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A "Hardy–Littlewood" approach to the S-unit equation

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This paper is concerned with the quantitative theory of the S-unit equation. Suppose K is an algebraic number field, finite dimensional over $\mathbb{Q}$, with ring of algebraic integers denoted o_K. We define a prime to be an equivalence class of non-trivial valuations on K. The infinite primes are those which contain archimedean valuations and will be denoted by S_∞. It is usual to normalise the valuations so that the product formula holds;

$$\prod_v |\alpha|_v = 1 \quad \text{for all } \alpha \in K^*, \tag{1}$$

where the product extends over all the valuations. Suppose S is a finite set of valuations of K which contains the archimedean valuations, $S \supset S_\infty$. Write U_S for the set of all $\alpha \in K^*$ with the property

$$|\alpha|_v = 1 \quad \text{for } v \notin S. \tag{2}$$

The set U_S forms a group under multiplication and the structure of U_S is well known;

$$U_S \cong T \times \mathbb{Z}^s. \tag{3}$$

Here T denotes a (finite) torsion group, consisting of the roots of unity inside K, also $|S| = s + 1$. We are going to assume throughout the paper that if $v \in S$ extends the valuation corresponding to the finite (rational) prime p then all of the w which extend p are also contained in S.

Let $c_0, \ldots, c_n$ denote non-zero algebraic numbers. There is a considerable amount of interest in the values taken by the expression

$$c_0x_0 + \ldots + c_nx_n, \tag{4}$$

where the x_i are allowed to run through the group of S-units of K. Several authors [4–7, 9, 10] have considered the qualitative theory, that is, where one sets the expression (4) to zero and considers the number of possible solutions. It becomes clear that one must forbid scaler multiples of (4). Thus, it is convenient to work inside the projective space $P^n(K)$. This is the set of all vectors $\underset{\sim}{x} = (x_0, \ldots, x_n)$, $x_i \in K$, where two such are identified if one is a non-zero multiple of the other.

In [4], it was proved that the equation

$$c_0 x_0 + \ldots + c_n x_n = 0, \tag{5}$$

has only a finite number of solutions $\underset{\sim}{x} = (x_0, \ldots, x_n) \in P^n(K)$, where each $x_i \in U_S$, provided there are no proper, vanishing subsums. That is,

$$c_{i_0} x_{i_0} + \ldots + c_{i_r} x_{i_r} \neq 0, \tag{6}$$

for all proper subsets $\{i_0, \ldots, i_r\} \subset \{0, \ldots, n\}$.

In the quantitative theory a very important role is played by the following definition. Given $\underset{\sim}{x} = (x_0, \ldots, x_n) \in P^n(K)$, define the projective height of $\underset{\sim}{x}$ to be

$$H(\underset{\sim}{x}) = \prod_v \max \{|x_0|_v, \ldots, |x_n|_v\}. \tag{7}$$

This is well-defined on $P^n(K)$ because it is independent of scalar multiples by the product formula (1). Observe that when each $x_i \in U_S$, one may rewrite (7) more simply as

$$H(\underset{\sim}{x}) = \prod_{v \in S} \max \{|x_0|_v, \ldots, |x_n|_v\}. \tag{8}$$

The aim of the theory is to make a comparison between the expression

$$N(\underset{\sim}{x}) = \prod_v |c_0 x_0 + \ldots + c_n x_n|_v, \tag{9}$$

and the height function $H(\underset{\sim}{x})$ in (8). To simplify the presentation of this paper suppose that $c_0 = 1$, $c_i \in \mathbb{Z}$, $i = 1, \ldots, n$ (thus, the product in (9) can run over the valuations of K). It follows from Evertse's Theorem 2 in [4] that, given $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that for all $\underset{\sim}{x} = (x_0, \ldots, x_n) \in P^n(K)$, $x_i \in U_S$, which satisfy condition (6), we have

$$N(\underset{\sim}{x}) > cH(\underset{\sim}{x})^{1-\varepsilon}. \tag{10}$$

In this paper we will introduce a new approach, which could be seen as a refinement of the result (10). It contains a technique not dissimilar from the Hardy–Littlewood method. The technique employs (10), also the theorem of Baker on linear forms in logarithms. The reader should note that (10) depends upon Schlickewei's generalisation of the Subspace Theorem of W. Schmidt.

The techniques we wish to introduce can perfectly well be exhibited in the case $n = 2$. Given the projective nature of the problem let $\underline{x} = (1, x_1, x_2)$, $x_i \in U_S$, so that (9) becomes,

$$N(\underline{x}) = \prod_{v \in S} |1 + c_1 x_1 + c_2 x_2|_v, \quad c_i \in \mathbb{Z}. \tag{11}$$

For the complex variable z, define

$$N(z) = \sum_{\underline{x}}{}' (\log N(\underline{x}))^{-z}, \tag{12}$$

where the "click" indicates that those $\underline{x}$ have been omitted, which give rise to a vanishing subsum. The result (10) guarantees that only finitely many $\underline{x}$ have $N(\underline{x}) = 1$ so they also can be omitted without harming the kind of result we seek. A similar assumption will prevail throughout the paper.

THEOREM: *The function $N(z)$ is analytic in the half-plane* $\mathrm{Re}(z) > 2s$, *where* $|S| = s + 1$; *it admits meromorphic continuation to* $\mathrm{Re}(z) > 2s - 2$ *where it is analytic apart from simple poles at* $z = 2s$ and $2s - 1$. *The pole at* $z = 2s$ *has residue*

$$\psi_1 \left(\frac{\omega_K}{R_K}\right)^2, \tag{13}$$

where ψ_1 is a positive rational number (a combinatorial constant, independent of the c_i) and ω_K, R_K denote (respectively) the number of roots of unity in K and the regulator of K. The pole at $z = 2s - 1$ has residue

$$\psi_2 \left(\frac{\omega_K}{R_K}\right)^2 + \psi_3 \left(\frac{\omega_K}{R_K}\right)^4 + \psi_4 \left(\frac{\omega_K}{R_K}\right)^2 \log\left(\prod_{v \in S} |c_1 c_2|_v\right). \tag{14}$$

This is not the first time that results from Diphantine Approximation have been used to give information about suitably defined complex series. For example, one can guarantee convergence of the Hasse–Weil ζ-function for an elliptic curve using the Hasse–Davenport bound for the number of solutions of equations over finite fields.

Our ideas are being presented in this way for three reasons. The first is that results such as these translate into results about counting functions via an appropriate Tauberian theorem. One such is the Delange-Ikahara Tauberian Theorem (see [8]). One may deduce the following.

COROLLARY: *Let* $N(q) = \# \{\underline{x}$: *condition* (6) *holds and* $N(\underline{x}) < q\}$. *Then the following asymptotic formula is valid,*

$$N(q) = \psi_5 (\log q)^{2S} + o((\log a)^{2S}), \quad \text{as } q \to \infty, \tag{15}$$

where ψ_5 is a constant which is independent of the c_i.

The second is that, in general application of the method, the "second pole" (in this case, the pole at $z = 2s - 1$) usually contains information of an arithmetic nature. This can be of quite a mysterious kind, see the example 2 in §3. The third reason is; we believe that the function $N(z)$ provides a suitable context for studying much harder and more general questions about the values taken by $N(\underline{x})$. For example, it is folklore that one does not expect $N(\underline{x})$ to take pure power values very often, up to a suitably defined notion of degeneracy. Our methods provide a context in which asymptotic results might be given.

The proof of the theorem depends, in part, upon a comparison made between $N(z)$ and the series defined by

$$H(z) = \sum_{\underline{x}} (\log H(\underline{x}))^{-z}, \quad z \in \mathbb{C}, \underline{x} = (1, x_1, x_2), \quad x_i \in U_S. \tag{16}$$

Write $H'(z)$ for the sub-series of those $\underline{x}$ which do *not* give rise to a vanishing sub-sum of (5). Note that the sub-series defined by

$$H(z) - H'(z), \tag{17}$$

where $\underline{x}$ runs over those $\underline{x}$ which *do* give rise to a vanishing subsum, is analytic in the half-plane $\text{Re}(z) > s$. Thus they are, in a very precise sense, negligible. The proof of this will become apparent when we compare the series (16), (17) with suitable integrals. See the remark after the proof of Corollary 1.

Note also that we could have introduced the refinement of letting x_1, x_2 run over fixed subgroups of U_S. Indeed, this is much closer to the approach in [3]. This too would have created notational difficulties but it does have an amusing effect upon the residue at $z = 2s - 1$. The factor $|c_1 c_2|_v$ becomes weighted according to the relative ranks of the subgroups of U_S.

The layout of the paper is as follows. In §1 a sequence of Lemmas is provided which express geometric results in terms of complex functions. In §2, some asymptotic formulae are obtained, based upon the geometric considerations of §1 and the results from the theory of Diophantine Approximation, to which reference has already been made. In §3, a short discussion is presented of how modifications to this general technique can be applied to some other classes of problems which give rise to an S-unit equation. The letters $\theta_1, \theta_2, \ldots$ will be used to denote a sequence of positive constants.

§1. Suppose $L_1^{v_1}, \ldots, L_1^{v_s}, L_2^{v_1}, \ldots, L_2^{v_s}$ are linear forms on $\mathbb{R}^s$ with $L_i^{v_j}$, $j = 1, \ldots, s$ linearly independent, for fixed i. Define $L_i^{v_{s+1}}$ by

$$\sum_{j=1}^{s+1} L_i^{v_j} = 0. \tag{18}$$

Extend $L_i^{v_j}$ to $V = \mathbb{R}^s \oplus \mathbb{R}^s$ by setting,

$$L_1^{v_j}(y_1) = L_1^{v_j}(y_1, y_2), \quad L_2^{v_j}(y_1, y_2) = L_2^{v_j}(y_2), \quad (y_1, y_2) \in \mathbb{R}^s \oplus \mathbb{R}^s. \tag{19}$$

Define: $M_j(y) = \max\{0, L_1^{v_j}(y), L_2^{v_j}(y)\}$,

$$M(y) = \sum_{j=1}^{s+1} M_j(y), \quad y \in \mathbb{R}^s \oplus \mathbb{R}^s,$$

$$I(z) = \int_V M(y)^{-z}\, \mathrm{d}y, \quad z \in \mathbb{C}, \tag{20}$$

where the integral is taken over $V = \mathbb{R}^{2s} - B$, B denoting an open ball about the origin.

To justify the introduction of these definitions, first observe that the function H, in (8), is invariant under the action of T in the following sense;

$$H(t_0 x_0, \ldots, t_n x_n) = H(x_0, \ldots, x_n) \text{ for all } t_i \in T. \tag{21}$$

Thus, in the sequel, we will work modulo T. Now write $V_{\mathbb{Z}}$ for the integer points in V. Choose bases for the S-units x_1, x_2. Then the logarithms of the absolute values of the units are linear forms in the exponents of the units. Identify these forms with the $L_i^{v_j}$ as above, (so that equation (18) becomes a restatement of the product formula (1)). Then log $H(\underline{x})$ becomes identified with $M(x)$, $x \in V_{\mathbb{Z}}$. We will tend to abuse this fact and switch between the two; a liberty granted by the decision to work modulo T.

LEMMA 1: *The function* $I(z)$ *is analytic for* $\mathrm{Re}(z) > 2s$ *with a meromorphic continuation to the whole plane. Here it is analytic apart from simple poles at* $z = 1, 2, \ldots, 2s$. *The residue of the pole at* $z = 2s$ *is* ψ_1/R_K^2.

Proof: This is trivial, the integral can be evaluated explicitly. The residue arises in the evaluation of the integral, as the Jacobian of a simple transformation. ■

Note that the dependence upon the choice of B is of a very trivial kind. If B is altered then the resultant change in $I(z)$ is to add to it an entire function.

COROLLARY 1: *The function* $N(z)$ *is analytic in the half-plane* $\mathrm{Re}(z) > 2s$.

Proof: By the result (10) it is sufficient to prove that the function $H(z)$ is analytic for $\mathrm{Re}(z) > 2s$. But this follows at once from Lemma 1 by the integral test. In fact, $H(z)$ (and therefore $N(z)$) forms an absolutely convergent series which is uniformly convergent on compact subsets of the half-plane $\mathrm{Re}(z) > 2s$. The analyticity follows from the usual theory about such series. ■

Note that the remark in the introduction concerning the half-plane of convergence for the series (17) is easily verified. A vanishing subsum can occur only in a "diagonal" way, that is, when $x_1 = \pm x_2$. Thus the integral test gives a comparison between (17) and an integral over $\mathbb{R}^s$.

LEMMA 2: *The function* $w_K^{-2}H(z) - I(z)$ *is analytic for* $\mathrm{Re}(z) > 2s - 1$.

Proof: Write c_x for the unit cube with center x. Then

$$I(z) = \sum_{x \in V_{\mathbb{Z}}} \int_{c_x} M(y)^{-z}\, \mathrm{d}y.$$

Now apply the mean-value theorem to each of the integrals. Also, observe the following quasi-linearity property of M:

$$M(x + \delta_x) = M(x) + o(1) \text{ uniformly for } \delta_x \in c_x. \tag{22}$$

This property (22) follows because the coefficients of the forms, also the length of δ_x are all uniformly bounded quantities. Thus, with our earlier identifications,

$$w_K^{-2}H(z) - I(z) = \sum_{x=\underline{x}} \{(\log H(\underline{x}))^{-z} - (M(x) + o(1))^{-z}\}.$$

Expand the inner bracket by the binomial theorem. The leading terms cancel and the lower order terms guarantee, together with the known convergence properties of $H(z)$, that $w_K^{-2}H(z) - I(z)$ is analytic for $\mathrm{Re}(z) > 2s - 1$. ∎

COROLLARY 2: *$H(z)$ has a meromorphic continuation to* $\mathrm{Re}(z) > 2s - 1$ *with only a simple pole at $z = 2s$, residue $\psi_1(w_K/R_K)^2$.* ∎

Next we will show that there is a massive non-uniformity in the distribution of the units. As usual, write $\underline{x} = (1, x_1, x_2)$ for a (projective) vector, where $x_1, x_2 \in U_S$. For $v \in S$, write

$$H_v(\underline{x}) = \max\{1, |x_1|_v, |x_2|_v\}. \tag{23}$$

Also, let $H_v^*(\underline{x})$ (respectively $H_v^{**}(\underline{x})$) denote the second (third) largest member of the set in (23). Repetitions are *not* excluded so we might have $H_v(x) = H_v^*(\underline{x})$ etc. For each $v \in S$ suppose constants $A_v > 0$, $B_v > 1$ are given. Define $U_0 = U_0(A_v, B_v)$ by

$$U_0 = \left\{\underline{x}\colon \forall v \in S \ \frac{H_v^*(\underline{x})}{H_v(\underline{x})} < A_v \exp(-B_v \log\log H(\underline{x}))\right\}, \tag{24}$$

$$H_0(z) = \sum_{\underline{x} \in U_0} (\log H(\underline{x}))^{-z}. \tag{25}$$

LEMMA 3: *$H(z) - H_0(z)$ is analytic in the half-plane* $\mathrm{Re}(z) > 2s - 1$.

The reader should notice the coincidence between the expression in (24) and the form of (38) in the statement of Theorem A.

Here, and in the sequel, we will use the following terminology. If $f(\sigma)$ and $g(\sigma)$ are two function of the real variable σ which take positive, real values then we will say that *g majorises f* if $f(\sigma) = O(g(\sigma))$, with the usual big O notation.

Proof of Lemma 3: Apply the integral test to the sum which defines $H(z) - H_0(z)$. Use the notation of Lemma 1 and write $\sigma = \text{Re}(z)$. Define M_j^* and M_j^{**} to satisfy

$$M_j^{**} \leqslant M_j^* \leqslant M_j \text{ for the set } \{0, L_1^{v_j}, L_2^{v_j}\}, \quad (j = 1, \ldots, s + 1).$$

If one of the inequalities in (24) is violated then (in "log space"),

$$\exists j, \quad M_j^{**} \leqslant M_j^* \leqslant M_j \leqslant M_j^* + \theta_1 + \theta_2 (\log M), \quad M = \sum_{j=1}^{s+1} M_j. \quad (26)$$

Thus the integral is majorised by a finite sum of integrals over subregions of V defined by inequalities (26). There are three cases to consider:

(a) $0 \leqslant L_2^{v_j} \leqslant L_1^{v_j} \leqslant L_2^{v_j} + \theta_1 + \theta_2 (\log M), \quad M_j = L_1^{v_j},$

(b) $L_2^{v_j} \leqslant 0 \leqslant L_1^{v_j} \leqslant \theta_1 + \theta_2 (\log M), \quad M_j = L_1^{v_j},$

(c) $L_2^{v_j} \leqslant L_1^{v_j} \leqslant 0 \leqslant L_1^{v_j} + \theta_1 + \theta_2 (\log M), \quad M_j = 0.$

For (b), write

$$0 \leqslant - L_2^{v_j} \leqslant L_1^{v_j} - L_2^{v_j} \leqslant - L_2^{v_j} + \theta_1 + \theta_2 (\log M).$$

Now make the change of variables;

$$-L_2^{v_j} = N_j^*, \quad L_1^{v_j} - L_2^{v_j} = N_j.$$

The region becomes larger if we replace M_j by $2N_j$. The (linear) change of variables maps the open ball about the origin into another open ball about the origin. We can enlarge the new open ball at the cost of adding an entire function. Similar remarks hold in case (c). Thus we compare with the integral of the function $(\Sigma N_j)^{-\sigma}$ over the region

$$\begin{gathered} 0 < \theta_3 \leqslant N_j^* \leqslant N_j \leqslant N_j^* + \theta_4 (\log N), \\ i \neq j, \quad 0 < \theta_3 \leqslant N_i^* \leqslant N_i, \quad N = \sum_{k=1}^{s} N_k. \end{gathered} \quad (27)$$

(The lower inequalities guarantee that an open ball has been removed about the origin).

To facilitate the integration write $N^* = N_j^* + \Sigma_{i \neq j} N_j$. Now the region becomes larger when we write, for some θ_5,

$$0 < \theta_3 \leqslant N_j^* \leqslant N_j \leqslant N_j^* + \theta_5 (\log N^*).$$

To see this, replace successively N_j by $N_j^* + \theta_4 (\log N)$ on the right hand side of (27). The result converges and we may take $\theta_5 = 2\theta_4$.

Perform the N_j integral first to give,

$$(1 - \sigma)^{-1}\{(N^* + \theta_5 \log N^*)^{1-\sigma} - N^{*1-\sigma}\}.$$

Expand the first bracket and cancel the leading terms to obtain

$$\theta_5 (\log N^*) N^{*-\sigma} + \text{lower order terms.}$$

Replace $\log N^*$ by $\theta_6 N^{*\varepsilon}$, $\varepsilon > 0$. The remaining integrations are straightforward and yield a function analytic in the half-plane $\sigma > 2s - 1 + \varepsilon$. ■

COROLLARY 3: *$H_0(z)$ admits meromorphic continuation to* $\mathrm{Re}(z) > 2s - 1$. *In this half-plane, the only singularity is a simple pole at $z = 2s$, with residue*

$$\psi_1 \left(\frac{w_K}{R_K}\right)^2. \quad ■$$

LEMMA 4: *The function $H(x)$ has meromorphic continuation to* $\mathrm{Re}(z) > 2s - 2$ *where it is analytic apart from simple poles at* $z = 2s$, $2s - 1$.

Proof: The idea is to reconsider the expression

$$w_K^{-2} H(z) - I(z) = \sum_{\underset{\sim}{x}} (\log H(\underset{\sim}{x}))^{-z} - \sum_{x \in V_{\mathbb{Z}}} \int_{c_x} M(y)^{-z}\, \mathrm{d}y. \tag{28}$$

As before, we abuse notation and identify $\underset{\sim}{x}$ with $x \in V_{\mathbb{Z}}$ and consequently $\log H(\underset{\sim}{x})$ with $M(x)$. (It will be convenient to alternate between the two). Change the variables in the integral in (28); write c for the unit cube with centre origin. Then

$$w_K^{-2}H(z) - I(z) = \sum_{\underline{x}=x\in V_{\mathbb{Z}}} \left\{(\log H(x))^{-z} - \int_c M(x+w)^{-z}\,\mathrm{d}w\right\}$$

$$= \sum_{\underline{x}\in U_0} + \sum_{\underline{x}\notin U_0}. \tag{29}$$

For $x \notin U_0$, use the mean-value theorem to give,

$$\sum_{\underline{x}\notin U_0} a(\underline{x})\,(\log H(\underline{x}))^{-z-1}.$$

where the $a(\underline{x})$ are uniformly bounded quantities. By Lemma 3, this series converges back to $\mathrm{Re}(z) > 2s - 2$.

For the sum $\Sigma_{\underline{x}\in U_0}$, use the fact that for $\underline{x} \in U_0$ with $\log H(\underline{x})$ sufficiently large, and for all $w \in c$,

$$M(x + y) = M(x) + M(w), \tag{30}$$

(and not just $M(x) + O(1)$ as in (23)). In other words, the same linear form is being used to define $M_j(x + w)$, $M_j(w)$ and $M_j(x)$.

Thus

$$\sum_{\underline{x}\in U_0} = \sum_{x=\underline{x}\in U_0} \left\{(\log H(\underline{x}))^{-z} - \int_c (M(x) + M(w))^{-z}\,\mathrm{d}w\right\}.$$

Expand the inner bracket and cancel the leading terms to obtain

$$\sum_{\underline{x}\in U_0} = H_0(z+1).\left\{-z\int_c M(w)\,\mathrm{d}w\right\}$$

$$+ \text{ something analytic for } \mathrm{Re}(z) > 2s - 2. \tag{31}$$

By Lemma 3, $H_0(z + 1)$ has meromorphic continuation to $\mathrm{Re}(z) > 2s - s$ with only a simple pole at $z = 2s - 1$. ■

COROLLARY 4: *The residue of the pole of $H(z)$ at $z = 2s - 1$ looks like,*

$$\theta_7\left(\frac{w_K}{R_K}\right)^2 + \theta_8\left(\frac{w_K}{R_K}\right)^4. \tag{32}$$

Proof: This arises by putting together the formulae (29) and (31). The first term appears in Corollary 2. The second comes from Corollary 2 and the evaluation of the integral inside the curly bracket in formula (31). ■

This non-uniformity of distribution persists. Given, as before, $A_v > 0$, $B_v > 1$ for $v \in S$, write $\tilde{U}_0$ for the set of all $\underline{x}$ such that,

$$\forall v \in S \quad \frac{H_v^{**}(\underline{x})}{H_v(\underline{x})} < A_v \exp\left(-B_v \log\log H(\underline{x})\right). \tag{33}$$

This gives rise to a series

$$\tilde{H}_0(z) = \sum_{\underline{x} \in \tilde{U}_0} (\log H(\underline{x}))^{-z}, \quad z \in \mathbb{C}. \tag{34}$$

LEMMA 5: *$H(z) - \tilde{H}_0(z)$ is analytic for* $\mathrm{Re}(z) > 2s - 2$.

Proof: Use the notation of the proof of Lemma 3. Write $\sigma = \mathrm{Re}(z)$. If one of the inequalities in (33) is violated then in "log space" we obtain an inequality, for some j (which might as well be 1),

$$M_1^{**} \leqslant M_1^* \leqslant M_1 \leqslant M_1^{**} + \theta_9 + \theta_{10} \log M.$$

As in the proof of Lemma 3, replace $M_1^* - M_1^{**}$ by N_1^*, $M_1 - M_1^{**}$ by N_1. Then the problem comes down to the study of

$$\int \left\{ \sum_{j=1}^{s} N_j \right\}^{-\sigma} \mathrm{d}N_1 \ldots \mathrm{d}N_1^* \ldots \mathrm{d}N_s^*$$

over the region:

$$0 < \theta_{11} \leqslant N_1^* \leqslant N_1 \leqslant \theta_{12} \log N, \quad N = \sum_{k=1}^{s} N_k, \tag{35}$$

$$i \neq 1 \quad 0 < \theta_{11} \leqslant N_i^* \leqslant N_i.$$

(The lower inequalities gurantee the removal of an open ball about the origin).

Suppose, without loss of generality, that $N_2 \geqslant \ldots \geqslant N_s$. Then

$$N_1 + \ldots + N_s \leqslant N_1 + (s-1)N_2 \leqslant (s-1)(N_1 + N_2).$$

Then the integral is majorised by

$$\int (N_1 + N_2)^{-\sigma},$$

over the region

$$0 < \theta_{11} \leqslant N_1^* \leqslant N_1 \leqslant \theta_{13} \log N_2, \tag{36}$$

$$N_3, \ldots, N_s, N_2^*, \ldots, N_s^* \leqslant N_2. \tag{37}$$

Perform the N_1 integral to give

$$(1 - \sigma)^{-1} \{(\theta_{13} \log N_2 + N_2)^{1-\sigma} - (N_1^* + N_2)^{1-\sigma}\}.$$

We may expand both brackets (the second because $N_1^* < \theta_{13} \log N_2$), and cancel the leading terms to give

$$(\theta_{13} \log N_2)N_2^{-\sigma} - N_1^* N_2^{-\sigma} + \text{lower order terms.}$$

Perform the N_1^* integral to yield

$$\begin{aligned} f(\sigma, N_2) \;=\; & (\theta_{13} \log N_2)^2 N_2^{-\sigma} - \theta_{11}(\theta_{13} \log N_2)^{-\sigma} \\ & - \tfrac{1}{2}(\theta_{13} \log N_2)^2 N_2^{-\sigma} + \tfrac{1}{2}\theta_{11}^2 N_2^{-\sigma}. \end{aligned}$$

Observe that $|f(\sigma, N_2)| < \theta_{14} N_2^{-\sigma+\varepsilon}$, $\varepsilon > 0$. Thus the integral is majorised by

$$\int N_2^{-\sigma+\varepsilon},$$

over the region $\theta_{11} \leqslant N_2^*, \ldots . N_s^*\ N_3, \ldots . N_s$. Perform the $N_2^*, \ldots, N_s^*$, $N_3, \ldots, N_s$ integrals to yield,

$$\int_{\theta_{11} \leqslant N_2 < \infty} (N_2 - \theta_{11})^{2s-3} N_2^{-\sigma+\varepsilon}.$$

Clearly this is well-defined for $\sigma > 2s - 2 + \varepsilon$. This completes the proof. ■

COROLLARY 5: *$\tilde{H}_0(z)$ has a meromorphic continuation to the half-plane* $\mathrm{Re}(z) > 2s - 2$ *where it is analytic apart from simple poles at* $z = 2s$ *and* $2s - 1$. *The residues are* (*respectively*) *as stated in Corollaries 2 and 4.* ■

§2. THEOREM A: *Suppose K is a number field and S is a finite set of valuations of K. Let U_S denote the group of S-units of K and suppose $\alpha_1, \ldots, \alpha_s$ are non-zero constants from K. Then for each $v \in S$ there are positive constants A'_v, B'_v with the property*

$$\forall x \in U_S \; |x - \alpha_v|_v > A'_v \exp(-B'_v \log \log H(\underline{x})), \tag{38}$$

provided $x \neq \alpha_v$, where $H(\underline{x})$ denotes the projective height of (7).

The proof of this theorem can be put together from the results in [1] and [11]; [1] for the archimedean $v \in S$ and [11] for the non-archimedean $v \in S$. The statement of (38) is readily transformed into one about linear forms in logarithms. The left hand side of (38) is greater than the same expression but with $\log |x|_v$ (respectively $\log |\alpha_v|_v$) replacing x (α_v). Also, observe that $\log H(\underline{x})$ is commensurate with the absolutely largest exponent of the units in U_S.

Note that for each $v \in S$ we have six possibilities for orderings, $H_v(\underline{x}) \geqslant H_v^*(\underline{x}) \geqslant H_v^{**}(\underline{x})$. As we run over $v \in S$, label all possibilities for all orderings by $i \in I$.

LEMMA 6: Recall the definitions of U_0 and $\tilde{U}_0$ from (24) and (33).

(i) $\log N(\underline{x}) = \log H(\underline{x}) + \theta_i + O(1/\log H(\underline{x}))$ for $i \in I$, $\underline{x} \in U_0$.

(ii) $\log N(\underline{x}) = \log H(\underline{x}) + O(\log \log H(\underline{x}))$ for all $\underline{x} \in \tilde{U}_0$.

(iii) $\log N(\underline{x}) > \theta_{15} \log H(\underline{x})$ for all $\underline{x}$ with (6) (no vanishing subsum).

Proof: (i) Given $i \in I$,

$$\log N(\underline{x}) = \sum_v \log |1 + c_1 x_1 + c_2 x_2|_v$$

$$= \sum_v \log H_v(\underline{x}) + \theta_i + \sum_v \log \left[1 + 0\left(\frac{H_v^*(\underline{x})}{H_v(\underline{x})}\right)\right],$$

where θ_i can represent one of 0, $\log |c_1|_v$ or $\log |c_2|_v$, depending upon which of 1, $|x_1|_v$ or $|x_2|_v$ is largest. Now (i) follows from the definition of U_0 and the fact that $B_v > 1$.

(ii) Given $v \in S$ suppose that $|c_1 x_1|_v \geqslant |c_2 x_2|_v \geqslant 1$, the other cases being entirely similar. Now we may suppose that

$$|x_1|_v^{-1} \leqslant A_v \exp(-B_v \log\log H(\underline{x})), \quad A_v > 0, \quad B_v > 1. \tag{39}$$

Then

$$\begin{aligned} |c_1 x_1 + c_2 x_2 + 1|_v &= |x_1|_v \left| c_1 + c_2 \frac{x_2}{x_1} + \frac{1}{x_1} \right|_v \\ &\geqslant |x_1|_v \left\{ \left| \frac{c_1}{c_2} + \frac{x_2}{x_1} \right|_v |c_2|_v - |x_1|_v^{-1} \right\} \\ &\geqslant |x_1|_v \{A_v' \exp(-B_v' \log\log H(\underline{x})) - |x_1|_v^{-1}\}, \end{aligned}$$

where the last inequality follows from Theorem A. Now choose $A_v = A_v'/2$, $B_v = B_v'$ in (39) (we may suppose that B_v' in the statement of Theorem A satisfies $B_v' > 1$). Then

$$|c_1 x_1 + c_2 x_2 + 1|_v \geqslant A_v'' H_v(\underline{x}) \exp(-B_v'' \log\log H(\underline{x})).$$

Now taking logs and summing over $v \in S$ gives the formula in (ii). (Note that the constants A_v, B_v in the definition (24) are chosen *after* the application of Baker's Theorem in the proof of Lemma 6. Thus, the choice of $\tilde{U}_0$ is determined by the proof of that Lemma).

(iii) This is a direct application of the result of Evertse in [4] (see (10)). ∎

Now we come to the proof of the main result. Begin by breaking up the sum which defines $N(z)$, according to Lemma 6. That is,

$$N(z) = \sum_{\underline{x}}{}' (\log N(\underline{x}))^{-z} = \sum_{\underline{x} \in U_0}{}' + \sum_{\underline{x} \in \tilde{U}_0 - U_0}{}' + \sum_{\underline{x} \notin \tilde{U}_0}{}' = S_1 + S_2 + S_3.$$

Note that S_1 requires further refinement according to Lemma 6(i). This depends upon the possible orderings of $|x_1|_v$, $|x_2|_v$, 1, $v \in S$. In fact they occur with equal frequency and we will assume this into the constant θ_i.

We will switch freely between the series $H(z)$ and $H'(z)$ (see (17)) having observed that they differ by a function which is analytic in the half-plane $\mathrm{Re}(z) > s$.

Write $\sigma = \text{Re}(z)$. For S_3 use part (iii) of Lemma 6 which gives $(\log N(\underline{x}))^{-1} = O((\log H(\underline{x}))^{-1})$. Then

$$|S_3| \leqslant \theta_{15}^{-\sigma} \sum_{\underline{x} \notin \bar{U}_0}' (\log H(\underline{x}))^{-\sigma}.$$

By Lemma 5 This converges for $\sigma > 2s - 2$; thus S_3 defines an analytic function in this half-plane.

For S_1 and S_2 expand the brackets

$$\left(\log H(\underline{x}) + \theta_i + \frac{a(\underline{x})}{\log H(\underline{x})}\right)^{-z}$$

and $(\log H(\underline{x}) + b(\underline{x}) \log \log H(\underline{x}))^{-z}$, (where $a(\underline{x}), b(\underline{x}) = O(1)$). Then

$$\begin{aligned} S_1 + S_2 = & \sum_{\underline{x} \in U_0}' (\log H(\underline{x}))^{-z} + \sum_{i \in I} \theta_i \sum_{\underline{x} \in U_0}' (\log H(\underline{x}))^{-z-1} \\ & + \sum_{\underline{x} \in U_0}' a(\underline{x})(\log H(\underline{x}))^{-z-2} + \sum_{\underline{x} \in \bar{U}_0 - U_0}' (\log H(\underline{x}))^{-z} \\ & + \sum_{\underline{x} \in \bar{U}_0 - U_0}' b(\underline{x}) \log \log H(\underline{x}) (\log H(\underline{x}))^{-z-1} \end{aligned}$$

By an earlier remark, we can happily remove the "clicks" from each of the summations; the only cost being the addition of a function which is analytic in the half-plane $\text{Re}(z) > s$.

The third term is analytic for $\text{Re}(z) > 2s - 2$ by comparison with $H(z + 2)$. In the fifth term replace $\log \log H(\underline{x})$ by $(\log H(\underline{x}))^{\varepsilon}$, $\varepsilon > 0$. Then Lemmas 3 and 5 apply to show that this series is analytic for $\text{Re}(z) > 2s - 2 + \varepsilon$.

Put the first and the fourth terms together to give

$$\sum_{\underline{x} \in \bar{U}_0} (\log H(x))^{-z}.$$

By Corollary 5 this series has a meromorphic continuation to $\text{Re}(z) > 2s - 2$ with simple poles at $z = 2s, 2s - 1$. For the second term apply Corollary 3 to obtain the meromorphic continuation to $\text{Re}(z) > 2s - 2$, with only a simple pole at $z = 2s - 1$.

To obtain the formulae (13) and (14) apply the Corollaries 2, 3, 4 and 5, regarding also the proof of part (i) of Lemma 6 which gives the form of the constant $\Sigma_{i\in I}\theta_i$ as a multiple of $\log(\Pi_{v\in S}|c_1c_2|_v)$. ■

§3. This technique is capable of great application. What follows are examples where modifications of the technique will give the result stated.

1. Suppose K is a finite extension of $\mathbb{Q}$ with ring of algebraic integers 0_K. Let $T: K \to \mathbb{Q}$ denote the trace map. One studies

$$T(z) = \sum_{\substack{x\in 0_K^* \\ T(x)\neq 0}} (\log T(x))^{-z}.$$

If 0_K^* has torsion free rank $r(>1$ say) then $T(z)$ is convergent for $\mathrm{Re}(z) > r$ with a meromorphic continuation to $\mathrm{Re}(z) > r - 2$. There are simple poles at $z = r, r - 1$ with residues:

$$\underline{z = r} \quad \theta_{16}\left(\frac{w_K}{R_K}\right), \quad \underline{z = r - 1} \quad \theta_{17}\left(\frac{w_K}{R_K}\right) + \theta_{18}\left(\frac{w_K}{R_K}\right)^2,$$

where the θ_i are positive rationals (combinatorial constants) and w_K, R_K denote respectively the number of roots of unity and the regulator of K.

More generally, one may study the solutions of a degenerate norm form equation in this way be examining the non-zero traces of the solutions (see [2] for details).

2. Given n a positive, square free integer, write $L = \mathbb{Q}(\zeta_n)$ for the nth cyclotomic field. Suppose $\mathbb{Q} \subset K \subset L$ with $\Gamma = \mathrm{Gal}(K|\mathbb{Q})$. It is known that the conjugates of $a = T_{L/K}(\zeta_n)$ under Γ form $\mathbb{Z}$-basis for the ring of integers of K. The set of all such generators is

$$T_{L/K}(\zeta_n)\cdot \mathbb{Z}\Gamma^*.$$

Here, one studies

$$N(z) = \sum_{x\in\mathbb{Z}\Gamma^*} (\log|N_{k|\mathbb{Q}}(a\cdot x)|)^{-z}, \quad z\in\mathbb{C}.$$

If r_Γ denotes the torsion free rank of $\mathbb{Z}\Gamma^*$ then $N(z)$ converges for $\mathrm{Re}(z) > r_\Gamma$ and admits a meromorphic continuation to $\mathrm{Re}(z) > r_\Gamma - 2$

with simple poles at $z = r_\Gamma, r_\Gamma - 1$. The pole at $z = r_\Gamma$ depends upon Γ only, while that at $z = r_\Gamma - 1$ looks like

$$\Gamma_1 + \Gamma_2 \log b,$$

where Γ_1, Γ_2 are (messy) group-ring constants. Here b is a positive integer which depends in a rather subtle way upon K. See [3] for more details.

3. Given $K|\mathbb{Q}$ and S one can also study $N(x) = \Pi_{v \in S}|1 + x|_v$ for $x \in U_S$. Here there are several advantages. Firstly, there is no serious non-vanishing sub-sum condition. Secondly there is no need to invoke the Subspace Theorem so the methods are entirely effective.

4. One of the motivations for the study of S-unit equations is the theory of linear recurrence sequences (see [9]). In the general case these provide a fertile testing ground for the techniques presented here. Not least because the coefficients c_i in (4) are no longer constants. This will be the subject of another paper.

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