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Introduction

In this article I study a class of schemes of finite type over a field, which I call Alexander schemes, introduced by Kleiman and Thorup under the name of $C_{\mathbb{Q}}$-orthocyclic schemes (see [Kleiman–Thorup], 2). I claim that the class of Alexander schemes is a reasonable answer to the question of what is the most natural general class of schemes that behave like smooth schemes from the point of view of intersection theory with rational coefficients.

There is now a well developed theory of Chow groups of possibly singular schemes of finite type over a field, extending the classical intersection theory on smooth quasiprojective varieties (see [Fulton]).

However, Chow groups of smooth schemes have many distinctive properties. For example, they have a natural commutative ring structure and they are contravariant for general morphisms (see [Fulton], Chapter 8).

It is a natural question to ask whether there are singular schemes whose Chow groups behave like Chow groups of smooth schemes. Unfortunately, there do not seem to be any interesting examples.

But if we consider instead Chow groups with rational coefficients, it has been known for a long time that on the quotient of a smooth quasiprojective

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scheme by a finite group there is an intersection product (see [Briney]). Recently Mumford and Gillet constructed an intersection product on Chow groups with rational coefficients of certain moduli spaces that locally in the étale topology are quotients of smooth schemes by finite groups (see [Mumford 2] and [Gillet]).

If X is an oriented n-dimensional topological manifold and Y is a closed subset of X, one form of Alexander duality says that the homomorphism from the local cohomology group $H_Y^k(X, \mathbb{Z})$ to the Borel–Moore homology group $H_{n-k}(Y, \mathbb{Z})$ obtained by taking cap products with the fundamental class of X in $H_n(X, \mathbb{Z})$ is an isomorphism.

In intersection theory, Chow groups are usually thought of as homology groups, and they have some properties in common with Borel–Moore homology groups. Let us indicate by $A_*(X)$ the Chow group of a scheme of finite type over a field. Then if Y is a subscheme of X, or, more generally, if $Y \to X$ is a morphism of finite type, Fulton and MacPherson introduced an analog of local cohomology, the bivariant group $A^*(Y \to X)$ ([Fulton], Chapter 17). If X is an n-dimensional smooth scheme and $Y \to X$ is a morphism of finite type then the homorphism from $A^k(Y \to X)$ to $A_{n-k}(Y)$ defined by the cap product with the fundamental class of X in $A_n(X)$ is an isomorphism (see [Fulton], Propositions 17.4.2 and 17.3.1).

From now on, let us consider only Chow groups and bivariant groups with rational coefficients. Then, following Kleiman and Thorup, I define an Alexander scheme as an equidimensional scheme X such that for any morphism $Y \to X$ of finite type this form of Alexander duality is satisfied, and a certain commutativity condition holds (Definition 2.1). It turns out that Alexander schemes have most of the formal intersection-theoretical properties of smooth schemes (Note 2.4). For example, if X is an Alexander scheme and $X \to X$ is the identity morphism, then $A^*(X \to X)$ has a natural commutative ring structure, and so the isomorphism of $A^*(X \to X)$ with $A_*(X)$ defines an intersection product on $A_*(X)$.

Although the definition of Alexander scheme is very abstract, it seems to have some geometric content. For example, Alexander schemes are geometrically unibranch (Proposition 2.5), and one can characterize Alexander schemes of dimension 1 and 2 over perfect fields geometrically. Precisely, a curve is an Alexander scheme if and only if it is geometrically unibranch (Corollary 2.10), and a surface over a perfect field is an Alexander scheme if and only if it is geometrically unibranch and all the components of the exceptional divisors on a resolution of its singularities are rational (Theorem 4.1).

Also, the formal properties of the bivariant theory are so strong that calculations are often possible (see Example 4.2 for some calculations on surfaces).

Many examples of singular Alexander schemes are obtained as a consequence of the following result: if Y is a normal scheme, X is an Alexander scheme and there exists a finite subjective morphism from X to Y, then Y is an Alexander scheme (Proposition 2.11). So, for example, the quotient of a smooth scheme by a finite group is an Alexander scheme.

A remarkable feature of the property of being an Alexander scheme is that in dimension 1 and 2 it is local in the étale topology. In Section 5 of this article I pose the problem of whether this is true in any dimension, and discuss the following result in this direction, proved in (Vistoli 2]. Say that a scheme has *quotient singularities* if locally in the étale topology it is the quotient of a smooth scheme by a finite group. Then in characteristic 0 a scheme with quotient singularities is an Alexander scheme.

The contents of this article form a part of my doctoral thesis. Thanks are due to my advisor M. Artin and to S. Kleiman for many very useful discussions.

1. Preliminaries

Fix a field k. In this section by a *scheme* we will always mean a scheme of finite type over k. If X is a scheme, we define

$$A_*(X) \quad = \quad (\text{Chow group of } X) \otimes \mathbb{Q}.$$

This group $A_*(X)$ is the Chow group of X with rational coefficients. In other words, if $Z_*(X)$ is the group of cycles on X with rational coefficients, graded by dimension, then

$$A_*(X) \quad = \quad Z_*(X)/(\text{Rational equivalence}).$$

The theory of the Chow group is developed in [Fulton]. The formal properties of the Chow group with integer coefficients are inherited by A_*. Let $f\colon X \to Y$ be a morphism of schemes. If f is proper we have the proper pushforward $f_*\colon A_*(X) \to A_*(Y)$ (see [Fulton], 1.4), if f is flat of constant fiber dimension we have the flat pullback $f\colon A_*(Y) \to A_*(X)$ ([Fulton], 1.7) and if f is a regular embedding we have the refined Gysin homomorphism $f^!\colon A_*(Y') \to A(X \times_Y Y')$ for all morphism of schemes $Y' \to Y$ ([Fulton], Chapter 6).

For each morphism of schemes $f\colon X \to Y$ we can define a group, denoted by $A^*(f)$ or $A^*(X \to Y)$, called the bivariant group of f, as in [Fulton], Chapter 17, using Chow groups with rational coefficients. An element α of

$A^p(X \to Y)$ associated to each morphism of schemes $g\colon Y' \to Y$ and each class of cycles y' in $A_k(Y')$ a class in $A_{k-p}(X \times_Y Y')$, denoted $\alpha_g \cap y'$ or simply $\alpha \cap y'$. This operation is required to commute with proper pushforward, flat pullback and Gysin homomorphism (see [Fulton], 17.1).

If $f\colon X \to X$ is the identity, we write $A^*(X)$ for $A^*(f)$.

Again, the formal properties of the bivariant group in the integral case extend to A^*. If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms of schemes for any two classes $\alpha \in A^*(X \to Y)$ and $\beta \in A^*(Y \to Z)$ we have a product $\alpha\beta \in A^*(X \xrightarrow{fg} Z)$ defined by composition. If f is proper there is also the pushforward $f_*\colon A^*(X \to Z) \to A^*(Y \to Z)$. Finally, if $f\colon X \to Y$ and $g\colon Y' \to Y$ are two morphisms there is the pullback $f^*\colon A^*(X \to Y) \to A^*(X \times_Y Y' \to Y')$. The properties given in [Fulton], 17.2 are satisfied. In particular, we have the projection formula: if $f\colon X \to Y$ and $g\colon Y' \to Y$ are two morphisms with g proper, and $h\colon X \times_Y Y' \to X$ is the projection, then

$$h_*(g^*\alpha \cap y') = \alpha \cap g_*y'$$

for any α in $A^*(X \to Y)$ and y' in $A_*(Y')$.

Also A^* becomes a functor from schemes to graded associative $\mathbb{Q}$-algebras.

Inside $A^*(X \to Y)$ we can define a subgroup $C^*(X \to Y)$ as follows (see [Kleiman–Thorup, p. 337).

(1.1) Definition: Take α in $A^*(X \to Y)$. We say that α is in $C^*(X \to Y)$ if, for every pair of morphisms $Y' \to Y$ and $Y'' \to Y'$, every class of cycles y' in $A_*(Y')$, and every β in $A^*(Y'' \to Y')$, we have

$$\alpha \cap (\beta \cap y') = \beta \cap (\alpha \cap y').$$

This subgroup C^* of A^* is closed under product, proper pushforward and pullback. If $f\colon X \to Y$ is a flat or locally complete intersection morphism, the orientation class [1] of f (see [Fulton], 17.4) lies in $C^*(f)$.

If Y has a smooth resolution of singularities, then $C^*(X \to Y) = A^*(X \to Y)$. More generally, this identity holds if there exists a proper surjective morphism $Y' \to Y$ with Y' smooth (see Lemma 2.6 and Proposition 2.2).

Next we prove some technical lemmas.

(1.2) Lemma: *Let $f\colon X \to Y$ be a proper and surjective morphism of schemes. Then*

(i) *The pushforward*

$$f_*\colon A_*(X) \to A_*(Y)$$

is surjective, and

(ii) *if* $U \to Y$ *is a morphism, the pullback*

$$f^*: A^* (U \to Y) \to A^* (U \times_Y X \to X)$$

is injective.

Proof: The surjectivity of $f_*: A_*(X) \to A_*(Y)$ follows from the surjectivity of $f_*: Z_*(X) \to Z_*(Y)$, which is clear. Now take β in $A^*(U \to Y)$ and suppose that $f^*\beta = 0$. To prove that $\beta = 0$ it is enough to prove that for any y in $A_*(Y)$ we have $\beta \cap y = 0$, by a base change argument. There exists x in $A_*(X)$ such that $f_*x = y$. But then $\beta \cap y = \beta \cap f_*x = g_*(f^*\beta \cap x) = 0$, and so $\beta = 0$.

(1.3) LEMMA: *Let* $f: X \to Y$ *be proper and surjective. Let* x *be in* $A_*(X)$ *such that* $f_*x = 0$ *in* $A_*(Y)$. *Then* x *is represented by a cycle* ξ *in* $Z_*(X)$ *with* $f_*\xi = 0$ *in* $Z_*(Y)$.

Proof: First we prove this fact. If v is in $Z_*(Y)$ and is rationally equivalent to 0, then there is a cycle u in $Z_*(X)$, u rationally equivalent to 0, such that $f_*u = v$. We may assume that $v = [\text{div}(r)]$, where r is a non-zero rational function on a subvariety W of Y. Let V be a subvariety of X contained in $f^{-1}(W)$, such that the induced morphism $V \to W$ is surjective and generically finite, say of degree n. Lift r to a rational function s on V. Then $f_*[\text{div}\,(s)] = [\text{div}\,(r^n)] = n[\text{div}\,(r)]$, because of [Fulton], Proposition 1.4. We can take $u = (1/n)[\text{div}\,(s)]$.

Now, let x be in $A_*(X)$ with $f_*x = 0$. Take a cycle w on X representing x. The cycle f_*w on Y is rationally equivalent to 0, and so there is a cycle w' in $Z_*(X)$ rationally equivalent to 0, such that $f_*w' = f_*w$. Then $\xi = w - w'$ represents x, and $f_*\xi = 0$.

(1.4) LEMMA: *Let* $f: X \to Y$ *be proper and surjective. Let* $U \to Y$ *be a morphism. If*

$$C^*(U \times_Y X \to X) = A^*(U \times_Y X \to X),$$

then

$$C^*(U \to Y) = A^*(U \to Y).$$

Proof: Let $q\colon Y' \to Y$ be a morphism. Form the fibre diagram

$$\begin{tikzcd}
T' \arrow[rrr] \arrow[dr, "r"] \arrow[ddd, "g'"'] & & & X' \arrow[dl] \arrow[ddd, "f'"] \\
& T \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] & \\
& U \arrow[r] & Y & \\
U' \arrow[rrr] \arrow[ur, "s"'] & & & Y' \arrow[ul, "q"]
\end{tikzcd} \tag{1.5}$$

Take β in $A^*(U \to Y)$, γ in $A^*(Y' \to Y)$ and y in $A_*(Y)$. Choose x in $A_*(X)$ such that $f_*x = y$. Then $\gamma \cap (\beta \cap y) = \gamma \cap (\beta \cap f_*x) = \gamma \cap g_*(f^*\beta \cap x) = g'_*(f^*\gamma \cap (f^*\beta \cap x))$. Similarly $\beta \cap (\gamma \cap y) = g'_*(f^*\beta \cap (f^*\gamma \cap x))$. But $f^*\beta$ is in $C^*(T \to X)$, and therefore $\gamma \cap (\beta \cap y) = \beta \cap (\gamma \cap y)$. By base change we conclude that β commutes with all bivariant classes, and therefore that $\beta \in C^*(U \to Y)$.

(1.6) LEMMA: *Let $f\colon X \to Y$ be proper and surjective, $U \to Y$ a morphism, $T = U \times_Y X$. Let α be in $A^*(T \to X)$. Then $\alpha = f^*\beta$ for some β in $A^*(U \to Y)$ if and only if the following condition holds. For any morphism $Y' \to Y$ construct the fiber diagram 1.5. If x' in $A_*(X')$ is such that $f'_*x' = 0$, then $g'_*(\alpha \cap x') = 0$ in $A_*(U')$.*

Proof: If $\alpha = f^*\beta$, then $g'_*(\alpha \cap x') = \beta \cap f'_*x' = 0$. Conversely, assume that the condition holds. We define β in $A^*(U \to Y)$ as follows. If $q\colon Y' \to Y$ is a morphism, $y' \in A_*(Y')$, choose x' in $A_*(X')$ such that $f'_*x' = y'$. Set $\beta \cap y' = g'_*(\alpha \cap x')$. The condition insures that $\beta \cap y'$ does not depend on x'. Let $q'\colon Y'' \to Y'$ be a proper morphism. Consider the fiber diagram

$$\begin{tikzcd}
T'' \arrow[rrrrr] \arrow[dr, "r"] \arrow[ddddd, "g''"'] & & & & & X'' \arrow[dl] \arrow[ddddd, "f''"] \\
& T' \arrow[rrr] \arrow[dr] \arrow[ddd, "g'"'] & & & X' \arrow[dl] \arrow[ddd, "f'"] & \\
& & T \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] & & \\
& & U \arrow[r] & Y & & \\
& U' \arrow[rrr] \arrow[ur] & & & Y' \arrow[ul] & \\
U'' \arrow[rrrrr] \arrow[ur, "s'"'] & & & & & Y'' \arrow[ul]
\end{tikzcd}$$

If $y'' \in A_*(Y'')$, $x'' \in A_*(X'')$ with $f''_* x'' = y''$, then $\beta \cap q'_* y'' = g'_*(\alpha \cap p'_* x'') = g'_* r'_*(\alpha \cap x'') = s'_* g''_*(\alpha \cap x'') = s'_*(\beta \cap y'')$.

The proof that β commutes with flat pullback and Gysin homomorphisms is analogous, using [Fulton], Proposition 1.7 and Theorem 6.2(a).

It is easy to check that $f^*\beta = \alpha$.

(1.7) LEMMA: *Let $f\colon X \to Y$ be an universal homeomorphism of schemes.*

(i) *The pushforward*

$$f_*\colon A_*(X) \to A_*(Y)$$

is an isomorphism.

(ii) *If $U \to Y$ is a morphism, $T = U \times_Y X$, then the pullback*

$$f^*\colon A^*(U \to Y) \to A^*(T \to X)$$

is an isomorphism.

(iii) *If $Y \to Z$ is a morphism, the pushforward*

$$f_*\colon A^*(X \to Z) \to A^*(Y \to Z)$$

is an isomorphism.

Proof: By Lemma 1.2 (i), $f_*\colon A_*(X) \to A_*(Y)$ is surjective. The push-forward of cycles $f_*\colon Z_*(X) \to Z_*(Y)$ is bijective. If x in $A_*(X)$ is such that $f_*x = 0$, Lemma 1.3 implies that $x = 0$. So f_* is also injective. This proves (i).

Let us prove (ii). The injectivity of f^* holds by Lemma 1.2 (ii). Its surjectivity is immediate from Lemma 1.7 and part (i).

To prove (iii), take a morphism $Z' \to Z$, and form the fiber diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & Z \end{array}$$

If $\alpha \in A^*(X \to Z)$ is such that $f_*\alpha = 0$ in $A^*(Y \to Z)$, then $f'_*(\alpha \cap z') = 0$ for any z' in $A_*(Z')$, and so $f_*\alpha = 0$ by part (i). If β is in $A^*(Y \to Z)$, we define $\alpha \cap z' = x'$, where z' is in $A_*(Z')$ and $x' \in A_*(X')$ is such that $f'_*x' = \beta \cap z'$. It is easy to check that α belongs to $A^*(X \to Z)$, and $f_*\alpha = \beta$ by definition.

Let G be a finite group operating on the left on a scheme X, with quotient $Y = X/G$. The action of G on X induces a right action of G on $A_*(X)$, by the formula

$$x \cdot g = g_X^* x,$$

where $g_X: X \to X$ indicates the morphism induced by $g \in G$. Also, if T is another G-scheme and $T \to X$ an equivariant morphism, G operates on the right on $A^*(T \to X)$ by the same formula. Let us denote by $A_*(X)^G$ and $A^*(T \to X)^G$ the groups of invariants.

(1.8) LEMMA: *In the situation above*

(i) *The pushfoward*

$$f_*: A_*(X)^G \to A_*(Y)$$

is an isomorphism.

(ii) *If $U \to Y$ is a morphism, $T = U \times_Y X$ the pullback, then*

$$f^*: A^*(U \to Y) \to A^*(T \to X)^G$$

is an isomorphism.

Proof: (see [Fulton], Example 1.7.6., and [Vistoli 1], Lemma 8): There is a bijective correspondence between close invariant subsets of X and closed subsets of Y. Hence the pushforward of cycles $f_*: Z_*(X)^G \to Z_*(Y)$ is bijective. Therefore $f_*: A_*(X)^G \to A_*(Y)$ is surjective. Let x be a class of cycles in $A_*(X)^G$ such that $f_*x = 0$. By Lemma 1.4 there is $\xi \in Z_*(X)$ representing x such that $f_*\xi = 0$ in $Z_*(Y)$. By averaging over G we can assume that ξ is in $Z_*(X)^G$. But then $\xi = 0$, and $x = 0$. This proves (i).

To prove (ii), notice that if $Y' \to Y$ is a morphism, $X' = Y' \times_Y X$, there is an induced action of G on X', but in positive characteristic it is false in general that $X'/G = Y'$. However, a quotient X'/G exists, and the induced morphism $X'/G \to Y'$ is a universal homeomorphism, so that the pushforward $A_*(X')^G \to A_*(Y')$ is still an isomorphism, by Lemma 1.8 (i).

We only have to prove that $f^*: A^*(Y \to Y) \to A^*(T \to X)^G$ is surjective, because injectivity follows from Lemma 1.2 (i). Take α in $A^*(T \to X)^G$ and

g in G. Let q': $Y' \to Y$ be a morphism. We have the fiber diagram

$$\begin{tikzcd}
T' \arrow[rrrrr] \arrow[dr, "g_{T'}"] \arrow[ddddd] & & & & & X' \arrow[dl, "g_X"'] \arrow[ddddd, "p"] \\
& T' \arrow[rrr] \arrow[dr] \arrow[ddd] & & & X' \arrow[dl] \arrow[ddd, "p"] & \\
& & U' \arrow[r] \arrow[d] & Y' \arrow[d, "q"] & & \\
& & U \arrow[r] & Y & & \\
& T \arrow[rrr] \arrow[ur] & & & X \arrow[ul, "f"'] & \\
T \arrow[rrrrr] \arrow[ur, "g_T"] & & & & & X \arrow[ul, "g_X"']
\end{tikzcd}$$

For any x' in $A_*(X')$ we have $\alpha_p \cap g^*_{X'} x' = g^*_{T'}(\alpha_{p\ g_X} \cap x') = g^*_{T'}(\alpha_{g_X\ p} \cap x') = g^*_{T'}((g^*_X\alpha)_p \cap x') = g^*_{T'}((\alpha\,.\,g)_p \cap x') = g^*_{T'}(\alpha_p \cap x')$. In other words, we have just checked that $\alpha\ \cap$: $A_*(X') \to A_*(T')$ is a G-equivariant homomorphism.

Now assume that x' in $A_*(X')$ is such that $f'_*x' = 0$. Then $f'_*(x'\,.\,g) = 0$ for all g in G, so that $f'_*(1/n\ \Sigma_g x'\,.\,g) = 0$, where n is the order of G. It follows that $1/n\ \Sigma_g x'\,.\,g = 0$, because of part (i). Hence $0 = h'_*(\alpha \cap 1/n\ \Sigma_g x'\,.\,g) = h'_*(1/n\ \Sigma_g(\alpha \cap x')\,.\,g) = h'_*(\alpha \cap x')$. From Lemma 1.7 we conclude that α is in the image of $A^*(U \to Y)$.

2. Alexander schemes

By a *scheme* we still mean a scheme of finite type over a fixed field k.

If X is a purely n-dimensional scheme we denote by $[X]$ the class in $Z_n(X)$ or in $A_n(X)$ associated with X, namely, the class of the cycle that contains every irreducible component of X with multiplicity equal to the length of the local ring of X at its generic point. (see [Fulton], 1.5).

For each morphism of schemes $T \to X$, with X equidimensional, we define the evaluation homomorphism

$$ev_X\colon A^*(T \to X) \to A_*(T)$$

by

$$ev_X(\alpha) \quad = \quad \alpha \cap [X].$$

The following definition is due to Kleiman and Thorup (see [Kleiman–Thorup], p. 21).

(2.1) DEFINITION: We say that a scheme X *satisfies commutativity* if

$$C^*(T \to X) \quad = \quad A^*(T \to X)$$

for all morphisms $T \to X$.

We say that a scheme X *satisfies Alexander duality*, or that it is an *Alexander scheme*, if it satisfies commutativity, is equidimensional and if

$$ev_X\colon A^*(T \to X) \to A_*(T)$$

is an isomorphism for all $T \to X$.

In the terminology of Kleiman and Thorup, an Alexander scheme is called $C_{\mathbb{Q}}$-orthocyclic.

(2.2) PROPOSITION: *Let $f\colon X \to Y$ be a smooth morphism of constant fiber dimension. If Y is an Alexander scheme, so is X.*

Proof: First note that if Y is equidimensional so is X. Consider the orientation class $[f] \in C^*(X \to Y)$. For each $T \to X$, composition with $[f]$ yields homomorphisms $A^*(T \to X) \to A^*(T \to Y)$ and $C^*(T \to X) \to C^*(T \to Y)$. The first one is an isomorphism, by [Fulton], Proposition 17.4.2. This result is proved by explicitly constructing an inverse homomorphism from $A^*(T \to Y)$ to $A^*(T \to X)$, which is immediately seen to carry $C^*(T \to Y)$ to $C^*(T \to X)$. Hence $C^*(T \to X) \to C^*(T \to Y)$ is an isomorphism too, and if Y satisfies commutativity so does X. The proof is concluded by observing that the diagram

$$\begin{tikzcd}
A^*(T \to X) \arrow[rr] \arrow[dr, "ev_X"'] & & A^*(X \to Y) \arrow[dl, "ev_Y"] \\
& A_*(T) &
\end{tikzcd}$$

commutes.

(2.3) PROPOSITION: *A regular equidimensional scheme satisfies Alexander duality.*

Proof: This is [Kleiman–Thorup], Proposition 3.9. We offer another proof for the case of a smooth scheme. By Proposition 2.2, it is enough to prove that $S = \mathrm{Spec}(k)$ is an Alexander scheme. That

$$ev_S\colon A^*(T \to S) \to A_*(T)$$

is an isomorphism is proved in [Fulton], Proposition 17.3.1. The inverse u: $A_*(T) \to A^*(T \to X)$ is constructed as follows. Let $Y \to S$ be a morphism, $t \in A_*(T)$ and $y \in A_*(Y)$. Then $u(t) \cap y$ is the exterior product $t \times y$ in $A_*(T \times Y)$. We need to prove that $u(t)$ is in $C^*(T \to S)$. It is enough to assume that $t = [V]$ for some subvariety V of T. Let j: $V \to T$ be the inclusion. Then $u(t) = j_*\mu$, where $\mu \in C^*(V \to S)$ is the orientation class determined by flat pullback. Therefore $u(t)$ belongs to $C^*(T \to S)$.

(2.4) *Note:* Intersection theory with rational coefficient on Alexander schemes is very similar to intersection theory on smooth schemes. Let Y be an Alexander scheme, f: $X \to Y$ and $Y' \to Y$ two morphisms. If y is in $A_*(Y')$ and x in $A_*(X)$, we can define $x\cdot_f y$ in $A_*(X \times_Y Y')$ by

$$x \cdot_f y = \alpha \cap y,$$

where $\alpha \in A^*(X \to Y)$ is such that $\alpha \cap [Y] = x$. This is not quite as refined as the operation defined in [Fulton], 8.1, for smooth schemes, but it sufficies for any purpose I can think of. The formal properties of [Fulton], Proposition 8.1, are all satisfied. In particular $A_*(Y)$ is in a natural way a commutative ring, and if $X \to Y$ is a morphism $A_*(X)$ becomes an $A_*(Y)$-module, in such a way that the projection formula is satisfied, as in [Fulton], Proposition 8.3.

(2.5) PROPOSITION: *An Alexander scheme is geometrically unibranch.*

Proof: Suppose X is not geometrically unibranch. Then there exists an étale morphism $Y \to X$ with Y connected, but not irreducible. Let n be the dimension of Y. If we show that $A^0(Y)$ is $\mathbb{Q}$, then ev_Y: $A^0(Y) \to A_n(Y)$ cannot be an isomorphism, because $A_n(Y)$ is generated by the irreducible components of Y, and so Y can not be an Alexander scheme. Therefore X is not an Alexander scheme, because of Proposition 2.2.

Let α be in $A^0(Y)$. If $V \to Y$ is a morphism with V integral, then

$$\alpha \cap [V] = r(V)[V]$$

for some rational number $r(V)$. We want to prove that $r(V)$ is independent of $V \to Y$. If g: $V \to W$ is a flat morphism of integral schemes over Y, then $r(V)[V] = \alpha \cap [V] = \alpha \cap g^*[W] = g^*(\alpha \cap [W]) = r(W)[W]$, and therefore $r(V) = r(W)$. By generic flatness, this also holds if we simply assume $V \to W$ dominant. In particular, by considering the closure of the image of $V \to Y$, we can restrict ourselves to proving that $r(W)$ is constant for all subvarieties W of Y. Let V and W be two subvarieties of Y with V properly contained in W. Let $\tilde{W}$ be the blow up of W along V, and call $\tilde{V}$ the exceptional divisor. Then the embedding j: $\tilde{V} \to \tilde{W}$ is regular, and therefore $\alpha \cap [\tilde{V}] = \alpha \cap j^*[\tilde{W}] = j^*(\alpha \cap [\tilde{W}]) = r(\tilde{W})[\tilde{V}] = r(W)[\tilde{V}]$. If V' is a component of $\tilde{V}$ dominating V, it follows that $r(V) = r(V') = r(W)$. Since Y is connected we conclude that $r(V)$ must be constant.

(2.6) LEMMA: *If there is a proper and surjective morphism from X to Y and X satisfies commutativity, so does Y.*

Proof: This is a consequence of Lemma 1.4.

(2.7) PROPOSITION: *Let f: $X \to Y$ be an universal homeomorphism. Then X satisfies Alexander duality if and only if Y does.*

(2.8) COROLLARY: *A scheme X satisfies Alexander duality if and only if X_{red} does.*

(2.9) COROLLARY: *Let X be a scheme, $\bar{X}$ its normalization (i.e., the normalization of X_{red}). Then X is an Alexander scheme if and only if $\bar{X}$ is an Alexander scheme and X is geometrically unibranch.*

Proof: The scheme X is geometrically unibranch if and only if the normalization morphism $\bar{X} \to X$ is a universal homeomorphism (see EGA IV, 6.15.5 and 6.15.6). Therefore the result follows from Propositions 2.5 and 2.7.

(2.10) COROLLARY: *A scheme of dimension 1 is an Alexander scheme if and only if it is geometrically unibranch.*

Proof of 2.7: The scheme X is equidimensional if and only if Y is. We may assume that X and Y are connected. Proposition 2.5 implies that if we assume that either X or Y is an Alexander scheme, then X and Y are irreducible. Hence $f_*[X] = d[Y]$ for some nonzero rational number d. If $U \to Y$ is a morphism, $T = U \times_Y X$ and g: $T \to U$ is the projection, then for any β in $A^*(X \to Y)$ we have $g_*(f^*\beta \cap [X]) = d(\beta \cap [Y])$, that is, the diagram

$$\begin{array}{ccc} A^*_*(T \to X) & \xrightarrow{ev_X} & A_*(T) \\ \uparrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g_*} \\ A^*(U \to Y) & \xrightarrow{ev_Y} & A_*(U) \end{array}$$

commutes up to a nonzero rational number. But f^* and g_* are isomorphisms, by Lemma 1.7, and therefore ev_X is an isomorphism if and only if ev_Y is an isomorphism.

We conclude the proof by showing the X satisfies commutativity if and only if Y does. If X satisfies commutativity, so does Y, by Lemma 2.6. Conversely, assume that Y satisfies commutativity. We want to prove that $C^*(T \to X) = A^*(T \to X)$ for every $T \to X$. First assume that $T = U \times_Y X$ some $U \to Y$. Then the pullback f^*: $A^*(U \to Y) \to A^*(T \to X)$ is an isomorphism and carries $C^*(U \to Y) = A^*(U \to Y)$ into $C^*(T \to X)$. Hence $C^*(T \to X) = A^*(T \to X)$. In general, set $T' = T \times_Y X$, and call h: $T' \to T$ the projection. Then h is an universal homeomorphism, and therefore h_*: $A^*(T' \to X) \to A^*(T \to X)$ is an isomorphism, by Lemma 1.7 (iii). Since h_* carries $C^*(T' \to X)$ into $C^*(T \to X)$ and $C^*(T' \to X) = A^*(T' \to X)$, this concludes the proof.

The next proposition gives us our first examples of Alexander schemes that are normal but not regular.

(2.11) PROPOSITION (cf. [Fulton], Example 17.4.10):

(i) *Let X be an Alexander scheme, G a finite group operating on X, in such a way that a geometric quotient X/G exists. Then X/G is an Alexander scheme.*

(ii) *Let f: $X \to Y$ be a finite surjective morphism. If X is an Alexander scheme and Y is geometrically unibranch, then Y is an Alexander scheme.*

Proof: Let us prove (i). Set $Y = X/G$. We can assume that Y is connected. But Y is unibranch, because X is unibranch, and so Y is irreducible. By Lemma 2.6, Y satisfies commutativity.

Let $U \to Y$ be a morphism, $T = U \times_Y X$, g: $T \to U$ the projection. Since Y is irreducible, $f_*[X]$ is a rational multiple of Y, and therefore the diagram

$$\begin{array}{ccc} A^*(T \to X)^G & \xrightarrow{ev_X} & A_*(T)^G \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle g_*} \\ A^*(U \to Y) & \xrightarrow{ev_Y} & A_*(U) \end{array}$$

commutes up to a rational number. But f^* and g_* are isomorphisms, because of Lemma 1.8, and therefore ev_Y is an isomorphism.

Now we prove (ii). Corollary (2.9) implies that we can take X and Y to be normal. We can also assume that X and Y are irreducible. Let E be the separable closure of $k(Y)$ in $k(X)$, and let X' be the normalization of Y in E. Then f: $X \to Y$ factors as $X \to X' \to Y$, and $X \to X'$ is a universal homeomorphism. By proposition 2.7, X' is an Alexander scheme, and so we can assume that $k(X)$ is separable over $k(Y)$. Let K be a Galois closure of $k(X)$ over $k(Y)$, and let X_0 be the normalization of Y in K. Let G be the Galois group of K over $k(Y)$, H the Galois group of K over $k(X)$. Then $X_0/G = Y$ and $X_0/H = X$. If $U \to Y$ is a morphism, $T = U \times_Y X$ and $T_0 = U \times_Y X_0$, we know that ev_X: $A^*(T \to X) \to A_*(T)$ is an isomorphism, and therefore ev_{X_0}: $A^*(T_0 \to X_0)^H \to A_*(T_0)^H$ is an isomorphism. We want to prove that ev_Y: $A^*(U \to Y) \to A_*(U)$ is an isomorphism, or, equivalently, that ev_{X_0}: $A^*(T_0 \to X_0)^G \to A_*(T_0)^G$ is an isomorphism. This follows from the following easy lemma, applied to ev_{X_0}.

(2.12) LEMMA: *Let G be a finite group, H a subgroup of G. Let V and W be two $\mathbb{Q}$-vector spaces on which G acts, p: $V \to W$ an equivariant linear transformation. If the restriction p: $V^H \to W^H$ is an isomorphism, then p: $V^G \to W^G$ is an isomorphism.*

Proof: Let V_0 and W_0 be the kernel and the cokernel of p: $V \to W$. If Γ is a finite group, the functor that sends a $\mathbb{Q}$-vector space Z with an action of Γ to Z^Γ is an exact functor. Hence V_0^H and W_0^H (resp. V_0^G and W_0^G) are the kernel and the cokernel of p: $V^H \to W^H$ (resp. p: $V^G \to W^G$). But if $V^H = W^H = 0$, then $V^G = W^G = 0$.

(2.13) PROPOSITION: *Let f: $X \to Y$ be a proper and surjective morphism of equidimensional schemes. Suppose that there exists $[f]$ in $C^d(X \to Y)$, $d = \dim(X) - \dim(Y)$, such that*

$$[f] \cap [Y] = [X].$$

If X satisfies Alexander duality, so does Y.

In particular this holds if f is flat or a locally complete intersection morphism.

Proof: Let $U \to Y$ be a morphism, $T = U \times_Y X$. We can define a homomorphism f_*: $A^*(T \to X) \to A^*(U \to Y)$ as follows (cf. [Fulton], p. 328).

Let $Y' \to Y$ be a morphism, and form the fiber diagram 1.3. If α is in $A^*(T \to X)$ and y' in $A_*(Y')$, define

$$f_*\alpha \cap y' = g'_*(\alpha \cap ([f] \cap y')).$$

One checks easily that $f_*\alpha$ is in $A^*(U \to Y)$.

Assume that X is an Alexander scheme. The scheme Y satisfies commutativity by Lemma 2.6.

Now take β in $A^*(U \to Y)$ and assume that $\beta \cap [Y] = 0$. Then $f^*\beta \cap [X] = \beta \cap ([f] \cap [Y]) = [f] \cap (\beta \cap [Y]) = 0$. Hence $f^*\beta = 0$, and by Lemma 1.2, $\beta = 0$. Therefore ev_Y: $A^*(U \to Y) \to A_*(U)$ is injective.

To prove surjectivity, let u be in $A_*(U)$. Choose t in $A_*(T)$ with $g_*t = u$. Since X is an Alexander scheme, $t = \alpha \cap [X]$ for some α in $A^*(T \to X)$. Hence $u = g_*t = g_*(\alpha \cap [X]) = g_*(\alpha \cap ([f] \cap [Y])) = f_*\alpha \cap [Y]$.

3. Change of base field

The purpose of this section is to prove the following theorem. Let $k \subset K$ be a field extension. If X is a scheme over k, write X_K for $X \times_k \operatorname{Spec}(K)$. Recall that K is called *separable* over k if every subfield of K containing k and finitely generated over k is separably generated over k. If k is perfect, every extension of k is separable.

(3.1) THEOREM: *Let X be a scheme of finite type over k.*

(i) *If K is algebraic over k and X_K is an Alexander scheme, then X is an Alexander scheme.*

(ii) *If K is separable over k, X_K satisfies commutativity and X is an Alexander scheme, then X_K is an Alexander scheme.*

Note that if K is finite over k part (i) is a consequence of Proposition 2.11 (ii).

We will use the following notation. If R and S are k-subalgebras of K, R contained in S, we let Ψ_R^S denote the canonical morphism from $\operatorname{Spec}(S)$ to $\operatorname{Spec}(R)$. If X_R is a scheme over R, we set

$$X_S = X_R \times_R \operatorname{Spec}(S).$$

Let R be a finitely generated k-algebra contained in K, and let X_R be a scheme of finite type over R. We want to define a homomorphism

$$\Psi_R^{K*}: A_*(X_R) \to A_*(X_K)$$

of degree equal to $-\dim(R)$. If V_R is a closed integral subscheme of X_R, set

$$\Psi_R^{K*}[V_R] = [V_K].$$

This defines

$$\Psi_R^{K*}: Z_*(X_R) \to Z_*(X_K).$$

If r is a rational function of V_R, call D the pullback of the divisor of r to V_R. So D is a principal Cartier divisor on V_K, and $\Psi_R^{K*}[\operatorname{div}(r)] = D.[V_R]$, where the last term is defined as in [Fulton], 2.3 From [Fulton], Proposition 2.3(e), it follows that $\Psi_R^{K*}[\operatorname{div}(r)]$ is rationally equivalent to 0, and therefore Ψ_R^{K*} passes to rational equivalence. If $R \subset S \subset K$, S finitely generated and flat over R, $p: X_S \to X_R$ is the canonical morphism, then p is flat, and

$$\Psi_R^{K*} = \Psi_S^{K*}p^*: A_*(X_R) \to A_*(X_S).$$

(3.2) LEMMA: *Let $f_R: X_R \to Y_R$ be a morphism of schemes of finite type over R, $f_K: X_K \to Y_K$ obtained by base change.*
(i) *If f_R is proper,*

$$f_{K*}\Psi_R^{K*} = \Psi_R^{K*}f_{R*}: A_*(X_R) \to A_*(Y_K).$$

(ii) *If f_R is flat,*

$$f_K^*\Psi_R^{K*} = \Psi_R^{K*}f_R^*: A_*(Y_R) \to A_*(X_K).$$

(iii) *If f_R is a regular embedding, $Y_R' \to Y_R$ a morphism of finite type, $X_R' = X_R \times_{Y_R} Y_R'$, then*

$$f_K^!\Psi_R^{K*} = \Psi_R^{K*}f_R^!: A_*(Y_R') \to A_*(X_R').$$

Proof: (i) and (ii) are straightforward, and they even hold at the level of cycles.

In view of (i), to prove (iii) it is enough to prove that, if Y_R' is integral, $f_K^!(\Psi_R^{K*}[Y_R']) = \Psi_R^{K*}f_R^![Y_R']$. Let N_R' be the pullback to X_R' of the normal bundle to X_R in Y_R, $s_R: X_R' \to N_R'$ the zero section, C_R' the normal bundle to X_R' in Y_R'. Then $f_R^![Y_R'] = s_R^*[C_R']$. Since the Gysin homomorphism s_R^* is the inverse of the flat pullback from $A_*(X_R')$ to $A_*(N_R')$ and we know that Ψ_R^{K*}

commutes with flat pullback, we have $\Psi_R^{K*} f_R^![Y_R'] = \Psi_R^{K*} s_R^*[C_R'] = s_K^* \Psi_R^{K*}[C_R'] = s_K^*[C_K']$. But K is flat over R and the formation of normal cones commutes with base change along flat morphisms, and so C_K' is the normal cone to X_K' in Y_K'. By [Fulton], Example 6.2.1, we have $s_K^*[C_K'] = f_K^![Y_K'] = f_K^! \Psi_R^{K*}[Y_R']$.

We can also define

$$\Psi_R^{K*}: A^*(X_R \to Y_R) \to A^*(X_K \to Y_K)$$

for any morphism $X_R \to Y_R$ of finite type over R. Let $Y_K' \to Y_K$ a morphism of schemes of finite type over K, and let y_K' be in $A_*(Y_K')$. Then there exists a finitely generated R-algebra S contained in K, a scheme Y_S' of finite type over S such that $Y_S' \times_S \operatorname{Spec}(K) = Y_K'$, and a class of cycles y_S' in $A_*(Y_S')$ such that $\Psi_S^{K*} y_S' = y_K'$. For any α in $A^*(X_R \to Y_R)$ define

$$(\Psi_R^{K*}\alpha) \cap y_K' = \Psi_S^{K*}(\alpha \cap y_S').$$

This makes sense, because Y_S' is of finite type over Y_R'. If S' is another finitely generated R-subalgebra of K, $Y_{S'}'$, an S'-scheme of finite type with $Y_{S'}' \times_{S'} \operatorname{Spec}(K) = Y_{K'}'$ and $y_{S'}' \in A_*(Y_{S'}')$ is such that $\Psi_{S'}^{K*} y_{S'}' = y_{K'}'$, then we can find finitely generated R-subalgebra T of K containing S and S', flat over S and S', such that:

(i) $Y_S' \times_S \operatorname{Spec}(T)$ is isomorphic to $Y_{S'}' \times_{S'} \operatorname{Spec}(T)$, and, using this isomorphism to identify them,

(ii) If $q: Y_S' \times_S \operatorname{Spec}(T) \to Y_S'$ and $q': Y_{S'}' \times_{S'} \operatorname{Spec}(T) \to Y_{S'}'$ denote the projections, then $q^* y_S' = q'^* y_{S'}'$.

Then, if we set $Y_T' = Y_S' \times_S \operatorname{Spec}(T) = Y_{S'}' \times_{S'} \operatorname{Spec}(T)$, $X_T' = X_R' \times_{Y_R} Y_T'$, and call $p: X_T' \to X_S'$ and $p': X_T' \to X_{S'}'$ the projections, we have $\Psi_S^{K*}(\alpha \cap y_S') = \Psi_T^{K*} p^*(\alpha \cap y_S') = \Psi_T^{K*}(\alpha \cap q^* y_S') = \Psi_T^{K*}(\alpha \cap q'^* y_{S'}') = \Psi_{S'}^{K*}(\alpha \cap y_{S'}')$. Hence $\Psi_R^{K*}\alpha$ is well defined.

The fact that $\Psi_R^{K*}\alpha$ belong to $A^*(X_K \to Y_K)$ is a consequence of Lemma 3.2 and of the following.

(3.3) LEMMA: *Let $X_K \to Y_K$ be a morphism of schemes of finite type over K which is proper, or flat, or a regular embedding of codimension d. Then there exists a finitely generated k-subalgebra R of K and a morphism $X_R \to Y_R$ of schemes of finite type over R with the same property which yields $X_K \to Y_K$ by change.*

Proof: Choose a finitely generated k-subalgebra T of K and a morphism $X_T \to Y_T$ of schemes of finite type over T which gives $X_K \to Y_K$ when pulled back to Spec(K). Then it is clearly enough to prove that there is an open subset V of Spec(T) such that $X_T \to Y_T$ restricted to V has the desired property.

For properness this follows from EGA IV, 9.6.1 (iv). For flatness it follows from EGA IV, 12.2.6 (ii) applied to the projective system of a fine open subschemes of Spec(T).

Suppose that $X_K \to Y_K$ is a regular embedding of codimension d. We can assume that Y_K is affine and the ideal of X_K in Y_K is generated by d functions $\bar{f}_1, \ldots, \bar{f}_d$ on Y_K. It follows from EGA IV, 9.6.2.I (i) that by localizing T we can assume Y_T affine. By localizing further, we can assume that $\bar{f}_1, \ldots, \bar{f}_d$ extend to functions $f_1, \ldots, f_d$ on Y_T, and the ideal of X_T in Y_T is generated by $f_1, \ldots, f_d$. Consider the Koszul complex of $f_1, \ldots, f_d$. Its cohomology groups are coherent sheaves on Y_T, whose supports do not intersect Y_K. Hence the union in Spec(T) of the images of the supports is a constructible set that does not contain the generic point. Therefore its complement contains an open subset of Spec (T), and in the inverse image of this open subset X_T is regularly embedded in Y_T.

If K is algebraic over k, we can also define

$$\Psi^K_{k*}\colon A_*(X_K) \to A_*(X)$$

for any scheme X of finite type over k.

If K is finite over k, the morphism $p\colon X_K \to X$ is finite. Given x_K in $A_*(X_K)$ we set

$$\Psi^K_{k*}x_K = \frac{1}{\deg(K/k)}\, p_*x_K.$$

It is clear that if $k \subset k' \subset K$ is an intermediate field, then

$$\Psi^K_{k*}x_K = \Psi^{k'}_{k*}(\Psi^K_{k'*}x_K),$$

and that

$$\Psi^K_{k*}\Psi^{K*}_k\colon A_*(X) \to A_*(X)$$

is the identity.

In general, if x_K is in $A_*(X_K)$ there exists an intermediate field $k \subset k' \subset K$ such that k' is finite over k and a class of cycles $x_{k'}$ in $A_*(X_{k'})$

with $\Psi_{k'}^{K*}x_{k'} = x_K$. Then we define

$$\Psi_{k*}^K x_K = \Psi_{k'*}^K x_{k'}$$

To check that this is well defined, fix another finitely generated intermediate subfield k'' and a class $x_{k''}$ in $A_*(X_{k''})$ with $\Psi_{k''}^{K*}x_{k''} = x_K$. There will be a subfield k_0 of K finite over k such that $k' \subset k_0$, $k'' \subset k_0$ and $\Psi_{k'}^{k_0*}x_{k'} = \Psi_{k''}^{k_0*}x_{k''}$ in $A_*(X_{k_0})$. Then $\Psi_{k*}^{k'}x_{k'} = \Psi_{k*}^{k'}\Psi_{k'*}^{k_0}\Psi_{k'}^{k_0*}x_{k'} = \Psi_{k*}^{k_0}\Psi_{k'}^{k_0*}x_{k'} = \Psi_{k*}^{k_0}\Psi_{k''}^{k_0*}x_{k''} = \Psi_{k*}^{k''}x_{k''}$.

(3.4) Lemma: *Let f: $X \to Y$ be a morphism of schemes of finite type over k, and let f_K: $X_K \to Y_K$ be obtained by base change.*

(i) *If f is proper and $x_K \in A_*(X_K)$, then*

$$\Psi_{k*}^K f_{K*}x_K = f_*\Psi_{k*}^K x_K.$$

(ii) *If $\alpha \in A^*(X \to Y)$ and $y_K \in A_*(X_K)$, then*

$$\Psi_{k*}^K(\Psi_k^{K*}\alpha \cap y_K) = \alpha \cap \Psi_{k*}^K y_K.$$

The proof is straighforward from the definitions.

We also define

$$\Psi_{k*}^K : A^*(X_K \to Y_K) \to A^*(X \to Y)$$

for a morphism $X \to Y$ of finite type over k, still with the hypothesis that K is algebraic over k.

If α_K is an $A^*(X_K \to Y_K)$, $Y' \to Y$ is a morphism of finite type and y' is in $A_*(Y')$, we set

$$(\Psi_{k*}^K\alpha_K) \cap y' = \Psi_{k*}^K(\alpha_K \cap \Psi_k^{K*}y').$$

(3.5) Lemma: (i) *$\Psi_{k*}^K\alpha_K$ is in $A^*(X \to Y)$.*
(ii) *If $\alpha_K \in C^*(X_K \to Y_K)$ then $\Psi_{k*}^K\alpha_K \in C^*(X \to Y)$.*

The proof is straightforward, using Lemma 3.4.

The following is a weak version of a criterion due to Kleiman and Thorup (see [Kleiman–Thorup], Proposition 3.3).

(3.6) Lemma: Let X be an equidimensional scheme of finite type over a field. Suppose that for all integral schemes V and all morphisms of finite type

$V \to X$ there exists α_V in $C^d(V \to X)$, $d = \dim(X) - \dim(V)$, such that

$$\alpha_V \cap [X] = [V].$$

Then X is an Alexander scheme.

Proof of 3.1: To prove (i), let $V \to X$ be a morphism of finite type with V integral. Since X_K is an Alexander scheme, there is α_K in $C^d(V \to X)$, $d = \dim(X_K) - \dim(V_K) = \dim(X) - \dim(V)$, such that $\alpha_K \cap [X_K] = [V_K]$. But then $(\Psi^K_{k*}\alpha_K) \cap [X] = \Psi^K_{k*}(\alpha \cap \Psi^{K*}_k[X]) = \Psi^K_{k*}(\alpha \cap [X_K]) = \Psi^K_{k*}[V_K] = [V]$, and $\Psi^K_{k*}\alpha$ belongs to $C^d(V \to X)$ because of Lemma 3.5. The conclusion follows from Lemma 3.6

Let us prove (ii). Let V_K be an integral scheme over K, $V_K \to X_K$ a morphism of finite type. There is a finitely generated k-algebra R contained in K such that $V_K \to X_K$ is obtained by base change from a morphism of finite type $V_R \to X_R$. By localizing, we can assume that V_R is integral and flat over R, and, since K is separable over k, that R is smooth. Then X_R is an Alexander scheme, by Proposition 2.2. Therefore there exists α_R in $C^d(V_R \to X_R)$, $d = \dim(X_R) - \dim(V_R) = \dim(X_K) - \dim(V_K)$, with $\alpha_R \cap [X_R] = [V_R]$. Then $(\Psi^{K*}_R\alpha_R) \cap [X_K] = (\Psi^{K*}_R\alpha_R) \cap \Psi^{K*}_R[X_R] = \Psi^{K*}_R(\alpha_R \cap [X_R]) = \Psi^{K*}_R[V_R] = [V_K]$. We have that $\Psi^{K*}_R\alpha$ belongs to $A^d(V_K \to X_K)$, and, because X_K satisfies commutativity, $C^d(V_K \to X_K) = A^d(V_K \to X_K)$. Therefore, by Lemma 3.6, X_K is an Alexander scheme.

4. Surfaces

We already know how to characterize geometrically Alexander schemes of dimension 1. We shall do the same for schemes of dimension 2 over a perfect field. Corollary 2.9 allows us to restrict our attention to normal surfaces.

(4.1) THEOREM: *A normal surface over a perfect field satisfies Alexander duality if and only if all the components of the exceptional divisors on some resolution of its singularities are rational curves.*

Here by a rational curve we mean an integral complete scheme C of dimension 1 with $H^1(\bar{C}, \mathcal{O}_{\bar{C}}) = 0$, where $\bar{C}$ is the normalization of C.

Proof: Suppose that the normal surface Y is an Alexander scheme, and let $f: X \to Y$ be a resolution of singularities. If k is the base field, let K be obtained by adding uncountable many indeterminates to k and taking the algebraic closure. Then by Theorem 3.1 (ii) Y_K is still an Alexander scheme, and since k is perfect, $X_K \to Y_K$ is still a resolution of singularities. If one of the exceptional divisors on X_K has a component which is not rational, the same is true of X. So we can assume that the base field is uncountable and algebraically closed.

Let Q be a singular point on Y, $E = E_1 \cup \ldots \cup E_n$ the exceptional divisor of Q with irreducible components $E_1, \ldots, E_n$. We want to prove that E_i's are rational. By blowing up we can assume that the E_i's are smooth. Let

$$\bar{E} = f^{-1}(Q) = X \times_Y Q.$$

Let $j\colon E \to \bar{E}$ the embedding, which is a homeomorphism, because $\bar{E}_{\mathrm{red}} = E$.

We will show that the pullback

$$f^*\colon A^2(Q \to Y) \to A^2(\bar{E} \to X)$$

is bijective.

It is injective by Lemma 1.2. Fix α in $A^2(\bar{E} \to X)$. Take a morphism $Y' \to Y$, and form the fiber diagram

$$\begin{tikzcd}
X'_Q \arrow[rrr] \arrow[dr] \arrow[ddd, "g'"'] & & & X' \arrow[dl] \arrow[ddd, "f'"] \\
& \bar{E} \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] & \\
& Q \arrow[r] & Y & \\
Y'_Q \arrow[rrr] \arrow[ur] & & & Y' \arrow[ul]
\end{tikzcd}$$

Choose x' in $A_*(X')$ such that $f'_*x' = 0$ in $A_*(Y')$. Then by Lemma 1.4 x' is represented by a cycle on X' with image 0 in $Z_*(Y')$. This means that the components of this cycle must map to the inverse image Y'_Q of Q in Y', beccause elsewhere f' is an isomorphism. Let V be such a component. The morphism $V \to X$ obtained from f' factors through $\bar{E}$, and hence through some E_i. Let $h\colon V \to E_i$ be the induced morphism. Since E_i is smooth, h is the composite of a flat morphism and a regular embedding, and therefore $\alpha \cap [V] = \alpha \cap h^*[E_i] = h^*(\alpha \cap [E_i])$. But $\alpha \cap [E_i] = 0$, because α is in $A^2(\bar{E} \to X)$ and the dimension of E_i is 1. Hence $\alpha \cap x' = 0$, and a fortiori $g'_*(\alpha \cap x') = 0$. By Lemma 1.6 α is in the image of f^*, and therefore f^* is bijective. Now have a diagram

$$\begin{array}{ccc} A^2(\bar{E} \to X) & \xrightarrow{ev_X} & A_0(\bar{E}) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle g_*} \\ A^2(Q \to Y) & \xrightarrow{ev_Y} & A_0(Q) \end{array}$$

which commutes, because $g_*(f^*\beta \cap [X]) = \beta \cap f_*[X] = \beta \cap [Y]$ for any β in $A^2(Q \to Y)$. We know that f^*, ev_X and ev_Y are isomorphisms. Hence $g_*: A_0(\bar{E}) \to A_0(Q) = \mathbb{Q}$ is an isomorphism. Also $j_*: A_0(E) \to A_0(\bar{E})$ is an ismorphism. But then $A_0(E) = \mathbb{Q}$, and this is impossible if E contains a nonrational curve. In fact the kernel of the pushforward

$$\oplus_i A_0(E_i) \to A_0(E)$$

is finite dimensional, generated by elements of the form $[e_i] - [e_j]$, where $e_i \in E_i$ and $e_j \in E_j$ are points that are identified in E (this follows, for example, from Lemma 1.4). On the other hand, if E_i has positive genus $A_0(E_i)$ must be uncountable. Let J be the group of rational points on the Jacobian of E_i. Then

$$A_0(E_i) = \mathbb{Q} \oplus (J \otimes \mathbb{Q}) = \mathbb{Q} \oplus (J/\text{torsion}).$$

But J is uncountable, because the base field is uncountable and algebraically closed, and its torsion subgroup is countable.

Now we prove the converse. Let $f: X \to Y$ be a resolution such that all the components of the exceptional divisors are rational. By Theorem 3.1 (i), we can assume that the base field is algebraically closed. We suppose for simplicity of notation that there is only one singular point Q on Y (the proof in the general case is completely analogous). Let $E = E_1 \cup \ldots \cup E_n$ be the exceptional divisor. By blowing up, we can assume that the E_i's are smooth and intersect transversally.

From Lemma 2.6 we know that Y satisfies commutativity. By Proposition 2.13, to prove that Y is an Alexander scheme it is enough to show that there exists β in $A^0(X \to Y)$ with

$$\beta \cap [Y] = [X].$$

Let $\bar{X} = X \times_Y X$ with second projection $p: \bar{X} \to X$. The homomorphism

$$ev_X: A^0(\bar{X} \xrightarrow{p} X) \to A_2(\bar{X})$$

is an isomorphism, and $\bar{X}$ has n^2+1 2-dimensional components: the image of the diagonal δ: $X \to \bar{X}$, which we call X_0, and $E_i \times E_j$ for i, $j = 1, \ldots, n$. Let α_0 and α_{ij}, $i, j = 1, \ldots, n$, in $A^0(\bar{X} \xrightarrow{p} X)$ be such that

$$\alpha_0 \cap [X] \;=\; [X_0] \text{ and } \alpha_{ij} \cap [X] \;=\; [E_i \times E_j].$$

Set $\bar{E} = f^{-1}(Q)$. Then $\bar{X} \times_X E_k = X \times_Y E_k = \bar{E} \times E_k$. Construct the fiber diagram

$$\begin{tikzcd}
E_i \times (E_j \cap E_k) \arrow[r] \arrow[d] & \bar{E} \times E_k \arrow[r] \arrow[d] & E_k \arrow[d] \\
E_i \times E_j \arrow[r, "\varphi"] & \bar{X} \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

We want to calculate $\alpha_0 \cap [E_k]$ and $\alpha_{ij} \cap [E_k]$ in $A_1(\bar{E} \times E_k)$. If δ: $X \to \bar{X}$ is the diagonal, $\alpha_0 = \delta_*(1)$, where 1 is the identity in $A^*(X)$ and δ_*: $A^*(X) \to A^*(\bar{X} \xrightarrow{p} X)$ is the pushforward. Therefore

$$\alpha_0 \cap [E_k] \;=\; [\delta(E_k)]$$

in $A_1(\bar{E} \times E_k)$. Since E_k is a rational curve, if e is any point on E_k we have

$$[\delta(E_k)] \;=\; [E_k \times e] + [e \times E_k]$$

in $A_1(E_k \times E_k)$, and hence also in $A_1(\bar{E} \times E_k)$.

The morphism $E_i \times E_j \to X$ obtained from the diagram above is the composite $E_i \times E_j \to E_j \to X$ of the second projection and the embedding. Hence if γ_{ij} in $A^0(E_i \times E_j \to X)$ is such that $\gamma_{ij} \cap [X] = [E_i \times E_j]$, then $\gamma_{ij} \cap [E_k] = [E_i] \times [E_j . E_k]$, where $[E_j . E_k]$ is the cycle intersection of $[E_j]$ and $[E_k]$, supported on $E_j \cap E_k$. Since $\varphi_* \gamma_{ij} = \alpha_{ij}$ and all points of E_k are rationally equivalent, we see that

$$\alpha_{ij} \cap [E_k] \;=\; (E_j . E_k)[E_i \times e],$$

where e is a point of E_k and $(E_j . E_k)$ denotes the intersection number of E_j and E_k.

Set

$$\alpha \;=\; \alpha_0 + \sum_{ij} r_{ij} \alpha_{ij}$$

for certain rational numbers r_{ij}, $i, j = 1, \ldots, n$; Then

$$\alpha \cap [E_k] = [E_k \times e] + [e \times E_k] + \sum_{ij} r_{ij} (E_j . E_k)[E_i \times e]$$

in $A_1(\bar{E} \times E_k)$. Choose the r_{ij}'s in such a way that

$$\sum_j r_{ij}(E_j . E_k) = \begin{cases} -1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}.$$

This is possible because of the nondegeneracy of the intersection matrix. In this case $\alpha \cap [E_k] = [e \times E_k]$. We will prove that $\alpha = f^*\beta$ for some β in $A^0(X \to Y)$ by using Lemma 1.7. Given a morphism $Y' \to Y$ let X' be $X \times_Y Y'$, and let f': $X' \to Y'$ be the projection. Suppose that x' in $A_*(X')$ is such that $f'_* x' = 0$. By Lemma 1.3 x' has a representative ξ' in $Z_*(X')$ with $f'_* \xi' = 0$ in $Z_*(Y')$. This means that the components of ξ' must map to the inverse image of Q in Y'. To check that the condition of Lemma 1.6 is satisfied we are therefore allowed to assume that Y' maps to Q in Y. Then $X' = Y' = \bar{E}$, and the fiber diagram becomes

$$\begin{tikzcd}
E \times Y' \times \bar{E} \arrow[rrr] \arrow[dr] \arrow[ddd, "g'"'] & & & Y' \times \bar{E} \arrow[dl] \arrow[ddd, "f'"] \\
& \bar{X} \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] & \\
& X \arrow[r, "f"] & Y & \\
\bar{E} \times Y' \arrow[rrr] \arrow[ur] & & & Y' \arrow[ul]
\end{tikzcd}$$

Let V be a component of $\xi' \in Z_*(Y' \times \bar{E})$. Then V will be contained in $Y' \times E_k$ for some k. Let h: $V \to E_k$ and H: $\bar{E} \times V \to \bar{E} \times E_k$ be the projections. Since E_k is smooth, h factors as a flat morphism followed by a regular embedding, and so $\alpha \cap [V] = \alpha \cap h^*[E_k] = H^*(\alpha \cap [E_k]) = H^*[e \times E_k] = [e \times V]$. Hence $\alpha \cap x' = [e] \times x'$, and $g'_*(\alpha \cap x') = g'_*([e] \times x') = [e] \times f'_* x' = 0$. Therefore $\alpha = f^*\beta$ for some β in $A^0(X \to Y)$, and $\beta \cap [Y] = \beta \cap f_*[X] = g_*(\alpha \cap [X]) = g_*([X_0] + \Sigma_{ij} r_{ij}[E_i \times E_j]) = [X]$.

Despite the very abstract character of the notion of Alexander scheme, the formal properties of the bivariant theory are so strong that it is often possible to compute pullbacks and products with ease. Here we give an example.

We call a normal surface over a perfect field satisfying the condition of Theorem 4.1 an Alexander surface.

Let Y be an Alexander surface, $f: X \to Y$ a resolution of singularities. We shall compute

$$f^*: A_1(Y) \to A_1(X)$$

and, in case Y is complete, the intersection pairing

$$A_1(Y) \otimes A_1(Y) \to \mathbb{Q}$$

obtained by composing the product $A_1(Y) \otimes A_1(Y) \to A_0(Y)$ with the degree map $A_0(Y) \to \mathbb{Q}$. Let $E_1, \ldots, E_n$ be the components of the various exceptional divisors. The intersection matrix $(E_i . E_j)$ is nonsingular.

Let C be an integral curve on Y, $\bar{C}$ the proper transform of C in X. We identify $A^*(X)$ and $A^*(Y)$ with $A_*(X)$ and $A_*(Y)$ via the evaluation maps. Then $f^*[C] = [C] \cap [X]$ in $A_1(X)$. We have $f_* f^*[C] = f_*([C] \cap [X]) = [C] \cap [Y] = [C]$. By Lemma 1.4 we see that

$$f^*[C] = [\bar{C}] + \sum_i \lambda_i [E_i]$$

in $A_1(X)$ for some rational numbers $\lambda_1, \ldots, \lambda_n$. Therefore $[C] \cap [E_j] = [\bar{C}] \cap [E_j] + \Sigma_i \lambda_1 [E_i] . [E_j]$ in $A_0(E_j)$. Call Q_j the image of E_j in Y. The pushforward of $[E_j]$ to $A_1(Q_j)$ is 0, and so the pushforward of $[C] \cap [E_j]$ to $A_0(Q_j)$ must be 0. If we indicated by $(A . B)$ the intersection number of the divisors A and B on X, defined when A or B is proper, we conclude that

$$(\bar{C} . E_j) + \sum_i \lambda_i (E_i . E_j) = 0.$$

This uniquely determines the λ_i's because the intersection matrix is nondegenerate.

Now if Y is complete, C and D are integral curves on Y, $\bar{C}$ and $\bar{D}$ the proper transforms in X, and the rational numbers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ are determined by the equations

$$(\bar{C} . E_j) + \sum_i \lambda_i (E_i . E_j) = 0 \text{ and } (\bar{D} . E_j) + \sum_i \mu_i (E_i . E_j) = 0.$$

we have

$$\begin{aligned}(C . D) &= \deg (f^*[C] . f^*[D]) = (\bar{C} . \bar{D}) + \sum_i (\lambda_i (\bar{C} . E_i) \\ &\qquad + \mu_i (\bar{D} . E_i)) + \sum_{ij} \lambda_i \mu_j (E_i . E_j) \\ &= (\bar{C} . \bar{D}) + \sum_j \mu_j (\bar{C} . E_j) = (\bar{C} . \bar{D}) + \sum_i \lambda_i (\bar{D} . E_i).\end{aligned}$$

This is exactly the pairing $A_1(Y) \otimes A_1(Y) \to \mathbb{Q}$ defined in [MumfordI]. IIb.

5. An open problem

There are a number of open questions concerning Alexander schemes. Here is what I think is by far the most interesting one: is the Alexander property local in the étale topology or in the Zariski topology?

A Zariski or étale open subset of an Alexander scheme is an Alexander scheme, by Proposition 2.2. So the problem for the étale topology becomes: if U is an Alexander scheme and $U \to X$ is an étale surjective morphism, is X an Alexander scheme?

We have seen that this is the case for curves and for surfaces over a perfect field (Corollary 2.10 and Theorem 4.1).

The problem in proving results in this direction is that at present there is no homological machinery to relate local and global intersection-theoretical properties of a scheme, so that to prove that a scheme satisfies Alexander duality one needs some sort of global construction. Perhaps one can give a local geometric characterization of Alexander schemes using a resolution of singularities, extending the characterization of Alexander surfaces. The first step would be to look at the case of threefolds.

The following result, contained in [Vistoli 2], has been mentioned in the introduction. We say that a scheme of finite type over a field has quotient singularities when locally in the étale topology it is the quotient of a smooth scheme by a finite group.

(5.1) THEOREM: *In characteristic* 0, *a scheme with quotient singularities satisfies Alexander duality.*

This is considerably more difficult to prove than any of the results of this article. Given a scheme X with quotient singularities, one constructs a smooth stack (in the sense of [Deligne–Mumford]) having X as a moduli space, proves a form of Alexander duality for Chow groups of smooth stacks and then relates the intersection theory on the stack with the intersection theory on X.

Added in proof

With the techniques of this paper one can easily prove that the problem above has a positive solution when X has a resolution of singularities, and isolated singularities.

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