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The hyperosculating points of surfaces in $\mathbb{P}^3$

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0. Introduction

Let $X \subset \mathbb{P}_k^3$ be a smooth surface over k, where k is an algebraically closed field of characteristic p, and $\mathscr{L} = \mathscr{O}(1)$ be the sheaf of linear sections of X. For any subspace $V' \subset V = H^0(X, \mathscr{O}(1))$, we have the following canonical homomorphisms

$$a_s: V_X' \to \mathscr{P}_X^s(1), \quad s = 0, 1, 2, \ldots,$$

where $\mathscr{P}_X^s(1)$ is the sheaf of the s^{th} principal parts of $\mathscr{L}$ on X. Many authors treated a_1 for the generic subspaces V' of dimensions 2 and 3. It resulted in a restudy for polar loci (see Kleiman [1] and Piene [2]). But it seems that there are few results about the higher a_s for the whole space V perhaps for lacking of "genericness". For the space V the relevant questions seem to be as follows.

(a) How to determine the least s which makes a_s injective? The existence of such an s was proved by Mount and Villamayor in [3] for a general case. In [5] we called such an s a coordinate gap number, denoted by $b_2(X)$.

(b) How to describe the scheme I, defined by the Fitting ideal $F^{n_{b_2}-4}$(coker a_{b_2}), where n_{b_2} is the rank of $\mathscr{P}_X^{b_2}$, In particular, if dim $I = 0$, how to get the 0-cycle expression of I, hence the number of its points (with corresponding multiplicities)? Indeed the points of I are the hyperosculating points in the sense of Pohl [4], which were called b_2-inflections in [6] so as to correspond to the curve case.

(c) What X has only a finite number of b_2-inflections? Corresponding to these questions, our main results are the following.

(a) $b_2 = 2$ or p^m for some $m \geqslant 1$; and $b_2 = p^m$ if and only if the defining polynomial for X can be written as

$$\sum_0^3 X_i F_i(X_0, X_1, X_2, X_3)^{p^m},$$

where p^m is the largest number for such a form.

(b) If X has a finite number of p^m-inflections then the 0-cycle expression of I is

$$(p^m + 1)[K_X]^2 + 4(p^m + 2)[K_X][\mathscr{L}] + 16[\mathscr{L}]^2 + p^{2m}c_2(\Omega_X) \cap [X],$$

and therefore the number of p^m-inflections is

$$((p^{2m} + p^m + 1)\deg^2 X - 4p^m(p^m + 1)\deg X + 6p^{2m})\deg X.$$

(c) If $b_2 = p^m$ and $\deg X = 1 + p^m$, then X has a finite number of p^m-inflections.

Furthermore, for a generic surface in $\mathbb{P}^3$ with $b_2 = p^m$ and $\deg X = 1 + kp^m$, the number of $b_2(X)$-inflections is finite.

Throughout this paper we always assume $p \neq 2$.

1. Coordinate gap number

First let us set up notations.

Let X be a non-degenerate, smooth surface in $\mathbb{P}^3$, and $\mathscr{L} = \mathscr{O}(1)$. We have the canonical homomorphisms

$$a_s\colon V_X \to \mathscr{P}^s_X(1), \quad s = 0, 1, 2, \ldots,$$

defined by

$$V_X = p_* q^* \mathscr{L} \to p_*(q^*\mathscr{L} \otimes \mathscr{O}_{\Delta^s}) = \mathscr{P}^s_X(1),$$

where $p, q\colon X \times X \to X$ denote the first and the second projections, respectively, Δ is the diagonal in $X \times X$, and $\mathscr{O}_{\Delta^s} = \mathscr{O}_{X \times X}/\mathscr{I}_\Delta^{s+1}$, $\mathscr{I}_\Delta$ is the ideal of definition for Δ in $X \times X$. For more details we refer to [7].

DEFINITION: The least integer s which makes a_s injective is said to be *coordinate gap number*, denoted by $b_2(X)$. Obviously, $b_2 \geqslant 2$.

PROPOSITION 1.1: *Let X be defined by the homogeneous polynomial $G(X_0, X_1, X_2, X_3)$. The necessary and sufficient conditions for $b_2 > 2$ are $G_{ij} = 0$ for all $0 \leqslant i, j \leqslant 3$, where $G_{ij} = \partial^2 G/\partial X_i \partial X_j$; in particular, we have $\deg G = 1 + kp$ for some integer $k \geqslant 1$.*

Proof: See Theorem 3.1 and its corollary in [5].

Then $b_2 > 2$ implies $p > 0$. Therefore, in what follows we always assume p is odd and positive since the case $b_2 = 2$ has been treated in [6].

LEMMA: *There is a coordinate system in* $\mathbb{P}^3$ *such that, if the polynomial for defining* X *is* $G = \Sigma_0^3 X_i F_i(X_0, \ldots, X_3)^{p^m}$, *then any two of the divisors* $[F_0]$, $[F_1]$, $[F_2]$, $[F_3]$ *on* X *have no common component.*

Proof: Since X is smooth, the linear system generated by $[F_i]$ is base-point free, so it determines a morphism

$$F = (F_0 : F_1 : F_2 : F_3) : X \to \check{\mathbb{P}}^3.$$

It is obvious that there are four linear independent planes in $\check{\mathbb{P}}^3$, such that any two of their intersections with $F(X)$ have no common component. Then pulling them back to X, we see that any two of $\Sigma_{j=0}^3 a_{ij} F_j$, $i = 0, \ldots, 3$, with $\det(a_{ij}) \neq 0$ for some constants a_{ij}, have no common component. Therefore, considering the transformations

$$T_i = \sum_0^3 a_{ij}^{p^m} X_j, \quad i = 0, \ldots, 3,$$

bringing G into G', we have

$$(F_i')^{p^m} = \partial G'/\partial T_i = \sum_j a_{ij}^{p^m} \partial G/\partial X_j = \left(\sum_j a_{ij} F_j\right)^{p^m},$$

and the proof is complete.

THEOREM 1.2: *If* $b_2 > 2$, *then* $b_2 = p^m$ *for some* m; *and* $b_2 = p^m$ *if and only if* $\deg X = 1 + kp^m$ *for some* $k \geq 1$, *and the polynomial for defining* X *can be written as*

$$G = \sum_0^3 X_i F_i(X_0, \ldots, X_3)^{p^m}$$

where $\deg F_i = k$ *and* p^m *is the largest exponential for such an expression.*

Proof: Since $b_2 > 2$, we have $G_{ij} = 0$; so we can write G in the above form, and the F_i's can be assumed to satisfy the assertion of the lemma since $b_2(X)$ and p^m are invariant under a nonsingular linear transform.

Let us calculate $b_2(X)$ as follows. Without loss of generality, we can assume $F_3 \neq 0$. On $U = (X_0 \neq 0) \cap (F_3 \neq 0)$ let us take $x = X_1/X_0$, $y = X_2/X_0, z = X_3/X_0$ to be the affine coordinates, and calculate $b_2(X)$ near a point $Q = (x_0, y_0, z_0) \in U$. For this purpose consider the completion R of the regular local ring $B = \mathcal{O}_{X,Q}$ with respect to its maximal ideal, which has a system of uniform parameters $\{x - x_0, y - y_0\}$. Since $(\mathcal{O}_X(1))_Q \cong \mathcal{O}_{X,Q}$ and $B \cong$ its image in R, we can identify V_Q with the image of $B \oplus Bx \oplus By \oplus Bz$ in $R \oplus Rx \oplus Ry \oplus Rz$, $\mathcal{P}^s_X(1)_Q$ with its image in P^s_R, and $a_{s,Q}$ with $d^s_R \cdot \phi$ by definition, where $\phi\colon B \oplus Bx \oplus By \oplus Bz \to B + Bx + By + Bz \subset R$, $d^s_R = \mathrm{id}_R + T^s_R\colon R \to P^s_R$, and T^s_R is the operator of s-truncated Taylor series (see [3]).

Now P^s_R is a free R-module with basis 1, $\mathrm{d}x$, $\mathrm{d}y$, . . . $(\mathrm{d}x)^s$, $(\mathrm{d}x)^{s-1}\mathrm{d}y$, . . . , $(\mathrm{d}y)^s$, and $a_{s,Q}(z) = d^s_R \cdot \phi(z)$ is uniquely expressed as

$$a_{s,Q}(z) \;=\; z + R_1\mathrm{d}x + R_2\mathrm{d}y + \sum_{i+j\geqslant 2}^{s} R_{ij}\mathrm{d}x^i\mathrm{d}y^j, \tag{$*$}$$

where R_i, $R_j \in R$, $s \gg 0$.

Moreover, on X we have

$$F_0^{p^m}(1, x, y, z) + xF_1^{p^m}(1, x, y, z) + \cdots \;=\; 0.$$

Setting $F_i(1, x, y, z) = f_i(x, y, z) \in B \subset R$, we get

$$\begin{aligned} 0 &= a_{s,Q}(f_0^{p^m} + xf_1^{p^m} + yf_2^{p^m} + zf_3^{p^m}) \\ &= (1 + T^s_R)(f_0^{p^m} + xf_1^{p^m} + yf_2^{p^m} + zf_3^{p^m}) \\ &= f_0^{p^m}(1 \otimes x, 1 \otimes y, 1 \otimes z) + (1 \otimes x)f_1^{p^m}(1 \otimes x, 1 \otimes y, 1 \otimes z) \\ &\quad + \cdots \\ &= f_0^{p^m}(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) + (x + \mathrm{d}x)f_1^{p^m}(x + \mathrm{d}x, \ldots) \\ &\quad + \ldots, \end{aligned}$$

where $\mathrm{d}x = 1 \otimes x - x \otimes 1$. Writing

$$\begin{aligned} f_i(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) &= f_i(x, y, z) + f_{i1}\mathrm{d}x + f_{i2}\mathrm{d}y + f_{i3}\mathrm{d}z \\ &\quad + f_{i11}\mathrm{d}x^2 + \ldots, \end{aligned}$$

we then have

$$0 = (f_0 + f_{01}\mathrm{d}x + f_{02}\mathrm{d}y + f_{03}\mathrm{d}z + \dots)^{p^m}$$
$$+ (x + \mathrm{d}x)(f_1 + f_{11}\mathrm{d}x + f_{12}\mathrm{d}y + f_{13}\mathrm{d}z + \dots)^{p^m}$$
$$+ (y + \mathrm{d}y)(f_2 + f_{21}\mathrm{d}x + f_{22}\mathrm{d}y + f_{23}\mathrm{d}z + \dots)^{p^m}$$
$$+ (z + \mathrm{d}z)(f_3 + f_{31}\mathrm{d}x + f_{32}\mathrm{d}y + f_{33}\mathrm{d}z + \dots)^{p^m}. \qquad (**)$$

Substituting (*) into (**) and identifying the coefficients of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}x^2$, $\mathrm{d}x\mathrm{d}y$, $\mathrm{d}y^2$, ... with zero, we obtain

(1) $R_1 f_3^{p^m} = -f_1^{p^m}$ and $R_2 f_3^{p^m} = -f_2^{p^m}$, so $R_1 = -f_1^{p^m}/f_3^{p^3}$ and $R_2 = -f_2^{p^m}/f_3^{p^m}$. Since $f_3^{p^m}$ is a unit in B, we have $R_i \in B$, $i = 1, 2$.
(2) $R_{ij} = 0$, where $1 \leqslant i + j < p^m$;
(3) $R_{i,p^m-i} = 0$, where $1 \leqslant i < p^m$;
(4) R_{0,p^m} and $R_{p^m,0}$ satisfy the following conditions:

$$R_1^{p^m}(f_{03}^{p^m} + xf_{13}^{p^m} + yf_{23}^{p^m} + zf_{33}^{p^m}) + (f_{01}^{p^m} + xf_{11}^{p^m} + yf_{21}^{p^m} + zf_{31}^{p^m})$$
$$+ f_3^{p^m} R_{p^m,0} = 0,$$
$$R_2^{p^m}(f_{03}^{p^m} + xf_{13}^{p^m} + yf_{23}^{p^m} + zf_{33}^{p^m}) + (f_{02}^{p^m} + xf_{12}^{p^m} + yf_{22}^{p^m} + zf_{23}^{p^m})$$
$$+ f_3^{p^m} R_{0,p^m} = 0.$$

Obviously, $R_{0,p^m}, R_{p^m,0} \in B = \mathcal{O}_{X,Q}$.

From (1)–(4), we see that $b_2(X) \geqslant p^m$. We conclude that $b_2 = p^m$. Otherwise, we would have $R_{0,p^m} = R_{p^m,0} = 0$ in $\mathcal{O}_{X,Q}$. Then there is a neighborhood $U' \subset U$ of Q such that R_{0,p^m} and $R_{p^m,0}$ vanish on U'. On the other hand, on U' we have

$$f_i(x, y, z) = F_i(1, X_1/X_0, \dots, X_3/X_0) = F_i(X_0, \dots, X_3)/X_0^k,$$

and

$$f_{ji}(x, y, z) = F_{ji}(X_0, \dots, X_3)/X_0^{k-1},$$

where $F_{ji}(X_0, \dots, X_3) = \partial F_j/\partial X_i$. Therefore it follows from (4) that on U' we have

$$F_i^{p^{2m}}\left(\sum_{j=0}^{3} X_j F_{j3}^{p^m}\right) - F_3^{p^{2m}}\left(\sum_{j=0}^{3} X_j F_{ji}^{p^m}\right) = 0, \quad i = 1, 2. \qquad (***)_i$$

But since X is irreducible, $(***)_i$, $(i = 1, 2)$ is valid on X. Moreover, let us verify that $(***)_i$ is valid for $i = 0, 3$: $(***)_3$ is an identity; as for $(***)_0$, multiplying X_i to $(***)_i$ $(i = 1, 2, 3)$, then summing them up, we obtain

$$\left(\sum_{i=1}^{3} X_i^{p^m} F_i^{p^{2m}}\right)\left(\sum_{j=0}^{3} X_j F_{ji}^{p^m}\right) - F_3^{p^m}\left(\sum_{i=1}\sum_{j=0} X_1^{p^m} X_j F_{ji}^{p^m}\right) = 0.$$

Since

$$\sum_{i=1} X_i^{p^m} F_i^{p^{2m}} = \left(\sum X_i F_i^{p^m}\right)^{p^m} = -X_0^{p^m} F_0^{p^m},$$

$$\sum_{i=1} X_i^{p^m} F_{ji}^{p^m} = \left(\sum X_i F_{ji}\right)^{p^m} = (kF_j - X_0 F_{j0})^{p^m} = k^{p^m} F_j^{p^m} - X_0^{p^m} F_{j0}^{p^m},$$

we have

$$-X_0^{p^m} F_0^{p^{2m}} \left(\sum_j X_j F_{ji}^{p^m}\right) - F_3^{p^m}\left(k^{p^m} \sum_j X_j F_j^{p^m} - X_0^{p^m} \sum_j X_i F_{j0}^{p^m}\right) = 0,$$

so $(***)_0$ is valid.

Now we claim that

$$\sum_{j=0}^{3} X_j F_{ji}^{p^m} = 0, \quad \text{for } i = 0, \ldots, 3. \tag{****}$$

In fact, from $(***)$ we see that $F_3^{p^m} = 0$ implies $F_i^{p^{2m}}(\Sigma_j X_j F_{ji}^{p^m}) = 0$; therefore, $[F_3^{p^{2m}}] \leqslant [F_i^{p^{2m}}(\Sigma_j X_j F_{ji}^{p^m})]$ as divisors on X. By the lemma, $[F_3^{p^{2m}}]$ and $[F_i^{p^{2m}}]$ $(i \neq 3)$ have no common components, hence $[F_3^{p^{2m}}] \leqslant [\Sigma_j X_j F_{ji}^{p^m}]$. On the other hand, since $\deg[F_3^{p^{2m}}] = (\deg X)(kp^{2m}) > \deg[\Sigma_j X_j F_{ji}^{p^m}] = (p^m(k-1)+1)\deg X$, we see that $(****)$ are valid on X, hence our claim since $\deg G > \deg(\Sigma_j X_j F_{ji}^{p^m})$.

The differentiating $(****)$ with respect to X_j, we obtain $F_{ji} = 0$ for all i, j, and therefore each F_i can be written as $H_i(X_0^p, \ldots, X_3^p)$, which contradicts our choice of p^m. The proof is complete.

2. p^m-inflections

Definition: Let the coordinate gap number of X be b_2. The Fitting ideal $F^{n_{b_2}-4}(\text{coker } a_{b_2})$ defines a subscheme I in X, called *b_2-inflection locus*. And a point of I is said to be a b_2*-inflection*; the multiplicity of an inflection is defined as the multiplicity of the point in I.

In [6] we have treated the 2-inflection locus already, at present we work only on $b_2 = p^m$, where $m \geqslant 1$. For this purpose we proceed in several steps as follows.

Step 1. The fundamental diagrams [7]

The following diagrams are fundamental for our purpose:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \Omega^1_X(\mathscr{L}) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathscr{K} & \longrightarrow & V_X & \xrightarrow{a_1} & \mathscr{P}^1_X(\mathscr{L}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle b_1} & & \\
0 & \longrightarrow & \mathscr{E} & \longrightarrow & V_X & \xrightarrow{a_0} & \mathscr{L} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \Omega^1_X(\mathscr{L}) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\qquad (\mathrm{A1})
$$

where a_0, a_1 are surjective since X is smooth; $\mathscr{K} = \ker a_1$ and $\mathscr{E} = \ker a_0$, hence the rows in (A1) are exact; moreover, the second column from the right is exact according to the structure of $\mathscr{P}^s_X(\mathscr{L})$, and by 5-lemma the second column from the left also is exact.

Similarly, we have

$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & s^2\Omega_X(\mathscr{L}) \\
 & & & & & & \downarrow \\
 & & & & V_X & \xrightarrow{a_2} & \mathscr{P}^2_X(\mathscr{L}) \\
 & & & & \| & & \downarrow{\scriptstyle b_2} \\
0 & \longrightarrow & \mathscr{K} & \xrightarrow{i} & V_X & \xrightarrow{a_1} & \mathscr{P}^1_X(\mathscr{L}) \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}
\qquad (\mathrm{A2})
$$

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & (S^m\Omega_X)(\mathscr{L}) \\ & & & & \downarrow \\ 0 & \longrightarrow & V_X & \xrightarrow{a_{pm}} & \mathscr{P}_X^{p^m}(\mathscr{L}) \\ & & \| & & \downarrow b_{pm} \\ & & V_X & \xrightarrow{a_{pm-1}} & \mathscr{P}_X^{p^m-1}(\mathscr{L}) \\ & & & & \downarrow \\ & & & & 0 \end{array} \qquad (\mathrm{A}_{p^m})$$

where S^m denotes the operator of the m-th symmetric product, Ω_X the sheaf of Kähler differentials.

In (A2), the homomorphism $a_2 \cdot i$ factors through $(S^2\Omega_X)(\mathscr{L})$. Moreover, $a_2 \cdot i(\mathscr{K}) = 0$. Otherwise, by the exactness of the right column, we would have that a_2 is injective, contradicting $p^m > 2$. Thus $\ker a_2 \cong \mathscr{K}$ and $im(a_2) \cong \mathscr{P}_X^1(\mathscr{L})$. Inductively proceeding up to (A_{p^m-1}), we have $\ker(a_{p^m-1}) \cong \mathscr{K}$ and $im(a_{p^m-1}) = \mathscr{P}_X^1(\mathscr{L})$. Finally, in (A_{p^m}), we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{K} & \xrightarrow{i} & (S^{p^m}\Omega_X)(\mathscr{L}) & \longrightarrow & \operatorname{coker} i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_X & \xrightarrow{a_{pm}} & \mathscr{P}_X^{p^m}(\mathscr{L}) & \longrightarrow & \operatorname{coker} a_{p^m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{P}_X^1(\mathscr{L}) & \xrightarrow{j} & \mathscr{P}_X^{p^m}(\mathscr{L}) & \longrightarrow & \operatorname{coker} j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since $h \cdot j = id.: \mathscr{P}_X^1(\mathscr{L}) \to \mathscr{P}_X^{p^m-1}(\mathscr{L}) \to \mathscr{P}_X^1(\mathscr{L})$, where $h = b_2 \cdot \cdots \cdot b_{p^m-1}$, the bottom row is splitting, and then (coker j) is locally free. By the property of Fitting ideal (see [9]), $F^{n_{p^m}-4}(\operatorname{coker} a_{p^m}) = F^{\binom{p^m+2}{2}-4}(\operatorname{coker} a_{p^m}) = F^{p^m}(\operatorname{coker} i)$. Locally, $\mathscr{K}$, Ω_X are free in an open set. Taking the open set as the one in Section 1, we may assume $\mathrm{d}x$, $\mathrm{d}y$ to be the basis of Ω_X, z to that of $\mathscr{K}$. Therefore

$$i(z) = \sum_{i+j=p^m} R_{ij}\mathrm{d}x^i\mathrm{d}y^j,$$

and $F^{p^m}(\operatorname{coker} i)$ is generated by $(R_{ij})_{i+j=p^m}$.

Step 2. Passing to E

Let $E = \operatorname{Proj} \bigoplus_{i \geqslant 0} S^i \mathscr{E}^\vee = \mathbb{P}(\mathscr{E}^\vee)$ and $Y = \mathbb{P}(\Omega_X(\mathscr{L})^\vee)$. We have a diagram

$$\begin{array}{ccc} & E & \xrightarrow{\ f\ } & Y \\ \swarrow^{\beta} & \downarrow{\scriptstyle \pi} & \swarrow_{\alpha} & \\ \check{\mathbb{P}}^3 & X & & \end{array} \tag{B}$$

where π and α are the structure morphism of E and Y, respectively; β is defined by the homomorphism: $0 \to \mathscr{E} \to V_X$ in (Al); f is a rational map determined by $\mathscr{E} \to \Omega_X(\mathscr{L})$ in (Al). Indeed, E is the incidence variety of X in $\mathbb{P}^3$, i.e., $E = \{(x, h) \in X \times \check{\mathbb{P}}^3 \mid x \in H$, where h is represented by $H\}$. Therefore E is the space of linear sections of X parameterized by $\check{\mathbb{P}}^3$, and β^{-1}(point) corresponds uniquely to a divisor $\in |[\mathscr{L}]|$.

On E, consider the composition

$$A : \pi^*\mathscr{K} \xrightarrow{\pi^* i} S^{p^m}\pi^*\Omega_X(\mathscr{L}) \longrightarrow S^{p^m}\Omega_E(\pi^*\mathscr{L}) \longrightarrow S^{p^m}\Omega_{E/\check{\mathbb{P}}^3}(\pi^*\mathscr{L}), \tag{1}$$

where $S^{p^m}\pi^*\Omega_X(\mathscr{L}) \to S^{p^m}\Omega_E(\pi^*\mathscr{L})$ is determined by the exact sequence

$$0 \to \pi^*\Omega_X \to \Omega_E \to \Omega_{E/X} \to 0; \tag{2}$$

$S^{p^m}\Omega_E \to S^{p^m}\Omega_{E/\check{\mathbb{P}}^3}$ is determined by the exact sequence

$$\beta^*\Omega_{\check{\mathbb{P}}^3} \to \Omega_E \to \Omega_{E/\check{\mathbb{P}}^3} \to 0. \tag{3}$$

The left side of (3) is injective at the smooth points of β. The scheme of singular locus of β is exactly that of singular points of the fiber of β (cf. [1], IIID), so it turns out to be $\mathbb{P}(\check{\mathscr{K}})$, i.e., the subscheme $\{(Q \in X$, the tangent plane section passing $Q)\} \subset E$, in other words, the projective conormal bundle of X in $\mathbb{P}^3$. Write $W = E - \mathbb{P}(\check{\mathscr{K}})$.

Now we take an open set $U \subset X$ small enough that $\{1, x, y, z\}$ is a basis of V_X when restricted in U, i.e., $V_U = \{\eta + \lambda x + \mu y + \nu z\}$; besides, we can assume that $\{x, y, z\}$ is a basis of $\mathscr{E}|_U$, and z is a basis of $\mathscr{K}|_U$; it follows that (x, y) is the local coordinate in U, $(\eta, \lambda, \mu, \nu)$ is the coordinate in $\pi^{-1}(U)$.

$\Omega_{E/\check{\mathbb{P}}^3}$ is locally free in $W \cap U$ with rank 1. Identifying E with a subscheme of $X \times \check{\mathbb{P}}^3 = \mathbb{P}(\check{V}_X)$, we have an exact sequence

$$\mathcal{O}(-E)|_E \xrightarrow{\ \psi\ } \Omega_{X \times \check{\mathbb{P}}^3/\check{\mathbb{P}}^3}|_E \to \Omega_{E/\check{\mathbb{P}}^3} \to 0$$

$\mathcal{O}_{X \times \check{\mathbb{P}}^3}(-E)$ is the ideal of definition for E in $X \times \check{\mathbb{P}}^3$, which can locally be expressed as $\eta + \lambda x + \mu y + \nu z$, hence $\psi(\eta + \lambda x + \mu y + \nu z) = \lambda\, dx + \mu dy + \nu\, dz$; since $a_1(z) = 0$ on E, we have $\Omega_{E/\check{\mathbb{P}}^3} = \mathcal{O}_E dx + \mathcal{O}_E dy \bmod (\lambda\, dx + \mu\, dy)$.

It follows from the discussion above that $A(z) = (\Sigma_{i+j=p^m} R_{ij}\lambda^j\mu^i t)$ and t is a basis of $\Omega_{E/\check{\mathbb{P}}^3}$, hence $F^0(\text{coker } A)$ is generated by $(\Sigma_{i+j=p^m} R_{ij}\lambda^j\mu^i)$ on $W \cap U$; therefore $F^0(\text{coker } A)$ defines a scheme J' with $\text{codim}_W J' = 1$.

By sequences (1) and (3), and applying Porteous' formula, we have

$$F^0(\text{coker } A)|_W \cong (\pi^*\mathscr{K} \otimes \Omega_{E/\check{\mathbb{P}}^3}^{-p^m} \otimes \pi^*\mathscr{L}^{-1})_W$$

$$\cong (\pi^*\mathscr{K} \otimes \Lambda^4\Omega_E^{-1} \otimes \Lambda^3\beta^*\Omega_{\check{\mathbb{P}}^3})^{p^m} \otimes \pi^*\mathscr{L}^{-1})_W.$$

Since $\Lambda^4\Omega_E =$ the canonical sheaf K_E of E and $\beta^*\Omega_{\check{\mathbb{P}}^3} \cong \beta^*\mathcal{O}(-4) \cong \mathcal{O}_E(-4)$, finally we have

$$F^0(\text{coker } A)|_W \cong (\pi^*\mathscr{K} \otimes K_E^{-p^m} \otimes \mathcal{O}(-4p^m) \otimes \pi^*\mathscr{L}^{-1})_W.$$

Step 3. Passing to Y

Let us consider the diagram (B) and the rational map f once more. The homomorphism $\mathscr{E} \to \Omega_X(\mathscr{L})$ can locally be expressed as $\lambda x + \mu y + \nu z \mapsto \lambda x + \mu y$, hence the map is a correspondence with $(x, y; \lambda, \mu, \nu) \mapsto (x, y; \lambda, \mu)$ and defined on $W = E - \mathbb{P}(\check{\mathscr{K}})$. The closure J of the scheme-theoretic inverse image of J' in Y is defined by ideal $(\Sigma_{i+j=p^m} R_{ij}\lambda^j\mu^i)$, the same form for defining J'; in other words, the scheme-theoretic image of J coincides with J' on W, i.e., $f^{-1}(J) = J'$.

In virtue of $\text{codim}_E \mathbb{P}(\check{\mathscr{K}}) = 2$ on E and well-known facts (e.g., see [8]), we have $\text{Pic } W \cong \text{Pic } E \cong \pi^*\text{Pic } X + \{\mathcal{O}_E(1)\}$; moreover, $\text{Pic } Y = \alpha^*\text{Pic } X + \{\mathcal{O}_Y(1)\}$ on Y. Therefore, it follows from $f^*(\alpha^*\text{Pic } X) = \pi^*\text{Pic } X$ and $f^*\mathcal{O}_Y(1) = \mathcal{O}_E(1)$ that f^* is an isomorphism (noting that α^* and π^* are splitting).

On the other side, we have the following exact sequences on E:

$$0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_E \longrightarrow \Omega_{E/X} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow \Omega_{E/X} \longrightarrow \pi^*\check{\mathscr{E}}(-1) \longrightarrow \mathcal{O}_E \longrightarrow 0, \tag{5}$$

and

$$0 \longrightarrow \mathscr{E} \longrightarrow V_X \longrightarrow \mathscr{L} \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{K} \longrightarrow V_X \longrightarrow \mathscr{P}^1_X(\mathscr{L}) \longrightarrow 0, \tag{A1}$$

$$o \longrightarrow \Omega_X(\mathscr{L}) \longrightarrow \mathscr{P}^1_X(\mathscr{L}) \longrightarrow \mathscr{L} \longrightarrow 0.$$

Therefore, we get

$$\mathscr{K} \cong (K_X \otimes \mathscr{L}^3)^{-1},$$

$$K_E \overset{(4)}{\cong} \pi^* K_X \otimes \Lambda^2 \Omega_{E/X} \overset{(5)}{\cong} \pi^* K_X \otimes \pi^* \Lambda^3 \check{\mathscr{E}} \otimes \mathscr{O}_E(-3)$$

$$\cong \pi^* K_X \otimes \pi^* \mathscr{L} \otimes \mathscr{O}_E(-3)$$

hence

$$\{f^*(\alpha^* K_X^{-(p^m+1)} \otimes \alpha^* \mathscr{L}^{-(p^m+4)} \otimes \mathscr{O}_Y(-p^m))\}_W$$

$$= \{\pi^* K_X^{-(p^m+1)} \otimes \pi^* \mathscr{L}^{-(p^m+4)} \otimes \mathscr{O}_E(-p^m)\}_W \cong F^0(\text{coker } A)|_W$$

This shows that the ideal of definition for J is isomorphic to

$$\alpha^* K_X^{-(p^m+1)} \otimes \alpha^* \mathscr{L}^{-(p^m+4)} \otimes \mathscr{O}_Y(-p^m).$$

Step 4. Passing to Z

Let $Z = \mathbb{P}(S^{p^m} \Omega_X(\mathscr{L})^\vee)$. We have an immersion $g\colon Y \to Z$ over X, defined by $\mathscr{O}_Y(p^m)$:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & X & \end{array}$$

Alternatively g is defined by the following composition of homomorphisms

$$\bigoplus_{i \geqslant 0} S^i(S^{p^m} \Omega_X(\mathscr{L})^\vee) \longrightarrow \bigoplus_{i \geqslant 0} S^{ip^m} \Omega_X(\mathscr{L})^\vee \longrightarrow \bigoplus_{j \geqslant 0} S^j \Omega_X(\mathscr{L})^\vee.$$

Locally, it turns out to be the Veronese morphism (cf. [10]).

Let us show that there exists a unique divisor D on Z such that $D \cap Y = J$. In fact, by Step 1, for every point $Q \in X$ there is a neighborhood U of Q such that J is defined by $(\Sigma_{i+j=p^m} R_{ij}\lambda^j\mu^i)$ on $\alpha^{-1}(U)$, so $g^*: \alpha^{-1}(U) \to \gamma^{-1}(U)$ can be expressed as $(x, y; \lambda, \mu) \mapsto (x, y; \lambda^{p^m}, \lambda^{p^m-1}\mu, \ldots, \mu^{p^m})$, and therefore $\Sigma R_{ij}\lambda^j\mu^i \mapsto \Sigma R_{ij}T_{ij}$ is an 1–1 correspondence between the set of p^m-forms in λ, μ and the set of 1-forms in T_{ij}. Then, on each $\alpha^{-1}(U)$ there exists such a D and these forms are invariant under linear transformations, so we can piece them together to get such an effective divisor D on Z as desired.

Now, the ideal $\mathcal{O}_Z(-D)$ can be determined as follows. Since $g^*(\gamma^* \operatorname{Pic} Z) = \alpha^* \operatorname{Pic} Y$ and $g^*\mathcal{O}_Z(1) = \mathcal{O}_Y(p^m)$, we see that $g^*: \operatorname{Pic} Z \to \operatorname{Pic} Y$ is injective by the same reasons as shown in Step 1; moreover

$$g^*(\gamma^*K_X^{-(p^m+1)} \otimes \gamma^*\mathscr{L}^{-(p^m+4)} \otimes \mathcal{O}_2(-1))$$

$$= \alpha^*K_X^{-(p^m+1)} \otimes \alpha^*\mathscr{L}^{-(p^m+4)} \otimes \mathcal{O}_Y(-p^m) = \mathcal{O}_Y(-J').$$

Consequently

$$\mathcal{O}_Z(-D) = \gamma^*K_X^{-(p^m+1)} \otimes \gamma^*\mathscr{L}^{-(p^m+4)} \otimes \mathcal{O}_Z(-1).$$

LEMMA: $F^{p^m}(\operatorname{coker} i) = F^{p^m-1}(\Omega_{D/X})$.

Proof: Consider the following exact sequence

$$\mathcal{O}_D(-D) \xrightarrow{\delta} \Omega_{Z/X}|_D \longrightarrow \Omega_{D/X} \longrightarrow 0,$$

where $\mathcal{O}_D(-D)$ is locally generated by $\Sigma R_{ij}T_{ij}$. In each $\{T_{rs} \neq 0\}$ let $t_{ij} = T_{ij}/T_{rs}$. Then $\delta(\Sigma R_{ij}t_{ij}) = d_{Z/X}(\Sigma R_{ij}t_{ij}) = \Sigma R_{ij}dt_{ij}$. Consequently $F^{p^m-1}(\Omega_{D/X})$ is generated by $(R_{ij})_{i+j=p^m}$ which is just $\gamma^*F^{p^m}(\operatorname{coker} i)$, as shown in Step 1.

Note that the lemma shows that $F^{p^m-1}(\Omega_{D/X})$ defines the scheme $\gamma^{-1}(I)$ by the properties of Fitting ideals (e.g. see [9]); this is a new starting-point of our argument.

Step 5. Pulling back to Y

Let $\bar{g} = g|_W: J \to D$. Then we have the exact sequence

$$0 \to \mathscr{I} \to \bar{g}^*\Omega_{D/X} \to \Omega_{J/X} \to 0 \tag{6}$$

which defines $\mathscr{J}$ as the kernel. On the other hand, in the following exact sequence $\mathcal{O}(-J)|_J \xrightarrow{\sigma} \Omega_{Y/X}|_J \to \Omega_{J/X} \to 0$ we have $\sigma = 0$. In fact, from Section 1, (1)–(4), we have locally $\mathcal{O}(-J) = (R\lambda^{p^m} + S\mu^{p^m})$, where $R = R_{0,p^m}$, $S = R_{p^m,0}$, so $\sigma(R(x, y)\lambda^{p^m} + S(x, y)\mu^{p^m} = d_{Y/X}(R\lambda^{p^m} + S\mu^{p^m}) = 0$. Consequently $\Omega_{Y/X}|_J \cong \Omega_{J/X}$, thus it is a locally free sheaf of rank 1. Using (6) we then have

$$\bar{g}^* F^{p^m-1}(\Omega_{D/X}) \cong F^{p^m-1}(\bar{g}^*\Omega_{D/X}) \cong F^{p^m-2}(\mathscr{J}). \tag{7}$$

Consider the fundamental diagrams for $Z \to X$ and $\mathcal{O}_Z(1)$. In a similar way to Step 1 we have

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \Omega_{Z/X}(1) & & \\
 & & & & & & \downarrow & & \\
 & & 0 & \longrightarrow & \gamma^*\gamma_*\mathcal{O}_Z(1) & \xrightarrow{a'_1} & \mathscr{P}^1_{Z/X}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \Omega_{Z/X}(1) & \xrightarrow{i'} & \gamma^*S^{p^m}\Omega_X(\mathscr{L})^\vee & \xrightarrow{a'_0} & \mathcal{O}_Z(1) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{A1$'$}$$

and (A2)$'$, (A3)$'$, . . . , (Ap^m)$'$.

Let $t_{ij} = T_{ij}/T_{0p^m}$ be the affine coordinates on $\gamma^{-1}(U) = U \times \mathbb{P}^{p^m}$. Then $\{t_{ij}\}$ is a basis of $\Omega_{Z/X}(1)$.

In (As)$'$ for any s, the homomorphism

$$a'_s \cdot i' : \Omega_{Z/X}(1) \to \mathscr{P}^s_{Z/X}(1)$$

factors through $\mathscr{H}_s(1)$, the kernel of the following composition

$$\mathscr{P}^s_{Z/X}(1) \xrightarrow{b'_s} \mathscr{P}^{s-1}_{Z/X}(1) \xrightarrow{b'_{s-1}} \cdots \xrightarrow{b'_1} \mathcal{O}_Z(1).$$

Twisting $a'_s \cdot i'$ with $\mathcal{O}(-1)$, we obtain

$$v_s : \Omega_{Z/X} \to \mathscr{H}_s.$$

In addition, we have the natural maps

$$g^* \mathcal{P}^s_{Z/X} \to \mathcal{P}^s_{Y/X}$$

(see [11]), which can be locally expressed as $(\mathrm{d}t_{ij})^r \mapsto T^r(\lambda^i)$ for $r \leqslant s$, where $T^r(\lambda^i) = (\lambda + \mathrm{d}\lambda)^i - \lambda^i$ is the r-truncated Taylor series of λ^i.

Let $\mathcal{T}_s$ be the kernel of

$$\mathcal{P}^s_{Y/X} \to \mathcal{P}^{s-1}_{Y/X} \to \cdots \to \mathcal{O}_Y.$$

Then we have $u_s\colon g^*\mathcal{H}_s \to \mathcal{T}_s$. Setting $\zeta_s = u_s \cdot g^* v_s$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & (g^*\Omega_{Z/X})|_J & \xrightarrow{\zeta} & \mathcal{T}_{p^m-1}|_J & & \\
 & & & & \| & & \downarrow \phi & & \\
 & & & & (g^*\Omega_{Z/X})|_J & \longrightarrow & \Omega_{Y/X}|_J & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & (g^*\Omega_{D/X})|_J & \longrightarrow & \Omega_{J/X} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where $\zeta = \zeta_{p^m-1}|_J$, ϕ is a self-evident surjective map, and the bottom row is just (6). Let us show that ζ is also surjective. In fact, rank $\Omega_{Z/X} =$ rank $\mathcal{T}_{p^m-1} + 1 = p^m$, and locally, a basis of $g^*\Omega_{Z/X}$ is $\{t_{ij}\}$, where $i + j = p^m$ and $i \neq 0$, a basis of $\mathcal{T}_{p^m-1}$ is $\{\mathrm{d}\lambda, (\mathrm{d}\lambda)^2, \ldots, (\mathrm{d}\lambda)^{p^m-1}\}$, moreover, $\zeta(t_{ij}) = T^{p^m-1}(\lambda^i) = \Sigma_{n\neq i}\binom{i}{n}\lambda^n \mathrm{d}\lambda^{i-n}$ for $i < p^m$ and $\zeta(t_{p^m,0}) = 0$, and therefore the matrix of ζ with respect to these bases is

$$\begin{pmatrix} 1 & & & \\ & 1 \, . & 0 & \\ & & \ddots & \\ & * & 1 & \\ & & & 0 \end{pmatrix}$$

Consequently ζ is surjective.

Let $\mathcal{U}$ be the kernel of ϕ, $\mathcal{N}$ the kernel of $\bar{g}^*\Omega_{Z/X}|_J \to \bar{g}^*\Omega_{D/X}$, and $\mathcal{S}$ the kernel of ζ. Then we have a homomorphism $\tau\colon \mathcal{S} \to \mathcal{J}$ with its kernel $\mathcal{M}$ and cokernel $\mathcal{K}$. Now, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{N} & \xrightarrow{\xi} & \mathscr{U} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I} & \longrightarrow & g^*\Omega_{Z/X}|_J & \xrightarrow{\zeta} & \mathscr{T}_{p^m-1}|_J & \longrightarrow & 0 \\
 & & | & & \| & & \downarrow & & \\
 & \tau & | & & g^*\Omega_{Z/X}|_J & \longrightarrow & \Omega_{Y/X}|_J & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{J} & \longrightarrow & g^*\Omega_{D/X}|_J & \longrightarrow & \Omega_{J/X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathscr{X} & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Moreover from the exact sequence

$$\mathcal{O}_D(-D) \xrightarrow{\delta} \Omega_{Z/X}|_D \longrightarrow \Omega_{D/X} \longrightarrow 0$$

we known that $\bar{g}^*\mathcal{O}_D(-D) = \mathcal{O}_J(-J) \to \mathscr{N}$ is surjective, hence ξ is zero. Indeed, it follows from Step 3 that $\mathcal{O}_J(-J)$ is generated locally by $R\lambda^{p^m} + S\mu^{p^m}$, so $\zeta(R\lambda^{p^m} + S\mu^{p^m}) = 0$, and therefore $\mathscr{U} = \mathscr{X}$ and $\mathscr{M} = \mathscr{N}$.

In the diagram above, let us break up the left column into two parts:

$$0 \to \operatorname{Im} \tau \to \mathscr{J} \to \mathscr{X} \to 0, \tag{a}$$

$$0 \to \mathscr{M} \to \mathscr{I} \to \operatorname{Im} \tau \to 0, \tag{b}$$

where the sheaf $\mathscr{X}$ is locally free, and (b) can be replaced by

$$\mathcal{O}_J(-J) \to \mathscr{I} \to \operatorname{Im} \tau \to 0. \tag{b$'$}$$

Then we can prove the following proposition:

PROPOSITION 2.1: $\alpha^{-1}(I)$ *is defined by* $F^0(\operatorname{Im} \tau)$.

Proof: From the lemma we see that $\gamma^{-1}(I)$ is defined by $F^{p^m-1}(\Omega_{D/X})$, so $\alpha^{-1}(I)$ is defined by $\bar{g}^*F^{p^m-1}(\Omega_{D/X}) = F^{p^m-1}(\bar{g}^*\Omega_{D/X})$. Moreover, applying F^{p^m-1} to the bottom row of the diagram above and noting rank $\Omega_{J/X} = 1$, we have $F^{p^m-1}(\bar{g}^*\Omega_{D/X}) = F^{p^m-2}(\mathscr{J})$. Now applying F^{p^m-2} to (a) and noting rank $\mathscr{X} = p^m-2$, we obtain what we want.

THEOREM 2.2: *If X has only a finite number of p^m-inflections, then, as a cycle,*

$$[I] = (p^m + 1)[K_X]^2 + 4(p^m + 2)[K_X][\mathscr{L}] + 16[\mathscr{L}] + p^{2m}c_2(\Omega_X) \cap [X],$$

where $[K_X]$ denotes the divisor associated with K_X, etc., $[K_X]^2$ is the intersection of divisors and c_i is the operator of i-th Chern class.

Consequently

$$\# I = \deg I = ((p^{2m} + p^m + 1) \deg^2 X - 4p^m(p^m + 1) \deg X + 6p^m) \deg X.$$

Proof: By our hypothesis, $\operatorname{codim}_J \alpha^{-1}(I) = 1$. Since (b)′ is the resolution of (Im τ), it follows from Proposition 2.1, that I is exactly the degeneracy locus of $\mathcal{O}_J(-J) \to \mathscr{I}$ with rank $\mathscr{I} = 1$, hence it has a "correct dimension". Then applying Porteous' formula to (b)′ (cf. [8] in the simplest case), we have

$$[\alpha^{-1}(I)] \sim (c_1(\mathcal{O}(J)) + c_1(\mathscr{I})) \cap [J]$$
$$= (c_1(\mathcal{O}(J)) + c_1(\mathscr{I}))c_1(\mathcal{O}(J)) \cap [Y].$$

Let us calculate $c_1(\mathscr{I})$. Since

$$0 \to \mathscr{I} \to (g^*\Omega_{Z/X})|_J \to \mathscr{T}_{p^m-1}|_J \to 0$$

is exact, we have

$$c_1(\mathscr{I}) = (c_1(g^*\Omega_{Z/X}) - c_1(\mathscr{T}_{p^m-1}))_J.$$

From

$$0 \to \Omega_{Z/X} \to S^{p^m}\Omega_X(\mathscr{L})^\vee(-1) \to \mathcal{O}_Z \to 0$$

and the splitting principle, we also have

$$g^*c_1(\Omega_{Z/X})|_J = (-\tfrac{1}{2}p^m(p^m + 1)\alpha^*c_1(\Omega_X) - p^m(p^m + 1)\alpha^*c_1(\mathscr{L}) - p^m(p^m + 1)c_1(\mathcal{O}_Y(1)))_J.$$

As for $c_1(\mathcal{T}_{p^m-1})$, using the sequence for defining $\mathcal{T}_{p^m-1}$ and the fundamental diagrams of Y/X, we obtain

$$\begin{aligned} c_1(\mathcal{T}_{p^m-1}) &= \tfrac{1}{2}p^m(p^m - 1)c_1(\Omega_{Y/X}) \\ &= -\tfrac{1}{2}p^m(p^m - 1)(\alpha^*c_1(\Omega_X) + 2\alpha^*c_1(\mathcal{L}) + 2c_1(\mathcal{O}_Y(1))). \end{aligned}$$

Moreover, by the last formula in Step 3, we see

$$c_1(\mathcal{O}(J)) = (p^m + 1)c_1(\alpha^*K_X) + (p^m + 4)c_1(\mathcal{L}) + p^m c_1(\mathcal{O}_Y(1)),$$

hence

$$\begin{aligned} [\alpha^{-1}(I)] &= (\alpha^*c_1(\Omega_X) + (4 - p^m)\alpha^*c_1(\mathcal{L}) \\ &\quad - p^m\alpha^*c_1(\mathcal{O}(1)))((p^m + 1)\alpha^*c_1(\Omega_X) \\ &\quad + (p^m + 4)\alpha^*c_1(\mathcal{L}) + p^m c_1(\mathcal{O}(1))) \cap [Y] \\ &= ((p^m + 1)\alpha^*c_1(\Omega_X)^2 + (-p^{2m} + 4p^m + 8)\alpha^*c_1(\Omega_X)\alpha^*c_1(\mathcal{L}) \\ &\quad + (16 - p^{2m})\alpha^*c_1(\mathcal{L})^2 - p^{2m}\alpha^*c_1(\Omega_X)c_1(\mathcal{O}(1)) \\ &\quad - 2p^{2m}c_1(\mathcal{O}(1))\alpha^*c_1(\mathcal{L}) - p^{2m}c_1(\mathcal{O}(1))^2) \cap [Y]. \end{aligned}$$

Since α is flat we have $[\alpha^{-1}(I)] = \alpha^*[I]$. So from the projection formula $\alpha_*(c_1(\mathcal{O}(1)) \cap \alpha^*[I]) = [I]$ we see

$$\begin{aligned} [I] &= ((p^m + 1)c_1(\Omega_X)^2 + (-p^{2m} + 4p^m + 8)c_1(\Omega_X)c_1(\mathcal{L}) \\ &\quad + (16 - p^{2m})c_1(\mathcal{L})^2 - p^{2m}s_1(\Omega_X(\mathcal{L}))c_1(\Omega_X) \\ &\quad - 2p^{2m}s_1(\Omega_X(\mathcal{L}))c_1(\mathcal{L}) - p^{2m}s_2(\Omega_X(\mathcal{L}))) \cap [X], \end{aligned}$$

where s_i is the operator of the i-th Segre class, hence

$$\begin{aligned} s_1(\Omega_X(\mathcal{L})) &= -(c_1(\Omega_X) + 2c_1(\mathcal{L})), \\ s_2(\Omega_X(\mathcal{L})) &= -c_1(\Omega_X)^2 + 3c_1(\Omega_X)c_1(\mathcal{L}) + 3c_1(\mathcal{L})^2 - c_2(\Omega_X). \end{aligned}$$

Writing $[K_X] = c_1(\Omega_X)$ and $[\mathscr{L}] = c_1(\mathscr{L})$, we have

$$[I] = (p^m + 1)[K_X]^2 + 4(p^m + 2)[K_X][\mathscr{L}] + 16[\mathscr{L}]^2 + p^{2m} c_2(\Omega_X) \cap [X].$$

Noting $c_2(\Omega_X) = (\deg^2 X - 4 \deg X + 6)c_1(\mathscr{L})^2$, finally we get

$$\begin{aligned} \# I = \deg I &= ((p^m + 1)(\deg X - 4)^2 + 4(p^m + 2)(\deg X - 4) \\ &\quad + 16 + p^{2m}(\deg^2 X - 4 \deg X + 6))\deg X \\ &= ((p^{2m} + p^m + 1)\deg^2 X - 4p^m(p^m + 1) + 6p^{2m})\deg X. \end{aligned}$$

The proof is complete.

EXAMPLE: Suppose that X is defined by

$$X_0^{p+1} + X_1^{p+1} + X_2^{p+1} + X_3^{p+1} = 0.$$

We shall prove in the next section that X has only a finite number of p-inflections. Then Theorem 2.2 tells us

$$\# I = (p + 1)(p^4 - p^3 + 2p^2 - p + 1).$$

It is interesting that this result can get verified by a direct calculation.

In affine coordinates the equation is

$$1 + x^{p+1} + y^{p+1} + z^{p+1} = 0.$$

According to the notations in Section 1 we have

$$R_{p0} = -x(z^{p^2-1} - x^{p^2-1})z^{-(p^2+p-1)},$$

$$R_{0p} = -y(z^{p^2-1} - y^{p^2-1})z^{-(p^2+p-1)}.$$

Then we have three groups of solutions as follows.

(1) $x = 0$, $y = 0$ and $z = e_i$ ($i = 1, \ldots, p + 1$), where e_i are the $(p + 1)$th roots of -1. By the symmetry of the equation, these solutions amount to $6(p + 1)$.

(2) $x = 0$, $z^{p^2-1} - y^{p^2-1} = 0$. Let $y = w_j z$ ($j = 1, \ldots, p^2 - 1$), where w_j are the $(p^2 - 1)$th roots of unity. Then we have whenever $(1 + w_j^{p+1}) \neq 0$, the equation has $(p + 1)$ solutions for z. Since such w_j's amount to $(p^2 - p - 2)$, this group consists of $4(p + 1)(p^2 - p - 2)$ elements by symmetry again.

(3) $x^{p^2-1} = y^{p^2-1} = z^{p^2-1} = 0$. Substituting $x = w_i z$ and $y = w_j z$ into the equation where w_i and w_j are the $(p^2 - 1)$th roots of unity, we have

$$1 + (1 + w_i^{p+1} + w_j^{p+1})z^{p+1} = 0.$$

If $(1 + w_i^{p+1} + w_j^{p+1}) \neq 0$, then there are $(p + 1)$ solutions for z. If $1 + w_i = 0$, then we have $(1 + w_i^{p+1} + w_j^{p+1}) \neq 0$ for every w_j, so the number of (w_i, w_j) with $1 + w_i^{p+1} = 0$ and $1 + w_i^{p+1} + w_j^{p+1} \neq 0$ is $(p + 1)(p^2 - 1)$. If $1 + w_i^{p+1} \neq 0$, the number of w_j with $1 + w_i^{p+1} + w_j^{p+1} = 0$ is $(p + 1)$, the number of (w_i, w_j) with $1 + w_i^{p+1} \neq 0$ and $1 + w_i^{p+1} + w_j^{p+1} \neq 0$ is $(p^2 - 1 - (p + 1))^2$. In sum, all these solutions amount to $(p + 1)^3(p^2 - 3p + 3)$.

Consequently the total number of solutions is the very same number as given by Theorem 2, 2.

3. Finiteness

Let us adopt all the notations in Section 1. Let $\deg X = 1 + kp^m$ and $b_2(X) = p^m$. Then by Theorem 1.2, for some coordinate system, X is defined by

$$G = \sum_i X_i F_i^{p^m}.$$

Suppose that C is irreducible and reduced curve on X, and $[F_i]^*$ $(i = 0, \ldots, 3)$ are divisors determined by sections of F_i on C. We say that C possesses the property (GX) if it satisfies the following conditions: (a) No two of divisors $[F_i]^*$ have common component point; (b) C is not contained in any $X_i = 0$ for $i = 0, \ldots, 3$.

LEMMA 1: *Suppose C possesses (GX) and is contained in the p^m-inflection locus of X. Then, on C we have*

$$\sum_j X_j F_{ji}^{p^m} = 0, \quad i = 0, \ldots, 3. \qquad (****)_i$$

Proof: Since $C \subset I$ by our hypothesis, we have $R_{0p^m}|_C = 0 = R_{p^m0}|_C$, hence by Section 1, (4), on C

$$F_i^{p^{2m}} \cdot \left(\sum_{j=0} X_j F_{3j}\right) - F_3^{p^{2m}} \cdot \left(\sum_j X_j F_{ij}\right) = 0, \quad i = 0, \ldots, 3. \qquad (***)_i$$

Then, nothing remains but to imitate the argument in Section 1, nearly verbatim.

LEMMA 2: *For the given C on X there is a coordinate system under which C possesses (GX) where G is the polynomial for defining X.*

Proof: The proof is also similar to that of the lemma in Section 1. It is sufficient to find out four independent planes such that the intersection of any two of them does not meet the image of C in $\mathbb{P}^3$. In addition, since such a choice is generic, pulling them back to $\mathbb{P}^3$ we can still choose coordinate planes, not containing C.

LEMMA 3: *If for some coordinate system we have* $(****)_i|_C = 0$ *for all i, so do we for any other system.*

Proof: Assume in coordinate $\{X_i\}$ we have

$$G = \sum_{j=0}^{3} X_j F_{ji}^{p^m} = 0, \quad i = 0, \ldots, 3$$

on C. Let

$$\begin{pmatrix} X_0 \\ \cdot \\ \cdot \\ \cdot \\ X_3 \end{pmatrix} = (a_{ij}) \begin{pmatrix} T_0 \\ \cdot \\ \cdot \\ \cdot \\ T_3 \end{pmatrix}$$

where $\det (a_{ij}) \neq 0$ and $(a_{ij})^{-1} = (b_{ij})$.

Then the polynomial G is transformed into

$$G' \doteq \sum_i \sum_j a_{ij} T_J F_i(\Sigma\, a_{0j} T_j, \ldots, \Sigma\, a_{3j} T_j)^{p^m},$$

so we have

$$F_j'(T_0, \ldots, T_3)^{p^m} = \partial G'/\partial T_j = \sum_k a_{kj} F_k^{p^m} = \left(\sum_k a_{kj}^{p^{-m}} F_k\right)^{p^m}$$

and

$$\sum_j T_j F'^{p^m}_{ji} = \sum_{jr} b_{jr} X_r \sum_k \left(a_{kj}^{p-m} \sum_s a_{si} F_{ks} \right)^{p^m},$$

$$= \sum_{j,k,r,s} b_{jr} a_{kj} a_{si}^{p^m} X_r F_{ks}^{p^m} = \sum_{k,r,s} \delta_{kr} a_{si}^{p^m} X_r F_{ks}^{p^m}$$

$$= \sum_{r,s} a_{si}^{p^m} X_r F_{rs}^{p^m} = \sum_s a_{si}^{p^m} \left(\sum_r X_r F_{rs}^{p^m} \right).$$

Hence the lemma.

THEOREM 3.1: *$C \subset I$ if and only if $(****)_i|_C = 0$ for $i = 0, \ldots, 3$.*

Proof: A direct consequence of lemma 1, 2, 3.

PROPOSITION 3.2: *If* $\deg X = 1 + p^m$ *and* $b_2(X) = p^m$, *then X has only a finite number of p^m-inflections.*

Proof: By our hypothesis, $F_i = \Sigma_{j=0}^3 a_{ij} X_j$, and $\det(a_{ij})$ is invertible. Then $(****)_i (i = 0, \ldots, 3)$ are expressed as

$$a_{00}^{p^m} X_0 + a_{10}^{p^m} X_1 + a_{20}^{p^m} X_2 + a_{30}^{p^m} X_3 = 0,$$

$$\ldots .$$

$$a_{03}^{p^m} X_0 + a_{13}^{p^m} X_1 + a_{23}^{p^m} X_2 + a_{33}^{p^m} X_3 = 0,$$

hence the solution of $(****)$ contains no curve.

The following examples show that there are surfaces which take curves as p^m-inflection locus even if p does not divide $k = \deg F_i$.

EXAMPLE: $G = X_0(X_0^p + X_0 X_1^{p-1})^{p^{m-1}} + X_1(X_0^2 X_1^{p-2} + X_1^p)^{p^{m-1}} + X_2^{p^m+1} + X_3^{p^m+1}$. Then $F_{j2} = F_{j3} = 0$ for all j, and the sections of planes $X_0 = 0$ and $X_1 = 0$ on X both are contained in I.

EXAMPLE: $G = X_0(X_0^2 + 2 \cdot 3^{p-m} X_0 X_1 + 3^{p-m} X_1^2)^{p^m} + 2X_0^{2p^m} X_1 + X_2(X_2^2 - 2 \cdot 3^{p-m} X_2 X_3 + 3^{p-m} X_3^2)^{p^m} + 2X_3 X_2^{2p^m}$. Then the line $\{X_0 = X_1, X_2 = X_3\}$ is contained in I.

In general, we have

THEOREM 3.3: *Under a fixed coordinate system, a generic surface with* $b_2 = p^m$ *and* deg $X = 1 + kp^m$ *has only a finite number of* p^m*-inflections.*

Proof: The surfaces with $b_2(X) = p^m$ and deg $X = 1 + kp^m$ are determined uniquely by $F_0, \dots, F_3$ up to a constant multiplication, thus the set of such surfaces can be parametrized by an open set S_{k,p^m} in $\mathbb{P}^N$, where $N = 4\binom{k+3}{3} - 1$.

Let Y_i be the surface defined by $(****)_i$. Then Y_i is smooth if and only if the four surfaces $F_{ji}(j = 0, \dots, 3)$ have no common point in $\mathbb{P}^3$; it is well known that this is an open condition of those coefficients in $F_{ji}(j = 0, \dots, 3)$, so hereafter we assume all the Y_i are smooth. Now fix Y_0 and take a point Q in Y_0. The section of a surface through Q, cut out by Y_0, is singular at Q if and only if the surface is tangent to Y_0 at Q. A general surface tangent to Y_0 at Q submits to 3 independent equations, so we see by dimension computations that the subset of S_{k,p^m} consisting of those surfaces to which the corresponding Y_1 are tangent to Y_0, has dimension $(4\binom{k-1+3}{3} - 1) + 5$. Since $\dim S_{k,p^m} - 4\binom{k+2}{3} - 4 = 2(k + 1)(k + 2) - 5 > 0$, it follows that almost every surfaces in S_{k,p^m} has the property that its Y_1, cut out by Y_0, gives a smooth section. Moreover, since the divisor on Y_0, determined by this section, is ample, the section is connected and consequently irreducible. It follows that the homogeneous ideal $(\Sigma X_j F_{j0}^{p^m}, \Sigma X_j F_{j1}^{p^m})$ is prime. Now we show that to a general point in S_{k,p^m} the corresponding Y_2 intersects its $Y_0 \cap Y_1$ in a finite number of points only. If to the contrary, the intersection would contain a curve, thus it coinciding with the intersection of Y_0 and Y_1 since the latter is irreducible. By Nullstellensatz we have $\Sigma X_j F_{j2}^{p^m} \in (\Sigma X_j F_{j0}^{p^m}, \Sigma X_j F_{j1}^{p^m})$, hence $\Sigma X_j F_{j2}^{p^m} = a \Sigma X_j F_{j0}^{p^m} + b \Sigma X_j F_{j1}^{p^m}$ where a and b are constants for the sake of degree.

Differentiating them with respect to X_j, we obtain $F_{j2}^{p^m} = aF_{j0}^{p^m} + bF_{j1}^{p^m}$, where $j = 0, \dots, 3$ and $(a, b) \neq 0$, thus resulting in $3\binom{k+2}{3} - 1$ relations among their coefficients. Since $\dim S_{k,p^m} - 3\binom{k+2}{3} + 1 > 0$, we conclude finally that the scheme defined by $(****)$ of a general point in S_{k,p^m} contains no curve. Therefore, our proof is complete by 3.1.

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