal valuation of F.*

Proof: Assume that v_y centres on X at $x \in X$ and let F be the irreducible component of $\pi^{-1}(y)$ which goes through x. By proposition 1.4 $v_F \leqq v_x \leqq v_y$ with respect to $\mathcal{O}_{X,x}$. Furthermore, by (b) of the same proposition $v_y \leqq v_F$ with respect to $\mathcal{O}_{Y,y}$. As $\mathcal{O}_{Y,y} \subseteq \mathcal{O}_{X,x}$, we have $v_F \leqq v_x \leqq v_y \leqq v_F$ with respect to $\mathcal{O}_{Y,y}$. Therefore $v_F = v_x = v_y$ and so $\overline{\{x\}} = F$.

DEFINITION 1.6: Let $\mathcal{O}$ be a local ring, α an ideal of $\mathcal{O}$ and $\mathcal{O}_v \supset \mathcal{O}$ a valuation ring. The value of v on α is $v(\alpha) = \min_{f \in \alpha} \{v(f)\}$ = minimum value v of a set of generators of α. $v(\alpha)$ does not depend on the ring $\mathcal{O}$ in the following sense: If $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{O}_v$, then $v(\alpha) = v(\alpha \cdot \mathcal{O}')$ because if $f_1, \ldots, f_n$ generate α, they also generate $\alpha \cdot \mathcal{O}'$.

PROPOSITION 1.7: *Let $\mathcal{O}$ be a local regular ring having a closed point x; let α be an ideal such that $(\alpha)_0$ has an irreducible component of codimension 1 and let v be a valuation with center x. If $v(\alpha) = 1$ then α is principal.*

Proof: Let H be the irreducible component of $(\alpha)_0$ of codimension 1. $\alpha = (f_1, \ldots, f_n) \subseteq \mathfrak{p}_H = (g)$, $1 = v(\alpha) = v(f_i) = v(g \cdot x_i) = v(g) + v(x_i)$. Therefore $v(x_i) = 0$ and so x_i is an invertible. Hence $g = f_i/x_i \in \alpha$ and $\alpha = (g)$.

COROLLARY 1.8: *Let $\mathcal{O}$ be a local regular ring and let α be an ideal of $\mathcal{O}$ whose radical is the maximal ideal $\mathfrak{m}_x$. Let π: $\bar{X} \to X =$ Spec $\mathcal{O}$ be the blowing up along $\mathfrak{m}_x$. If, for each point $y \in \pi^{-1}(x)$, $v_y(\alpha) = 1$, then $\alpha = \mathfrak{m}_x$.*

Proof: By the previous proposition $\alpha \cdot \mathcal{O}_{\bar{X}}$ is a principal ideal at each point of $\bar{X}$. Let $f_1, \ldots, f_k$ a system of generators of α. As $v_x(\alpha) = 1$ we can assume that $v_x(f_i) = 1$ for each i. Let $H_i = (f_i)_0$ and $\bar{H}_i$ the strict transform of H_i by π. We have that: $\alpha \cdot \mathcal{O}_{\bar{X}} = (g) \cdot (\bar{f}_1, \ldots, \bar{f}_k)$ where $g = 0$ is the local equation of $\pi^{-1}(x)$ and $\bar{f}_i = 0$ is the local equation of $\bar{H}_i$. Since $\alpha \cdot \mathcal{O}_{\bar{X}} = (g)$ one has that $\bar{H}_1 \cap \ldots \cap \bar{H}_k = \phi$. Now, $\bar{H}_i \cap \pi^{-1}(x)$ is a hypersurface in $\pi^{-1}(x) \simeq \mathbb{P}_{n-1}$ whose degree is equal to $v_x(f_i) = 1$. Therefore $\bar{H}_1 \cap \pi^{-1}(x), \ldots, \bar{H}_k \cap \pi^{-1}(x)$ are hyperplane without any point in common. So there exists n of them: $\bar{H}_1 \cap \pi^{-1}(x), \ldots, \bar{H}_n \cap \pi^{-1}(x)$ which do not meet. This says that $f_1, \ldots, f_n$ are linearly independent on m/m^2. Therefore $f_1, \ldots, f_n$ is a system of generators of m. Q.E.D.

DEFINITION 1.9: Let $\mathcal{O}_x$ be a local ring having a closed point x and let v be a valuation with center x. Let $X_1 \xrightarrow{\pi} \operatorname{Spec} \mathcal{O}$ be the blowing up at x and $\mathcal{O}_{x_1}$ the local ring of X_1 at x_1 (the center of v on X_1). One has $\mathcal{O}_x \hookrightarrow \mathcal{O}_{x_1} \hookrightarrow \mathcal{O}_v$. We repeat this with $\mathcal{O}_{x_1}$ and so on; one has $\mathcal{O}_x \hookrightarrow \mathcal{O}_{x_1} \hookrightarrow \mathcal{O}_{x_2} \hookrightarrow \cdots \subset \mathcal{O}_v$. This process is called the factorization of the pair $(\mathcal{O}_x, \mathcal{O}_v)$ by monoidal transformations.

THEOREM 1.10: *Let π: $X \to Y$ be a proper and birational morphism where X is normal. Let $\mathcal{O}$ be the local ring of Y at y and v a valuation with center y. If v centres on X at a point x of codimension* 1, *then the factorization of the pair $(\mathcal{O}, \mathcal{O}_v)$ is finite.*

Proof: π is a morphism of finite type and so, locally at x, it is $\mathcal{O} \to \mathcal{O}[f_1/f, \ldots, f_n/f] = B, f_i, f \in \mathcal{O}$ and $\mathcal{O}_v = B_x$. Therefore, it is sufficient to show that $f_s/f \in \mathcal{O}_{x_i}$, for some i. Let $\alpha = (f, f_s)$. We have to prove that $\alpha \cdot \mathcal{O}_{x_1}$ is principal for some i. $\alpha \cdot \mathcal{O}_{x_1} \subset m_y \cdot \mathcal{O}_{x_1} = (g_1)$. Therefore $\alpha \cdot \mathcal{O}_{x_1} = (g_1) \cdot \alpha_1$. If $\alpha \cdot \mathcal{O}_{x_1}$ is not principal, then $\alpha_1 \subseteq m_{x_1}$. $\alpha_1 \cdot \mathcal{O}_{x_2} \subset m_{x_2}\mathcal{O}_{x_2} = (g_2)$ and $\alpha_1 \cdot \mathcal{O}_{x_2} = (g_2) \cdot \alpha_2$ and so on; we have $\alpha \cdot \mathcal{O}_{x_i} = (g_1) \cdot (g_2) \cdot \cdots \cdot (g_i) \cdot \alpha_i$, $v(\alpha) = v(\alpha \cdot \mathcal{O}_{X_i}) = v(g_1) + v(g_2) + \cdots + v(g_i) + v(\alpha_i)$. Therefore for some i, $v(\alpha_i) = 0$ and hence $\alpha \cdot \mathcal{O}_{x_i} = (g_1 \cdot \cdots \cdot g_i)$.

REMARK: The only valuations which will be used in this work will be those of the valuations of the above theorem.

2. Dualizing sheaf for a birational morphism

We use the theorems of the general theory of duality. These theorems and the notations used have appeared in [6].

If π: $X' \to X$ is a birational and proper morphism, we will call locus of π the closed set of points where π is not an isomorphism and we will call exceptional fibre of π, π^{-1} (locus of π).

THEOREM 2.1: *Let X be a regular scheme and let $\bar{X} \xrightarrow{\pi} X$ be a proper and birational morphism where $\bar{X}$ is normal. If $D^{\cdot}_{\bar{X}/X}$ is the dualizing complex of π, then there exists an open set $U \subset \bar{X}$ containing the points of codimension* 1 *of $\bar{X}$ such that*
(a) $\mathscr{H}^i(D^{\cdot}_{\bar{X}/X})|_U = 0$ *for* $i \neq 0$ and $\mathscr{H}^0(D^{\cdot}_{\bar{X}/X})|_U = \omega_{\bar{X}/X}$ *is an invertible sheaf.*
(b) *If f: $\bar{X}' \to \bar{X}$ is another proper and birational morphism where $\bar{X}'$ is normal and U and U' are as in* (a), *then*

$$\omega_{\bar{X}/X|_V} = \omega_{\bar{X}'/\bar{X}} \otimes f^* \omega_{\bar{X}/X|_V} \quad \text{where} \quad V = U' \cap f^{-1}(U).$$

Proof: We can assume, locally at $\bar{X}$, that one has the commutative diagram:

$$\begin{tikzcd} \bar{X} \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, "\pi"'] & \mathbb{P}^d_X \arrow[dl] \\ X & \end{tikzcd}$$

If U is the set of regular points of $\bar{X}$ we have that i, restricted to U, is a regular immersion. By the theory of duality for immersions one obtains, on U, that

$$\mathscr{H}^i(D^{\cdot}_{\bar{X}/\mathbb{P}^d_X}) = \begin{cases} 0, & i \neq d \\ \text{an invertible sheaf } \omega_{\bar{X}/\mathbb{P}^d_X}, & i = d \end{cases}$$

Since $D^{\cdot}_{\bar{X}/\mathbb{P}^d_X} \underline{\underline{\otimes}}\ \underline{\underline{i}}^* D^{\cdot}_{\mathbb{P}^d_X/X}$ in derived category and

$$\mathscr{H}^i(D^{\cdot}_{\mathbb{P}^d_X/X}) = \begin{cases} 0 \text{ for } i \neq -d \\ \omega_{\mathbb{P}^d_X/X}, \text{ an invertible sheaf for } i \neq -d \end{cases}$$

one can conclude (a).
(b) Is deduced from the following formula in derived category

$$D^{\cdot}_{\bar{X}/X} = D_{\bar{X}/\bar{X}} \underline{\underline{\otimes}}\ \underline{\underline{f}}^* D^{\cdot}_{\bar{X}/X}$$

THEOREM 2.2: *Let π: $\bar{X} \to X$ be as above. Then*
(a) *The dualizing sheaf $\omega_{\bar{X}/X}$* defines a unique divisor $K_{\bar{X}/X}$ *on U whose support is contained in the exceptional fibre of π.*
(b) *If f: $\bar{X}' \to \bar{X}$ is another morphism as in the previous theorem, then* $K_{\bar{X}'/X} = K_{\bar{X}'/\bar{X}} + f^* K_{\bar{X}/X}$ on $V = U' \cap f^{-1}(U)$.

Proof: (a) As $\omega_{\bar{X}/X} = \mathcal{O}_{\bar{X}}$ out of the exceptional fibre of π, there exists a divisor $K_{\bar{X}/X}$ associated to $\omega_{\bar{X}/X}$ whose support is contained in the exceptional fibre. If $K'_{\bar{X}/X}$ is another divisor in the conditions of $K_{\bar{X}/X}$ then $K'_{\bar{X}/X} - K_{\bar{X}/X} = D(f)$ where f has neither zeros nor poles in $\bar{X}$ outside the exceptional fibre. Since the locus of π has codimension $\geqq 2$ we have that f has neither zeros nor poles in X. That is to say it is an invertible and so $D(f) = 0$.

(b) By part (b) of the previous theorem $K_{\bar{X}'/X} + f^* \cdot K_{\bar{X}/X}$ is a divisor associated to $\omega_{\bar{X}/X}$ whose support is contained in the exceptional fibre of $\pi \circ f$. By (a) one concludes.

3. Definition of $v(D)$ = value of divisor D for a valuation v

Let X be a regular scheme and let D be a Cartier divisor of X. Let Σ be the function field of X and v a valuation of Σ. Let $\bar{X} \xrightarrow{\pi} X$ be a proper and birational morphism where $\bar{X}$ is normal and v centres, on $\bar{X}$, at a point x_v of codimension 1. We define $v(D)$ = coefficient of $(\pi^*D + K_{\bar{X}/X})$ on x_v, $K_{\bar{X}/X}$ being the divisor associated to $\omega_{\bar{X}/X}$ (dualizing sheaf of π).

THEOREM 3.1: *$v(D)$ does not depend on the chosen scheme $\bar{X}$.*

Proof: Let $\bar{X}'$ be another scheme in the same conditions as $\bar{X}$. We can assume that there exists a morphism $f: \bar{X}' \to \bar{X}$ since one can compare both schemes with the normalization of the graph of the birational transformation existing between $\bar{X}$ and $\bar{X}'$.

Let x'_v and x_v be the centers of v on $\bar{X}'$ and $\bar{X}$ respectively. By part (b) of theorem 2.2 we obtain that

$$\text{coef}\,(K_{\bar{X}'/X}) \text{ on } x'_v = \text{coef}\,(f^*K_{\bar{X}/X}) \text{ on } x'_v + \text{coef}\, K_{\bar{X}'/\bar{X}} \text{ on } x'_v$$

Since as f is an isormophism at x'_v one has that

$$\text{coef}\,(f^*\bar{D}) \text{ on } x'_n = \text{coef}\, \bar{D} \text{ on } x_v, \text{ for any divisor } \bar{D} \text{ of } \bar{X}, \text{ and}$$

$$\text{coef}\,(K_{\bar{X}'/\bar{X}}) \text{ on } x'_n = 0$$

Therefore:

$$\text{coef}\,(\pi^*D + K_{\bar{X}/X}) \text{ on } x_v = \text{coef}\,(f^*\pi^*D) \text{ on } x'_v + \text{coef}\,(f^*K_{\bar{X}/X})$$

$$\text{on } x'_n = \text{coef}\,(f^* \cdot \pi^*D + K_{\bar{X}'/X}) \text{ on } x'_v \qquad \text{Q.E.D.}$$

Properties of $v(D)$

We shall call the divisor zero on X, O_X.

(a) The compute of $v(D)$ is local; that is to say, if v centres at $x \in X$ and $Ui\colon \to X$ is an open set of X containing x then $v(D) = v(i^*D)$.

(b) If v centres on X at a subscheme H of codimension 1 then $v(D) =$ coefficient of D on H.

(c) $v(O_X) = v(K_{\bar{X}/X})$ where $\pi\colon \bar{X} \to X$ is such that v centres on $\bar{X}$ at a point of codimension 1.

(d) $v(D_1 + D_2) = v(D_1) + v(D_2) - v(O_X)$.

(e) If $f\colon X' \to X$ is a proper birational morphism where X' is normal and v centres on X' at a point where $K_{X'/X}$ is defined, then

$$v(D) = v(f^*D) - v(O_{X'}) + v(K_{X'/X})$$

In particular, for $D = O_X$, $v(O_X) = v(K_{X'/X})$.

(f) If $f \in \Sigma$ and $D(f)$ is the divisor of zeros and poles of f, then

$$v(D(f)) = v(f) + v(O_X)$$

Proof: (a), (b) and (c) are proved by the definition.

(d) Let $\bar{X} \to X$ be such that v centres on $\bar{X}$ at a point of codimension 1. By (b) and (c). $v(D_1 + D_2) = v(\pi^*D_1 + \pi^*D_2 + K_{\bar{X}/X}) = v(\pi^*D_1 + K_{\bar{X}/X} + \pi^*D_2 + K_{\bar{X}/X} - K_{\bar{X}/X}) = v(D_1) + v(D_2) - v(O_X)$.

(e) Let $\bar{X} \xrightarrow{\pi} X'$ be such that v centres on $\bar{X}$ at a point of codimension 1 $v(D) = v(\pi^*f^*D + K_{\bar{X}/X}) = v(\pi^*f^*D + K_{\bar{X}/X'} + \pi^*K_{\bar{X}/X}) = v(f^*D) + v(\pi^*K_{X'/X}) = v(f^*D) + v(\pi^*K_{X'/X} + K_{\bar{X}/X'} - K_{\bar{X}/X'}) = v(f^*D) + v(K_{X'/X}) - v(K_{\bar{X}/X'}) = v(f^*D) + v(K_{X'/X}) - v(O_{X'})$.

(f) Let $\bar{X} \xrightarrow{\pi} X$ be such that v centres on $\bar{X}$ at a point of codimension 1 $v(D(f)) = v(\pi^*D(f) + K_{\bar{X}/X}) = v(D(f)) + v(K_{\bar{X}/X}) = v(f) + v(O_X)$. Since every Cartier divisor is locally $D(f)$, the properties (a) and (f) permit one to reduce the computation of $v(D)$ to the computation of $v(O_X)$. Furthermore, we can get, by blowing ups, v to centre at a point of codimension 1 (theorem 1.10). So, by property (e), in order to compute $v(O_X)$ it suffices to compute $K_{X'/X}$ where $X' \to X$ is the blowing up a point.

PROPOSITION 3.2: *Let* $\mathcal{O}$ *be a regular local ring with a closed point* x. *If* $p\colon \bar{X} \to X = \operatorname{Spec} \mathcal{O}$ *is the blowing up at* x *then* $K_{\bar{X}/X} = (n-1)E$ *where* $E = p^{-1}(x)$ *and* $n = \dim \mathcal{O}$.

Proof: E is the divisor associated with the sheaf $\mathcal{O}_{\bar{X}}(1)$. As $\omega_{\bar{X}/X}$ is equal to $\mathcal{O}_{\bar{X}}$ outside E, one has that $\omega_{\bar{X}/X} = \mathcal{O}_{\bar{X}}(K) . p^{-1}(x) = \mathbb{P}_K^{n-1} =$ projective

$n - 1$ space over $K = \operatorname{Spec} \mathcal{O}/m_x$. On the one hand $\omega_{p^{-1}(x)/K} = \mathcal{O}_{\mathbb{P}_K^{n-1}}(-n)$, and on the other; $\omega_{E/\operatorname{Spec} K} = j^*\omega_{\bar{X}/X} \otimes \omega_{E/\bar{X}}$ where $j\colon E \hookrightarrow \bar{X}$ is the canonical embedding. $\omega_{E/\bar{X}} = (p_E/p_E^2)^* = \mathcal{O}_E(-1)$. Therefore $\mathcal{O}_E(-n) = \mathcal{O}_E(K) \otimes \mathcal{O}_E(-1) = \mathcal{O}_E(K - 1)$.

COROLLARY 3.3: *In the conditions of the theorem, if v_x is the normal valuation of x then $v_x(O_X) = \dim \mathcal{O} - 1$.*

Proof: $v_x = v_E$ by proposition 1.5 and so, $v_x(O_X) = v_x(K_{\bar{X}/X}) = v_E((n - 1)E) = n - 1$.

COROLLARY 3.4: *In the conditions of the corollary above, if $D = \Sigma_{i=1}^r n_i H_i$ is a divisor of X then $v_x(D) = \Sigma_{i=1}^r n_i m_x H_i + n - 1$ where $m_x H_i =$* multiplicity of H_i *at x.*

Proof: $D = D(f)$ where $f = f_1^{n_1} \cdot \cdots \cdot f_r^{n_r}$ and $(f_i)_0 = H_i$. By the property (f), $v_x(D) = v_x(f) + v_x(O_x) = \Sigma_i n_i v_x(f_i) + n - 1$.

PROPOSITION 3.5: *Let X be a regular scheme and let v be a valuation having a center $x \in X$. If the codimension of x is n then $v(O_X) \geqq (n - 1)v(m_x)$ the equality being given if and only if $v = v_x$.*

Proof: We can assume that $X = \operatorname{Spec} \mathcal{O}$ and let $\bar{X} \xrightarrow{p} X$ be the blowing up at x. $v(O_X) = v(K_{\bar{X}/X}) = v((n - 1)E)$ where $E = p^{-1}(x) . m_x \cdot \mathcal{O}_{\bar{X}} = (f) = p_E$. By property (f) $v((n - 1)E) = (n - 1)v(f) + v(O_{\bar{X}}) = (n - 1)v(m_x) + v(O_{\bar{X}})$.

If v centres on $\bar{X}$ at a point of codimension 1 then $v = v_x$ (proposition 1.5) and $v(O_{\bar{X}}) = 0$. If $v \neq v_x$ we repeat the process and obtain $v(O_{\bar{X}}) = (K - 1)v(m_{\bar{x}}) + v(O_{\bar{X}'})$ where $\bar{x}$ is the center of v on $\bar{X}$ and $\bar{X}'$ in the blowing up of Spec $\mathcal{O}_{\bar{X},\bar{x}}$ at $\bar{x}$ and so on. We repeat the process up to centres at a point of codimension 1 and we have then finished.

PROPOSITION 3.6: Let $\pi\colon X \to Y$ *be a proper and birational morphism between regular schemes whose exceptional fibre is formed by the hypersurfaces $H_1, \ldots, H_r$. For each point $y \in Y$ the following is satisfied.*

$$\dim F_y = \sum_{x \in H_i} v_{H_i}(O_Y) \cdot m_x H_i$$

$F_y = \overline{\{x\}}$ being the irreducible component of $\pi^{-1}(y)$ at which v_y centres on X and $m_x H_i =$ multiplicity of H_i at x.

Proof: If n = codimension of y and K = codimension of x then

$$n - 1 = v_y(O_Y) = v_y(K_{X/Y}) = v_y\left(\sum_{i=1}^{r} v_{H_i}(O_Y)H_i\right)$$

$$= \sum_{x \in H_i} v_{H_i}(O_Y) \cdot m_x H_i + K - 1$$

The last equality is held by corollary 3.4.

4. Regularity of the centers of blowing up

PROPOSITION 4.1: *Let $\pi: X' \to X$ be a proper and birational morphism between regular schemes such that the exceptional fibres are equidimensional. For any $x \in$ locus of π, the normal valuation of x, v_x, centres on X', precisely at the irreducible component of $\pi^{-1}(x)$, F_x, such that:*
(a) *There exists a unique irreducible component H of the exceptional cycle going through F_x.*
(b) *H is regular at the generic point of F_x.*
(c) *The normal valuation of H coincides with the normal valuation of $\pi(H)$; that is $v_H = v_{\pi(H)}$.*

Proof: Since the exceptional fibres are equidimensional, the irreducible components $Z_1, \ldots, Z_s$ of the locus of π are equidimensional and all hypersurfaces H_i of the exceptional cycle fulfill the expressions $\pi(H_i) = Z_j$. Assume the codimension of $Z_i = d$, and so the dimension of the exceptional fibres are $d - 1$. By proposition 3.6 one has:

$$d - 1 = \dim F_x = \sum_{F_x \subseteq H_i} v_{H_i}(O_X) \cdot m_{F_x} H_i$$

By proposition 3.5, $v_{H_i}(O_X) \geqq$ codimension of $\pi(H_i) - 1 = d - 1$. Therefore, there can only be one hypersurface, H, of the exceptional cycle containing F_x. Furthermore, $m_{F_x}H = 1$ and $v_H(O_X) = d - 1$.

Part (c) is deduced from proposition 3.5.

Conversely: $v_{F_x}(O_X) = v_{F_x}(K_{X'/X}) = v_{F_x}(\Sigma\, v_{H_i}(O_X)H_i) = \Sigma_{F_x \subset H_i} v_{H_i}(O_X) \cdot m_{F_x}H_i + v_{F_x}(O_{X'})$. Since (a), (b) and (c) are satisfied, one has that

$$v_{F_x}(O_X) = v_{\pi(H)}(O_X) + v_{F_x}(O_{X'}) = d - 1 + K - 1$$

where d = codimension of $\pi(H)$ and K = codimension of F_x.

Therefore $v_{F_x}(O_X) = d - 1 + n - (d - 1) - 1 = n - 1$ where $n = \dim \mathcal{O}_{X,x}$. By proposition 3.5 we conclude that $v_{F_x} = v_x$.

PROPOSITION 4.2: *Let $\pi: X' \to X$ be a proper and birational morphism between regular schemes and let $x \in X$. If the normal valuation of x, v_x, centres on X', at x', then $m_x \cdot \mathcal{O}_{X',x'} = m_{x'}$.*

Proof: By corollary 1.8, it suffices to prove that $v_{\bar{x}}(m_x) = 1$ for each $\bar{x} \in p^{-1}(x')$, $p: \bar{X} \to \operatorname{Spec} \mathcal{O}_{X',x'}$ being the blowing up along $m_{x'}$. Since the problem is local, we can assume that $X = \operatorname{Spec} \mathcal{O}_{X,x}$. Let $n = \dim \mathcal{O}_{X,x}$, $K =$ codimension of x' in $X' = \dim \mathcal{O}_{X',x'}$ and $s =$ codimension of $\bar{x}$ in $\bar{X} = \dim \mathcal{O}_{\bar{X},\bar{x}}$.

One has $n \geqq K \geqq s$ because $\pi(x') = x$ and $p(\bar{x}) = x'$. Assume that $K_{X'/X}$, the relative dualizing divisor of π is, locally at x', the divisor of zeros of g and assume that $f = 0$ is the local equation, at $\bar{x}$, of $p^{-1}(x')$. We know, by proposition 1.5, that $v_x = v_{x'}$,

$$n - 1 = v_x(O_X) = v_{x'}(K_{X'/X}) = v_{x'}(g) + v_{x'}(O_{X'})$$
$$= v_{x'}(g) + K - 1$$

That is to say $v_{x'}(g) = n - K$, (1).
By proposition 3.5

$$(n - 1)\, v_{\bar{x}}(m_x) < v_{\bar{x}}(O_X) = v_{\bar{x}}(K_{X'/X}) = v_{\bar{x}}(g) + v_{\bar{x}}(O_{X'})$$
$$= v_{\bar{x}}(g) + v_{\bar{x}}(K_{\bar{X}/X'}) = v_{\bar{x}}(g) + v_{\bar{x}}(D(f^{k-1}))$$
$$= v_{\bar{x}}(g) + v_{\bar{x}}(f^{k-1}) + v_{\bar{x}}(O_{\bar{X}}) = v_{\bar{x}}(g) + K - 1 + s - 1 \qquad (*)$$

If $\bar{g} = 0$ is the local equation of the strict transform of $g = 0$ by p and $m =$ multiplicity of $g = 0$ at x', then one has that $g = f^m \cdot \bar{g}$ and $v_{\bar{x}}(g) = m + v_{\bar{x}}(\bar{g})$.

$v_{\bar{x}}(\bar{g}) = m_{\bar{x}}(\bar{g})_0 \leqq m_{\bar{x}}[(\bar{g})_0 \cap (f)_0] \leqq$ degree of $[(\bar{g})_0 \cap (f)]_0$ in $p^{-1}(x) \simeq \mathbb{P}^{k-1} = m_{x'}(g)_0 = v_{x'}(g) = m$.
Therefore $v_{\bar{x}}(g) = m + v_{\bar{x}}(\bar{g}) \leqq 2 \cdot v_{x'}(g)$.
Going back to $(*)$ we have that

$$(n - 1)\, v_{\bar{x}}(m_x) < 2 \cdot v_{x'}(g) + K - 1 + s - 1 \overset{(1)}{=} 2(n - K) + K - 1$$
$$+ s - 1 \leqq 2(n - K) + 2(K - 1) = 2(n - 1)$$

Therefore $v_{\bar{x}}(m_x) = 1$.

LEMMA 4.3: *Let $\mathcal{O} \hookrightarrow \mathcal{O}'$ be a finite morphism between complete local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively. Assume that $\mathcal{O}'$ is regular, $\mathcal{O}$ is integrally closed and $\mathfrak{m} \cdot \mathcal{O} = \mathfrak{m}'$. Then $\mathcal{O}$ is regular and $\mathcal{O} \hookrightarrow \mathcal{O}'$ is a faithfully flat morphism.*

Proof: We will prove this by induction on $n = \dim \mathcal{O}$. For $n = 0$ there is nothing to say and suppose the theorem is true until $n - 1$. Let $f \in \mathfrak{m}$ and $f \notin \mathfrak{m}'^2$. $f \cdot \mathcal{O}' \cap \mathcal{O} = (f) = \mathfrak{p}$. Indeed: Let $x \in \mathcal{O}$ such that $x \in f \cdot \mathcal{O}'$. We have that $x = f \cdot t$, $t \in \mathcal{O}'$. Therefore $x/f \in \mathcal{O}'$ and $\mathcal{O} \hookrightarrow \mathcal{O}[x/f] \hookrightarrow \mathcal{O}'$. Since $\mathcal{O} \hookrightarrow \mathcal{O}[x/f]$ is a finite morphism and as $\mathcal{O}$ is integrally closed we conclude that $x/f \in \mathcal{O}$. If $\mathcal{O}/f$ is integrally closed, applying induction over the morphism $\mathcal{O}/f \hookrightarrow \mathcal{O}'/f$ one has that $\mathcal{O}/f$ is regular and so $\mathcal{O}$ is regular. Let us show that $\mathcal{O}/f$ is integrally closed: Let $\bar{\mathcal{O}}_1$ be the integral closure of $\mathcal{O}/f$. We have $\mathcal{O}/f \hookrightarrow \bar{\mathcal{O}}_1 \hookrightarrow \mathcal{O}'/f$ and $\mathcal{O}_1 = \pi^{-1}(\bar{\mathcal{O}}_1)$, being $\pi\colon \mathcal{O}' \to \mathcal{O}'/f$ the canonical projection. As $\pi^{-1}(0) = f \cdot \mathcal{O}' \subset \mathcal{O}_1$ we can deduce that $\mathfrak{p}_1 = f \cdot \mathcal{O}_1$ is a prime ideal and $\bar{\mathcal{O}}_1 = \mathcal{O}_1/f$. Tensoring by $\mathcal{O}_{\mathfrak{p}}$ (localized at $\mathfrak{p}$). $\mathcal{O}_{\mathfrak{p}} \to \mathcal{O}_{1\mathfrak{p}}$ we obtain a finite morphism such that $\mathfrak{p} \cdot \bar{\mathcal{O}}_{1\mathfrak{p}} = (f) = \mathfrak{p}_1$. By Nakayama $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{1\mathfrak{p}}$ and so $\mathcal{O}$ and $\mathcal{O}_1$ are birational. Therefore $\mathcal{O} = \mathcal{O}_1$. In order to prove that $\mathcal{O} \hookrightarrow \mathcal{O}'$ is a faithfully flat morphism it suffices to prove that $\mathcal{O}'$ is a free $\mathcal{O}$-module. Let $K = \dim_{\mathcal{O}/\mathfrak{m}} \mathcal{O}/\mathfrak{m}'$. By Nakayama's lemma $\mathcal{O}'$ has K generators. Therefore, there exists the exact sequence: $0 \to N \xrightarrow{\varphi} \mathcal{O} \oplus^{K\text{-th}} \cdots \oplus \mathcal{O} \to \mathcal{O}' \to 0$. Tensoring this sequence by $\mathcal{O}/\mathfrak{m}^n$ we have $N/\mathfrak{m}^n N \xrightarrow{\varphi_n} \oplus_k \mathcal{O}/\mathfrak{m}^n \to \mathcal{O}'/\mathfrak{m}'^n \to 0$, (1). If $N_n = \operatorname{Im} \varphi_n$, taking lengths we obtain:

$$K \cdot \ell(\mathcal{O}/\mathfrak{m}^n) = \ell(N_n) + \ell_{\mathcal{O}}(\mathcal{O}'/\mathfrak{m}'^n)$$

$$\ell_{\mathcal{O}}(\mathcal{O}'/\mathfrak{m}'^n) = \ell_{\mathcal{O}'}(\mathcal{O}'/\mathfrak{m}'^n) \cdot \dim_{\mathcal{O}/\mathfrak{m}} \mathcal{O}'/\mathfrak{m}'$$

Therefore:

$$K \cdot \ell(\mathcal{O}/\mathfrak{m}^n) = \ell(N_n) + K \cdot \ell_{\mathcal{O}'}(\mathcal{O}'/\mathfrak{m}'^n)$$

$\ell_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^n) = \ell_{\mathcal{O}'}(\mathcal{O}'/\mathfrak{m}'^n)$ because Samuel's polynomial is the same for all local regular rings with the same dimension.

Therefore $\ell(N_n) = 0$ and so $\varphi_n = 0$. Taking $\varprojlim$ in the exact sequence (1) we obtain

$$\hat{N} = N \xrightarrow{\hat{\varphi}} \varprojlim N_n \to \oplus \hat{\mathcal{O}} = \oplus \mathcal{O} \to \hat{\mathcal{O}}' \to 0$$

That is to say $\hat{\varphi} = |\varphi_n| = \varphi = 0$. Q.E.D.

THEOREM 4.4: *Let $X' \xrightarrow{\pi} X$ be a proper and birational morphism between regular schemes such that the exceptional fibres are equidimensional. For each $x \in X$, there exists an irreducible component Z of the locus of π which is regular at x.*

Proof: For each $x \in X$, let F_x be the irreducible component of $\pi^{-1}(x)$ at which v_x centres. Let H be the irreducible component of the exceptional fibre given by proposition 4.1. We will prove that $Z = \pi(H)$ is regular at x. Let $x' \in X'$ be the center of v_x on X'. So $\overline{\{x'\}} = F_x$ and $v_{x'} = v_x$. Let $\mathcal{O}$ and $\mathcal{O}'$ be the local ring of X and X' at x and x' respectively. We know that $m_x\mathcal{O}' = m_{x'}$ by proposition 4.2.

Firstly, we will show that we can suppose that $\mathcal{O}$ is complete. If we complete with respect to the ideal m_x in the morphism $\mathcal{O}/p_Z \hookrightarrow \mathcal{O}'/p_H$ we obtain $\widehat{\mathcal{O}/p_Z} \hookrightarrow \mathcal{O}'/p_H \otimes_{\mathcal{O}} \hat{\mathcal{O}} \hookrightarrow \widehat{\mathcal{O}'/p_H}$ where $\widehat{\mathcal{O}'/p_H}$ is the completion of $\mathcal{O}'/p_H$ with respect to the ideal $m_{x'}$. Since H is regular at x' we have that $\widehat{\mathcal{O}'/p_H}$ is integral and so $\widehat{\mathcal{O}/p_Z}$ is also integral. As $\mathcal{O}/p_Z$ is regular if and only if $\widehat{\mathcal{O}/p_Z}$ is regular, taking the morphism $X' x_{\mathrm{Spec}\,\mathcal{O}}$ Spec $\hat{\mathcal{O}} \xrightarrow{\pi}$ Spec $\hat{\mathcal{O}}$ instead of π we can suppose that $\mathcal{O}$ is complete.

By proposition 4.1, there exists a closed point $y \in F_x$ where H and F_x are regular and $m_x \cdot \mathcal{O}_{X',y} = p_{F_x}$. If dim $F_x = K$, let Z_1 be a regular subscheme of H of codimension K which meets F_x transversally at y. Restricting the morphism π to Z_1, we obtain a finite morphism $\bar{\pi}$: $Z_1 \to Z$. Applying lemma 4.3 we conclude if we prove that Z is integrally closed.

Let $\bar{Z}$ be the integral closure of Z. We have $Z_1 \to \bar{Z} \to Z$ where $\bar{Z} \xrightarrow{\varphi} Z$ and $Z_1 \to \bar{Z}$ are finite morphisms. Let $\bar{x} = \varphi^{-1}(x)$. By lemma 4.3 $\mathcal{O}_{\bar{Z},\bar{x}} \hookrightarrow \mathcal{O}_{Z_1,y}$ is a faithfully flat morphism. Therefore $m_y = m_{\bar{x}} \otimes_{\mathcal{O}_{\bar{Z}}} \mathcal{O}_{Z_1} = m_x \cdot \mathcal{O}_{\bar{Z}} \otimes_{\mathcal{O}_{\bar{Z}}} \mathcal{O}_{Z_1}$ and so $m_{\bar{x}} = m_x \cdot \mathcal{O}_{\bar{Z}}$. By Nakayama's lemma we finish if $\mathcal{O}_Z/m_x = \mathcal{O}_{\bar{Z}}/m_{\bar{x}}$. In order to see this it suffices to prove that $\mathcal{O}'/m_{x'}$ has no algebraic elements over $\mathcal{O}/m_x$. If v_x is the normal valuation of x then $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{O}_{v_x}$ and $K = \mathcal{O}/m_x \subset \mathcal{O}'/m' \subset \mathcal{O}_{v_x}/p_{v_x}$.

Let $m_x = (x_1, \ldots, x_n)$ be a minimal system of generators of m_x, $\mathcal{O} \to \mathcal{O}[x_2/x_1, \ldots, x_n/x_1] = \mathcal{O}_1 \subset \mathcal{O}_{v_x}$ and $p_{v_x} \cap \mathcal{O}_1 = (x_1) = m_x \cdot \mathcal{O}_1$. Therefore $\mathcal{O}_{v_x}/p_{v_x} = K(x_2/x_1, \ldots, x_n/x_1)$. Q.E.D.

5. Theorems of factorization

DEFINITION 5.1: We shall say that the birational morphism π: $X' \to X$ factors locally through a blowing up along a regular subscheme if for every $x \in X$ there exists an open set U and a regular subscheme Z containing x such that $\pi^{-1}(U) \xrightarrow{\pi} U$ factors through the blowing up at $U \cap Z$.

THEOREM 5.2: *Let π: $X' \to X$ be a proper and birational morphism between regular schemes. Let $H_1, \ldots, H_s$ be the hypersurfaces of the exceptional cycle of π and let v_i be the valuation centred at H_i with center, on X, z_i. Assume*:
(a) *The exceptional fibres of π are equidimensional and of dimension $d - 1$.*
(b) *For every i, the factorization of the pair $(\mathcal{O}_{X,z_i}, \mathcal{O}_{v_i})$ is by local regular rings of the same dimension d. Then, π factors locally through a blowing up along a regular subscheme.*

Proof: We can assume that $X = \operatorname{Spec} \mathcal{O}$ where $\mathcal{O}$ is a local regular ring having a closed point x. Let Z be the regular component of the locus of π given in theorem 4.4. We know that $\pi(H) = Z$ where H is just the irreducible component of the exceptional cycle which goes through the center of v_x on X'. We will prove by induction on $n = \dim \mathcal{O}$, that π factors through the blowing up along Z. For $n = 1$ there is no problem.

Let $X_1 \xrightarrow{p} X$ the blowing up along Z and let Γ be the graph of the birational transform between X' and X_1

$$\begin{array}{ccc} \Gamma & \xrightarrow{p_2} & X' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_1 & \xrightarrow{p} & X \end{array}$$

We have to prove that p_2 is an isormorphism. p_2 is an isomorphism precisely at the points $x' \in X'$ such that $\not{p}_Z \cdot \mathcal{O}_{X',x'}$ is principal.

Let $x \neq y \in X$ such that $x \in \overline{\{y\}} \subseteq Z$. The morphism π': $X' \times_X T \to T = \operatorname{Spec} \mathcal{O}_{X,y}$ holds the assumptions of the theorem and $\dim \mathcal{O}_{X,y} < n$. So, by hypothesis of induction, $\not{p}_Z \cdot \mathcal{O}_{X',x'}$ is principal for each $x' \in X'$ such that $\pi(x') = y$. Therefore, we can assume that the locus of p_2 is contained in $\pi^{-1}(x)$. Let Z_1 be an irreducible component of the locus of p_2 of codimension $\geqq 2$, such that $p_2^{-1}(Z_1)$ has codimension 1. Let $\bar{H}$ be an irreducible component of $p_2^{-1}(Z_1)$ and let $Z_2 = p_1(\bar{H})$. We have the following commutative diagram:

$$\begin{array}{ccc} \bar{H} & \xrightarrow{p_2} & Z_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z_2 & \xrightarrow{p} & x \end{array}$$

and $\bar{H} \subseteq Z_1 \times_x Z_2$.

$\dim Z_2 < n - 1$. Indeed: If $\dim Z_2 = n - 1$ then $Z = x$, $Z_2 = p^{-1}(x)$ and $V_{Z_2} = V_{\bar{H}}$. By proposition 1.5, $v_x = V_{p^{-1}(x)} = V_{Z_2}$ and v_x centres on X'

at a point of codimension 1 (notice $Z = x$). But $V_{Z_2} = V_{\bar{H}}$ centres on X' at Z_1 and one obtains a contradiction.

Since $\bar{H}$ is contracted on Z_2, we have that p_1 is not an isomorphism at Z_2. Besides, $n - 1 = \dim \bar{H} \leqq \dim Z_1 + \dim Z_2$. Therefore $\dim Z_2 \geqslant n - d$ since $Z_1 \subseteq \pi^{-1}(x)$ and so $\dim Z_2 \geqq 1$. (*)

Let H' be the irreducible component of the exceptional cycle of p_1 which goes through the center of V_{Z_2} on Γ. Let $p_2(H') = \bar{Z}_1$.

$\dim \bar{Z}_1 < n - 1$. Indeed: if $\dim \bar{Z}_1 = n - 1$ then $V_{\bar{Z}_1} = V_{H'}$ and since $\bar{Z}_1$ is an irreducible component of the exceptional cycle of π, by hypothesis (b), $V_{\bar{Z}_1}$ centres on X and on X_1 at points of codimension d. So, $V_{\bar{Z}_1}$ centres on X at Z and on X_1 at the subscheme of codimension $d, p_1(H')$. But $Z_2 \subseteq p_1(H')$ and by (*) we conclude that $\dim Z_2 = n - d$ and $p_1(H') = Z_2$. As V_{Z_2} centres on X at x, which is the center of $V_{H'}$ on X, we obtain that $Z = x$ and $d = n$. But then $\dim Z_2 = 0$ and this contradicts (*).

As H' is contracted on $\bar{Z}_1$, one has that p_2 is not an isomorphism at the points of $\bar{Z}_1$. Let $y \in \bar{Z}_1$ be the point at which V_{Z_2} centres. On one hand $V_{Z_2}(O_X) = V_{Z_2}(K_{X_1/X}) \overset{E = p^{-1}(Z)}{=\!=\!=} V_{Z_2}[(d - 1)E] = d - 1 + V_{Z_2}(O_{X_1}) = d - 1 + \operatorname{cod} Z_2 - 1 \leqq 2(d - 1)$, and the other $V_{Z_2}(O_X) = V_{Z_2}(K_{X'/X}) = V_{Z_2}(\Sigma_{i=1}^{s} V_{H_i}(O_X)H_i) = \Sigma_{y \in H_i} V_{H_i}(O_X) \cdot V_{Z_2}(f_i) + V_{Z_2}(O_{X'})$ where $f_i = 0$ is the local equation of H_i at y.

By proposition 3.5, $V_{Z_2}(O_{X'}) \geqq \operatorname{cod} y - 1$ and $V_{H_i}(O_X) \geqq d - 1$. Besides, $\operatorname{cod} y - 1 \geqq 1$ because y is a point where p_2 is not an isomorphism. Therefore, $\Sigma_{y \in H_i} V_{H_i}(O_X) \cdot V_{Z_2}(f_i) < 2(d - 1)$, and so there exists a unique hypersurface H_1 such that $y \in H_1$. Furthermore, $V_y(f_1) \leqq V_{Z_2}(f_1) = 1$ and $V_{H_1}(O_X) < 2(d - 1)$. But $V_{H_1}(O_X) = V_{H_1}(K_{X_1/X}) = (d - 1) \cdot V_{H_1}(\not{p}_Z) + V_{H_1}(O_{X_1})$. By hypothesis (b) and proposition 3.5, $V_{H_1}(O_{X_1}) \geqq d - 1$. Therefore $V_{H_1} = V_{p^{-1}(Z)} = V_Z$ and so $H_1 = H$ (proposition 4.1). Hence $\not{p}_Z \cdot \mathcal{O}_{X',y} \subseteq \not{p}_H$ is an ideal of codimension 1. But $V_{Z_2}(\not{p}_Z \cdot \mathcal{O}_{X',y}) = V_{Z_2}(\not{p}_Z) = V_{Z_2}(\not{p}_Z \cdot \mathcal{O}_{X_1}) = V_{Z_2}(\not{p}_E) = 1$ and by proposition 1.7 we obtain a contradiction.

THEOREM 5.3: Let π: $X' \to X$ *be a proper and birational morphism between regular schemes whose fibres are of dimension* $\leqq 1$. *Then* π *factors, locally, through a blowing up along a regular subscheme of codimension 2.*

Proof: One has to verify conditions (a) and (b) of the above theorem. In this case the exceptional fibres have dimension 1 and so $\mathcal{O}_{X,z_i}$ is of dimension 2 for each v_i. Therefore, condition (b) is satisfied because $\dim \mathcal{O}_j \leqslant \dim \mathcal{O}_{X,z_i}$ for each ring $\mathcal{O}_j$ of the factorization of the pair $(\mathcal{O}_{X,z_i}, \mathcal{O}_{v_i})$.

THEOREM 5.4: *Let π: $X' \to X$ be a proper birational morphism between regular schemes whose locus is a closed point x. If $v_1, \ldots, v_n$ are the valuations with centers of codimension 1 on X' then π is a composition of blowing ups at closed points if and only if the factorization of the pair $(\mathcal{O}_{X,x}, \mathcal{O}_{v_i})$ is by local regular rings of the same dimension.*

Proof: This is a corollary of theorem 5.2 noticing that there is only one exceptional fibre: $\pi^{-1}(x)$ and noticing that one can factor successively by blowing up at points because condition b) is always satisfied.

THEOREM 5.5: *In the hypothesis of theorem 5.2. If π: $X' \to X$ is, furthermore, a projective morphism then π is the composition of the blowing up along regular centers.*

Proof: It is sufficient to prove that π factors through the blowing up along a regular center since one is then able to factor the morphism successively until no irreducible components of the exceptional cycle exist.
If $X' \xrightarrow{\pi} X$ is a projective morphism, then one has the commutative diagram:

$$\begin{tikzcd} X' \arrow[r, phantom, "\overset{i}{\subseteq}"] \arrow[d] & \mathbb{P}^n_X \arrow[dl] \\ X & \end{tikzcd}$$

If Z is an irreducible component of the locus of π then we call $n_Z =$ min $(m_Z H$ being $H = \pi(\bar{H})$ and $\bar{H}$ the intersection of X' with a hyperplane of $\mathbb{P}^n_X$).

Let $\{Z_1, \ldots, Z_n\}$ be the irreducible components of the locus of π and let Z_1 be such that $n_{Z_1} \geqq n_{Z_i}$ for $i = 1, \ldots, n$. We shall prove that π factors through the blowing up along Z_1. Let $x \in Z_1$. We know by theorem 5.2 that π factors, locally at x, through the blowing up along Z_2. In order to conclude we have to show that $Z_1 = Z_2$.

Assume that $Z_1 \neq Z_2$ and let U be an open set such that $\pi^{-1}(U) \xrightarrow{\pi} U$ factors through the blowing along $Z_2 \cap U$. We have:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi"'] \arrow[dr, "\bar{\pi}"] & \\ U & X_2 \arrow[l, "p"'] \end{tikzcd}$$

where p is the blowing up along $Z_2 \cap U$.

Let $F = p^{-1}(x)$ and $\bar{F}$ = strict transform of F by $\bar{\pi}$. We take a generic hyperplane, $\bar{H}$, not containing $\bar{F}$ such that $m_{Z_2}(\pi(\bar{H} \cap X')) = n_{Z_2}$. Let $H' = \bar{\pi}(\bar{H} \cap X')$, $\pi(\bar{H} \cap X') = H$, $\bar{Z}_1$ = strict transform of Z_1 by p and $x' \in \bar{Z}_1 \cap F$ a closed point. Since H' does not contain F we have that $m_{Z_2}H = m_x H$. Furthermore $m_x H$ = degree of $F \cap H'$ in $\mathbb{P}_k^{d-1} = F$ which is larger than $m_{x'}H'$. Therefore:

$$m_{Z_2}H > m_{x'}H' \geqq m_{\bar{Z}_1}H' = m_{Z_1}H \geqq n_{Z_1}$$

Hence one obtains $n_{Z_2} > n_{Z_1}$ and this contradicts the choice of Z_1.

THEOREM 5.6: *Let π: $X' \to X$ be a proper and birational morphism such that the reduced exceptional cycle H is simple. Then, $\pi(H) = Z$ is regular and π is the blowing up along Z.*

Proof: By the theorem 4.4 and 5.2 it is sufficient to show that the exceptional fibres of π are equidimensional. Let $x \in Z$ and F an irreducible component of $\pi^{-1}(x)$. If d = cod Z then dim $F \geqq d - 1$ because π: $H \to Z$ is a proper morphism. We have that $V_F(O_X) = V_F(K_{X'/X}) = V_F(V_H(O_X) \cdot H) \overset{V_Z = V_H}{=\!=\!=} V_F((d-1)H) = d - 1 + \text{cod } F - 1$. By proposition 3.5, $V_F(O_X) \geqq n - 1$ where $n = \dim \mathcal{O}_{X,x}$. Therefore cod $F \geqq n - d + 1$ and so dim $F \leqq d - 1$. Q.E.D.

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