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Cohomology of desingularization of moduli space of vector bundles

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Introduction

Let X be a smooth projective curve over $\mathbb{C}$, and let U_0 be the moduli space of rank 2 semistable vector bundles on X with trivial determinant. When X is non-hyperelliptic, a canonical desingularization of U_0 has been constructed by Narasimhan and Ramanan [5] using the Hecke correspondence. Let H_0 denote this desingularization. The main result proved in this paper is the following.

THEOREM 1: *Let X be a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 3$ which is non-hyperelliptic and let U_0 be the moduli space of rank 2 semistable vector bundles on X with trivial determinant. The cohomology group $H^3(H_0, \mathbb{Z})$ of the Narasimhan-Ramanan desingularization H_0 of U_0 is torsion-free and of rank 2g.*

The motivation for proving that H^3 is torsion-free is as follows. The moduli space U_0 is known to be unirational, but it is not known whether it is rational. As the torsion subgroup of $H^3(V, \mathbb{Z})$ is a birational invariant of a smooth projective variety (see [1]), it was of interest to determine $H^3(H_0, \mathbb{Z})$ for some non-singular model H_0 of U_0.

Narasimhan and Ramanan state that (see [5, page 292]) the desingulization H_0 can be blown down along certain projective fibrations to obtain another non-singular model of U_0. A proof of this is given in §4 below (Prop. 4A2). This new desingularization has the same third cohomology group as the original desingularization.

The arrangement of this paper is as follows. Let K be the singular locus of U_0. Let $Z \subset H_0$ be the complement of the fibres of $H_0 \to U_0$ over the points of K of the form $j \oplus j^{-1}$ where j is an element of order 2 in the Jacobian. For any $x \in X$, there is a natural conic bundle C over Z which degenerates into pairs of lines over the divisor Y which is the inverse image of K in Z. Sections 1 to 3 are devoted mainly to proving that the total space of this conic bundle is smooth. This allows us to apply the result proved in [6] which relates the topological Brauer class of the $\mathbb{P}^1$-bundle on $Z-Y$ determined by C to the cohomology class in $H^2(Y, \mathbb{Z})$ which is associated to the 2-sheeted cover of the degeneration locus Y determined by C.

In §4 we show that multiplication by $c_1(N_{Y,Z})$ gives an injective map from $H^2(Y, \mathbb{Q})$ to $H^4(Y, \mathbb{Q})$ where $N_{Y,Z}$ is the normal bundle to Y in Z. Finally in §5 we complete the proof of Theorem 1.

Remark on notation: In this paper, we have followed the notation used in [5]. In particular, for a point x on the curve X, L_x denotes the line bundle defined by the divisor x, and U_x denotes the moduli space of all rank 2 stable vector bundles E on X for which det E is isomorphic to L_x. The moduli space of all rank 2 stable vector bundles on X whose determinants are isomorphic to L_x for some $x \in X$ is denoted by U_X.

§1. The morphism from H_0 to Hilb (U_x)

Let $H_0 \subset$ Hilb (U_X) be the desingularization of U_0, and let $x \in X$ be any point. As a subscheme of Hilb (U_X), H_0 parametrizes a flat family $W \subset U_X \times H_0$ of subschemes of U_X. By [5] Lemmas 7.9 and 7.12 for any point $y \in H_0$, the Hilbert polynomial of the subscheme W_x^y of U_x with respect to the ample generator of Pic U_x is $2m + 1$ where $W_x^y = W^y \cap U_x$ where W^y is the fibre of W over y. As H_0 is an integral scheme, it follows from the constancy of the Hilbert polynomial that the family $W_x = W \cap (U_x \times H_0)$ of subschemes of U_x parameterized by H_0 is a flat family.

The above flat family has a classifying morphism ϕ: $H_0 \to$ Hilb (U_x). It follows from [5] §7 that ϕ is injective. In this section, we prove (Lemma 1.3) that if $l = l_1 \cup l_2$ is a limit Hecke cycle (see §7 of [5] for the precise definition) over a non-nodal point of $K \subset U_0$, then the differential $d\phi$ of ϕ: $H_0 \to$ Hilb (U_x) is injective at the point $l \in H_0$.

LEMMA 1.1: *Let S be a non-singular variety, T a non-singular curve, and P: $S \to T$ a smooth morphism. Let $V \subset S$ be a 2-dimensional closed reduced subscheme such that either (i) V is irreducible and the morphism $V \to T$ is smooth, or (ii) $V = V_1 \cup V_2$ is a union of exactly two irreducible components each of dimension 2, and the morphisms $V_1 \to T$, $V_2 \to T$, and $V_1 \cap V_2 \to T$ are smooth where $V_1 \cap V_2$ is the scheme theoretic intersection of the reduced schemes V_1 and V_2 which we assume is a non-singular curve.*

Let $t \in T$ be a closed point and let S_t (resp. V_t) denote the fibre of S (resp. V) over t. Then

(1) *V is a local complete intersection in S.*

(2) *V_t is a local complete intersection in S_t.*

(3) *The normal bundle N_{V_t,S_t} of V_t in S_t is the restriction to V_t of the normal bundle $N_{V,S}$ of V in S.*

Proof of Lemma 1.1: As V_1 and V_2 are nonsingular subvarieties of the non-singular variety S such that the scheme $V_1 \cap V_2$ is a non-singular variety, the reduced scheme $V = V_1 \cup V_2$ is a local complete intersection by [5] Lemma 8.2. This proves (1).

If V is irreducible, by assumption the morphism $V \to T$ is smooth. Hence V_t is a nonsingular variety in particular a local complete intersection. If $V = V_1 \cup V_2$ then as above, both $V_{1,t}$ and $V_{2,t}$ are non-singular varieties. As $V_1 \cap V_2 \to T$ is also smooth, it follows that the scheme theoretic intersection of $V_{1,t}$ and $V_{2,t}$ is a regular scheme and hence the scheme V_t (which is reduced) is a local complete intersection as above. This proves (2).

To prove (3) we prove the dual statement about conormal bundles, namely, that $N^*_{V_t,S_t} \simeq N^*_{V,S} | V_t$. A local section of $N^*_{V,S} | V_t$ is locally defined by a local section of $N^*_{V,S}$ which is a differential l-form on S which kills vectors tangent to V. This defines a 1-form on S_t which kills vectors tangent to V_t. This defines a morphism of vector bundles from $N^*_{V,S} | V_t$ to $N^*_{V_t,S_t}$. From the hypothesis of this lemma it is easy to see that this morphism is injective on each fibre hence an isomorphism as these vector bundles are of the same rank.

LEMMA 1.2: *Let P: $S \to T$ be as in lemma 1.1. Let $W \subset S \times R$ be a flat family of subschemes of S parameterized by a variety R, such that for each point $r \in R$, the subscheme $W^r \subset S$ satisfies the conditions imposed on $V \subset S$ in lemma 1.1. Let $t \in T$ be a point and let the family $W_t = W \cap (S_t \times R)$ of subschemes of S_t parameterized by R again be flat. Let θ: $R \to$ Hilb (S) and ϕ: $R \to$ Hilb (S_t) be the classifying morphisms for the families W and W_t. For $r \in R$, let*

$$\Psi_r: T_{\theta(r)} \text{ Hilb } (S) \to T_{\phi(r)} \text{ Hilb } (S_t)$$

be the restriction map from $H^0(W^r, N_{W^r,S})$ to $H^0(W^r_t, N_{W^r_t,S_t})$ where $N_{W^r_t,S_t} \simeq N_{W^r,S} | W^r_t$ by lemma 1.1. Then the following diagram of vector spaces is commutative.

$$\begin{tikzcd} T_r R \arrow[r, "d\theta"] \arrow[dr, "d\phi"'] & T_{\theta(r)} \text{ Hilb } (S) \arrow[d, "\Psi_r"] \\ & T_{\phi(r)} \text{ Hilb } (S_t) \end{tikzcd}$$

Proof of Lemma 1.2: By functoriality, it is enough to prove a similar statement where the variety R is replaced by Spec $k[\varepsilon]/(\varepsilon^2)$. In this set up, the lemma follows immediately by taking a suitable affine open cover of S and looking at the defining equations of the family with respect to this open cover.

Now let H_0 be the desingularization of U_0 and let ϕ: $H_0 \to \text{Hilb}\,(U_x)$ be the morphism defined at the beginning of this section.

LEMMA 1.3: *Let $l = l_1 \cup l_2$ be a limit Hecke cycle over a non-nodal singular point $\langle j \oplus j^{-1} \rangle \in U_0$, and let l be represented by a point $y \in H_0$. Then the differential map*

$$d\phi_y\colon T_y H_0 \to T_{\phi(y)}\,\text{Hilb}\,(U_x)$$

is injective.

Proof: Let P: $S \to T$ be the map det: $U_X \to X$. Then the hypothesis of lemma 1.2 is satisfied by the restricted universal family W parameterized by H_0, and so $d\phi_y$ is the restriction map from

$$H^0\,(l, N_{l,U_X}) \to H^0\,(l_x, N_l, U_X \,|\, l_x)$$

The injectivity of this map follows from lemma 1.4 below.

LEMMA 1.4: *Any global section of N_{l,U_X} which restricts to zero an $l_x = l \cap U_x$ is identically zero.*

For the proof of lemma 1.4, we need the following

LEMMA 1.5: *Let U be a non-singular variety and let Y_1, Y_2 be closed non-singular surfaces in U which intersect transversally along a non-singular curve X. Let $Y = Y_1 \cup Y_2$ (which is a local complete intersection). Then there is a short exact sequence of sheaves on Y_1 as follows.*

$$0 \to N_{Y_1,U} \to N_{Y,U} \,|\, Y_1 \to \tilde{N}_{X,Y_1} \otimes \tilde{N}_{X,Y_2} \to 0$$

In particular, any global section of $N_{Y,U} \,|\, Y_1$ which vanishes identically over X is the image of a global section of $N_{Y_1,U}$.

Proof of Lemma 1.4: Let $N^1 = N_{l,U_X} \,|\, l_1$ and $N^2 = N_{l,U_X} \,|\, l_2$. Then we have the following exact sequence on l.

$$0 \to N_{l,U_X} \to \tilde{N}^{-1} \oplus \tilde{N}^{-2} \to \widetilde{N_{l,U_X} \,|} X \to 0$$

Let $\theta \in H^0(l, N_{l,U_X})$ which restricts to zero on $l_x = l \cap U_x$. Then θ defines sections θ_1 and θ_2 of N^1 and N^2 respectively such that $\theta_1 = 0$ on $l_{1,x} = l_1 \cap U_x$, $\theta_2 = 0$ on $l_{2,x}$ and $\theta_1 = \theta_2$ on $X = l_1 \cap l_2$. By lemma 1.5 we have an exact sequence.

$$0 \to N_{l_1,U_x} \to N^1 \to \tilde{N}_{X,l_1} \otimes \tilde{N}_{X,l_2} \to 0$$

Now $l_1 = P(E_1)$ and $l_2 = P(E_2)$ where the vector bundles E_1 and E_2 occur in exact sequences

$$0 \to j^2 \to E_1 \to \mathcal{O}_X \to 0$$

$$0 \to j^{-2} \to E_2 \to \mathcal{O}_X \to 0$$

In $l = l_1 \cup l_2$, the bundles $P(E_1)$ and $P(E_2)$ are identified along the sections defined by j^2 and j^{-2} respectively. Hence, $N_{X,l_1} \simeq j^{-2}$ and $N_{X,l_2} \simeq j^2$. Hence we get the short exact sequence

$$0 \to N_{l_1,U_X} \to N^1 \to \mathcal{O}_X \to 0$$

Now, $\theta_1 \in H^0(l_1, N^1)$ vanishes over $l_{1,x} = l_1 \cap U_x$. Hence the image of θ_1 in $H^0(X, \mathcal{O}_X)$ must vanish at $x \in X$ and hence be identically zero. Hence by exactness of the sequence, $\theta_1 \in H^0(l_1, N^1)$ is the image of an element $\theta_1' \in H^0(l_1, N_{l_1,U_X})$. Now, θ_1' must also be zero on $l_{1,x}$. Hence to complete the proof of lemma 1.4, it is enough to show that any global section of N_{l_1,U_X} which is zero on $l_{1,x} = l_1 \cap U_x$ is identically zero.

The bundle N_{l_1,U_X} occurs in the following exact sequence.

$$0 \to N_{l_1,P(D_l)} \to N_{l_1,U_X} \to N_{P(D_l),U_X} | l_1 \to 0$$

Hence we have to prove that any global section of $N_{l_1,P(D_l)}$ and any global section of $N_{P(D_l),U_X} | l_1$ which vanish over $l_{1,x}$ are identically zero. Now, $N_{l_1,P(D_l)}$ is isomorphic to a direct sum of $g - 2$ copies of τ where τ is the hyperplane bundle on $l_1 = P(E_1)$. As E_1 occurs in the exact sequence

$$0 \to j^2 \to E_1 \to \mathcal{O}_X \to 0$$

it follows that $\dim H^0(l_1, \tau) = 1$, and the non-zero sections vanish precisely over $X \subset l_1$. Hence, any section of $N_{l_1,P(D_l)}$ which vanishes over $l_{1,x}$ is identically zero. We now consider $N_{P(D_l),U_X} | l_1$. We have the following exact sequence on $P(D_j)$ (see [5], lemma 6.22)

$$0 \to \text{trivial} \to N_{P(D_j),U_X} \to \tau^{-1} \otimes \pi^* F(j) \to 0$$

where "trivial" denotes a trivial vector bundle, π: $P(D_j) \to X$ is the projection, and $F(j)$ is a vector bundle on X. Restricting attention to $l_1 \subset P(D_j)$, we get

$$0 \to \text{trivial} \to N_{P(D_j),U_X} \mid l_1 \to \tau^{-1} \otimes \pi_1^* F(j) \to 0$$

where π_1: $l_1 \to X$. Now, it is obvious that any global section of $\tau^{-1} \otimes \pi_1^* F(j)$ is identically zero, while any global section of the trivial bundle is constant. Hence, any global section of $N_{P(D_j),U_X} \mid l_1$ which is zero on $l_{1,x}$ is identically zero. This completes the proof of lemma 1.4.

Proof of Lemma 1.5: By [5] Lemma 8.2, there is an exact sequence

$$0 \to N^*_{Y,U} \mid Y_1 \to N^*_{Y_1,U} \to \tilde{N}^*_{X,Y_2} \to 0$$

We apply the functor *Hom* $(-, \mathcal{O}_{Y_1})$ to the above sequence to get the short exact sequence

$$0 \to N_{Y_1,U} \to N_{Y,U} \mid Y_1 \to \mathscr{C} \to 0$$

where $\mathscr{C}$ is the cokernel sheaf of the first morphism. Note that a non-zero section of $\mathscr{C}$ is defined by a homomorphism α: $N^*_{Y,U} \mid Y_1 \to \mathcal{O}_{Y_1}$ which does not extend to a homomorphism from $N^*_{Y_1,U}$ to $\mathcal{O}_{Y_1}$. Hence it follows that the support of $\mathscr{C}$ lies inside X. Now, let $P \in X$ and let f be a regular function on Y_1 defined in a neighbourhood of P which locally defines the divisor X. Then there exists a local basis $(e_1, \ldots, e_{n-2})$ for $N^*_{Y_1,U}$ and a local basis $(e'_1, \ldots, e'_{n-2})$ for $N^*_{Y,U}$ in a neighbourhood of P such that the homomorphism $N^*_{Y,U} \mid Y_1 \to N^*_{Y_1,U}$ is defined by $e'_1 \to f \cdot e_1$ and $e'_i \to e_i$ for $2 \leqslant i \leqslant n-2$, where $n = \dim U$. Note that when restricted to X, e_1 defines a local frame $\bar{e}_1$ of N^*_{X,Y_2} by the exactness of

$$0 \to N^*_{Y,U} \mid Y_1 \to N^*_{Y_1,U} \to \tilde{N}^*_{X,Y_2} \to 0.$$

Let α be a local homomorphism from $N^*_{Y,U} \mid Y_1$ to $\mathcal{O}_{Y_1}$ such that

$$\alpha(e'_1) = g_1, \ldots, \alpha(e'_{n-2}) = g_{n-2}$$

where the g_i are regular functions defined locally. To α we associate a local section $\bar{\alpha}$ of $L_X \otimes \tilde{N}_{X,Y_2}$ defined as follows where L_X is the line bundle on Y_1 defined by the divisor X. Note that f^{-1} is a local frame for the line bundle L_X. Then we define $\bar{\alpha} = g_1 \cdot f^{-1} \otimes \bar{e}_1$. Now, α extends to a local homomorphism from $N^*_{Y_1,U}$ to $\mathcal{O}_{Y_1}$ if and only if g_1 vanishes along X. Hence we locally get a well-defined isomorphism between $\mathscr{C}$ and $L_X \otimes \tilde{N}_{X,Y_2}$. It is easy to see that these local isomorphisms patch together to give a global isomorphism between $\mathscr{C}$ and $L_X \otimes \tilde{N}_{X,Y_2}$. Now by the adjunction formula, $L_X | X \simeq N_{X,Y_1}$, and hence $L_X \otimes \tilde{N}_{X,Y_2} \simeq \tilde{N}_{X,Y_1} \otimes \tilde{N}_{X,Y_2}$ which completes the proof of the lemma.

§2. The special Hecke cycle

A. Preliminaries

Let $x \in X$ be fixed. Then for a general $j \in J$, we can identify a point of H_0 over $\langle j \oplus j^{-1} \rangle \in U_0$ which represents a limit Hecke cycle having some special property. In this section, we define such a special Hecke cycle l (see definition 2.8) and then explicitly determine the normal bundle N_{l_x,U_x} of its restriction $l_x = l \cap U_x$ in U_x (see lemma 2.11).

A crucial part of the argument in this section is the following technical lemma about the normal bundle of the union of two smooth curves which intersect transversally inside a smooth variety. We shall use this lemma for

$$l_x = l_{1,x} \cup l_{2,x} \subset U_x.$$

LEMMA 2.1: *Let U be a smooth variety and L_1 and L_2 smooth closed curves in U meeting transversally at a single point s. Let the tangent space $T_s U$ have a direct sum decomposition*

$$T_s U = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

where $V_1 = T_s L_1$ and $V_2 = T_s L_2$. Let the normal bundles of L_1 and L_2 in U have direct sum decompositions as follows

$$N_{L_1,U} = F_2 \oplus F_3 \oplus \cdots \oplus F_m$$

$$N_{L_2,U} = G_1 \oplus G_3 \oplus \cdots \oplus G_m.$$

Under the quotient map

$$\varphi_1\colon T_sU \to (N_{L_1,U})_s$$

let the images of $V_2, \ldots, V_m$ be contained respectively in the subspaces $(F_2)_s, \ldots, (F_m)_s$ of $(N_{L_1,U})_s$, and let the induced maps $V_i \to (F_i)_s$ for $i \neq 1$ be isomorphisms. Similarly, let the images of $V_1, V_3, \ldots, V_m$ under the quotient map $\varphi_2\colon T_sU \to (N_{L_2},U)_s$ be contained respectively in the subspaces $(G_1)_s$, $(G_3)_s, \ldots, (G_m)_s$ of $(N_{L_2,U})_s$, and let the induced maps $V_i \to (G_i)_s$ for $i \neq 2$ be isomorphisms.

Under the above hypothesis, the normal bundle of $L_1 \cup L_2$ in U has the following direct sum decomposition. Let for $i \geqslant 3$, $F_i \vee G_i$ denote the bundle on $L_1 \cup L_2$ obtained by identifying $(F_i)_s$ and $(G_i)_s$ through their isomorphisms with V_i.

Let $F_1(s)$ and $F_2(s)$ denote the line bundles $F_2(s) = F_2 \otimes \mathcal{O}_{L_1}(s)$ and $G_1(s) = G_1 \otimes \mathcal{O}_{L_2}(s)$. Then

$$N_{L_1\cup L_2,U} \simeq (F_2(s) \vee G_1(s)) \oplus \left(\bigoplus_{i\geqslant 3} F_i \vee G_i\right)$$

where $F_2(s) \vee G_1(s)$ is the line bundle on $L_1 \cup L_2$ obtained by some identification between the fibres of $F_2(s)$ and $G_1(s)$ at s.

Proof of Lemma 2.1: We shall prove the dual statement about the conormal bundle of $L_1 \cup L_2$, namely,

$$N^*_{L_1\cup L_2,U} \simeq F_2^*(-s) \vee G_1^*(-s) \oplus \left(\bigoplus_{i\geqslant 3} F_i^* \vee G_i^*\right)$$

We have an exact sequence

$$0 \to N^*_{L_1\cup L_2,U}|L_1 \to N^*_{L_1,U} \to \widetilde{T_s^*L_2} \to 0$$

Now, $N^*_{L_1,U} \simeq F_2^* \oplus F_3^* \oplus \cdots \oplus F_m^*$, and the map $N^*_{L_1,U} \to \widetilde{T_s^*L_2}$ maps F_i^* to zero for $i \geqslant 3$. Hence $N^*_{L_1\cup L_2,U}|L_1$ is canonically isomorphic to $\mathscr{F} \oplus F_3^* \oplus \cdots \oplus F_m^*$ where $\mathscr{F}$ is the kernel sheaf of the map $F_2^* \to \widetilde{T_s^*L_2}$, which is just $F_2^*(-s)$, the sheaf of all section of F_2^* which vanish at s. We thus get an isomorphism

$$f\colon F_2^*(-s) \oplus F_3^* \oplus \cdots \oplus F_m^* \to N^*_{L_1\cup L_2,U}|L_1$$

Note that at the point s, the image of (F_i^*) under f_s for $i \geqslant 3$ in $(N^*_{L_1 \cup L_2, U})_s$ is precisely the subspace of $(N^*_{L_1 \cup L_2, U})_s$ corresponding to $V_i^* \subset T_s^* U$. Similarly, we have an isomorphism

$$g\colon G_1^*(-s) \oplus G_3^* \oplus \cdots \oplus G_m^* \to N_{L_1 \cup L_2, U} \,|\, L_2$$

such that for $i \geqslant 3$ the image of $(G_i^*)_s$ under g_s in $(N_{L_1 \cup L_2, U})_s$ is precisely the subspace corresponding to $V_i^* \subset T_s^* U$.

Hence under the isomorphism $g_s^{-1} \cdot f_s$ between the fibres over s of $F_2^*(-s) \oplus F_3^* \oplus \cdots \oplus F_m^*$ and $G_1^*(-s) \oplus G_3^* \oplus \cdots \oplus G_m^*$, the image of $(F_i^*)_s$ $(i \geqslant 3)$ is precisely $(G_i^*)_s$. Now it is easy to see that the induced maps from $(F_2^*(-s))_s$ to $(G_i^*)_s$ are all zero for $i \geqslant 3$. Hence under $g_s^{-1} \cdot f_s$, $(F_2^*(-s))_s$ maps isomorphically onto $(G_1^*(-s))_s$. Hence $N^*_{L_1 \cup L_2, U}$ is isomorphic to the direct sum

$$F_2^*(-s) \vee G_1^*(-s) \oplus \left(\bigoplus_{i \geqslant 3} F_i^* \vee G_i^* \right).$$

This proves the lemma.

B. The special Hecke cycle

In this subsection we define the special Hecke cycle. Let $D_{j,x} = H^1(X, j^2 \otimes L_x^{-1})$ for $j^2 \neq 1$, and let $P(D_{j,x})$ be imbedded in U_x as in the §6 of [5]. $P(D_{j,x})$ parameterises a family of triangular bundles, and hence the imbedding $P(D_{j,x}) \to U_x$ has a lift $\overline{P(D_{j,x})} \to H_x$, where $H_x \to U_x$ is the dual projective Poincaré bundle. Let $s(j,x) = P(D_{j,x}) \cap P(D_{j^{-1},x})$, and $\bar{s}(j,x) = \overline{P(D_{j,x})} \cap \overline{P(D_{j^{-1},x})}$. Let J' be the subset of J where $j^2 \neq 1$. When x is fixed, we get morphisms $\mu\colon J' \to U_x$ and $\bar{\mu}\colon J' \to H_x$ where $\mu(j) = \mu(j^{-1}) = s(j, x)$ and $\bar{\mu}(j) = \bar{\mu}(j^{-1}) = \bar{s}(j, x)$. Note that $\bar{\mu}$ is the lift of μ to H_x.

Remark 2.2: It can be checked that the differential of μ is everywhere injective on J'. As $\bar{\mu}$ is the lift of μ, it follows that $d\bar{\mu}$ is also everywhere injective on J'.

LEMMA 2.3: *Consider the morphism $J \to U_0$ which is defined by the family $R \oplus R^{-1}$ of vector bundles on X parameterized by J where $R \to X \times J$ is the Poincaré line bundle. Then the open subset of J on which the differential of this map is injective is a non-empty set.*

Proof: Let $K \subset U_0$ be the singular set with reduced subscheme structure. Then the morphism $J \to U_0$ factors through $K \to U_0$. Let $\phi \neq K_* \subset K$ be the smooth open part of K and let $J_* \subset J$ be its inverse image. As $J \to K$ is surjective, $J_* \to K_*$ is surjective. Now, $\dim J_* = \dim K_* = g$, and both J_* and K_* are smooth. Hence there exists a point in J_* at which the map $J_* \to K_*$ is of maximal rank. This proves the lemma.

Definition 2.4: Let $x \in X$ be fixed. By (2.2)–(2.3), there exists a non-empty sub-set of J such that

$$j^2 \neq 1,\ H^0(X, j^2 \otimes L_x) \ = \ 0,\ H^0(X, j^{-2} \otimes L_x) \ = \ 0,$$

the differential of the map $\mu\colon J \to U_x$ sending $j \mapsto s(j, x)$ is injective (in particular the differential of $\bar{\mu}\colon J \to H_x$ is injective) and the differential of the map $J \to U_0\colon j \mapsto \langle j \oplus j^{-1}\rangle$ is injective for any j in this open subset of J. We will call such a j a *general point* of J.

Lemma 2.5: *Let $x \in X$ be fixed and let j be a general point of J (see 2.4). Then the tangent space to H_x at the point $\bar{s} = \bar{s}(j, x)$ has the following direct sum decomposition.*

$$T_{\bar{s}}H_x \ = \ T_{\bar{s}}\overline{P(D_{j,x})} \oplus T_{\bar{s}}\overline{P(D_{j^{-1},x})} \oplus \mathrm{d}\bar{\mu}(T_jJ)$$

Proof: We have a commutative diagram

$$\begin{tikzcd} J' \arrow[rr, "\bar{\mu}"] \arrow[dr, "\nu"'] & & H_x \arrow[dl, "h"] \\ & U_0 & \end{tikzcd}$$

where $\nu(j) = \langle j \oplus j^{-1}\rangle$. Now as $j \in J$ is a general point, $\mathrm{d}\nu$ is injective at j. Hence by the commutativity of the above triangle, $\mathrm{d}h$ is injective on the image $\mathrm{d}\mu(T_jJ)$ of T_jJ inside $T_{\bar{s}}H_x$. On the other hand, $\mathrm{d}h$ is zero on both $T_{\bar{s}}\overline{P(D_{j,x})}$ and $T_{\bar{s}}\overline{P(D_{j^{-1},x})}$, which in turn are linearly independent as shown in [5] §6. Hence the three subspaces are linearly independent in $T_{\bar{s}}H_x$. As their dimensions add up to $3g - 2$ which is the dimension of $T_{\bar{s}}H_x$, the lemma is proved.

Lemma 2.6: *For a general j,* let $L_j \subset T_{\bar{s}}H_x$ *be the kernel of the differential map $T_{\bar{s}}H_x \to T_sU_x$ at the point $\bar{s} = \bar{s}(j, x)$. Then there exists a general $j \in J$ such that the projection of L_j on each of the three direct summands of $T_{\bar{s}}H_x$ given by lemma 2.5 is non-zero.*

Remark 2.7: Though we do not need it here, with some extra calculations we can in fact prove that the conclusion of lemma 2.6 is true for all general j.

Proof of Lemma 2.6: $P(D_{j,x})$ and $P(D_{j^{-1},x})$ intersect transversally inside U_x. This shows that L_j is not contained in $T_{\bar{s}}\overline{P(D_{j,x})} \oplus T_{\bar{s}}\overline{P(D_{j^{-1},x})}$. Hence the projection of L_j on $d\bar{\mu}(T_j J)$ is always non-zero for a general j. Also, as $d\mu$ is injective by remark 2.2, L_j is not contained in $d\bar{\mu}(T_j J)$. Hence the projection of L_j on $T_{\bar{s}}\overline{P(D_{j,x})} \oplus T_{\bar{s}}\overline{P(D_{j^{-1},x})}$ is always non-zero for a general j. Now, the condition that the projection of L_j on $T_{\bar{s}}\overline{P(D_{j,x})}$ is zero is a closed condition in the Zariski topology. By the irreducibility of the open subset of J of general points j, we see that if the lemma is false, then either $\operatorname{im} L_j \subset T_{\bar{s}}\overline{P(D_{j,x})}$ for all general j or $\operatorname{im} L_j \subset T_{\bar{s}}\overline{P(D_{j^{-1},x})}$ for all general j. Note that the open subset $V \subset J$ of all general j, is stable under $j \to j^{-1}$.

Hence we get a section of the canonical map of J to the Kummer variety over a non-empty open set (namely the image of V). This contradiction proves the lemma.

Now choose a general j satisfying lemma 2.6, and let $\bar{u}_1 \in T_{\bar{s}}\overline{P(D_{j,x})}$, $\bar{u}_2 \in T_{\bar{s}}\overline{P(D_{j^{-1},x})}$ and $\bar{u}_3 \in d\bar{\mu}(T_j J)$ be the projections of a non-zero vector in L_j. Note that $\bar{u}_1 + \bar{u}_2 + \bar{u}_3 \in L_j$. Hence, if $u_1, u_2, u_3 \in T_s U_x$ are the images of $\bar{u}_1, \bar{u}_2, \bar{u}_3$ respectively, then each u_i is non-zero, but $u_1 + u_2 + u_3 = 0$. Let L_1 and L_2 be lines in $P(D_{j,x})$ and $P(D_{j^{-1},x})$ respectively, each passing through $s = P(D_{j,x}) \cap P(D_{j^{-1},x})$, given by the tangent vectors $u_1 \in T_s P(D_{j,x})$ and $u_2 \in T_s P(D_{j^{-1},x})$. By §6 of [5], there exists a unique limit Hecke cycle l such that $l_x = L_1 \cup L_2$ where l_x denotes $l \cap U_x$.

DEFINITION 2.8: The limit Hecke cycle l above will be called a special Hecke cycle and $l_x = L_1 \cup L_2$ will be called a restricted special Hecke cycle.

Remark 2.9: Note that L_1 and L_2 are defined by the tangent vectors u_1, $u_2 \in T_s U_x$ above. The vector u_3 above, which satisfies $u_1 + u_2 + u_3 = 0$, is tangent to the image of μ: $J' \to U_x$.

Remark 2.10: The lines $L_1 \subset P(D_{j,x})$ and $L_2 \subset P(D_{j^{-1}})$ have the following direct description (which we do not use). L_1 is given by the 2-dimensional kernel of the linear map $H^1(X, j^2 \otimes L_x^{-1}) \to H^1(X, j^2 \otimes L_x)$ and a similar description holds for L_2. The vector $u_3 \in d\mu(T_j J)$ lies in the image under $d\mu$ of the kernel of the map $H^1(X, \mathcal{O}_X) \to H^1(X, L_x)$ where $H^1(X, \mathcal{O}_X) = T_j J$.

C. Normal bundle of the restricted special Hecke cycle

LEMMA 2.11: *Let $x \in X$ be fixed and let $j \in J$ be a general point for this x. Let $l_x \subset U_x$ be the restricted special Hecke cycle corresponding to (x, j), with*

$l_x = L_1 \cup L_2$, where each L_i is a line in the projective space P_i. Then the normal bundle N_{l_x, U_x} of l_x in U_x has the following direct sum decomposition into subbundles.

$$N_{l_x, U_x} \simeq A \oplus B \oplus C \oplus D$$

where

$$A \simeq \mathcal{O}_{L_1}(1) \vee \mathcal{O}_{L_2}(1)$$

$$B \simeq \overset{g-2}{\oplus} (\mathcal{O}_{L_1}(1) \vee \mathcal{O}_{L_2}(-1))$$

$$C \simeq \overset{g-2}{\oplus} (\mathcal{O}_{L_1}(-1) \vee \mathcal{O}_{L_2}(1)),$$

and $D \simeq \oplus^{g-1} \mathcal{O}_{l_x}$ is the trivial bundle of rank $g - 1$.

Proof: Choose splittings of the tangent spaces of $P_1 = P(D_j) \cap U_x$ and $P_2 = P(D_{j-1}) \cap U_x$ at the point $s = P_1 \cap P_2$ as follows. Let $T_sP_1 = V_1 \oplus V_3$ and let $S_sP_2 = V_2 \oplus V_4$ where $V_1 = T_sL_1$ and $V_2 = T_sL_2$ while V_3 and V_4 are arbitrary supplements. Let $u_3 \in \mathrm{d}\mu(T_jJ)$ be as in remark 2.9, and choose an arbitrary splitting

$$\mathrm{d}\mu(T_jJ) = \langle u_3 \rangle \oplus V_5$$

Then by lemma 2.6, we get a direct sum decomposition of T_sU_x as follows.

$$T_sU_x = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$$

Note that $\dim V_1 = \dim V_2 = 1$, $\dim V_3 = \dim V_4 = g - 2$ and $\dim V_5 = g - 1$. Now the normal bundle $N_{P_i U_x}$ of $P_i (i = 1, 2)$ in U_x occurs in the exact sequence (see [5] lemma 6.22 and the proof of lemma 7.10)

$$0 \to H^1(X, \mathcal{O}_X)_{P_i} \to N_{P_i, U_x} \to \mathcal{O}_{P_i}(-1)^{\oplus g-2} \to 0$$

such that under the canonical map $T_sU_x \to (N_{P_i, U_x})_s$, the subspace $\mathrm{d}\mu(T_jJ) \subset T_sU_x$ maps identically on $H^1(X, \mathcal{O}_X) \subset (N_{P_i, U_x})_s$. Hence the splitting of $\mathrm{d}\mu(T_jJ)$ as $\langle u_3 \rangle \oplus V_5$ induces a splitting of the trivial bundle $H^1(X, \mathcal{O}_X)_{P_i}$ on P_i for $i = 1, 2$, and we denote these splittings by $H^1(X, \mathcal{O}_X)_{P_1} \simeq \hat{F}_2 \oplus \hat{F}_5$ and $H^1(X, \mathcal{O}_X)_{P_2} \simeq \hat{G}_1 \oplus \hat{G}_5$ where the bundles $\hat{F}_2, \hat{F}_5, \hat{G}_1, \hat{G}_5$ are all trivial.

Now, as $H^1(\mathbb{P}^n, \mathcal{O}(1)) = 0$ for any projective space, we can choose a splitting of the exact sequence

$$0 \to H^1(X, \mathcal{O}_X)_{P_i} \to N_{P_i,U_x} \to \mathcal{O}_{P_i}(-1)^{\oplus g-2} \to 0$$

for $i = 1, 2$ to give direct sum decompositions as follows.

$$N_{P_1,U_x} \simeq \hat{F}_2 \oplus \hat{F}_5 \oplus \mathcal{O}_{P_1}(-1)^{\oplus g-2}$$

and

$$N_{P_2,U_x} \simeq \hat{G}_1 \oplus \hat{G}_5 \oplus \mathcal{O}_{P_1}(-1)^{\oplus g-2}$$

Now, the normal bundles N_{L_i,U_x} for $i = 1, 2$ fit in the exact sequence

$$0 \to N_{L_i,P_i} \to N_{L_i,U_x} \to N_{P_i,U_x}|L_i \to 0$$

As $N_{L_i,P_i} \simeq \oplus^{g-2}\mathcal{O}_{L_i}(1)$, the above exact sequence splits, and so we can choose a direct sum decomposition

$$N_{L_i,U_x} \simeq \mathcal{O}_{L_i}(1)^{\oplus g-2} \oplus N_{P_i,U_x}|L_i$$

Let F_2, F_5 be the restricted bundles $\hat{F}_2|L_1$ and $\hat{F}_5|L_1$ on L_1 while let $G_1 = \hat{G}_1|L_2$ and $G_5 = \hat{G}_5|L_2$. Note that F_2, F_5, G_1, G_5 are trivial bundles. Now define bundles F_3 and F_4 on L_1 and G_3, G_4 on L_2 by $F_3 = \mathcal{O}_{L_1}(1)^{\oplus g-2}$, $F_4 = \mathcal{O}_{L_1}(-1)^{\oplus g-2}$,

$$G_3 = \mathcal{O}_{L_2}(-1)^{\oplus g-2}, \quad \text{and} \quad G_4 = \mathcal{O}_{L_2}(1)^{\oplus g-2}$$

Then we get the direct sum decompositions

$$N_{L_1,U_x} \simeq F_2 \oplus F_3 \oplus F_4 \oplus F_5$$

while

$$N_{L_2,U_x} \simeq G_1 \oplus G_3 \oplus G_4 \oplus G_5.$$

Now from the definitions of the V_i's, F_i's and G_i's above it is immediate that the hypothesis of lemma 2.1 is satisfied. Hence by lemma 2.1, the normal bundle of $l_x = L_1 \cup L_2$ in U_x has the direct sum decomposition

$$N_{l_x, U_x} \simeq F_2(s) \vee G_1(s) \oplus F_3 \vee G_3 \oplus F_4 \vee G_4 \oplus F_5 \vee G_5.$$

$$F_2(s) \vee G_1(s) \cong \mathcal{O}_{L_1}(1) \vee \mathcal{O}_{L_2}(1) = A$$

while it is obvious that

$$F_3 \vee G_3 \cong (\mathcal{O}_{L_1}(1) \vee \mathcal{O}_{L_2}(-1))^{\oplus g-2} = B,$$

$$F_4 \vee G_4 \cong (\mathcal{O}_{L_1}(-1) \vee \mathcal{O}_{L_2}(1))^{\oplus g-2} = C$$

and

$$F_5 \vee G_5 \cong \mathcal{O}_{l_x}^{\oplus g-1} = D$$

This completes the proof of the lemma.

§3. The conic bundle over the desingularization

We first recall the definition of a conic bundle as given in §4 of [5].

DEFINITION 3.1: A scheme C together with a very ample line bundle h is said to be a *conic* if its Hilbert polynomial is $2m + 1$. A scheme C over T is said to be a *conic bundle* over T if with respect to a line bundle on C its fibres are conics and the morphism $C \to T$ is proper and faithfully flat.

For example, let Hilb $(U_x)'$ be the open subset of Hilb (U_x) where the Hilbert polynomial is $2m + 1$. Then the correspondence variety $\mathscr{C} \subset U_x \times$ Hilb $(U_x)'$ is a conic bundle over Hilb $(U_x)'$.

Remark 3.2: Let E be a rank 3 vector bundle on T together with a global section q of the projective bundle $P(S^2E^*)$ where S^2E^* is the second symmetric power of E^*. This section q defines a subscheme $C \subset P(E)$ such that $C \to T$ is a conic bundle as defined above. By [5] Remark 4.4 (iv), every conic bundle $C \to T$ arises in the above manner, where a possible canonical choice for the vector bundle E is the first direct image $R^1(\pi)_*\omega^2$ where π: $C \to T$ is the projection and ω is the relative dualizing sheaf on C over T. In the subsequent discussion we assume that every conic bundle C is defined as a subscheme of $P(E)$ where $E = R^1(\pi_*)\omega^2$. However, it can be shown that (though we do not need it) the *type* of the conic bundle (see below) is independent of the choice of the vector bundle E.

Note that a global section of $P(S^2E^*)$ is the same as a line subbundle $L \subset S^2E^*$ which defines a quadratic form q on E with values in L^*. The discriminant det q of q is a section of the line bundle $(\Lambda^3E^*)^2 \otimes L^{-3}$. The degeneration locus $Y \subset T$ of the conic bundle $C \to T$ is defined by the vanishing of det q. At a point $t \in Y$, the fibre C_t is a double line or a pair of distinct lines depending respectively on whether rank q is 1 or 2.

DEFINITION 3.3: Let t be a smooth variety and $C \to T$ be a conic bundle for which the degeneration locus Y is an irreducible divisor. Let C degenerate into pairs of distinct lines at all points of Y. Then the *type* $\tau(C)$ of the conic bundle is by definition the multiplicity of vanishing of the discriminant det q over the divisor Y.

Consider the morphism ϕ: $H_0 \to$ Hilb $(U_x)'$ defined in §1 where Hilb $(U_x)'$ is the open subset of Hilb (U_x) for which the Hilbert polynomials is $2m + 1$. The correspondence variety over Hilb $(U_x)'$ is a conic bundle $\mathscr{C}$ and hence its pull-back $\phi^*\mathscr{C}$ to H_0 is a conic bundle on H_0. The degeneration locus of $\phi^*\mathscr{C}$ is the closed subset $\pi^{-1}(K)$ which is the inverse image of the singular locus $K \subset U_0$ of U_0 under the map π: $H_0 \to H_0$. Let $K_0 \subset K$ be the set of nodal points of K. Let $Z \subset H_0$ be the open subset $Z = H_0 - \pi^{-1}(K_0)$. Let the conic bundle $C \to Z$ be the restriction of $\phi^*\mathscr{C} \to H_0$ to $Z \subset H_0$. Then the degeneration locus for C is the irreducible divisor $Y = \pi^{-1}(K - K_0) \subset Z$.

LEMMA 3.5: *The degeneration locus Y is a non-singular variety.*

Proof: Let $R \to X \times J'$ be the Poincaré line bundle, where $J' \subset J$ equals $\{j | j^2 \neq 1\}$. Then the first direct images $R^1(p_2)_*R^2$ and $R^1(p_2)_*R^{-2}$ are locally free sheaves on J', each of rank $g - 1$. Let $\tilde{Y} \to J'$ be the $P^{g-2} \times P^{g-2}$ bundle which is the fibred product of the corresponding projective bundles. As described in [5] §7, each point of $\tilde{Y}$ determines a limit Hecke cycle $l = l_1 \cup l_2 \subset U_X$ in a canonical manner, and so $\tilde{Y}$ parametrizes an algebraic family of subschemes of U_X each of which is a limit Hecke cycle and hence has the same Hilbert polynomial. As $\tilde{Y}$ is an integral scheme, the above family is flat and so we get a morphism f: $\tilde{Y} \to$ Hilb (U_X). It is easy to see that Y is the image of this morphism. As $\tilde{Y}$ is non-singular, to prove the non-singularity of Y it is enough to show that the tangent level map df is injective. Let g: $\tilde{Y} \to$ Hilb (U_x) be the composite map $g = \phi \circ f$ where ϕ: $H_0 \to$ Hilb (U_x) is as defined earlier. Then it is enough to prove that dg is injective. since the one point union $P(D_{j,x}) \cup P(D_{j^{-1},x})$ imbeds in U_x, it is obvious that dg is injective along the fibres of $\tilde{Y} \to J'$. On the other hand, let $b \in \tilde{Y}$ be over $j \in J'$, and let $v \in T_b \tilde{Y}$ project to a non-zero vector $w \in T_j J'$. Let the image $g(b)$ of b in Hilb (U_x) be the restricted Hecke cycle denoted by $L_1 \cup L_2$. Note that $L_1 \cap L_2$ is the point $s(j, x) = \mu(j)$ defined in §2. Then $v \in T_b \tilde{Y}$ corresponds to an infinitesimal deformation of $L_1 \cup L_2$ such that the corresponding infinitesimal displacement of the intersection point $\mu(j)$ in U_x is the tangent vector d$\mu(w)$, which is non-zero by §2. Hence the infinitesimal deformation of $L_1 \cup L_2$ is non-zero. Hence dg is injective, which proves the lemma.

Over Y, the conic bundle $C \to Z$ degenerates into pairs of distinct lines. This defines a 2-sheeted covering of Y. The following lemma is obvious and we omit its proof.

LEMMA 3.6: *The 2-sheeted cover of the degeneration locus Y defined by the conic bundle $C \to Z$ is not split. In fact, it has the same topological structure as the pull-back of the 2-sheeted cover $J' \to K'$ (where $J' = \{j \in J \mid j^2 \neq 1\}$ and $K' = K - K_0$) under the map π: $Y \to K'$.*

Next consider the restriction C_{Z-Y} of $C \to Z$ to the open set $Z - Y$, over which it defines a P^1-bundle. Under π: $H_0 \to U_0$ the open subset $Z - Y$ maps isomorphically onto the open set $U_0 - K$. Let H_x be the dual projective Poincaré bundle on U_x where $U_x = U_x \times \{x\} \subset U_x \times X$ and consider the Hecke map h: $H_x \to U_0$ as defined in [5] §5.

LEMMA 3.7: *There exists an isomorphism of varieties from C_{Z-Y} to $H_x - h^{-1}$ (K) which makes the following diagram commutative*

$$\begin{tikzcd} C_{Z-Y} \arrow[r, "\sim"] \arrow[d] & H_x - h^{-1}(K) \arrow[d] \\ Z-Y \arrow[r, "\sim", "\pi"'] & U_0 - K \end{tikzcd}$$

Proof: Let C_1 be the correspondence variety over the Hilbert scheme Hilb (U_X), which contains H_0 as a closed subscheme $H_0 \subset$ Hilb (U_X). Let H be the dual projective Poincaré bundle on U_X, with the Hecke map h: $H \to U_0$. Then it follows from the identification of $H_0 \subset$ Hilb (U_X) with an appropriate closed subset of the relative Hilbert scheme Hilb (H, U_0) (see [5] §5 onward) that the restriction of the correspondence variety C_1 over Hilb (U_X) to the subset $Z - Y$ is canonically isomorphic to the fibration h: $H - h^{-1}$ $(K) \to U_0 - K$ under the isomorphism π: $Z - Y \xrightarrow{\sim} U_0 - K$. Now by definition of the morphism ϕ: $H_0 \to$ Hilb (U_x), we just have to intersect the family $H - h^{-1}$ $(K) \subset U_X \times (U_0 - K)$ with the subset

$U_x \times (U_0 - K)$ to obtain a family of subvarieties of U_x parameterized by $U_0 - K \simeq Z - Y$ whose classifying map is $\phi|Z - Y$. Now, $(H - h^{-1}(K)) \cap (U_x \times (U_0 - K))$ is $H_x - h^{-1}(K)$ which proves the lemma.

The remaining part of this section is devoted to providing the following result about the conic bundle C on Z. For the proof of this, we need all the lemmas proved in §1 and 2.

THEOREM 3.8: *The total space of the conic bundle C over Z is smooth.*

Proof: The theorem follows immediately from the following two lemmas (Lemmas 3.9 and 3.10).

LEMMA 3.9: *Let C be a conic bundle over a smooth variety Z for which the degeneration locus is a smooth irreducible divisor Y over which C degenerates into pairs of distinct lines. Let $L_1 \cup L_2$ be one of these degenerate conics. Let $L_1 \cap L_2 = \{s\}$. If C is non-singular at the point $s \in C$, then C is non-singular.*

LEMMA 3.10: *Let $l_x \subset U_x$ be the restricted special Hecke cycle for a general $j \in J$, which is represented by a point $y \in Y$, where Y is the degeneration locus of $C \to Z$. Let $l_x = L_1 \cup L_2$ and let $\{s\} = L_1 \cap L_2$. Then the variety C is smooth as s.*

Proof of Lemma 3.9: Let $\hat{Y} \subset C$ be the canonical section of the conic bundle over the degeneration locus Y, so that $\hat{Y}$ is exactly the set of all meeting points of the pairs of lines. Then, as is true for any conic bundle over a smooth variety such that the degeneration locus is also smooth over which it degenerates into pairs of distinct lines, the singular locus of C is contained in $\hat{Y}$. Since C is smooth at $s \in \hat{Y}$, it now follows that the image $\pi(s)$ of s under $\pi\colon C \to Z$ has an open neighbourhood V in Z such that the total space of the restriction of C to V is smooth. Hence by Proposition 3 of [6] the restriction of C to V is of type 1. Now it follows from the definition of type that $\tau(C) = 1$. Hence by Proposition 3 of [6], C is non-singular.

The remaining part of this section is devoted to the proof of Lemma 3.10.

LEMMA 3.11: *Let U be a smooth projective variety and let $W \subset U \times S$ be a flat family of closed subschemes of U parameterized by a variety S such that each fibre W_t is a local complete intersection for $t \in S$. Let $t \in S$ be a point and let P be a point on W_t. Then the Zariski tangent space $T_P W \subset T_P U \times T_t S$ to the total space W at P is the fibred product $T_P U \times_{N_P} T_t S$ where N_P is the fibre at P of the normal bundle $N_{W_t,U}$ of W_t in U and the fibred product is taken* *with respect to the canonical map $T_P U \to N_P$ and the composite map*

$$T_t S \to H^0(W_t, N_{W_t,U}) \to N_P$$

Proof: Lemma 3.11 follows from a standard argument which we sketch for the sake of completeness. It is enough to prove a corresponding statement in which the parameter variety S is replaced by Spec $k[\varepsilon]/\varepsilon^2$. Let W be the flat infinitesimal deformation of a local complete intersection $W_0 \subset U$ parameterized by Spec $k[\varepsilon]/\varepsilon^2$ which corresponds to an element $\xi \in H^0(W_0, N_{W_0,U})$. Then by looking at the defining equations for $W_0 \subset U$ in an affine neighbourhood of P in U, it is easy to see that the sections of W over Spec $k[\varepsilon]/\varepsilon^2$ which specialize (for $\varepsilon = 0$) to $P \in W_0$ are in a linear bijection with the elements of $T_P U$ which map to $\xi_P \in N_P$ under the canonical map $T_P U \to N_P$. It follows that the tangent space to W at $P \in W_0$ is the inverse image of $\langle \xi_P \rangle$ under $T_P U \to N_P$. This is the desired result over Spec $k[\varepsilon]/\varepsilon^2$.

LEMMA 3.12: *Let E_0, E_1, E_2 be vector spaces with linear maps $f_1: E_1 \to E_0$ and $f_2: E_2 \to E_0$.* Let *E_3 be another vector space with a linear map $g: E_3 \to E_2$ such that*

(i) *g is injective*

(ii) *There exists a vector $\eta \in E_2$ such that $\eta \notin g(E_3)$ and $f_2(\eta) = 0$. Then we have*

$$\dim (E_1 \times_{E_0} E_2) > \dim (E_1 \times_{E_0} E_3).$$

Proof: Because $g: E_3 \to E_2$ is injective, we have an inclusion $E_1 \times_{E_0} E_3 \subset E_1 \times_{E_0} E_2$. Now as $f_2(\eta) = 0$, the pair $(0, \eta) \in E_1 \times_{E_0} E_2$. But $\eta \notin g(E_3)$, and hence $E_1 \times_{E_0} E_3$ is a proper subspace. This proves the inequality as all spaces are assumed to be finite dimensional.

We can now prove Lemma 3.10 by assuming the following two lemma which are proved later.

LEMMA 3.13: *Let $l_x \subset U_x$ be the restricted special Hecke cycle for a general $j \in J$, and let $s \in l_x$ be the intersection point $L_1 \cap L_2$ where $l_x = L_1 \cup L_2$. Let N_s be the fibre of the normal bundle N_{l_x,U_x} at s. Let f_1: $T_s U_x \to N_s$ and f_2: $H^0(l_x, N_{l_x,U_x}) \to N_s$ be the canonical maps. Then the fibred product $T_s U_x \times_{N_s} H^0(l_x, N_{l_x,U_x})$ is of dimension $3g - 1$ where $g =$ genus(X).*

LEMMA 3.14: *Let $l \subset U_X$ be the special Hecke cycle for a general j. Consider the restriction map*

$$g: H^0(l, N_{l,U_X}) \to H^0(l_x, N_{l_x,U_x})$$

where $N_{l_x, U_x} \simeq N_{l, U_X}|l_x$ *as proved before. Then there exists* $\eta \in H^0(l_x, N_{l_x, U_x})$ *such that at* $s \in l_x$, $\eta(s) = 0 \in N_s$, *and* η *does not lie in the image of* g.

Proof of Lemma 3.10: The dimension of H_0 is $3g - 3$ and so the dimension of C is $3g - 2$. Hence to prove the non-singularity of C at s, it is enough to prove that $\dim T_s C < 3g - 1$. Now, by lemma 3.11, $T_s C$ is the fibred product $T_s U_x \times_{N_s} T_y Z$. Now, $T_y Z \simeq H^0(l, N_{l, U_X})$ and the map $T_y Z \to N_s$ is the composite map

$$H^0(l, N_{l, U_X}) \to H^0(l_x, N_{l_x, U_x}) \to N_s$$

By Lemma 1.3 of §1, the map

$$H^0(l, N_{l, U_X}) \to H^0(l_x, N_{l_x, U_x})$$

is injective. By lemma 3.12 and 3.14, we get the inequality

$$\dim T_s C < \dim T_s U_x \times_{N_s} H^0(l_x, N_{l_x, U_x}).$$

Hence by Lemma 3.13, $\dim T_s C < 3g - 1$. This proves the lemma.

Proof of Lemma 3.13: Let $f_1 : E_1 \to E_0$ and $f_2 : E_2 \to E_0$ be vector space homomorphisms. Then

$$\dim(E_1 \times_{E_0} E_2) = \dim(\operatorname{im}(f_1) \cap \operatorname{im}(f_2)) + \dim\ker(f_1) + \dim\ker(f_2)$$

as follows from the short exact sequence

$$0 \to \ker(f_1) \oplus \ker(f_2) \to E_1 \times_{E_0} E_2 \to \operatorname{im}(f_1) \cap \operatorname{im}(f_2) \to 0$$

In the present situation, f_1 is the canonical map $T_s U_x \to N_s$ where N_s is the fibre of the normal bundle of $l_x = L_1 \cup L_2$ at s. Hence the kernel of f_1 is of dimension 2 as it is the subspace $T_s L_1 \oplus T_s L_2$, while with respect to the decomposition

$$N_{l_x, U_x} \simeq A \oplus B \oplus C \oplus D$$

given by Lemma 2.11, it is easy to see that the image of $T_s U_x \to N_s$ is $B_s \oplus C_s \oplus D_s$. Also, it follows immediately from Lemma 2.11 that the kernel of the natural map $f_2 : H^0(l_x, N_{l_x, U_x}) \to N_s$ is of dimension $2g - 2$, while the image of f_2 is $A_s \oplus D_s$. It follows that $\operatorname{im}(f_1) \cap \operatorname{im}(f_2) = D_s$, which is of dimension $g - 1$. Hence dim $T_s U_x \times_{N_s} H^0$ (l_x, N_{l_x, U_x}) equals $g - 1 + 2 + 2g - 2 = 3g - 1$. This proves the lemma.

To complete the proof of Theorem 3.8 only lemma 3.14 remains to be proved. We prove it by explicitly constructing the required element η.

Construction of η: By Lemma 2.11, $N_{l_x, U_x} \simeq A \oplus B \oplus C \oplus D$ where $A \simeq \mathcal{O}_{L_1}(1) \vee \mathcal{O}_{L_2}(1)$. Let η be a non-zero section of A such that η restricts to zero on L_2 (such an η is unique upto a non-zero constant multiple.)

Proof of Lemma 3.14: If possible, let η as constructed above be the restriction to l_x of some element $\eta' \in H^0$ (l, N_{l, U_X}). We show this leads to a contradiction. Since by definition η vanishes on $L_2 = l_{2,x}$, the restriction of η' to $l_{2,x}$ is zero. Hence by the proof of Lemma 1.4 of §1, η' vanishes all over l_2. Hence η' is an element of $\Gamma_0 \subset H^0$ (l, N_{l, U_X}), where Γ_0 denotes the set of all global sections of N_{l, U_X} which restrict to zero on l_2. Let $\Gamma_{0,x} \subset H^0$ (l_x, N_{l_x, U_x}) denote the set of all global sections of N_{l_x, U_x} which restrict to zero on $l_{2,x}$. Assume for the moment that there exist linear maps g and g_x which make the following diagram commute,

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{g} & H^0(l_1 \cap l_2, N_{l_1 \cap l_2, l_2}) \\ \downarrow & & \downarrow \\ \Gamma_{0,x} & \xrightarrow[g_x]{} & T_s L_2 \end{array} \tag{3.15}$$

where the vertical maps are the restriction maps. Moreover, assume that $g_x(\eta) \neq 0$. Now, $l_1 \cap l_2 \simeq X$, and $N_{l_1 \cap l_2, l_2} \simeq j^2$. Hence $H^0(l_1 \cap l_2, N_{l_1 \cap l_2, l_2}) = 0$. Hence $g(\eta') = 0$. This is a contradiction as $\eta' \mapsto \eta$ and $g_x(\eta) \neq 0$.

It now only remains to define the maps g and g_x which make the above diagram commute, such that $g_x(\eta) \neq 0$. For this consider the following general situation. Let U be a non-singular variety and let V_1, V_2 be smooth subvarieties whose scheme theoretic intersection $V_0 = V_1 \cap V_2$ is a smooth variety which is a divisor in both V_1 and V_2. Let V be the reduced scheme $V_1 \cup V_2$. By [5] lemma 8.2, V is a local complete intersection. Lemma 1.5 goes through in this case also, and we get a short exact sequence

$$0 \to N_{V_1, U} \to N_{V, U} | V_1 \to \tilde{N}_{V_0, V_1} \otimes \tilde{N}_{V_0, V_2} \to 0$$

In particular, any section of $N_{V,U} | V_1$ which vanishes over V_0 is the image of a section of $N_{V_1, U}$. Note however, that the image of a section of $N_{V_1, U}$ in $N_{V,U} | V_1$ does not necessarily vanish over V_0. We now describe a necessary and sufficient condition on a section $\sigma \in \Gamma$ $(V_1, N_{V_1, U})$ for its image $\sigma' \in \Gamma$

$(V_1, N_{V,U} | V_1)$ to vanish everywhere on the divisor $V_0 \subset V_1$. Consider the geometric vector bundles $N_{V_1,U} \to V_1$, and $N_{V_0,V_2} \to V_0$. Then there is a canonical imbedding of schemes $i: N_{V_0,V_2} \to N_{V_1,U}$ which makes the following diagram commute

$$\begin{array}{ccc} N_{V_0,V_2} & \xhookrightarrow{i} & N_{V_1,U} \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & V_1 \end{array}$$

Note that i is deduced by the composite morphism

$$TV_2 | V_0 \hookrightarrow TU | V_0 \longrightarrow N_{V_1,U} | V_0$$

which factors through N_{V_0,V_2}. Then following the proof of Lemma 1.5 we see that a section $\sigma \in H^0(V_1, N_{V_1,U})$ maps to a section of $H^0(V_1, N_{V,U} | V_1)$ which vanishes over V_0 if and only if $\sigma | V_0$ lies inside $i(N_{V_0,V_2}) \subset N_{V_1,U}$. We thus get a canonical linear map $g: \Gamma_0 \to H^0(V_0, N_{V_0,V_2})$ where $\Gamma_0 \subset H^0(V_1, N_{V,U} | V_1)$ is the subspace of all sections which vanish over V_0.

Now let the hypothesis of Lemma 1.1 of §1 be satisfied (with U in place of S). Then for any point $t \in T$ we have canonical isomorphisms $N_{V_t,S_t} \simeq N_{V,S} | V_t$ and $N_{V_{0,t},V_{2,t}} \simeq N_{V_0,V_2} | V_{0,t}$. These isomorphisms induce linear maps

$$H^0(V_1, N_{V,S} | V_1) \to H^0(V_{1,t}, N_{V_t,S_t} | V_{1,t})$$

and $H^0(V_0, N_{V_0,V_2}) \to H^0(V_0, N_{V_{0,t},V_{2,t}})$. The first of these maps induces a map $\Gamma_0 \to \Gamma_{0,t}$ where $\Gamma_{0,t} \subset H^0(V_{1,t}, N_{V_t,S_t} | V_{1,t})$ is the subspace of all sections which vanish over $V_{0,t}$. We also have canonical linear maps

$$g: \Gamma_0 \to H^0(V_0, N_{V_0,V_2}) \quad \text{and} \quad g_t: \Gamma_{0,t} \to H^0(V_{0,t}, N_{V_{0,t},V_{2,t}})$$

as defined in the last paragraph. It is immediate from the definition of these maps that the following diagram is commutative.

$$\begin{array}{ccc} \Gamma_0 & \longrightarrow & H^0(V_0, N_{V_0,V_2}) \\ \downarrow & & \downarrow \\ \Gamma_{0,t} & \longrightarrow & H^0(V_{0,t}, N_{V_{0,t},V_{2,t}}) \end{array}$$

Finally, note that Γ_0 is canonically isomorphic to the subspace of $H^0(V, N_{V,S})$ consisting of all sections which vanish everywhere on V_2 while similarly, $\Gamma_{0,t}$ is canonically isomorphic to the subspace of H^0 (V_t, N_{V_t,S_t}) consisting of all sections which vanish everywhere on $V_{2,t}$.

We apply the above discussion to the case where $S \to T$ is the map det: $U_X \to X$, $t \in T$ is the point $x \in X$, and $V \subset S$ is the special Hecke cycle l. This gives us the desired commutative diagram (3.15). From the definition of the map g_x it is clear that $g_x(\eta) \neq 0$ for the given η. This completes the proof of Lemma 3.14, and hence that of Theorem 3.8.

§4. Normal bundles of some divisors in H_0

In §4-A, we determine the normal bundles N_{T_k,H_0} of the divisors T_k which are contained in the fibres of $H_0 \to U_0$ over nodal points $k \in K$. (Prop 4.A.3). From this we deduce, in particular, that H_0 can be blown down along the fibres of the $\mathbb{P}^5$-fibrations $T_k \to G$ to obtain another non-singular model for U_0 (Proposition 4.A.2).

In §4-B, we determine the restriction of the normal bundle $N_{Y,Z}$ to a fibre of Y over a non-nodal point of K (Prop. 4.B.5). We use it in §4.C to prove a technical lemma (lemma 4.C.1) about the Chern class of $N_{Y,Z}$.

Section 4A

We first define the *geometric normal bundle* $\mathbb{N}_{U,V}$ of a smooth subvariety U of a scheme V. Let I be the ideal sheaf of U in V, and let $S(I/I^2)$ be the symmetric algebra of I/I^2. Then $\mathbb{N}_{U,V}$ is by definition of the linear bundle Spec $S(I/I^2)$ over U. The fibre of the scheme $\mathbb{N}_{U,V}$ over a point $P \in U$ is isomorphic to $T_P V/T_P U$. In case V is also smooth, $\mathbb{N}_{U,V}$ is just the total space of the usual normal bundle $N_{U,V}$.

Let $k \in K \subset U_0$ be a nodal point of K. The fibre of $H_0 \to U_0$ over k is the union $S_k \cup T_k$ where S_k is a $\mathbb{P}^{g-2}$ bundle over the Grassmannian of lines in $\mathbb{P}^{g-1}$ while T_k is canonically isomorphic to the space of all conics in PH^1 $(X, \mathcal{O}_X)$ (which is a $\mathbb{P}^5$ bundle over the Grassmannian of planes in a $\mathbb{P}^{g-1}$). Let $W = H^1(X, \mathcal{O}_X)$, and let $G =$ Grass $(3, W)$ be the space of 3 dimensional linear subspaces of W. Consider the tautological rank 3 subbundle E of the vector bundle $W \times G$ on G. Then for each nodal point $k \in K$, T_k is isomorphic to the total space of the projective bundle $P(S^2E^\vee)$ on G, where $S^2E^\vee$ is the rank 6 vector bundle of quadratic forms on E.

PROPOSITION 4.A.1: *The restriction of the normal bundle N_{T_k,H_0} to a fibre of $T_k \to G$ is isomorphic to $\mathcal{O}_{\mathbb{P}^5}(-1)$.*

The above proposition immediately implies the following.

PROPOSITION 4.A.2: *The variety H_0 can be blown down along the fibrations $T_k \to G$ (one for each node k) to obtain another smooth variety.*

Proposition 4.A.1 follows from the next proposition by restricting the line bundle N_{T_k, H_0} to a fibre of $T_k \to G$.

PROPOSITION 4.A.3: *Let π: $P(S^2E^\vee) \to G$ be the projection. Then under the canonical isomorphism of T_k with $P(S^2E^\vee)$, N_{T_k, H_0} is isomorphic to the line bundle $\pi^*\Lambda^3E \otimes \mathcal{O}_{P(S^2E^\vee)}(-1)$.*

Proof: Let $C \to P(S^2E^\vee)$ be the bundle of all conics in $PH^1(X, \mathcal{O}_X)$. Note that the total space C of this conic bundle is a subvariety of the fibre product $P(E) \times_G P(S^2E^\vee)$. On $P(E)$, we have the tautological exact sequence

$$0 \to \mathcal{O}_{P(E)}(-1) \to \pi^*_{P(E)}E \to Q \to 0$$

where Q is the quotient bundle (rank $Q = 2$). This gives an isomorphism

$$\mathcal{O}_{P(E)}(1) \otimes Q^\vee \xrightarrow{\sim} Q \otimes \pi^*_{P(E)}\Lambda^3E^\vee \qquad \ldots\ldots\ldots \qquad (1)$$

Let π_1 and π_2 be the projections of the fibred product $P(E) \times_G P(S^2E^\vee)$ on to the two factors. Then we have a natural morphism

$$\pi_1^*E \otimes \pi_1^*E \otimes \pi_2^*S^2E^\vee \to \mathcal{O}_{P(E)\times_G P(S^2E^\vee)}$$

When restricted to $C \subset P(E) \times_G P(S^2E^\vee)$, it induces a morphism

$$\pi_1^*\mathcal{O}_{P(E)}(-1) \otimes \pi_1^*Q \otimes \pi_2^*\mathcal{O}_{P(S^2E^\vee)}(-1)|_C \to \mathcal{O}_C.$$

Let the restrictions of $\pi_1^*\mathcal{O}_{P(E)}(-1)$ and $\pi_2^*\mathcal{O}_{P(S^2E^\vee)}(-1)$ to C be denoted respectively by $\mathcal{O}_C(-1, 0)$ and $\mathcal{O}_C(0, -1)$, so that the above morphism can be written as

$$\mathcal{O}_C(-1, 0) \otimes Q_C \otimes \mathcal{O}_C(0, -1) \to \mathcal{O}_C$$

where Q_C denotes the pullback of Q under $C \to P(E)$. This induces a morphism.

$$\mathcal{O}_C(0, -1) \to Q_C^\vee \otimes \mathcal{O}_C(1, 0).$$

Composing this morphism with the earlier isomorphism (1), we get a morphism

$$\mathcal{O}_C(0, -1) \to Q_C \otimes \Lambda^3 E_C^\vee$$

where E_C is the pullback of E under $C \to G$. This induces a morphism

$$\Lambda^3 E_C \otimes \mathcal{O}_C(0, -1) \to Q_C$$

Now, taking direct image under

$$\pi_2 \colon C \to P(S^2 E^\vee),$$

we get a morphism

$$\pi^* \Lambda^3 E \otimes \mathcal{O}_{P(S^2 E^\vee)}(-1) \to (\pi_2)_* Q_C \qquad \ldots . \qquad (2)$$

of sheaves on $P(S^2 E^\vee)$. Now, it is easy to check that for any conic $C_u \subset PH^1(X, \mathcal{O}_X)$ (which is the fibre of $C \to P(S^2 E^\vee)$ over a point u), dim $H^0(C_u, Q_{C_u}) = 4$. Hence $(\pi_2)_* Q_C$ is locally free of rank 4. For any conic C_u, (2) gives a linear map

$$\varphi_u \colon (\pi^* \Lambda^3 E \otimes \mathcal{O}_{P(S^2 E^\vee)}(-1))_u \to H^0(C_u, Q_{C_u}),$$

whose domain is a 1-dimensional vector space. By using explicit equations for C_u in terms of coordinates on $PH^1(X, \mathcal{O}_X)$, it can be checked that the map φ_u is injective, and further, its image is contained in the kernel of the map

$$H^0(C_u, Q_{C_u}) \to H^0(C_u, \mathcal{O}_{P(E)_u}(-1) \otimes N_{C_u, P(E)_u})$$

(where $P(E)_u \subset PH^1(X, \mathcal{O}_X)$ is the 2-plane containing C_u) which is induced by the composite map

$$Q_{C_u} \xrightarrow{\sim} \mathcal{O}_{P(E)_u}(-1) \otimes TP(E)_u|_{C_u} \to \mathcal{O}_{P(E)_u}(-1) \otimes N_{C_u, P(E)_u}$$

Now, let N_C be the normal bundle to C in $P(E) \times_G P(S^2 E^\vee)$, and let $T^\pi_{P(E)}$ be the tangent bundle to $P(E)$ along the fibres of $P(E) \to G$. Let the pullback of $T^\pi_{P(E)}$ to C again be denoted by $T^\pi_{P(E)}$. Note that $\mathcal{O}_C(-1, 0) \otimes T^\pi_{P(E)} \simeq Q_C$. The morphism $T^\pi_{P(E)} \to N_C$ induces a morphism $Q_C \to \mathcal{O}_C(-1, 0) \otimes N_C$. Taking direct image under $\pi_2 \colon C \to P(S^2 E^\vee)$, we get a morphism

$$(\pi_2)_* Q_C \to (\pi_2)_* (\mathcal{O}_C(-1, 0) \otimes N_C).$$

Now, for any $u \in P(S^2E^\vee)$, the restriction of N_C to C_u is isomorphic to $N_{C_u,P(E)_u}$, and it is easy to check that for all u,

$$\dim H^0(C_u, \mathcal{O}_{P(E)_u}(-1) \otimes N_{C_u,P(E)_u}) = 3.$$

Hence, $(\pi_2)_*(\mathcal{O}_C(-1, 0) \otimes N_C)$ is a vector bundle of rank 3.

We thus have the morphisms

$$\pi^*\Lambda^3E \otimes \mathcal{O}_{P(S^2E^\vee)}(-1) \to (\pi_2)_*Q_C \to (\pi_2)_*(\mathcal{O}_C(-1, 0) \otimes N_C)$$

of vector bundles on $P(S^2E^\vee)$ such that the first morphism is everywhere injective, while the composite is identically zero. Now, it follows from the proof of Proposition 8.5 of [5] that the geometric normal bundle $\mathbb{N}_{T_k,H_2}$ to $T_k\,(\simeq P(S^2E^\vee))$ in the scheme H_2 (which contains H_0) is canonically isomorphic to the inverse image of the zero section of the vector bundle morphism

$$(\pi_2)_*Q_C \to (\pi_2)_*(\mathcal{O}_C(-1, 0) \otimes N_C).$$

Hence, the line bundle $\pi^*\Lambda^3E \otimes \mathcal{O}_{P(S^2E^\vee)}(-1)$ is a subbundle of $\mathbb{N}_{T_k,H_2}$. On the other hand, the normal bundle N_{T_k,H_0} is also a subbundle of $\mathbb{N}_{T_k,H_2}$. By lemma 8.9 of [5], $\mathbb{N}_{T_k,H_2}$ is of rank 1 over the open subset of T_k corresponding to non-degenerate conics. Hence the line subbundles $\pi^*\Lambda^3 \otimes \mathcal{O}_{P(S^2E^\vee)}(-1)$ and N_{T_k,H_0} coincide over an open subset, and hence coincide everywhere. This completes the proof of proposition 4.A.3.

Section 4B

Let $C \to Z$ be the conic bundle of §3, with degeneration locus Y. The fibre F_k of $Y \to K$ over a non-nodal point $k \in K$ is isomorphic to $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$.

PROPOSITION 4.B.1: *The restriction of $N_{Y,Z}$ to F_k is isomorphic to the line bundle $\mathcal{O}(-1, -1)$ on $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \simeq F_k$.*

The proof of the proposition depends on the following lemmas.

LEMMA 4.B.2: *Let $y \in Y$, and let $\hat{y} \in \hat{Y}$ lie over y, where $\hat{Y} \subset C$ is the canonical section of C over Y. Then the image of the differential map* $d\pi$: *$T_yC \to T_yZ$ is equal to T_yY.*

Proof: As C is smooth (§3) one has $\dim T_{\hat{y}}C = \dim C$. As $\hat{Y} \to Y$ is an isomorphism T_yY is contained in the image of $d\pi$. As $\hat{y}$ is the intersection point of the pair of lines $L_1 \cup L_2$ which is the fibre of C over y, the kernel of $d\pi$ is at least 2 dimensional. Now, $\dim T_{\hat{y}} C = \dim T_y Y + 2$. Hence the lemma follows.

LEMMA 4.B.3: *Let C_y denote the fibre of C over $y \in Y$, which is a pair of lines in U_x meeting at a point s. Then the subspace $T_y Y \subset T_y Z$ is precisely the kernel of the composite map*

$$T_y Z \to H^0(C_y, N_{C_x, U_x}) \to \frac{(N_{C_x,U_x})_s}{\mathrm{im}\ T_s U_x}$$

Further, the induced map

$$(N_{Y,Z})_y \to (N_{C_x,U_x})_s/\mathrm{im}\ T_s U_x$$

is an isomorphism

Proof: Consider $C \subset U_x \times Z$. Then $\hat{y} = (y, s)$, and $T_{\hat{y}} C$ is the fibred product of $T_s U_x$ and $T_y Z$ over $(N_{C_x,U_x})_s$. Hence a vector $v \in T_y Z$ has a lift to $T_{\hat{y}} C$ if and only if its image in $(N_{C_x,U_x})_s$ is contained in the image of $T_s U_x$. But by lemma 4B2, $v \in T_y Z$ has a lift to $T_{\hat{y}} C$ if and only if $v \in T_y Y$. This proves the first assertion of the lemma and the induced map

$$(N_{Y,Z})_y \to (N_{C_x,U_x})_s/\mathrm{im}\ T_s U_x$$

is injective. As both these spaces are 1-dimensional, the above map is an isomorphism, which proves the lemma.

Remark 4.B.4: It is easy to see that there is a line bundle L on Y whose fibre at y is $(N_{C_x,U_x})_s/\mathrm{im}\ T_s U_x$. The above lemma infact shows that there is a canonical isomorphism $N_{Y,Z} \to L$.

Now, let L_1, L_2 be smooth curves in a smooth variety U, which meet transversally at a point $P \in U$. Then $(N_{L_1 \cup L_2, U})_P/\mathrm{im}\ T_P U$ is canonically dual to the kernel K_P of the linear map $N^*_{L_1 \cup L_2, U} \to T^*_P U$. An element of K_p is represented by an element $f \in \mathcal{O}_{P,U}$ which vanishes over $L_1 \cup L_2$ and which satisfies $df(P) = 0$. As $df = 0$, for any $u, v \in T_P U$ we can define the Hessian $u(v(f)) \in k$ of f with respect to u, v. This gives a well-defined linear map

$$T_P U \otimes T_P U \otimes K_P \to k,$$

which induces a linear map

$$T_P L_1 \otimes T_P L_2 \to (N_{L_1 \cup L_2, U})P/\mathrm{im}\ T_P U$$

which we call the *Hessian map*. The following lemma is easy to check using local coordinates.

LEMMA 4.B.5: *The Hessian map* $T_pL_1 \otimes T_pL_2 \to (N_{L_1 \cup L_2, U})_P/\mathrm{im}\, T_P V$ *is an isomorphism.*

Proof of Proposition 4.B.1: Let $k = \langle j \oplus j^{-1}\rangle$, be a non-nodal point of K. Then the points of F_k represent the degenerate conics of the form $L_1 \cup L_2 \subset U_x$, where L_1 and L_2 are lines in the subspaces $P_1 = P(D_{j,x})$ and $P_2 = P(D_{j^{-1},x})$ of U_x. Both L_1 and L_2 pass through the point $s = P_1 \cap P_2$. Hence, F_k is canonically isomorphic to $P(T_sP_1) \times P(T_sP_2)$. Note that the fibre of the line bundle $\mathscr{O}(-1, -1)$ on the above product over the point of F corresponding to $L_1 \cup L_2$ is $T_sL_1 \otimes T_sL_2$. Hence the proposition follows from lemmas 4.B.3 and 4.B.5.

Section 4C

In this subsection we prove the following.

PROPOSITION 4.C.1: *Let* $C \in H^2(Y, \mathbb{Q})$ *denote the rational Chern class of the line bundle* $N_{Y,Z}$. *Then the map* $H^2(Y, \mathbb{Q}) \to H^4(Y, \mathbb{Q})$ *which sends* η *to* $C \cup \eta$ *is injective.*

Proof: All cohomology groups considered here are with coefficients $\mathbb{Q}$. Consider the 2-sheeted cover $\tilde{Y} \to Y$; the map $H^2(Y) \to H^2(\tilde{Y})$ is injective. Note that $\tilde{Y}$ is the fibre product $E_1 \times_{J'} E_2$ where E_1 and E_2 are the $\mathbb{P}^{g-2}$ bundles whose fibres over j are $PH^1(X, j^2)$ and $PH^1(X, j^{-2})$ respectively. Let $\theta_1, \theta_2 \in H^2(\tilde{Y})$ denote the rational chern classes of the pullbacks to $\tilde{Y}$ of the line bundles $\mathscr{O}_{E_1}(1)$ and $\mathscr{O}_{E_2}(1)$. The natural $\mathbb{Z}/2$-action on $\tilde{Y}$ interchanges θ_1 and θ_2. By the Leray–Hirsch theorem, $H^*(\tilde{Y})$ is freely generated as a module over $H^*(J')$ by the elements $\theta_1^{m_1}\theta_2^{m_2} \in H^{2m_1+2m_2}(\tilde{Y})$ for $0 \leqslant m_1, m_2 \leqslant g - 2$. Hence the elements of $H^2(\tilde{Y})$ which are $\mathbb{Z}/2$-invariant have the form $a + \lambda(\theta_1 + \theta_2)$, where $a \in H^2(J')$ and $\lambda \in \mathbb{Q}$.

Now, it follows from Prop. 4.B.1 that the pullback to $\tilde{Y}$ of the chern class $C = C_1(N_{Y,Z}) \in H^2(Y)$ has the form

$$\tilde{C} = a - (\theta_1 + \theta_2)$$

where $a \in H^2(J')$. We shall show that for any $\eta \in H^2(Y)$, if $C \cup \eta = 0$ then $\eta = 0$. Let $\tilde{\eta} \in H^2(\tilde{Y})$ be the pullback of η. As $\tilde{\eta}$ is invariant $\tilde{\eta} = b + \lambda(\theta_1 + \theta_2)$ for some $b \in H^2(J')$ and $\lambda \in \mathbb{Q}$. Then $\tilde{\eta} \cup \tilde{C} = a \cup b + (\lambda a - b)(\theta_1 + \theta_2) - \lambda(\theta_1 + \theta_2)^2$. If $\tilde{\eta} \cup \tilde{C} = 0$, then by successively equating to zero the coefficient of $\theta_1\theta_2$ and $\theta_1 + \theta_2$, we see that $\tilde{\eta} = 0$. Hence $\eta = 0$.

Remark 4.C.2. From the description of the normal bundle N_{T_k,H_0} given by Proposition 4.A.1, it is easy to check that cup product with the integral Chern class of N_{T_k,H_0} defines an injective map from $H^2(T_k, \mathbb{Z})$ to $H^4(T_k, \mathbb{Z})$.

§5. Determination of $H^3(H_0)$

In this section we complete the proof of our main theorem that $H^3(H_0) \simeq \mathbb{Z}^{2g}$. All homology and cohomology groups are with coefficients $\mathbb{Z}$ unless otherwise indicated.

LEMMA 5.1: *Let $P \to B$ be a topological $\mathbb{P}^1$ bundle over a path connected base space B with $H^1(B) = 0$. Then the homomorphism $H^3(B) \to H^3(P)$ is surjective, and its kernel is generated by the toplogical Brauer class of the $\mathbb{P}^1$ bundle.*

Proof: Applying the Leray spectral sequence and using that $H^0(B) = \mathbb{Z}$, $H^1(B) = 0$, it is easy to check that the homomorphism $H^3(B) \to H^3(P)$ is surjective, and its kernel is generated by a single element γ which is the image of the generator of $H^0(B, H^2(\mathbb{P}^1))$ under the transgression map $E_3^{0,2} \to E_3^{3,0}$. Since twice the generator of $H^0(B, H^2(\mathbb{P}^1))$ comes from the Chern class of the tangent bundle along fibres of P, we see that $2\gamma = 0$. Hence, $\{0, \gamma\}$ is the kernel of $H^3(B) \to H^3(P)$. Let $\beta \in H^3(B)$ be the topological Brauer class of $P \to B$. The $\mathbb{P}^1$-bundle on P which is the pull back of $P \to B$ has a canonical global section, hence is topologically banal. By the functoriality of β, it follows that β lies in the kernel of $H^3(B) \to H^3(P)$, i.e., $\beta \in \{0, \gamma\}$. Finally, when $\beta = 0$, the $\mathbb{P}^1$-bundle $P \to B$ is topologically banal, and hence the map $H^3(B) \to H^3(P)$ is an inclusion. Hence when $\beta = 0$, we must have $\gamma = 0$. It follows that $\beta = \gamma$, which proves the lemma.

Next, as in §3, let H_x be the dual projective Poincaré bundle on $U_x(= U_x \times \{x\} \subset U(2, 1) \times X)$, and let h: $H_x \to U_0$ be the Hecke map.

LEMMA 5.2: *We have $H^3(H_x - h^{-1}(K)) \simeq \mathbb{Z}^{2g}$.*

Proof: By Alexander duality, $H_i(H_x, H_x - h^{-1}(K)) \simeq H^{n-i}(h^{-1}(K))$ where $n = \dim_{\mathbb{R}} H_x = 6g - 4$. Substituting this in the long exact homology sequence of the pair $(H_x, H_x - h^{-1}(K))$ we get the exact sequence

$$H_4(H_x) \to H^{n-4}(h^{-1}(K)) \to H_3(H_x - h^{-1}(K)) \to H_3(H_x)$$
$$\to H^{n-3}(h^{-1}(K)) \to H_2(H_x - h^{-1}(K)) \to H_2(H_x) \tag{1}$$

Now, $h^{-1}(K)$ is an irreducible variety of dimension $2g - 1$, so $\dim_{\mathbb{R}} h^{-1}(K) = 4g - 2$. Hence $H^{n-3}(h^{-1}(K)) = 0$ for $g \geqslant 3$ and $H^{n-4}(h^{-1}(K)) = 0$ for $g > 3$. When $g = 3$, $\dim_{\mathbb{R}} h^{-1}(K) = n - 4$. Observe that if V is an irreducible subvariety of dim r of a smooth projective variety W, then the map $H_{2r}(V) \to H_{2r}(W)$ is injective, and hence the map $H^{2r}(W) \to H^{2r}(V)$ has a torsion cokernel. Hence when $g = 3$, the map $H_4(H_x) \to H^{n-4}(h^{-1}(K))$ (which is just the natural map $H^{n-4}(H_x) \to H^{n-4}(h^{-1}(K))$ by Poincaré duality) has a torsion cokernel.

Hence, from (1) we see that

$$H_2(H_x - h^{-1}(K)) \subset H_2(H_x) \tag{2}$$

while there is a short exact sequence

$$0 \to T \to H_3(H_x - h^{-1}(K)) \to H_3(H_x) \to 0 \ldots . \tag{3}$$

where T is a torsion group. As H_x is the projective bundle given by a vector bundle on U_x the homology of H_x is the Kunneth product of the homologies of U_x and $\mathbb{P}^1$. Now, U_x is simply connected, rank $H_3(U_x) = 2g$, and $H_2(U_x)$ is torsion free since as is easily seen, U_x is rational (see also Atiyah & Bott [2]). Hence by the exact sequences (2) and (3), $H_2(H_x - h^{-1}(K))$ is torsion free and rank $H_3(H_x - h^{-1}(K)) = 2g$. Now an application of the universal coefficient theorem completes the proof of the lemma.

LEMMA 5.3: *The cohomology group $H^3(Z - Y)$ is of rank $2g$, and its torsion subgroup is generated by the topological Brauer class β of the restriction of the conic bundle.*

Proof: By lemma 3.7, $H_x - h^{-1}(K)$ is isomorphic to the total space of the restriction of C to $Z - Y$. As U_x is simply connected, as H_x is a $\mathbb{P}^1$-bundle on U_x, and as $h^{-1}(K)$ is of real codimension at least 4 in H_x, it follows that $H_x - h^{-1}(K)$ and hence $Z - Y$ are simply connected. Now the lemma follows immediately from lemmas 5.1 and 5.2.

LEMMA 5.4: *The Chern class $\alpha \in H^2(Y)$ of the line bundle associated to the 2-sheeted covering of the degeneration locus Y of the conic bundle C is non-zero. Moreover, $H^1(Y) = 0$.*

Proof: By lemma 3.6, the 2-sheeted covering of Y is not split, hence it determines a non-zero element $\alpha^* \in H^1(Y, \mathbb{Z}/2)$. The element $\alpha \in H^2(Y)$ is the image of α^* under the connecting morphism for the cohomology sequence for the short exact coefficients sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$. Hence to prove that $\alpha \neq 0$, it suffices to prove that $H^1(Y, \mathbb{Z}) = 0$. Note that topologically Y is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ bundle over the space $K - K_0$ where K is homeomorphic to the Kummer variety and K_0 is the set of its nodal points. Moreover, $H^1(K - K_0) = 0$ (see [7]). Hence it follows immediately by the Leray spectral sequence for the fibration $Y \to K - K_0$ that $H^1(Y) = 0$.

The proof of the next lemma makes crucial use of the main result proved in [6].

LEMMA 5.5: *We have $H^3(Z) \simeq \mathbb{Z}^{2g}$.*

Proof: Consider the Gysin sequence

$$H^1(Y) \to H^3(Z) \to H^3(Z - Y) \to H^2(Y) \to H^4(Z)$$

Since by lemma 5.4, $H^1(Y) = 0$, $H^3(Z) \to H^3(Z - Y)$ is injective. By Theorem 3.8, the total space of the conic bundle $C \to Z$ is smooth. Hence by Corollary 1 of [6], the topological Brauer class $\beta \in H^3(Z - Y)$ maps to the Chern class $\alpha \in H^2(Y)$ of the line bundle associated to the 2-sheeted covering of Y. Since by lemma 5.4, $\alpha \neq 0$, β does not lie in the subgroup $H^3(Z)$ of $H^3(Z - Y)$. Since by lemma 5.3, $\{0, \beta\}$ is the torsion subgroup of $H^3(Z - Y)$ it follows $H^3(Z)$ is torsion free. Now, we have a commutative triangle

$$\begin{tikzcd}
H^2(Y, \mathbb{Q}) \arrow[rr] \arrow[dr] & & H^4(Z, \mathbb{Q}) \arrow[dl] \\
& H^4(Y, \mathbb{Q}) &
\end{tikzcd}$$

where the map $H^2(Y, \mathbb{Q}) \to H^4(Y, \mathbb{Q})$ is given by cup product with $C_1(N_{Y,Z})$. Since this map is injective by Proposition 4.C.1 the map $H^2(Y, \mathbb{Q}) \to H^4(Z, \mathbb{Q})$ is injective, so that $H^3(Z, \mathbb{Q}) \simeq H^3(Z - Y, \mathbb{Q})$. The lemma now follows from lemma 5.3.

Now consider the projection $\pi: H_0 \to U_0$. By theorem 8.14 of [5], the fibre over a nodal point $k \in K \subset U_0$ is the union $S_k \cup T_k$ where S_k is a $\mathbb{P}^{g-2}$-bundle over Grass $(2, g)$ while T_k is a $\mathbb{P}^5$-bundle over Grass $(3, g)$. Note that $\dim S_k = 3g - 6$, and $\dim T_k = 3g - 4$. Let T be the union of all T_k as k varies over the nodal points of K (2^{2g} copies in all).

Note that Z is open in $H_0 - T$, and $(H_0 - T) - Z = \bigcup_k (S_k - T_k)$ which is of complex dimension $3g - 6$, hence of real codimension 6 in $H_0 - T$. Hence $H^3(H_0 - T) \simeq H^3(Z) \simeq \mathbb{Z}^{2g}$.

Next, consider the Gysin sequence

$$H^1(T) \to H^3(H_0) \to H^3(H_0 - T) \to H^2(T) \to H^4(H_0)$$

Note that $H^1(T) = 0$. By a combination of remark 4.C.2 and an argument used in the proof of lemma 5.5, the map $H^2(T) \to H^4(H_0)$ is injective. Hence, $H^3(H_0) \simeq H^3(H_0 - T) \simeq \mathbb{Z}^{2g}$, which proves theorem 1.

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