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Harmonic diffeomorphisms, minimizing harmonic maps and rotational symmetry

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Abstract. We study harmonic diffeomorphisms between $B^n \backslash \{a_1, \ldots, a_k\}$ and a Riemannian manifold $\mathscr{U}$. For $n = 2$ and $k = 0$, we prove that such diffeomorphisms are minimizing harmonic maps, we generalize this result by replacing B^2 by any Riemannian surface. For $n \geqslant 3$ we give a sufficient condition for such diffeomorphisms to be a minimizing harmonic map. We apply it in the case where $\mathscr{U}$ is a hypersurface of revolution in $\mathbf{R}^{n+1}$. This allows us to prove that, under some conditions, an equivariant harmonic map is necessarily minimizing.

Introduction

In this paper we consider a harmonic diffeomorphism $\bar{U}$ between the unit ball of $\mathbf{R}^n$, B^n, provided with the Euclidean metric c and a Riemannian manifold $\mathscr{U}$. We study the maps into $\mathscr{U}$ using the chart $\bar{U}^{-1}$ on $\mathscr{U}$. Let $g = (g_{ij})$ be the metric coefficients of $\mathscr{U}$ in the chart $\bar{U}^{-1}$, then the Euler equation which expresses that $\bar{U}$ is harmonic is

$$\text{grad}\,[\text{tr}(g)] = 2\,\text{div}\,(g).$$

We will integrate this equation.

We will prove that, if $n = 2$, then $\bar{U}$ is a minimizing harmonic map. Our idea in that case is to decompose g as the sum of two metrics g' and g'' such that g' is a conformal metric of c, (B^2, g'') has nonpositive Gauss curvature and the identity, *Id*, is harmonic between (B^2, c) and (B^2, g''). It then follows that *Id* is a minimizing harmonic map between (B^2, c) and (B^2, g') and that, by a theorem of Hartman [Ha1], Id is also a minimizing harmonic map between (B^2, c) and (B^2, g''). Therefore $\bar{U}$ is a minimizing harmonic map between (B^2, c) and $\mathscr{U}$. We generalize this result by replacing (B^2, c) by any Riemannian surface.

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For $n \geqslant 3$, we will find only sufficient conditions to assure that $\bar{U}$ is a minimizing harmonic map. Our key tool in that case is an adaptation of the projection and averaging procedure introduced in [CG]. We also use, as in [CG], the coarea formula of Federer whose importance in the framework of harmonic map was made clear by Almgren, Browder and Lieb [ABL].

In fact this approach holds also if we assume that $\bar{U}$ has a finite number of punctual singularities, and then $\mathcal{U}$ is diffeomorph to B^n minus the set of singularities.

Then given a target manifold $\mathcal{U}$ and a harmonic map $\bar{U}$ which is a diffeomorphism between B^n and $\mathcal{U}$, we want to know if the corresponding g_{ij} verify the sufficient conditions obtained in the first section. That is the subject of the second section. In this part we assume that $\mathcal{U}$ and $\bar{U}$ are $S0(n)$-equivariant, i.e. rotationally symmetric. We find particularly interesting results in the cases $n = 3$ and $n = 4$. We show here that if $\bar{U}$ is harmonic and if the metric coefficients of $\mathcal{U}$ verify some monotonicity condition then $\bar{U}$ is minimizing harmonic.

These properties are exploited in the third section for $n = 3$. We give some example of minimizing harmonic map. We conclude with the following result. Let $\mathcal{U}$ be the Riemannian manifold with boundary obtained by putting on a ball $B^3(l)$ of radius l (where l is a positive real number) the metric

$$ds^2_{\mathcal{U}} = g_{\|}(|x|)(d|x|)^2 + g_{\perp}(|x|)[(dx)^2 - (d|x|)^2].$$

Here $g_{\|}$ and $g_{\perp}$ are maps of class C^2 on $(0, l]$. We let $b(r) = r^2 g_{\perp}(r)$, we assume that b is of class C^2 on $[0, l]$, that $b'(r)$ is positive on $(0, l)$ and that $b'(0) = 0$. Then if $b'(l) > 0$ or if $b'(l) = 0$ and $b''(l) < -\frac{1}{4}$ there exists a regular $S0(3)$-equivariant minimizing map $\bar{u}$ (with respect to the boundary conditions $u(x) = lx$ on ∂B^3). And if $b'(l) = 0$, $-\frac{1}{4} \leqslant b''(l) < 0$ and $b'(s) \leqslant b''(l)(s - l)$ for any s in $[0, l]$, $lx/|x|$ is minimizing.

Notations

n is an integer greater than 1. The canonical orthonormal basis of $\mathbf{R}^n$ is denoted $(e_1, \ldots, e_n)$, the Euclidean metric c and the scalar product $\langle \, , \rangle$. For any point a of $\mathbf{R}^n$ and any positive real number ϱ, $B^n(a, \varrho)$ is the open ball of center a and of radius ϱ in $\mathbf{R}^n$; we note $B^n = B^n(0, 1)$ and $S_r^{n-1} = \partial B^n(0, r)$.

For any Borelian subset A of $\mathbf{R}^n$ and any r in $[0, n]$, $\mathcal{H}^r(A)$ is the Hausdorff measure of A of dimension r and $|A|$ the Lebesgue's measure of A. All the derivatives are taken in the distribution sense.

For any open subset Ω of $\mathbf{R}^n$, $C_c^1(\Omega, \mathbf{R})$ is the set of maps of class C^1 from Ω to $\mathbf{R}$ with compact support in Ω.

1. The general approach

Let us suppose that $\mathscr{U}$ is a manifold of class C^1 provided with a Riemannian metric of class C^0 and that $\bar{U}$ is a C^1-diffeomorphism between $B^n_* = B^n \backslash \{a_1, \ldots, a_k\}$ and $\mathscr{U}$, where $a_1, \ldots, a_k$ are distinct points called singularities; k belongs to $\mathbf{N}$ and when $n = 2$ we will assume that there is no singularity. We give to each point u of $\mathscr{U}$ the coordinates $y^i = \langle \bar{U}^{-1}(u), e_i \rangle$. Then using this chart the metric on $\mathscr{U}$ will be described by continuous map g_{ij} from B^n_* to $\mathbf{R}$ such that for every x in B^n_*, $(g_{ij}(x))$ is a positive definite n-dimensional two-order symmetric tensor. At a point u of $\mathscr{U}$ such that $\bar{U}(y) = u$, we let using Einstein's convention

$$ds^2_{\mathscr{U}}(u) = g_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j.$$

Furthermore we make the following hypothesis on the g_{ij} coefficients:

For any singularity a_l in B^n, let $B^n(a_l, r_l)$ be a neighbourhood of this singularity which does not meet any other singularity or any point of ∂B^n. There exist strictly positive constants K_1, K_2, K_3 and K_4 such that

$$\forall x \in B^n \backslash \left[\bigcup_l B^n(a_l, r_l)\right], \forall y \in \mathbf{R}^n, K_1 |y|^2 \leqslant g_{ij}(x) y^i y^j \leqslant K_2 |y|^2. \tag{1.1}$$

$$\forall l, \forall x \in B^n(a_l, r_l), \forall y \in \mathbf{R}^n, \text{ if } y \text{ is orthogonal to } (x - a_l) \tag{1.2}$$

$$K_3 |y|^2 \leqslant |(x - a_l)|^2 g_{ij}(x) y^i y^j \leqslant K_4 |y|^2$$

and $\int_{B^n(a_l, r_l)} g_{ii}(x)\,\mathrm{d}x < +\infty$.

Moreover we assume sometimes that there exist continuous maps h_l from $(0, r_l]$ to $[\delta, +\infty)$ where δ is a positive constant, such that h_l are in $L^1([0, r_l], \mathbf{R})$ and

$$\forall l, \forall x \in B^n(a_l, r_l), \forall y \in \mathbf{R}^n, \text{ if } y \text{ is parallel to } (x - a_l) \tag{1.3}$$

$$K_3 [h_l(|x - a_l|)]^2 |y|^2 \leqslant g_{ij}(x) y^i y^j \leqslant K_4 [h_l(|x - a_l|)]^2 |y|^2.$$

We remark that the above hypothesis on g_{ij} are quite reasonable. Indeed (1.1), (1.2) and (1.3) are satisfied if $\mathscr{U}$ is the interior of a compact C^1-Riemannian manifold with boundary. For this reason we will call (1.1), (1.2) and (1.3) compactness conditions.

We define the set $H^1_*(B^n, \mathscr{U})$ to be the set of maps U from B^n into $\mathscr{U}$ such that if $Y = \bar{U}^{-1} \circ U$, then

$$Y \text{ belongs to } H^1(B^n, B^n) \tag{1.4}$$

$$\text{in the trace sense, } Y(x) = x \text{ for a.e. } x \text{ on } \partial B^n \tag{1.5}$$

$$E(U) = \tilde{E}(Y) = \frac{1}{2}\int_{B^n} g_{ij}[Y(x)] \frac{\partial Y^i(x)}{\partial x^\beta} \frac{\partial Y^j(x)}{\partial x^\beta} \mathrm{d}x < +\infty. \tag{1.6}$$

$E(U)$ will be called the energy of U.

An obvious consequence of (1.2) is then that $\bar{U}$ belongs to $H^1_*(B^n, \mathscr{U})$ since

$$E(\bar{U}) = \frac{1}{2}\int_{B^n} g_{ii}(x)\,\mathrm{d}x < +\infty. \tag{1.7}$$

Let $C^1_*(B^n, \mathscr{U})$ be the set of maps U from B^n to $\mathscr{U}$ which are in $H^1_*(B^n, \mathscr{U})$ and which satisfy

there exists a map Z of class C^1 from B^n into B^n such that $Y(x) = Z(x)$ for a.e. x in B^n (1.8)

and

$Z^{-1}\{a_1, \ldots, a_k\}$ is finite and at every point of this set, ∇Z is invertible. (1.9)

In the following we will identify Z with Y.

We will say that $\bar{U}$ is a minimizing harmonic map if $\bar{U}$ is in $H^1_*(B^n, \mathscr{U})$ and satisfies

$$\forall U \in H^1_*(B^n, \mathscr{U}),\ E(U) \geqslant E(\bar{U}). \tag{1.10}$$

We will say that $\bar{U}$ is a C^1_*-minimizing harmonic map if $\bar{U}$ is in $C^1_*(B^n, \mathscr{U})$ and satisfies

$$\forall U \in C^1_*(B^n, \mathscr{U}),\ E(U) \geqslant E(\bar{U}). \tag{1.11}$$

Finally $\bar{U}$ will be called weakly harmonic if for any φ in $C_c^1(B^n, \mathbf{R}^n)$ the following limit exists and is zero

$$E'(\bar{U}, \varphi) = \lim_{\lambda \to 0} \frac{\tilde{E}(Id + \lambda\varphi) - \tilde{E}(Id)}{\lambda}. \tag{1.12}$$

We now give the Euler equations of a weakly harmonic map.

PROPOSITION 1: *$\bar{U}$ is weakly harmonic if and only if the following equalities hold on B^n*

$$\frac{\partial}{\partial x^\gamma}\left(\sum_i g_{ii}(x)\right) = \sum_i \frac{\partial}{\partial x^i}(g_{i\gamma}(x) + g_{\gamma i}(x)) \quad for\ \gamma = 1, \ldots, n. \tag{1.13}$$

If $\bar{U}$ is a C_*^1-minimizing harmonic map then $\bar{U}$ is weakly harmonic.

Proof: Let φ be in $C_c^1(B^n, \mathbf{R}^n)$ and let us choose λ small enough to ensure that $Y_\lambda: x \mapsto x + \lambda\varphi(x)$ is a C^1-diffeomorphism of B^n and that $\bar{U} \circ Y_\lambda$ belongs to $C_*^1(B^n, \mathscr{U})$. Let X_λ be the inverse of Y_λ then we have, using again Einstein's convention

$$\tilde{E}(Y_\lambda) = \frac{1}{2}\int_{B^n} g_{ij}[Y_\lambda(x)]\left(\delta_\beta^i + \lambda\frac{\partial\varphi^i(x)}{\partial x^\beta}\right)\left(\delta_\beta^j + \lambda\frac{\partial\varphi^j(x)}{\partial x^\beta}\right)\mathrm{d}x$$

We let

$$x = X_\lambda(y).$$

We have

$$\mathrm{d}x = |\det[(\nabla Y_\lambda)(X_\lambda(y))]^{-1}|\,\mathrm{d}y = \left|1 - \lambda\frac{\partial\varphi^\gamma}{\partial x^\gamma}[X_\lambda(y)] + \mathcal{O}(\lambda^2)\right|\mathrm{d}y.$$

$$\begin{aligned}\tilde{E}(Y_\lambda) = \frac{1}{2}\int_{B^n} g_{ij}(y)&\left[\delta_\beta^i + \lambda\frac{\partial\varphi^i(y)}{\partial x^\beta} + \lambda\left(\frac{\partial\varphi^i}{\partial x^\beta}(X_\lambda(y)) - \frac{\partial\varphi^i}{\partial x^\beta}(y)\right)\right]\\
&\times\left[\delta_\beta^j + \lambda\frac{\partial\varphi^j}{\partial x^\beta}(y) + \lambda\left(\frac{\partial\varphi^j}{\partial x^\beta}(X_\lambda(y)) - \frac{\partial\varphi^j}{\partial x^\beta}(y)\right)\right]\\
&\times\left[1 - \lambda\frac{\partial\varphi^\gamma}{\partial x^\gamma}(y) + \lambda\left(\frac{\partial\varphi^\gamma}{\partial x^\gamma}(y) - \frac{\partial\varphi^\gamma}{\partial x^\gamma}[X_\lambda(y)]\right) + \mathcal{O}(\lambda^2)\right]\mathrm{d}y\end{aligned}$$

$$= \frac{1}{2}\int_{B^n} g_{ii}(y)\,\mathrm{d}y$$

$$+ \frac{1}{2}\int_{B^n} \lambda \left[g_{ij}(y)\left(\frac{\partial \varphi^i}{\partial x^j}(y) + \frac{\partial \varphi^j}{\partial x^i}(y)\right) - g_{ii}(y)\frac{\partial \varphi^\gamma}{\partial x^\gamma}(y)\right]\mathrm{d}y$$

$$+ \frac{1}{2}\int_{B^n} \lambda \left[g_{ij}(y)\left(\frac{\partial \varphi^i}{\partial x^j}[X_\lambda(y)] - \frac{\partial \varphi^i}{\partial x^j}(y) + \frac{\partial \varphi^j}{\partial x^\beta}[X_\lambda(y)] - \frac{\partial \varphi^j}{\partial x^\beta}(y)\right) - g_{ii}(y)\left(\frac{\partial \varphi^\gamma}{\partial x^\gamma}[X_\lambda(y)] - \frac{\partial \varphi^\gamma}{\partial x^\gamma}(y)\right)\right]\mathrm{d}y$$

$$+ \mathcal{O}(\lambda^2).$$

The first term is $\tilde{E}(Y_0) = \tilde{E}(Id)$, the second one divided by λ is constant and the third one divided by λ tends to zero when λ tends to zero because of Lebesgue's theorem. Hence $E'(\bar{U}, \varphi)$ exists and

$$E'(\bar{U}, \varphi) = \frac{1}{2}\int_{B^n}\left[g_{ij}(x)\left(\frac{\partial \varphi^i}{\partial x^j}(x) + \frac{\partial \varphi^j}{\partial x^i}(x)\right) - g_{ii}(x)\frac{\partial \varphi^\gamma}{\partial x^\gamma}(x)\right]\mathrm{d}x.$$

And $E'(\bar{U}, \varphi)$ is zero for every test map φ in $C_c^1(B^n, \mathbf{R}^n)$ if and only if (1.13) is true in B^n.

Now if we suppose that $\bar{U}$ is a C^1_*-minimizing harmonic map it is obvious that $E'(\bar{U}, \varphi)$ is zero for every φ in $C_c^1(B^n, \mathbf{R}^n)$, this proves the second assertion of the theorem. Q.E.D.

The equations (1.13) have a simple integration.

THEOREM 2: *Let us suppose that $\mathcal{U}$ has no singularity i.e. $B^n_* = B^n$. Then the map g_{ij} of class C^0 from B^n to the set of the symmetric positive definite two-order tensors, is solution of* (1.3) *if and only if*

- *If $n = 2$, there exist maps*
  - □ *λ of class C^0 from B^2 to $(0, +\infty)$*
  - □ *φ holomorphic from B^2 to $\mathbf{C}$ such that*

$$\forall x \in B^2, \quad \lambda(x) > |\varphi(x)|$$

$$g(x) = \begin{bmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{bmatrix} + \begin{bmatrix} -\mathrm{Re}\,\varphi(x) & \mathrm{Im}\,\varphi(x) \\ \mathrm{Im}\,\varphi(x) & \mathrm{Re}\,\varphi(x) \end{bmatrix} \tag{1.14}$$

The value of the energy of the corresponding harmonic map $\bar{U}$ is then $E(\bar{U}) = \int_{B^2} \lambda(x)\,\mathrm{d}x$.

- *If* $n \geqslant 3$, *there exist maps*
  - □ $G_{ij} = G_{ji}$ *of class* C^2 *from* B^n *to* $\mathbf{R}$, *where* $i, j \in \{1, \ldots, n\}$, *and* $i \neq j$.
  - □ c_i *of class* C^0 *from* B^{n-1} *to* $\mathbf{R}$, *where* $i \in \{1, \ldots, n\}$ *such that if we note* $G_{ii}(x) = (x^i)^2 c_i(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$.

$$\begin{cases} \text{if } i \neq j \; g_{ij}(x) = \dfrac{\partial^2 G_{ij}(x)}{\partial x^i\,\partial x^j} \\ \quad g_{ii}(x) = \displaystyle\sum_p \frac{\partial^2}{(\partial x^p)^2}\left[\frac{1}{n-2}\left(\sum_j G_{jp}\right) - G_{ip}\right], \end{cases} \tag{1.15}$$

with the conditions which express that $(g_{ij}(x))$ *is positive definite.*

Proof: (a) Case $n = 2$. In that case, the result is classical (see e.g. [EL] (10.5) or [J] Lemma 1.1). The equations (1.13) becomes:

$$\begin{cases} \dfrac{\partial}{\partial x^1}(g_{22} - g_{11}) = \dfrac{\partial}{\partial x^2}(2g_{12}) \\ \dfrac{\partial}{\partial x^2}(g_{11} - g_{22}) = \dfrac{\partial}{\partial x^1}(2g_{12}) \end{cases}$$

We state $a = (g_{22} - g_{11})/2$, $b = g_{12}$, then

$$\begin{cases} \dfrac{\partial a}{\partial x^1} = \dfrac{\partial b}{\partial x^2} \\ -\dfrac{\partial a}{\partial x^2} = \dfrac{\partial b}{\partial x^1} \end{cases}$$

Hence if $\varphi(x) = a(x) + ib(x)$, we conclude that φ is holomorphic. If we note $\lambda(x) = \frac{1}{2}\,\mathrm{tr}\,[g(x)]$, we obtain then (1.14). The condition $\lambda(x) > |\varphi(x)|$ expresses that $g(x)$ is positive definite.

(b) Case $n \geqslant 3$. We let

$$t_i = \left[\sum_p g_{pp}\right] - 2g_{ii}.$$

Then (1.13) gives

$$\frac{\partial t_\gamma}{\partial x^\gamma} = 2\left[\sum_{i\neq\gamma}\frac{\partial g_{i\gamma}}{\partial x^i}\right] \quad \text{for } \gamma = 1, \ldots, n. \tag{1.16}$$

For $i \neq \gamma$, let us pose

$$G_{i\gamma}(x) = \int_0^{x^i}\int_0^{x^\gamma} g_{i\gamma}(x^1, \ldots, x^{i-1}, t^i, x^{i+1}, \ldots, x^{\gamma-1}, t^\gamma, x^{\gamma+1}, \ldots, x^n)\,\mathrm{d}t^i\,\mathrm{d}t^\gamma.$$

Hence

$$\frac{\partial^2 G_{i\gamma}(x)}{\partial x^i\,\partial x^\gamma} = g_{i\gamma}(x).$$

From (1.16)

$$\frac{\partial t_\gamma}{\partial x^\gamma} = 2\left[\sum_{i\neq\gamma}\frac{\partial^3 G_{i\gamma}}{(\partial x^i)^2(\partial x^\gamma)}\right].$$

We have then

$$t_\gamma = 2\left[\sum_{i\neq\gamma}\frac{\partial^2 G_{i\gamma}}{(\partial x^i)^2}\right] + 4c_\gamma(\widehat{x^\gamma})$$

where $\widehat{x^\gamma} = (x^1, \ldots, x^{\gamma-1}, x^{\gamma+1}, \ldots, x^n)$. So if we write $G_{ii}(x) = (x^i)^2 c_i(\hat{x}^i)$ we obtain

$$t_\gamma = 2\left[\sum_i\frac{\partial^2 G_{i\gamma}}{(\partial x^i)^2}\right].$$

Now we invert the relations $t_i = [\Sigma_l\, g_{ll}] - 2g_{ii}$ by

$$g_{ii} = \frac{1}{2}\left[\frac{1}{n-2}\sum_l t_l - t_i\right].$$

Then (1.15) follows. The converse is straightforward. Q.E.D.

We now investigate the minimality of $\bar{U}$. We start with the case $n = 2$ (and $k = 0$). We prove

THEOREM 3: *$\bar{U}$ is a minimizing harmonic map if $n = 2$.*

Remark: The proof will show that, if g is regular up to the boundary of B^2, then $\bar{U}$ is the unique minimizing harmonic map which agrees with $\bar{U}$ on ∂B^2.

Proof of Theorem 3: We use the notations of theorem 2. First we remark that we may assume φ regular on $\overline{B^2}$; indeed $\bar{U}$ will be minimizing if $\bar{U}$ restricted to $B^2(0, 1 - \varepsilon)$ is minimizing for any ε in $(0, 1)$. Let ω: $\overline{B^2} \to (0, +\infty)$ be defined by

$$\omega(x^1, x^2) = (2 - (x^1)^2 - (x^2)^2)^{-4}.$$

For ε small enough but positive we have (see (1.1))

$$\lambda^2 > |\varphi|^2 + \varepsilon\omega.$$

We choose such a ε, and we define the metric on B^2

$$g' = (\lambda - (|\varphi|^2 + \varepsilon\omega)^{1/2})(\mathrm{d}x)^2$$

and the metric on $\overline{B^2}$

$$g'' = (|\varphi|^2 + \varepsilon\omega)^{1/2}(\mathrm{d}x)^2 - \mathrm{Re}\,(\varphi(\mathrm{d}z)^2)$$

with $z = x^1 + ix^2$.

Clearly $g = g' + g''$. The map *Id* is a conformal map between (B^2, c) and (B^2, g') hence is minimizing between these two Riemannian surfaces (for the energy defined with c and g'; see e.g., [EL] (10.3)). Moreover, see Appendix B, $(\overline{B^2}, g'')$ has a negative Gauss curvature. Proceeding as in Appendix C we can construct on $\mathbf{R}^2$ a metric σ such that $\sigma = g''$ on $\overline{B^2}$, $(\mathbf{R}^2, \sigma)$ is complete, the Gauss curvature of $(\mathbf{R}^2, \sigma)$ is negative and $(\mathbf{R}^2, \sigma)$ satisfies Morrey's uniformity condition, see the remark in Appendix C. Then a theorem due to Morrey [M] asserts that there exists a map v: $(\overline{B^2}, c) \to (\mathbf{R}^2, \sigma)$ which is the identity on ∂B^2 and is minimizing. Then using a theorem due to Hartman [Ha 1] (more precisely its extension to manifolds with boundary, see [S1] theorem 2.10) we have, since the Gauss curvature of $(\mathbf{R}^2, \sigma)$ is negative and since v and *Id* are homotopic, $v = Id$. Hence *Id* is a minimizing map from (B^2, c) into (B^2, g''); moreover *Id* is also minimizing from (B^2, c) into (B^2, g') hence *Id* is minimizing from (B^2, c) into $(B^2, g' + g'') = (B^2, g)$.

Q.E.D.

The method used can be extended to more general Riemannian surfaces than (B^2, c). Indeed it allows to prove the following

THEOREM 3′: *Let (M, h) and (N, g) be two Riemannian compact surfaces of class C^∞ possibly with boundary. Then any smooth harmonic diffeomorphism between (M, h) and (N, g) is minimizing in its homotopy class. Moreover, if ∂M is non-empty or if the genus of M is strictly larger than* 1, *then such a diffeomorphism is the unique minimizing map in its homotopy class.*

Remark: Of course when $\partial M \neq \emptyset$ the homotopies are required to agree with u on ∂M. When ∂M is empty, Jost and Schoen have proved in [JS] the existence of a harmonic diffeomorphism between (M, h) and (N, g) if M and N are homeomorphic surfaces.

Proof of Theorem 3′: We start with the case $\partial M = \emptyset$. If the genus of M is 0 the result is well known. Therefore we assume that the genus of M is positive. Then there exists, on M, a metric h_0, in the conformal class of h, the Gauss curvature of which is nonpositive. In isothermal charts we have

$$h = \varrho \, \mathrm{d}z \, \mathrm{d}\bar{z}$$

$$h_0 = \omega^{1/2} \, \mathrm{d}z \, \mathrm{d}\bar{z}.$$

Let u be a harmonic diffeomorphism between (M, h) and (N, g) and let ε be a positive number. Let h'' be the metric defined on M by

$$h'' = (|\varphi|^2 + \varepsilon\omega)^{1/2} \, \mathrm{d}z \, \mathrm{d}\bar{z} - \mathrm{Re}\,(\varphi(\mathrm{d}z)^2)$$

where, with $z = x^1 + ix^2$,

$$2\varphi = -|u_{x^1}|_g^2 + |u_{x^2}|_g^2 + 2i\langle u_{x^1}, u_{x^2}\rangle_g.$$

We recall, see e.g. [EL] (10.5) or [J] Lemma 1.1, that $\varphi(\mathrm{d}z)^2$ is holomorphic; we note also that h'' is independent of the choice of the isothermal coordinates.

We can define also, for ε small enough, another metric on M by

$$h' = \lambda \, \mathrm{d}z \, \mathrm{d}\bar{z} - (|\varphi|^2 + \varepsilon\omega)^{1/2} \, \mathrm{d}z \, \mathrm{d}\bar{z}$$

where

$$\lambda = \tfrac{1}{2}(|u_{x^1}|_g^2 + |u_{x^2}|_g^2);$$

again h' is independent of the choice of the isothermal coordinates. The map u^{-1}: $N \to M$ and the metrics h' and h'' on M induce metrics, denoted g' and g'', on N. Clearly we have $g = g' + g''$. The map u is a conformal map between (M, h) and (N, g') hence is minimizing – for the energy defined by h and g' – in its homotopy class, see e.g. [EL] (10.3). We remark also that u is harmonic between (M, h) and (N, g''), since $\varphi(dz)^2$ is holomorphic. Moreover the Gauss curvature of (N, g'') is nonpositive (see Appendix B). Hence it follows from a theorem due to Eells and Sampson [ES] first corollary of page 158, that, in the homotopy class of u, there is a harmonic map v: $(M, h) \to (N, g'')$ whose energy is an absolute minimum (one could alternatively use a theorem due to Sacks-Uhlenbeck [SU] and Lemaire [Le1], noting that $\pi_2(N) = 0$). Now, still because the Gauss curvature of N is nonpositive, from a theorem due to Hartman, [Ha1] theorem E, u and v considered as maps from (M, h) into (N, g'') have the same energy. We conclude that u: $(M, h) \to (N, g' + g'')$ has minimum energy in its homotopy class. Uniqueness when the genus of M is strictly larger than 1 follows from the above decomposition of g and [Ha1] corollary p. 675.

In the case $\partial M \neq \phi$ we just sketch the proof since the arguments are quite similar to those used above and in the proof of theorem 3. As above we decompose g as the sum of two metrics g' and g'' such that u is conformal (hence minimizing) between (M, h) and (N, g'), u is harmonic between (M, h) and (N, g'') and the Gauss curvature of (N, g'') is negative. It then follows from Appendix C, [Le2] and [S1] Theorem 2.10 (one can alternatively use [S2], see [EL] (12.11)) that u is minimizing in its homotopy class from (M, h) into (N, g''). Hence u is minimizing in its homotopy class from (M, h) into (N, g). Uniqueness comes from the above decomposition of g and [S1] Theorem 2.10. Q.E.D.

Remark: It follows from Theorem 3′ that if u is a smooth harmonic map (not necessarily a diffeomorphism) between a Riemannian surface (M, h) and a Riemannian manifold (not necessarily of dimension 2) (N, g) then the energy of $u \circ \theta$ is not less than the energy of u, for any smooth map θ from M to M which is homotopic to the identity, with fixed boundary data if ∂M is non-empty. Indeed first note that the identity is a harmonic diffeomorphism between (M, h) and $(M, u_*(g) + \varepsilon h)$ for ε positive. Then apply Theorem 3′ and let ε goes to zero.

Now let us turn to the case $n \geqslant 3$.

Let A be any topological space which is provided with a Borelian measure $dm(\alpha)$. Let π be a continuous map from $A \times \overline{B^n}$ into $\mathbf{R}^2$. We note $\pi_\alpha = \pi(\alpha_1, \cdot)$. We assume that for any α in A π_α is a horizontal conformal map (see e.g. [B1] p. 122 for a definition) whose fibres are intersections of $(n-2)$-area minimizing surfaces with $\overline{B^n}$.

Remark: The maps π_α are harmonic morphisms (see [BCD] or [BE]). One can find interesting results – and examples – concerning harmonic morphisms in [B1], [B2] and [BW].

Our next theorem comes directly from the method introduced in [CG].

THEOREM 4: *Let us assume that*

- $\bar{U}$ *is a* C^1*-diffeomorphism between* B^n_* *and* $\mathcal{U}$ *and the* g_{ij} *are of class* C^0 *on* B^n_* *and satisfy* (1.1) *and* (1.2).
- *There exists a measurable function f from* $A \times \mathbf{R}^2$ *to* $[0, +\infty]$ *such that for every x in* B^n_* *and every y in* $\mathbf{R}^n$

$$\sum_{i,j} g_{ij}(x) y^i y^j = \int_A |\nabla \pi_\alpha(x) \cdot y|^2 f(\alpha, \pi_\alpha(x))\, \mathrm{d}m(\alpha). \tag{1.17}$$

Then $\bar{U}$ *is a* C^1_**-minimizing harmonic map.*

EXAMPLE: A is the set of Euclidean rotations in $\mathbf{R}^n$, $S0(n)$, $dm(\alpha)$ is the Haar measure on $S0(n)$, and for every R in $S0(n)$ we pose $\pi_R(x) = (\langle Rx, e_1\rangle, \langle Rx, e_2\rangle)$.

Proof of Theorem 4: We follow [CG]. Let U be in $C^1_*(B^n, \mathcal{U})$ and let $Y = \bar{U}^{-1} \circ U$. From (1.17) we have using Einstein's convention

$$g_{ij}[Y(x)] \frac{\partial Y^i}{\partial x^\beta}(x) \frac{\partial Y^j}{\partial x^\beta}(x) = \int_A \sum_\beta \left| \nabla \pi_\alpha[Y(x)] \frac{\partial Y}{\partial x^\beta}(x) \right|^2 f(\alpha, \pi_\alpha[Y(x)])\, \mathrm{d}m(\alpha).$$

for a.e. x in B^n_*. So we can write

$$E(U) = \frac{1}{2} \int_{B^n} \int_A |\nabla(\pi_\alpha \circ Y)|^2(x) f(\alpha, \pi_\alpha(Y(x)))\, \mathrm{d}m(\alpha)\, \mathrm{d}x, \tag{1.18}$$

and using Fubini's theorem

$$E(U) = \frac{1}{2} \int_A \int_{B^n} |\nabla(\pi_\alpha \circ Y)|^2(x) f(\alpha, \pi_\alpha(Y(x)))\, \mathrm{d}x\, \mathrm{d}m(\alpha).$$

Let $J_2(\pi_\alpha(Y(x))$ be the absolute value of the 2-determinant of the restriction of $\nabla(\pi_\alpha \circ Y)$ to the orthogonal complement of the kernel of $\nabla(\pi_\alpha \circ Y)$ (see

[F], p. 1.7.6. for a more precise definition). As it was remarked in [ABL] we have the inequality

$$|\nabla(\pi_\alpha \circ Y)(x)|^2 \geqslant 2J_2(\pi_\alpha \circ Y)(x)$$

Then

$$E(U) \geqslant \int_A \int_{B^n} J_2(\pi_\alpha \circ Y)(x) f(\alpha, \pi_\alpha[Y(x)])\, \mathrm{d}x\, \mathrm{d}m(\alpha).$$

We now proceed as in [ABL] to give a lower bound of

$$\int_{B^n} J_2(\pi_\alpha \circ Y)(x) f(\alpha, \pi_\alpha(Y(x)))\, \mathrm{d}x.$$

We apply the coarea formula of Federer to transform the right hand term of this inequality (see [F], Theorem 3.2.12, p. 249).

$$E(U) \geqslant \int_A \int_{\mathbf{R}^2} \mathscr{H}^{n-2}((\pi_\alpha \circ Y)^{-1}(z)) f(\alpha, z)\, \mathrm{d}z\, \mathrm{d}m(\alpha).$$

Using Sard's theorem $(\pi_\alpha \circ Y)^{-1}(z)$ is empty or a $(n - 2)$-submanifold of $\overline{B^n}$ for a.e. z in $\mathbf{R}^2$, the boundary of which is contained in ∂B^n. Hence it follows from (1.5) that $\partial(\pi_\alpha \circ Y)^{-1}(z) = \partial\pi_\alpha^{-1}(z)$ for a.e. z in $\mathbf{R}^2$. But $\pi_\alpha^{-1}(z)$ is a minimizing $(n - 2)$-submanifold in B^n and this implies

$$\mathscr{H}^{n-2}((\pi_\alpha \circ Y)^{-1}(z)) \geqslant \mathscr{H}^{n-2}(\pi_\alpha^{-1}(z)) \quad \text{for a.e. } z.$$

Hence we obtain

$$E(U) \geqslant \int_A \int_{\mathbf{R}^2} \mathscr{H}^{n-2}(\pi_\alpha^{-1}(z)) f(\alpha, z)\, \mathrm{d}z\, \mathrm{d}m(\alpha)$$

$$E(U) \geqslant \int_A \int_{B^n} J_2(\pi_\alpha)(x) f(\alpha, \pi_\alpha(x))\, \mathrm{d}x\, \mathrm{d}m(\alpha),$$

and since π_α is horizontally conformal

$$E(U) \geqslant \frac{1}{2} \int_A \int_{B^n} |\nabla\pi_\alpha(x)|^2 f(\alpha, \pi_\alpha(x))\, \mathrm{d}x\, \mathrm{d}m(\alpha). \tag{1.19}$$

Finally, using first (1.18) with $\bar{U}$, and then (1.19) we have

$$E(U) \geqslant E(\bar{U}).$$

Q.E.D.

An immediate consequence of theorem 4 is

THEOREM 5: *Let us assume that*

- *$\bar{U}$ is a C^1-diffeomorphism between B^n_* and $\mathcal{U}$. The g_{ij} are of class C^0 on B^n_* and satisfy* (1.1), (1.2) *and* (1.3).
- *There exists a measurable map f from $A \times \mathbf{R}^2$ to $[0, +\infty]$ such that for every x in B^n_* and every y in $\mathbf{R}^n$*

$$g_{ij}(x)y^iy^j = \int_A |\nabla\pi_\alpha(x)y|^2 f(\alpha, \pi_\alpha(x))\,dm(\alpha). \tag{1.17}$$

Then $\bar{U}$ is a minimizing harmonic map.

Proof: Since we have the hypothesis (1.3) we can apply the Appendix A of this paper. Then it suffices to apply Theorem 4 to obtain (1.10). Q.E.D.

II. Applications to some problems with symmetry

In this section we assume that $n \geqslant 3$ and we study the special case where the Riemannian manifold $\mathcal{U}$ and the map $\bar{U}$ are $S0(n)$-equivariant as follows.

We can define an action of $S0(n)$ on $\mathcal{U}$ by

$$\forall R \in S0(n), \quad \forall u \in \mathcal{U}, \quad Ru = \bar{U}\{R[\bar{U}^{-1}(u)]\}.$$

Then we will say that $\mathcal{U}$ and $\bar{U}$ are $S0(n)$-equivariant if the metric g_{ij} of $\mathcal{U}$ defined in the above section is invariant under the action of $S0(n)$.

Hence for every u in $\mathcal{U}$ and for every v in $T_u\mathcal{U}$, if we note $x = \bar{U}^{-1}(u)$, $y = (\nabla\bar{U})^{-1}(u)v$,

$$|v|^2_{\mathcal{U}} = g_{\|}(x)|y_{\|}|^2 + g_{\perp}(x)|y_{\perp}|^2$$

where $y_{\|} = \langle y, (x/|x|)\rangle$, $y_{\perp} = y - (x/|x|)\langle(x/|x|), y\rangle$, and $g_{\|}$ and $g_{\perp}$ are strictly positive functions, are continuous on B^n_*, and depend uniquely on $r = |x|$.

It is obvious in this case that either we have no singularity, and $B^n_* = B^n$, or we have one singularity which is 0 and $B^n_* = B^n\backslash\{0\}$. We will give applications of Theorem 4 by choosing:

- A is $S0(n)$
- $dm(\alpha)$ is the Haar-measure on $S0(n)$, $d\sigma(R)$.
- For every R in $S0(n)$,

$$\pi_R(x) = (\langle Rx, e_1\rangle, \langle Rx, e_2\rangle).$$

In this case, we can write (1.17) as follows

$$g_{ij}(x) = \int_{SO(n)} \langle \pi_R e_i, \pi_R e_j \rangle f(R, \pi_R(x))\, \mathrm{d}\sigma(R). \tag{2.1}$$

First let us write equation (1.13) which express that $\bar{U}$ is weakly harmonic: For every $\gamma = 1, \ldots, n$,

$$\frac{\partial}{\partial x^\gamma}\left[\sum_i g_{ii}\right] = \frac{\partial}{\partial x^\gamma}[ng_\perp + g_\parallel - g_\perp] = (n-1)g'_\perp \frac{x^\gamma}{r} + g'_\parallel \frac{x^\gamma}{r},$$

and

$$\begin{aligned} 2\left[\sum_i \frac{\partial g_{i\gamma}}{\partial x^i}\right] &= 2\left[\sum_i \frac{\partial[g_\perp \delta_{i\gamma} + (g_\parallel - g_\perp)(x^i x^\gamma/r^2)]}{\partial x^i}\right] \\ &= 2\left[g'_\perp \frac{x^\gamma}{r} + (g_\parallel - g_\perp)\frac{x^\gamma}{r^2} + \sum_i \left[(g'_\parallel - g'_\perp)\frac{(x^i)^2 x^\gamma}{r^3}\right.\right. \\ &\qquad \left.\left. + (g_\parallel - g_\perp)\frac{x^\gamma}{r^2} - 2(g_\parallel - g_\perp)\frac{(x^i)^2 x^\gamma}{r^4}\right]\right] \\ &= 2[g'_\parallel + (n-1)(g_\parallel - g_\perp)]\frac{x^\gamma}{r^2}. \end{aligned}$$

Hence (1.13) becomes

$$rg'_\parallel + 2(n-1)g_\parallel = (n-1)[rg'_\perp + 2g_\perp] \quad \text{on } (0, 1), \tag{2.2}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^{2(n-1)}g_\parallel] = (n-1)r^{2(n-2)}\frac{\mathrm{d}}{\mathrm{d}r}[r^2 g_\perp] \quad \text{on } (0, 1).$$

We let $a(r) = r^2 g_\perp(r)$. The $(n-1)$-dimensional measure of the image of the sphere of radius rS_r^{n-1} by $\bar{U}$ is $|S^{n-1}|a(r)^{(n-1)/2}$. Hence a and $g_\perp$ are of class C^1 on $(0, 1)$ since $\mathscr{U}$ is a manifold of class C^1 and $\bar{U}$ is of class C^1. It follows from the above equation that the weak derivative of $r^{2(n-1)}g_\parallel$ is equal a.e. to a continuous map. Hence a, $g_\perp$ and $g_\parallel$ are of class C^1 on $(0, 1)$. We have proved:

PROPOSITION 6: *If $\mathcal{U}$ and $\bar{U}$ are $S0(n)$-equivariant, if the g_{ij} are continuous and if $\bar{U}$ is weakly harmonic on B_*^n then $g_\parallel$ and $g_\perp$ are of class C^1 on* $(0, 1)$ *and satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^{2(n-1)}g_\parallel] = (n-1)r^{2(n-2)}\frac{\mathrm{d}}{\mathrm{d}r}[r^2 g_\perp]. \tag{2.3}$$

Remark: If there exists an isometric embedding σ of $\mathcal{U}$ into $\mathbf{R}^{n+1}$ such that $\sigma(\mathcal{U})$ is a hypersurface of revolution and such that the image by σ of the action of $S0(n)$ on $\mathcal{U}$ is the natural action of $S0(n)$ on $\sigma(\mathcal{U})$, then at a point u of $\mathcal{U}$, $a(r)$ represents the square of the distance of $\sigma(u)$ to the axis of revolution of $\sigma(\mathcal{U})$.

PROPOSITION 7: *Let us suppose that*

- *$\bar{U}$ is a C^1-diffeomorphism between B_*^n and $\mathcal{U}$ and the g_{ij} are of class C^0 on B_*^n and verify* (1.1) *and* (1.2).
- *$\bar{U}$ and $\mathcal{U}$ are $S0(n)$-equivariant.*
- *There exists a nonnegative integrable map φ of class C^0 on* $(0, 1]$ *such that for every r in* $(0, 1)$

$$r^2 g_\perp(r) = \frac{n-2}{n-1}\int_0^r \left(2 - \frac{t^2}{r^2}\right)\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t \tag{2.4}$$

$$r^2 g_\parallel(r) = (n-2)\int_0^r \frac{t^2}{r^2}\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t. \tag{2.5}$$

Then $\bar{U}$ is a C_*^1-minimizing harmonic map.

Proof: Let $\mathcal{U}_1$ and U_1 be respectively a Riemannian manifold and a C^1-diffeomorphism between B_*^n and $\mathcal{U}_1$ such that the metric on $\mathcal{U}_1$ is given in the chart U_1^{-1} by

$$\gamma_{ij}(x) = \int_{S0(n)} \langle \pi_R e_i, \pi_R e_j\rangle \varphi(|\pi_R x|)\,\mathrm{d}\sigma(R).$$

Let us show that this metric is invariant under the action of $S0(n)$. For every $\tilde{R}$ in $S0(n)$, we have,

$$\gamma_{ij}(\tilde{R}x)(\tilde{R}y)^i(\tilde{R}y)^j = \int_{S0(n)} \langle \pi_R e_i, \pi_R e_j\rangle\langle \tilde{R}y, e_i\rangle\langle \tilde{R}y, e_j\rangle \varphi(|\pi_R \tilde{R}x|)\,\mathrm{d}\sigma(R)$$

$$= \int_{SO(n)} \langle \pi_{R\tilde{R}} \tilde{R}^{-1} e_i, \pi_{R\tilde{R}} \tilde{R}^{-1} e_j \rangle \langle y, \tilde{R}^{-1} e_i \rangle \langle y, \tilde{R}^{-1} e_j \rangle \varphi(|\pi_{R\tilde{R}} x|) \, d\sigma(R)$$

$$= \gamma_{ij}(x) y^i y^j.$$

The metric on $\mathscr{U}_1$ is then given by two strictly positive functions $\gamma_\parallel$ and $\gamma_\perp$ on B^n_* which depend only on $r = |x|$:

$$|v|^2_{\mathscr{U}_1} = [\gamma_\parallel(r) - \gamma_\perp(r)] \left(y \frac{x}{r} \right)^2 + \gamma_\perp(r) y^2.$$

To compute $\gamma_\parallel$ and $\gamma_\perp$, it suffices to evaluate them at the point the coordinates of which are re_1 in the chart U_1^{-1}

$$\gamma_\parallel(r) = \int_{SO(n)} \langle \pi_R e_1, \pi_R e_1 \rangle \varphi(r|\pi_R e_1|) \, d\sigma(R) \tag{2.6}$$

$$\gamma_\parallel(r) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |\pi_{Id} \omega|^2 \varphi(r|\pi_{Id} \omega|) \, ds(\omega). \tag{2.7}$$

We need the following lemma.

LEMMA 8: *Let us assume that h is a real map of class C^0 on* [0, 1], *then*

$$\int_{S^{n-1}} h[((\omega^1)^2 + (\omega^2)^2)^{1/2}] \, ds(\omega) = 2\pi(n-2)|B^{n-2}| \int_0^1 t(1-t^2)^{(n-4)/2} h(t) \, dt. \tag{2.8}$$

PROOF OF LEMMA 8: Let us assume first that h is C^1 on [0, 1] and let us write $F(x) = h[((x^1)^2 + (x^2)^2)^{1/2}]$. We have

$$\int_{B^n} \sum_i x^i \frac{\partial F}{\partial x^i}(x) \, dx = \int_{S^{n-1}} F(\omega) \, ds(\omega) - n \int_{B^n} F(x) \, dx. \tag{2.9}$$

The left hand term in (2.9) is

$$L = \int_{B^n} h'[((x^1)^2 + (x^2)^2)^{1/2}][(x^1)^2 + (x^2)^2]^{1/2} \, dx$$

$$= 2\pi |B^{n-2}| \int_0^1 t^2(1-t^2)^{(n-2)/2} h'(t) \, dt$$

by setting $t = [(x^1)^2 + (x^2)^2]^{1/2}$. If we integrate by parts

$$L = 2\pi|B^{n-2}| \int_0^1 (nt^2 - 2)\, t(1 - t^2)^{(n-4)/2} h(t)\, \mathrm{d}t.$$

The right hand term is

$$R = \int_{S^{n-1}} F(\omega)\, \mathrm{d}s(\omega) = n2\pi|B^{n-2}| \int_0^1 t(1 - t^2)^{(n-2)/2} h(t)\, \mathrm{d}t.$$

Then (2.9) and the expressions of L and R give (2.8). The extension of (2.5) to the case where the map h is only continuous follows by density. □

End of the proof of Proposition 7: Using Lemma 8 in (2.7) we find

$$\gamma_{\|}(r) = \frac{2\pi(n-2)|B^{n-2}|}{|S^{n-1}|} \int_0^1 t(1 - t^2)^{(n-4)/2} t^2 \varphi(rt)\, \mathrm{d}t$$

$$\gamma_{\|}(r) = \frac{n-2}{r^2} \int_0^r \left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} \frac{t^2}{r^2}\, t\varphi(t)\, \mathrm{d}t.$$

Now we remark that

$$\begin{aligned} \gamma_{\|} + (n-1)\gamma_{\perp} &= \int_{SO(n)} \sum_i \langle \pi_R e_i, \pi_R e_i \rangle\, \varphi(r|\pi_R e_1|)\, \mathrm{d}\sigma(R) \\ &= 2 \int_{SO(n)} \varphi(r|\pi_R e_1|)\, \mathrm{d}\sigma(R) \\ &= \frac{2}{|S^{n-1}|} \int_{S^{n-1}} \varphi(r|\pi_0 \omega|)\, \mathrm{d}s(\omega) \\ &= (n-2) \int_0^1 2t(1 - t^2)^{(n-4)/2} \varphi(rt)\, \mathrm{d}t. \end{aligned}$$

Therefore

$$\gamma_{\perp}(r) = \frac{n-2}{(n-1)r^2} \int_0^r \left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} \left(2 - \frac{t^2}{r^2}\right) \varphi(t)\, t\, \mathrm{d}t.$$

Hence we deduce from this that $\gamma_{\perp} = g_{\perp}$ and $\gamma_{\|} = g_{\|}$ and so $g_{ij} = \gamma_{ij}$ on B^n_*. We can then apply Theorem 4 to conclude. Q.E.D.

PROPOSITION 9: *Let us suppose that*

- $\bar{U}$ *is a* C^1*-diffeomorphism between* B^n_* *and* $\mathcal{U}$ *and the* g_{ij} *are of class* C^0 *on* B^n_* *and verify* (1.1) *and* (1.2).
- $\bar{U}$ *and* $\mathcal{U}$ *are* $SO(n)$*-equivariant.*
- $\bar{U}$ *is weakly harmonic.*
- *There exists a nonnegative and continuous map* φ *on* (0, 1] *such that*

– *Either*

$$\begin{cases} a(r) = r^2 g_\perp(r) = \dfrac{n-2}{n-1}\displaystyle\int_0^r \left(2 - \frac{t^2}{r^2}\right)\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t. \\ \lim\limits_{r\to 0} r^{2(n-1)} g_\parallel(r) = 0. \end{cases} \tag{2.4}$$

– *Or*

$$\begin{cases} r^2 g_\parallel(r) = (n-2)\displaystyle\int_0^r \frac{t^2}{r^2}\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t \\ \lim\limits_{r\to 0} a(r) = \lim\limits_{r\to 0} \dfrac{n-2}{n-1}\displaystyle\int_0^r \left(2 - \frac{t^2}{r^2}\right)\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t < +\infty. \end{cases} \tag{2.5}$$

Then $\bar{U}$ *is a* C^1_**-minimizing harmonic map.*

Proof: The idea of the proof is to show that the hypothesis of Proposition 7 are in fact satisfied in both cases.

Let $\gamma_\parallel$ and $\gamma_\perp$ with

$$r^2\gamma_\perp(r) = \frac{n-2}{n-1}\int_0^r \left(2 - \frac{t^2}{r^2}\right)\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t$$

$$r^2\gamma_\parallel(r) = (n-2)\int_0^r \frac{t^2}{r^2}\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t$$

These maps are of class C^1 on (0, 1] since φ is continuous. We will show that they satisfy (2.2) or

$$r[\gamma_\parallel - (n-1)\gamma_\perp]' = 2(n-1)(\gamma_\perp - \gamma_\parallel) \tag{2.11}$$

But $\gamma_\parallel - (n-1)\gamma_\perp = (-2/r^2)(n-2)\int_0^r (1 - t^2/r^2)^{(n-2)/2} t\varphi(t)\,\mathrm{d}t$, hence

$$(\gamma_\parallel - (n-1)\gamma_\perp)' = \frac{1}{r^3}2(n-2)\int_0^r \left(2 - n\frac{t^2}{r^2}\right)\left(1 - \frac{t^2}{r^2}\right)^{(n-4)/2} t\varphi(t)\,\mathrm{d}t.$$

We find that $r(\gamma_{\parallel} - (n-1)\gamma_{\perp})'$ is precisely $2(n-1)(\gamma_{\perp} - \gamma_{\parallel})$. This proves (2.11) which can be rewritten as

$$\frac{d}{dr}[r^{2(n-1)}\gamma_{\parallel}(r)] = (n-1)r^{2(n-2)}\frac{d}{dr}[r^2\gamma_{\perp}(r)].$$

But since $\bar{U}$ is weakly harmonic, $g_{\parallel}$ and $g_{\perp}$ satisfy the same equation (2.3). Now it is easy to show that $(g_{\parallel}, g_{\perp}) = (\gamma_{\parallel}, \gamma_{\perp})$. Indeed:

- **First case**. We know that $g_{\perp} = \gamma_{\perp}$ and $\lim_{r\to 0} r^{2(n-1)}g_{\parallel}(r) = 0$. Hence we will have $g_{\parallel} = \gamma_{\parallel}$ if we check that $\lim_{r\to 0} r^{2(n-1)}\gamma_{\parallel}(r) = 0$. But

$$r^{2(n-1)}\gamma_{\parallel}(r) = r^{2(n-1)}(n-2)\int_0^1 t^2(1-t^2)^{n-4/2}t\varphi(t)\,dt$$

$$r^{2(n-1)}\gamma_{\parallel}(r) \leqslant (n-2)r^{2(n-1)}\int_0^1 (2-t^2)(1-t^2)^{n-4/2}t\varphi(t)\,dt$$

$$r^{2(n-1)}\gamma_{\parallel}(r) \leqslant (n-1)r^{2(n-2)}a(r).$$

The conclusion follows from the fact that a is bounded (see (1.2)).

- **Second case**. Similar argument □

Now we give an interesting consequence of the above proposition in the case $n = 3$.

THEOREM 10: *Let us suppose that*

- *$\bar{U}$ is a C^1-diffeomorphism between B^3_* and $\mathcal{U}$ and the g_{ij} are of class C^0 on B^3_* and verify* (1.1) *and* (1.2).
- *$\bar{U}$ is $S0(3)$-equivariant.*
- *$\bar{U}$ is weakly harmonic.*
- *$a(r) = r^2 g_{\perp}(r)$ is a nondecreasing function on* (0, 1]. *Then $\bar{U}$ is a C^1_*-minimizing harmonic map.*

Proof: Let us remark first that we can always suppose that $a(0) = 0$. Indeed since a is a positive and increasing map on (0, 1], $a(r)$ admits a limit $a(0)$ when r is tending to zero. If $a(0)$ is strictly positive, let us consider the map from B^3 to the 2-dimensional sphere of radius $\sqrt{a(0)}$, $S^2_{\sqrt{a(0)}}$ which associates to each x different from zero $\sqrt{a(0)}\,x/|x|$. This map is – modulo an isometry – the harmonic tangent map to $\bar{U}$ at zero, $T_0\bar{U}$. But it is known (see [BCL], [Li] or [CG]) that $T_0\bar{U}$ is a minimizing map. Hence for every U in $H^1_*(B^3, \mathcal{U})$, if $Y = \bar{U}^{-1}\circ U$,

$$\int_{B^3}\frac{a(0)}{|Y|^2}|\nabla Y_{\perp}(x)|^2\,dx \geqslant \int_{B^3}\frac{2a(0)}{r^2}\,dx. \tag{2.13}$$

It suffices to show that

$$\int_{B^3}\left\{\left[g_\perp(|Y|)-\frac{a(0)}{|Y|^2}\right]|(\nabla Y)_\perp(x)|^2+g_\|(|Y|)|(\nabla Y)_\|(x)|^2\right\}\mathrm{d}x$$

$$\geqslant \int_{B^3}\left[2\left(g_\perp(r)-\frac{a(0)}{r^2}\right)+g_\|(r)\right]\mathrm{d}x \tag{2.14}$$

to obtain $E(U)\geqslant E(\bar{U})$ by summing (2.13) and (2.14).

Hence we suppose now $a(0) = 0$ and a continuous on [0, 1]. And let us assume for the moment being that there exists a positive bounded measure $\mathrm{d}\mu$ on [0, 1] such that

$$a(r) = \frac{1}{2}\int_{[0,r]}\left(2-\frac{t^2}{r^2}\right)\left(1-\frac{t^2}{r^2}\right)^{-1/2}\mathrm{d}\mu(t). \tag{H}$$

for almost every r (in the sense of Lebesgue's measure).

Let us note K the map from $[0,+\infty)$ to $[0,+\infty)$ with

$$\begin{cases}\forall x\in[0,1),\ K(x)=\dfrac{1}{2}\dfrac{2-x^2}{\sqrt{1-x^2}}\\[2mm] \forall x\in[1,+\infty),\ K(x)=0\end{cases}$$

Then (H) is equivalent to

$$a(r) = \int_{[0,r]} K\left(\frac{t}{r}\right)\mathrm{d}\mu(t) \quad \text{for a.e. } r \text{ in } [0,1]. \tag{2.15}$$

We regularize the measure $\mathrm{d}\mu$ by introducing a map η of class C^1 from $\mathbf{R}$ to [0, 1] whose support is contained in $(-1, 0)$ and with weight

$$\int_{\mathbf{R}}\eta(x)\,\mathrm{d}x = 1.$$

Then we denote $\eta_\varepsilon(x) = (1/\varepsilon)\eta(x/\varepsilon)$ for ε in (0, 1). We extend $\mathrm{d}\mu$ outside [0, 1] by 0 and we define a measurable map l_ε by

$$l_\varepsilon(t) = \int_{[te^{-\varepsilon},t]}\eta_\varepsilon\left(\log\frac{x}{t}\right)\frac{\mathrm{d}\mu(x)}{x} \quad \text{if } t>0.$$

We let

$$dm_\varepsilon(t) \;=\; l_\varepsilon(t)\,dt.$$

Let a_ε be defined by

$$a_\varepsilon(s) \;=\; \int_0^{+\infty} K\left(\frac{t}{s}\right) dm_\varepsilon(t). \tag{2.16}$$

Then

$$a_\varepsilon(s) \;=\; \int_0^{+\infty}\left[\int_{x\in(0,+\infty)} K\left(\frac{t}{s}\right)\eta_\varepsilon\left(\log\frac{x}{t}\right)\frac{d\mu(x)}{x}\right]dt.$$

Writing $(t, x) = (x/v, x)$ we have $d\mu(x)\,dt = (x/v^2)\,d\mu(x)\,dv$ and

$$\begin{aligned} a_\varepsilon(s) \;&=\; \int_0^{+\infty}\left[\int_{x\in(0,+\infty)} K\left(\frac{x}{sv}\right)\eta_\varepsilon(\log v)\,\frac{x}{v^2}\,\frac{d\mu(x)}{x}\right]dv \\ &=\; \int_0^{+\infty}\left[\int_{x\in(0,+\infty)} K\left(\frac{x}{sv}\right)d\mu(x)\right]\frac{\eta_\varepsilon(\log v)}{v^2}\,dv \end{aligned}$$

Now we remark that since $K(t/s) \geqslant 1$, the equality (2.15) implies that $a(r) \geqslant \mu([0, r])$ for a.e. r and hence $\mu(\{0\}) = \lim_{r\to 0}\mu([0, r]) = 0$. Therefore by (2.15) for a.e. v in $[0, 1]$

$$\int_{x\in(0,+\infty)} K\left(\frac{x}{sv}\right)d\mu(x) \;=\; \int_{x\in[0,+\infty)} K\left(\frac{x}{sv}\right)d\mu(x) \;=\; a(sv).$$

Hence we obtain

$$a_\varepsilon(s) \;=\; \int_{e^{-\varepsilon}}^1 a(sv)\,\eta_\varepsilon(\log v)\,\frac{dv}{v^2} \tag{2.17}$$

Let us introduce the coefficients $g_{\varepsilon\|}$ and $g_{\varepsilon\perp}$ on $(0, 1]$ by

$$g_{\varepsilon\perp}(r) \;=\; \int_{e^{-\varepsilon}}^1 g_\perp(rv)\,\eta_\varepsilon(\log v)\,dv, \tag{2.18}$$

$$g_{\varepsilon\|}(r) \;=\; \int_{e^{-\varepsilon}}^1 g_\|(rv)\,\eta_\varepsilon(\log v)\,dv. \tag{2.19}$$

We observe that

$$\lim_{r\to 0} r^4 g_{\varepsilon\|}(r) \;=\; 0.$$

Indeed, since $\bar{U}$ is weakly harmonic, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^4 g_{\|}(r)] \;=\; 2r^2 a'(r) \geqslant 0.$$

Hence if $\lambda = \lim_{r\to 0} r^4 g_{\|}(r)$ were strictly positive, then

$$g_{\|}(r) \geqslant \frac{\lambda}{r^4} \quad \text{on} \quad (0, 1].$$

But this is not possible because of the hypothesis (1.2) on $g_{\|}$. So $\lim_{r\to 0} r^4 g_{\|}(r) = 0$, and it is easy to see that this implies that $\lim_{r\to 0} r^4 g_{\varepsilon\|}(r) = 0$ using (2.19).

We observe also that

$$g_{\varepsilon\perp}(r) \;=\; \frac{a_\varepsilon(r)}{r^2}.$$

Now if $\mathscr{U}_\varepsilon$ is a $S0(3)$-equivariant Riemannian manifold defined using metric coefficients $g_{\varepsilon\|}$ and $g_{\varepsilon\perp}$ on B^3_* and if $\bar{U}_\varepsilon$ is the corresponding C^1-diffeomorphism between B^3_* and $\mathscr{U}_\varepsilon$, it is easy to verify that $g_{\varepsilon\|}$ and $g_{\varepsilon\perp}$ are of class C^1 on B^3_*, that (1.1) and (1.2) are true, and that $\bar{U}_\varepsilon$ is weakly harmonic. Hence $\mathscr{U}_\varepsilon$ and $\bar{U}_\varepsilon$ satisfy all the hypothesis of Proposition 9 with $\varphi(t) = [l_\varepsilon(t)]/t$, and $\bar{U}_\varepsilon$ is a C^1_*-minimizing harmonic map.

Let us write this: for any map U_ε of $C^1_*(B^3_*, \mathscr{U}_\varepsilon)$, let Y be $\bar{U}_\varepsilon^{-1} \circ U_\varepsilon$, then

$$\int_{B^3} [g_{\varepsilon\perp}[|Y(x)|]|(\nabla Y)_\perp(x)|^2 + g_{\varepsilon\|}[|Y(x)|]|(\nabla Y)_\|(x)|^2]\,\mathrm{d}x$$
$$\geqslant \int_{B^3} [2g_{\varepsilon\perp}(r) + g_{\varepsilon\|}(r)]\,\mathrm{d}x. \tag{2.20}$$

We fix Y and ε_0 strictly positive. Let us remark that

$$\bar{U}_\varepsilon \circ Y \in C^1_*(B^3, \mathscr{U}_\varepsilon) \Leftrightarrow \bar{U} \circ Y \in C^1_*(B^3, \mathscr{U})$$
$$\Leftrightarrow Y \in C^1(B^3, B^3) \text{ and satisfies (1.9).}$$

For α strictly positive we let $\omega_\alpha = Y^{-1}(B^3(0, \alpha))$. Since $\bar{U} \circ Y$ belongs to $C^1_*(B^3, \mathscr{U})$ in the case where $B^3_* = B^3 \backslash \{0\}$, $Y^{-1}(\{0\})$ is a finite number of points of B^3 where ∇Y is invertible. We assume for the sake of commodity that $Y^{-1}(\{0\}) = \{0\}$. (But this does not change the conclusions). Then we apply the local inverse mapping theorem and if we choose α small enough the restriction Y_α of Y to ω_α is a bilipschitz C^1-diffeomorphism between ω_α and B^3_α. Then if we let $y = Y_\alpha(x)$

$$\begin{aligned} d_\alpha := \Big| \int_{\omega_\alpha} & [g_\perp(|Y(x)|)|(\nabla Y)_\perp(x)|^2 + g_\parallel(|Y(x)|)|(\nabla Y)_\parallel(x)|^2 \\ & - g_{\varepsilon\perp}(|Y(x)|)|(\nabla Y)_\perp(x)|^2 + g_{\varepsilon\parallel}(|Y(x)|)|(\nabla Y)_\parallel(x)|^2]\,\mathrm{d}x \Big| \\ = \Big| \int_{B^3(0,\alpha)} & [(g_\perp(r) - g_{\varepsilon\perp}(r))|(\nabla Y)_\perp(Y_\alpha^{-1}(y))|^2 \\ & + (g_\parallel(r) - g_{\varepsilon\parallel}(r))|(\nabla Y)_\perp(Y_\alpha^{-1}(y))|^2]|\det Y_\alpha^{-1}(y)|\,\mathrm{d}y \Big| \\ \leqslant \int_{B^3(0,\alpha)} & [g_\perp(r) + g_{\varepsilon\perp}(r) + g_\parallel(r) + g_\parallel(r)]\,C_1\,\mathrm{d}y \end{aligned}$$

where C_1 is a positive constant which does not depend on α. Hence because of (2.18) and (2.19) if we choose α small enough

$$d_\alpha \leqslant \frac{\varepsilon_0}{2}. \tag{2.21}$$

But the metric coefficients $g_\parallel$ and $g_\perp$ are bounded in the C^0-topology on $B^3 \backslash B^3(0, \alpha/2)$, this implies that if ε belongs to $(0, \log 2)$ the coefficients $g_{\varepsilon\parallel}$ and $g_{\varepsilon\perp}$ are bounded on $B^3 \backslash B^3(0, \alpha)$, and if ε tends to zero $g_{\varepsilon\perp}[|Y(x)|]$ and $g_{\varepsilon\parallel}[|Y(x)|]$ tend everywhere to $g_\perp[|Y(x)|]$ and to $g_\parallel[|Y(x)|]$ respectively. Hence we can apply Lebesgue's theorem and we conclude that if ε is small enough

$$\begin{aligned} \Big| \int_{B^3 \omega_\alpha} & [g_{\varepsilon\perp}[|Y(x)|]|(\nabla Y)_\perp(x)|^2 + g_{\varepsilon\parallel}[|Y(x)|]|(\nabla Y)_\parallel(x)|^2]\,\mathrm{d}x \\ & - \int_{B^3 \backslash \omega_\alpha} [g_\perp[|Y(x)|]|(\nabla Y)_\perp(x)|^2 + g_\parallel[|Y(x)|]|(\nabla Y)_\parallel(x)|^2]\,\mathrm{d}x \Big| \leqslant \frac{\varepsilon_0}{2} \end{aligned} \tag{2.22}$$

Hence it comes from (2.21) and (2.22) that, denoting $U = \bar{U} \circ Y$,

$$|E(U_\varepsilon) - E(U)| \leqslant \varepsilon_0$$

for ε small enough. Of course the same inequality holds with $\bar{U}_\varepsilon$ and $\bar{U}$. We show by this way that (2.20) implies

$$\forall U \in C^1_*(B^3, \mathscr{U}),\ E(U) \geqslant E(\bar{U}). \tag{1.11}$$

Now to finish the proof of Theorem 10 we prove (H).

LEMMA 11: *Let b be in $W^{1,1}((0,\ 1),\ \mathbf{R})$. If b is nondecreasing then there exists a positive bounded measure $\mathrm{d}\mu$ on $[0,\ 1]$ such that*

$$b(s) = \int_{[0,s]} K\left(\frac{t}{s}\right) \mathrm{d}\mu(t) \tag{2.23}$$

for almost every s in the sense of Lebesgue's measure.

Proof: Since $\int_{[0,s]} K(t/s)\,\mathrm{d}\delta_0(t) = 1$ for s in $(0,\ 1]$, where δ_0 is the Dirac measure at zero, we may assume that $b(0) = 0$. We fix ε strictly positive, we consider $K_\varepsilon(x) = K(x/[1 + \varepsilon])$, and we first show that there exists a positive map λ_ε in $L^1((0,\ 1),\ \mathbf{R})$ such that

$$b(s) = \int_0^s K_\varepsilon\left(\frac{t}{s}\right) \lambda_\varepsilon(t)\,\mathrm{d}t. \tag{2.24}$$

Now we take the derivative of (2.24)

$$b'(s) = K_\varepsilon(1)\left[\lambda_\varepsilon(s) - \int_0^s \frac{t}{s^2}\frac{K_\varepsilon'(t/s)}{K_\varepsilon(1)}\lambda_\varepsilon(t)\,\mathrm{d}t\right] \tag{2.25}$$

Let H_ε be the map from $(0,\ 1) \times (0,\ 1)$ to $[0,\ +\infty)$ defined by

$$\begin{cases} \text{if } \ t < s,\ H_\varepsilon(t,\ s) = \dfrac{t}{s^2}\dfrac{K_\varepsilon'(t/s)}{K_\varepsilon(1)} \\ \text{if } \ t \geqslant s,\ H_\varepsilon(t,\ s) = 0. \end{cases}$$

Then we note T_ε the operator on $L^1((0,\ 1),\ \mathbf{R})$ which is defined by,

$$\forall \lambda \in L^1((0,\ 1),\ \mathbf{R}),\ (T_\varepsilon\lambda)(s) = \int_0^1 H_\varepsilon(t,\ s)\lambda(t)\,\mathrm{d}t.$$

We remark that $K'(x) = \frac{1}{2}x^3/\sqrt{1 - x^2} \geqslant 0$, and consequently $K_\varepsilon'(x)$ is positive, H_ε and T_ε are positive, i.e. T_ε maps nonnegative functions in nonnegative functions.

Let us show that T_ε is an operator from $L^1((0, 1), \mathbf{R})$ to $L^1((0, 1), \mathbf{R})$ of norm strictly smaller than 1. For any λ in $L^1((0, 1), \mathbf{R})$,

$$(T_\varepsilon \lambda)(s) = \int_0^1 x \frac{K_\varepsilon'(x)}{K_\varepsilon(1)} \lambda(sx)\,\mathrm{d}x.$$

We have

$$\| T_\varepsilon(\lambda) \|_{L^1} = \int_0^1 \left| \int_0^1 x \frac{K_\varepsilon'(x)}{K_\varepsilon(1)} \lambda(sx)\,\mathrm{d}x \right| \mathrm{d}s$$

$$\| T_\varepsilon(\lambda) \|_{L^1} \leqslant \int_0^1 \int_0^1 x \frac{K_\varepsilon'(x)}{K_\varepsilon(1)} |\lambda(sx)|\,\mathrm{d}x\,\mathrm{d}s = \| T_\varepsilon(|\lambda|) \|_{L^1}$$

and

$$\| T_\varepsilon(|\lambda|) \|_{L^1} = \int_0^1 \frac{K_\varepsilon'(x)}{K_\varepsilon(1)} \left[\int_0^x |\lambda(t)|\,\mathrm{d}t \right] \mathrm{d}x$$

$$= \left[\frac{K_\varepsilon(x)}{K_\varepsilon(1)} \int_0^x |\lambda(t)|\,\mathrm{d}t \right]_0^1 - \int_0^1 \frac{K_\varepsilon(x)}{K_\varepsilon(1)} |\lambda(x)|\,\mathrm{d}x$$

$$= \int_0^1 \frac{K_\varepsilon(1) - K_\varepsilon(x)}{K_\varepsilon(1)} |\lambda(x)|\,\mathrm{d}x \leqslant \left[1 - \frac{1}{K_\varepsilon(1)} \right] \| \lambda \|_{L^1}.$$

Hence

$$\| T_\varepsilon(\lambda) \|_{L^1} \leqslant \left[1 - \frac{1}{K_\varepsilon(1)} \right] \| \lambda \|_{L^1},$$

and since $[b'(s)]/[K_\varepsilon(1)] = (1 - T_\varepsilon)(\lambda_\varepsilon(s))$, we can invert the equation (2.25) by

$$\lambda_\varepsilon = \left[\sum_0^{+\infty} T_\varepsilon^P \right] \left(\frac{b'}{K_\varepsilon(1)} \right),$$

because $\Sigma_0^{+\infty} T_\varepsilon^P$ converges in $\mathscr{L}(L^1((0, 1), \mathbf{R}))$.

Furthermore the solution λ_ε is nonnegative since T_ε and b' are positive. By integrating (2.25), we find that λ_ε is the solution of (2.24). Now we try to pass to the limit when ε tends to zero. Let μ_ε be the positive measure with

$d\mu_\varepsilon(t) = \lambda_\varepsilon(t)\,dt$. We have

$$\mu_\varepsilon([0,\ 1]) \leqslant b(1).$$

Hence we can extract a subsequence of ε, which we still denote ε for the sake of convenience, such that μ_ε converges weakly to a bounded positive measure μ, i.e.

$$\forall \varphi \in C^0([0,\ 1],\ \mathbf{R}),\ \int_{[0,1]} \varphi(t)\,d\mu_\varepsilon(t) \to \int_{[0,1]} \varphi(t)\,d\mu(t). \tag{2.26}$$

Let us fix ε_0 positive and let us suppose that $\varepsilon < \varepsilon_0$, then

$$b(s) = \int_0^s K_\varepsilon\left(\frac{t}{s}\right) d\mu_\varepsilon(t) \geqslant \int_0^s K_{\varepsilon_0}\left(\frac{t}{s}\right) d\mu_\varepsilon(t).$$

Let η be in $C^0([0,\ 1],\ [0,\ 1])$ with support in $[0,\ s)$ then

$$b(s) \geqslant \int_{[0,1]} K_{\varepsilon_0}\left(\frac{t}{s}\right) \eta(t)\,d\mu_\varepsilon(t),$$

and the limit when ε tends to zero gives using (2.26)

$$b(s) \geqslant \int_{[0,1]} K_{\varepsilon_0}\left(\frac{t}{s}\right) \eta(t)\,d\mu(t).$$

Now we pass to the limit when ε_0 tends to zero, using Lebesgue's theorem

$$b(s) \geqslant \int_{[0,1]} K\left(\frac{t}{s}\right) \eta(t)\,d\mu(t).$$

This inequality is true for every η with the above hypothesis. Hence

$$b(s) \geqslant \int_{[0,s)} K\left(\frac{t}{s}\right) d\mu(t)$$

and

$$b(s) \geqslant \int_{[0,s]} K\left(\frac{t}{s}\right) d\mu(t) \quad \text{for every } s \text{ such that } \mu(\{s\}) = 0. \tag{2.27}$$

Now we have

$$\int_0^1 b(s)\,\mathrm{d}s = \int_0^1 \left[\int_0^s K_\varepsilon\left(\frac{t}{s}\right)\mathrm{d}\mu_\varepsilon(t)\right]\mathrm{d}s \tag{2.28}$$

$$= \int_0^1 \left[\int_t^1 K_\varepsilon\left(\frac{t}{s}\right)\mathrm{d}s\right]\mathrm{d}\mu_\varepsilon(t) = \int_{[0,1]} c_\varepsilon(t)\,\mathrm{d}\mu_\varepsilon(t)$$

where

$$c_\varepsilon(t) = \int_t^1 K_\varepsilon\left(\frac{t}{s}\right)\mathrm{d}s = \frac{1}{1+\varepsilon}\int_{t/(1+\varepsilon)}^{1/(1+\varepsilon)} \frac{tK(y)}{y^2}\,\mathrm{d}y.$$

Let $c(t) = \int_t^1 [tK(y)]/y^2\,\mathrm{d}y$; it is easy to see that c_ε and c are continuous on [9, 1] (c_ε and c are extended at zero by their limits) and that c_ε converges to c in the C^0-topology. Then we pass to the limit in (2.28):

$$\int_0^1 b(s)\,\mathrm{d}s = \int_{[0,1]}\left[\int_t^1 \frac{tK(x)}{x^2}\,\mathrm{d}x\right]\mathrm{d}\mu(t)$$

$$= \int_0^1 \int_{t\in[0,s]} K\left(\frac{t}{s}\right)\mathrm{d}\mu(t)\,\mathrm{d}s$$

It follows from this and from (2.27) that

$$b(s) = \int_{[0,s]} K\left(\frac{t}{s}\right)\mathrm{d}\mu(t)$$

for a.e. s in the sense of the Lebesgue's measure.

This terminates the proof of Lemma 11 and Theorem 10. Q.E.D.

THEOREM 12: *Let us suppose that $\mathcal{U}$ and $\bar{U}$ verify all the hypothesis of Theorem* 10 *with the supplementary condition that* (1.3) *is true.*

Then $\bar{U}$ is a minimizing harmonic map.

Proof: Same arguments as in the proof of Theorem 5. Q.E.D.

We have another interesting application of Theorem 4 in the case $n = 4$.

THEOREM 13: *Let us suppose that*

- *$\bar{U}$ is a C^1-diffeomorphism between B^n_* and $\mathcal{U}$ and the g_{ij} are continuous and verify* (1.1) *and* (1.2).

- *$\bar{U}$ and $\mathcal{U}$ are $S0(4)$-equivariant.*
- *$\bar{U}$ is weakly harmonic.*
- *$r^4 g_\parallel(r)$ is a nondecreasing map on* $(0, 1]$
- $\lim_{r\to 0} r^2 g_\perp(r)$ *exists and belongs to* $[0, +\infty)$, $\lim_{r\to 0} r^2 g_\parallel(r) = 0$.

Then $\bar{U}$ is a C^1_*-minimizing harmonic map.

Proof: Using the same argument as in the proof of Theorem 10 we shall suppose that $\lim_{r\to 0} r^2 g_\perp(r) = 0$.

We will use Proposition 9 by proving that there exists a nonnegative continuous map φ on $(0, 1]$ such that

$$\begin{cases} r^2 g_\parallel(r) = 2\int_0^r \dfrac{t^2}{r^2}\, t\varphi(t)\,\mathrm{d}t \\ \lim\limits_{r\to 0} r^2 g_\perp(r) = \lim\limits_{r\to 0} \dfrac{4}{3}\int_0^r \left(2 - \dfrac{t^2}{r^2}\right) t\varphi(t)\,\mathrm{d}t < +\infty. \end{cases}$$

The first condition is $r^4 g_\parallel(r) = 2\int_0^r t^3\varphi(t)\,\mathrm{d}t$. We take $\varphi(r) = (1/2r^3)(\mathrm{d}/\mathrm{d}r)[r^4 g_\parallel(r)]$, which is nonnegative since $r^4 g_\parallel(r)$ is nondecreasing.

We know that $\lim_{r\to 0} r^2 g_\perp(r) = 0$ we must verify that

$$\lim_{r\to 0}\int_0^r \left(2 - \frac{t^2}{r^2}\right) t\varphi(t)\,\mathrm{d}t = 0.$$

But

$$\begin{aligned} \int_0^r \left(2 - \frac{t^2}{r^2}\right) t\varphi(t)\,\mathrm{d}t &\leqslant 2\int_0^r t\varphi(t)\,\mathrm{d}t \\ &\leqslant \int_0^r \frac{1}{t^2}\frac{\mathrm{d}}{\mathrm{d}t}[t^4 g_\parallel(t)]\,\mathrm{d}t \\ &\leqslant [t^2 g_\parallel(t)]_0^r + \int_0^r t g_\parallel(t)\,\mathrm{d}t \end{aligned}$$

and our conditions follows from $\lim_{r\to 0} r^2 g_\parallel(r) = 0$.

This achieves the proof of Theorem 13. Q.E.D.

THEOREM 14: *Let us suppose that $\bar{U}$ and $\mathcal{U}$ verify all the hypothesis of Theorem* 13 *further* (1.3), *then $\bar{U}$ is a minimizing harmonic map.*

Proof: See the proof of Theorem 13 and apply Theorem 5. Q.E.D.

III. Consequences and examples

We now establish results in the cases $n = 3$ and $n = 4$ using Theorems 12 and 14.

We consider a $SO(n)$-equivariant Riemannian manifold $\mathcal{U}$ of dimension n, C^1-diffeomorphic to B^n or $B^n\backslash\{0\}$, i.e. there exists a C^1-diffeomorphism X between $\mathcal{U}$ and B^n_* where B^n_* may be B^n or $B^n\backslash\{0\}$ such that the metric at a point u of $\mathcal{U}$ is given using two strictly positive functions $\gamma_\perp$ and $\gamma_\parallel$ of class C^0 on B^n_* which depend only on r by

$$\mathrm{d}s^2_{\mathcal{U}}(u) = \gamma_\perp(|X(u)|)\left(\mathrm{d}X - \left\langle \mathrm{d}X, \frac{X}{|x|}\right\rangle \frac{X}{|X|}\right)^2 + \gamma_\parallel(|X(u)|)\left\langle \mathrm{d}X, \frac{X}{|X|}\right\rangle^2.$$

(a) If $B^n_* = B^n\backslash\{0\}$.

We will assume that our Riemannian manifold $\mathcal{U}$ satisfy the compactness conditions (1.1), (1.2) and (1.3) i.e. there exists strictly positive constants K_1 and K_2 such that

$$\forall r \in (0,1) \quad K_1 \leqslant r^2\gamma_\perp(r) \leqslant K_2. \tag{3.1}$$

$\int_0^1 \sqrt{\gamma_\parallel(t)}\,\mathrm{d}t < +\infty$ and the closure of $\gamma_\parallel([\frac{1}{2}, 1))$ is a compact subset of $(0, +\infty)$. (3.2)

$$\int_{B^n} [(n-1)\gamma_\perp(r) + \gamma_\parallel(r)]\,\mathrm{d}x = |S^{n-1}| \times \int_0^1 [(n-1)\gamma_\perp(r) + \gamma_\parallel(r)]\,r^{n-1}\,\mathrm{d}r < +\infty. \tag{3.3}$$

We remark that (3.2) expresses that any meridian curve M_ω has a finite arc-length where $M_\omega = X^{-1}\{r\omega/r \in (0,1)\}$ and ω belongs to S^{n-1}.

This allows us to consider the map s in $C^1((0,1),(0,l))$ where $l = \int_0^1 \sqrt{\gamma_\parallel(t)}\,\mathrm{d}t$ with

$$s(r) = \int_0^r \sqrt{\gamma_\parallel(t)}\,\mathrm{d}t. \tag{3.4}$$

Then we shall call S the C^1-diffeomorphism between B^n_* and $B^n(0,l)\backslash\{0\} = B^n_*(l)$ defined by

$$S(x) = \frac{s(|x|)}{|x|}\,x. \tag{3.5}$$

And we shall call L the C^1-diffeomorphism $S \circ X$ between $\mathcal{U}$ and $B^n_*(l)$.

(b) If $B^n_* = B^n$.

We still assume (3.2) and let $l = \int_0^1 \sqrt{\gamma_{\|}(t)}\,dt$; s given by (3.4) belongs to $C^1([0, 1), [0, l))$ and S defined by (3.5) is a C^1-diffeomorphism between B^n and $B^n(0, l) = B^n_*(l)$. Then $L = S \circ X$ is a C^1-diffeomorphism between $\mathscr{U}$ and $B^n_*(l)$. Let us remark that in this case L is the inverse of the exponential map from $T_0\mathscr{U}$ to $\mathscr{U}$.

The metric coefficients $\tilde{\gamma}_{\|}$ and $\tilde{\gamma}_{\perp}$ on $\mathscr{U}$ in the chart L are then given by

$$\forall x \in B^n_*(l),\ \tilde{\gamma}_{\|}(|x|) \quad = \quad 1$$

$$\forall x \in B^n_*(l),\ |x|^2\tilde{\gamma}_{\perp}(|x|) \quad = \quad s^{-1}(|x|)^2\gamma_{\perp}[|S^{-1}(x)|].$$

Clearly, $\tilde{\gamma}_{\perp}$ has the same behavior as $\gamma_{\perp}$ described by (3.1). We remark that

$$|S^{n-1}| \int_0^l [2\tilde{\gamma}_{\perp}(r) + \tilde{\gamma}_{\|}(r)]\,r^{n-1}\,dr \leqslant |S^{n-1}| \int_0^1 (2K_2 + 1)\sqrt{\gamma_{\|}(t)}\,dt < +\infty.$$

Hence we shall suppose that $X = L$ and that $\gamma_{\|}(r) = \tilde{\gamma}_{\|}(r) = 1$ on $B^n_*(l)$.

An important case for $\mathscr{U}$ is described as follows.

Let $\mathscr{V}$ be a compact Riemannian manifold of class C^1 with boundary, which is diffeomorphic to $\bar{B}^n$ or to $\bar{B}^n \backslash B^n(0, \frac{1}{2})$ and $SO(n)$-equivariant. Then we can find a C^1-diffeomorphism between the interior of $\mathscr{V}$, $\mathring{\mathscr{V}}$ and B^n_*, which will verify (3.1), (3.2) and also (3.3) when $\mathscr{V}$ is diffeomorphic to $\overline{B^n} \backslash \{0\}$, and such that $\gamma_{\|}(|x|) = 1$. Then we take $\mathscr{U} = \mathring{\mathscr{V}}$.

EXAMPLE A: I is a closed bounded interval of $\mathbf{R}$, Γ is an imbedding of I into the meridian half-plane $P_+ = (0, +\infty) \times \mathbf{R}$ of class C^1, and $\mathscr{V}$ is the hypersurface of revolution of $\mathbf{R}^{n+1}$ of generatrix $\gamma[I]$. A parametrization of $\mathscr{V}$ is given by $\Gamma_I(r)(\omega^1 e_1 + \cdots + \omega^n e_n) + \Gamma_{II}(r) e_{n+1}$ where $\omega = (\omega^1, \ldots, \omega^n)$ belongs to S^{n-1}, r belongs to I and $\Gamma(r) = (\Gamma_I(r), \Gamma_{II}(r))$.

EXAMPLE B: F is a $SO(n)$-equivariant map of class C^1 from $\overline{B^n}$ to $\mathbf{R}$, i.e.

$$\forall R \in SO(n), \quad \forall x \in \overline{B^n}, \quad F(Rx) = F(x).$$

And $\mathscr{V}$ is the graph of F, $\{(x, F(x))(x \in \overline{B^n})\}$.

We have for $n = 3$.

THEOREM 15: *Let $\mathscr{U}$ be a $S0(3)$-equivariant Riemannian manifold of dimension* 3 *which is C^1-diffeomorphic to B^3_* with the conditions* (3.1), (3.2) *and also* (3.3) *when $\mathscr{U}$ is diffeomorphic to $\overline{B^3} \backslash \{0\}$. Let us assume that $r^2\gamma_{\perp}(r)$ is nondecreasing.*

Let $\bar{U}$ be a $S0(3)$-equivariant map from B^3 to $\mathcal{U}$ of class C^1 on B^3_* with

$$\begin{cases} L \circ \bar{U}(x) = lx & \text{for } x \text{ in } \partial B^3 \\ \lim_{x\to 0} L \circ \bar{U}(x) = 0 & \\ |L \circ \bar{U}(x)| \not\equiv 0 & \text{in } B^3(0, \delta)\backslash\{0\} \text{ for positive.} \end{cases} \tag{3.6}$$

Then if $\bar{U}$ is weakly harmonic, $\bar{U}$ is a minimizing harmonic map.

Remark: In Example B the condition $r^2\gamma_\perp(r)$ is increasing is always true.

Proof of Theorem 15: Let $\bar{Y}$ be $L \circ \bar{U}$. We let $r = |x|$, $\bar{R}(|Y(x)|) = |\bar{Y}(x)|$. The energy of $\bar{U}$ is, with $b(r) = r^2\gamma_\perp(r)$.

$$E(\bar{U}) = \frac{1}{2}\int_{B^3}\left[\left|\frac{\partial \bar{Y}}{\partial r}\right|^2 + \gamma_\perp(|\bar{Y}(x)|)\right]\mathrm{d}x$$

$$E(\bar{U}) = \frac{|S^2|}{2}\int_0^1 [(\bar{R}'(r))^2 + 2b[\bar{R}(r)]r^{-2}]r^2\,\mathrm{d}r.$$

We now use the fact that $\bar{U}$ is weakly harmonic. Let us remark that since $\mathcal{U}$ is a C^1-Riemannian manifold b is of class C^1. We take a map φ in $C^1_c((0, 1), \mathbf{R})$ and we let $\phi(x) = \varphi(|x|)(x/|x|)$, then

$$E'(\bar{U}, \phi) = |S^2|\int_0^1 [\bar{R}'(r)\varphi'(r) + b'[\bar{R}(r)]\varphi(r)r^{-2}]r^2\,\mathrm{d}r = 0$$

In fact since $\bar{R}'$ is of class C^1 we can extend by density the definition of $E'(\bar{U}, \phi)$ to the case where φ is integrable with compact support in $(0, 1)$ and φ' is a bounded measure and for such a φ, $E'(\bar{U}, \phi)$ is still zero.

From (3.6) and the fact that $\bar{R}$ is of class C^1 we deduce that $\bar{R}$ is surjective and that $\bar{R}'$ is strictly positive in a neighbourhood of zero. Let δ be in $(0, 1)$ and r_0 be in $(0, \delta)$ such that $\bar{R}'(r_0) > 0$ for any r in $(r_0, 1)$ we consider the test map with φ given by

$$\begin{cases} \forall r \in (0, r_0), & \varphi(r) = 0 \\ \forall r \in [r_0, r], & \varphi(r) = 1 \\ \forall r \in (r, 1), & \varphi(r) = 0 \end{cases}$$

Then $E'(\bar{U}, \phi) = |S^2|[\bar{R}'(r_0)r_0^2 - \bar{R}'(r)r^2 + \int_{r_0}^r b'[\bar{R}(r)]\,\mathrm{d}r] = 0.$

This implies since b is increasing

$$\bar{R}'(r_0)\,r_0^2 \leqslant \bar{R}'(r) \leqslant \bar{R}'(r_0) + \frac{1}{r_0^2}\int_{r_0}^{1} b'[\bar{R}(r)]\,\mathrm{d}r$$

hence

$$\int_{r_0}^{1} b'[\bar{R}(r)]\,\mathrm{d}r = \int_{r_0}^{1} \bar{R}'(r)\,b'[\bar{R}(r)]\,\frac{\mathrm{d}r}{\bar{R}'(r)} \leqslant \frac{1}{\bar{R}'(r_0)\,r_0^2}\,(b(l) - b[\bar{R}(r_0)])$$

Finally this leads to

$$\bar{R}'(r_0)\,r_0^2 \leqslant \bar{R}'(r) \leqslant \bar{R}'(r_0) + \frac{1}{r_0^4\,\bar{R}'(r_0)}\,b(l). \tag{3.7}$$

Since $\bar{R}$ is a C^1-diffeomorphism between $(0, 1)$ and $(0, l)$, $\bar{U}$ is a C^1-diffeomorphism between B^3_* and $\mathscr{U}$.

Hence we can use the chart $\bar{U}^{-1}$ on $\mathscr{U}$, and we verify easily that the condition (3.1), (3.2) and (3.3) are always true using this chart. Particularly the second assertion of (3.2) follows from (3.7).

We can use Theorem 14, (note that $a(r) = b(R(r))$ is increasing) this achieves the proof. Q.E.D.

We now give another consequence of Theorem 12.

PROPOSITION 16: *Let Σ be the Riemannian manifold constructed by putting on $\mathbf{R} \times S^2$ the metric $(\mathrm{d}s)^2 = (\mathrm{d}t)^2 + \gamma(t)(\mathrm{d}\omega)^2$ where (t, ω) belongs to $\mathbf{R} \times S^2$ and where γ is a function of class C^1 from $\mathbf{R}$ to $(0, +\infty)$. Let $\bar{u}$ be a $SO(3)$-equivariant map from B^3 to Σ which we will express using a function s in $C^1((0, 1], \mathbf{R})$ by $\bar{u}(r\omega) = (s(r), \omega)$. We suppose that*

$$(\gamma \circ s)' \text{ is positive on } (0, 1] \text{ and } s((0, 1]) \text{ is bounded in } \mathbf{R}. \tag{3.8}$$

$$\inf_{(0,1]} \gamma \circ s(r) = \inf_{y \in \mathbf{R}} \gamma(y). \tag{3.9}$$

$$\text{If } \quad \gamma \circ s(r) = \gamma(y) \quad \text{then} \quad |\gamma'[s(r)]| \geqslant |\gamma'(y)|. \tag{3.10}$$

Then if $\bar{u}$ is weakly harmonic, $\bar{u}$ is minimizing harmonic, i.e. minimizes the energy among the maps from B^3 to Σ which agree with $\bar{u}$ on ∂B^3.

Proof: We can always suppose that $s((0, 1]) = [0, l)$ because of (3.8), where l is in $(0, +\infty)$. We define a map f from $\mathbf{R}$ to $[0, l)$ by

- if $y \in [0, l]$, $f(y) = y$.
- if $y \notin [0, l]$ and if $\gamma(y) \in \gamma([0, l])$.

Since $\gamma \circ s$ is strictly increasing γ is a diffeomorphism between $[0, l)$ and $\gamma([0, l))$ and $f(y)$ is the unique real of $[0, l]$ such that

$$\gamma[f(y)] = \gamma(y).$$

• if $y \notin [0, l]$ and if $\gamma(y) \geqslant \gamma(0)$, $f(y) = 0$.
Then f is well defined and Lipschitz because of (3.9) and (3.10).
If $\gamma(y)$ belongs to $\gamma([0, l])$, $\gamma[f(y)] = \gamma(y)$ implies

$$\gamma'[f(y)]f'(y) = \gamma'(y)$$

and this implies $|f'(y)| \leqslant 1$ because of (3.10).
If $\gamma(y)$ belongs to $[g(0), +\infty)$, $|f'(y)| = 0 \leqslant 1$.
In both cases one verifies too that $\gamma(f(y)) \leqslant \gamma(y)$. Now we take φ a map from B^3 to Σ which agrees with $\bar{u}$ on ∂B^3 then if $\varphi = (\varphi_1, \varphi_2) \in \mathbf{R} \times S^2$,

$$E(\varphi) = \frac{1}{2}\int_{B^3} [|\nabla\varphi_1|^2 + \gamma[\varphi_1(x)]|\nabla\varphi_2|^2]\,dx.$$

Then letting $\tilde{\varphi}$ be $(f \circ \varphi_1, \varphi_2)$ we have

$$E(\tilde{\varphi}) = \frac{1}{2}\int_{B^3} [|f'(\varphi_1(x))]^2|\nabla\varphi_1|^2 + \gamma(f \circ \varphi_1(x))|\nabla\varphi_2(x)|^2]\,dx$$

$$E(\tilde{\varphi}) \leqslant E(\varphi).$$

Hence it suffices to verify that $\bar{u}$ minimizes the energy functional among the maps from B^3 to the image of $\bar{u}$, $[0, l) \times S^2$. And this follows from Theorem 12. Q.E.D.

EXAMPLE 1: The following example has been constructed by Gulliver and White in [GW]. Here $n = 3$ and Σ is a Riemannian manifold as in Proposition 16 with

$$\gamma(t) = 1 + \frac{t^4}{4} + \frac{t^6}{2}.$$

Then we put $\bar{u}(r\omega) = (s(r), \omega)$ with

$$s(r) = (1 - 2\log r)^{-1/2}.$$

In [GW] Gulliver and White ask if $\bar{u}$ is a minimizing harmonic map. It follows from Proposition 16 that the answer is yes. We recall that the interest of $\bar{u}$ is that the blow-up sequence $\bar{u}_\lambda(x) = \bar{u}(\lambda x)$ tends to its homogeneous tangent map more slowly than any positive power of λ when λ tends to zero, see [GW].

EXAMPLE 2: This is essentially the same example as the above example but in dimension 4. We take

$$\gamma(t) = 1 + \frac{t^4}{3} + \frac{t^6}{3}$$

and $s(r) = (1 - 2 \log r)^{-1/2}$.

Then a variant of Proposition 16 in the case of the dimension 4 can be proved without difficulty using Theorem 14, and we can show that $\bar{u}$ is a minimizing harmonic map if we verify that $r^4 g_{\|}(r)$ is increasing (where $g_{\|}$ has the sense of Theorem 14), i.e. we must show that

$$r^4(s'(r))^2 \text{ is increasing.}$$

But it is easy to compute that $(d/dr)[r^4(s'(r))^2] \geqslant 0$.

Now we present an analysis of $SO(3)$-equivariant harmonic maps into a $SO(3)$-equivariant manifold of dimension 3 of class C^2. The method is a straightforward adaptation of the method of [JK]. This study together with the other results of this section will give in some cases a minimizing solution.

THEOREM 17: *Let $\mathscr{V}$ be a compact Riemannian manifold of class C^2 with boundary of dimension* 3, *which is* $S0(3)$*-equivariant and* C^2*-diffeomorphism to* $\overline{B^3}$. *We describe $\mathscr{V}$ using the metric given by $\gamma_{\|}(s) = 1$ and $\gamma_\perp$ on the closed ball $\overline{B^3(l)}$ as presented at the beginning of this section. Here we assume that $\gamma_\perp$ belongs to $C^2([0, l], (0, +\infty))$ and we suppose that, $b'(s)$ is strictly positive on $(0, l)$ where $b(s) = s^2\gamma_\perp(s)$. Then*

- *If either $b'(l) > 0$ or*

$$b'(l) = 0 \quad \text{and} \quad b''(l) < -\tfrac{1}{4} \tag{3.11}$$

there exists a $SO(3)$-equivariant minimizing harmonic map $\bar{u}$ of class C^2 from $\overline{B^3}$ to $\mathscr{V}$ with $\bar{u}(x) = lx$ on ∂B^3.

- *If*

$$\begin{cases} b'(l) = 0, \ -\tfrac{1}{4} \leqslant b''(l) < 0 \quad \text{and} \\ \forall s \in [0, l], \ b'(s) \leqslant b''(l)(s - l) \end{cases} \tag{3.12}$$

$u_*(x) = l(x/|x|)$ minimizes the energy among the maps which agree with u_* on ∂B^3.

Proof: We try a $SO(3)$-equivariant harmonic map $\bar{u}$ of class C^2 on $\overline{B^3}$ with finite energy. We pose $\bar{u}(r\omega) = \bar{R}(r)\omega$ where $\bar{R}$ is suppose to be a map of class C^2 from $[0, 1]$ to $[0, l]$. Then as proved in Theorem 15 $\bar{R}$ must satisfy

$$\int_0^1 [\bar{R}'(r)\varphi'(r) + b'[\bar{R}(r)]\varphi(r)r^2]r^2\,\mathrm{d}r = 0$$

for every φ in $C_c^1((0, 1), \mathbf{R})$. And since $\bar{R}$ is of class C^2 this implies

$$r^2R''(r) + 2rR'(r) = b'[R(r)] \quad \text{on} \quad (0, 1). \tag{3.13}$$

with the boundary conditions

$$R(0) = 0, \quad R(1) = l \tag{3.14}$$

and the finite energy condition

$$E(u) = 2\pi\int_0^1 \left[\bar{R}'(r)^2 + \frac{2b[\bar{R}(r)]}{r^2}\right] r^2\,\mathrm{d}r < +\infty. \tag{3.15}$$

(a) We make the change of variable $t = \log r$ and define the map λ of class C^1 from $(-\infty, 0)$ to $[0, l]$ with

$$\lambda(t) = \bar{R}(r).$$

Then the required equations (3.13), (3.14) and (3.15) become respectively

$$\ddot{\lambda} + \dot{\lambda} = b'(\lambda) \quad \text{on} \quad (-\infty, 0) \tag{3.16}$$

$$\lim_{t\to-\infty} \lambda(t) = 0; \quad \lambda(0) = l \tag{3.17}$$

$$E(u) = 2\pi\int_{-\infty}^0 [\dot{\lambda}^2 + 2b(\lambda)]e^t\,\mathrm{d}t < +\infty. \tag{3.18}$$

where the dots denote the derivative with respect to t.

The appropriate way to study (3.16), (3.15) and (3.16) is to write this equation in the phase space. We let

$$x(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \lambda(t) \\ \dot{\lambda}(t) \end{pmatrix}$$

then $x(t)$ is an integral curve of the vector field

$$f\begin{pmatrix}\xi\\ \zeta\end{pmatrix} = \begin{pmatrix}\zeta\\ b'(\xi) - \zeta\end{pmatrix}$$

i.e.

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}p\\ b'(q) - p\end{pmatrix}. \tag{3.19}$$

Since $\gamma_\perp$ is bounded from above and from below and since $b(s) = s^2 g_\perp(s)$, we have $b(0) = b'(0) = 0$ but $b''(0) = 2\gamma_\perp(0) > 0$.

This implies that the point $\binom{0}{0}$ is critical and the linearized system at this point is

$$\begin{pmatrix}\dot\xi\\ \dot\zeta\end{pmatrix} = \begin{pmatrix}0 & 1\\ b''(0) & -1\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}. \tag{3.20}$$

The eigenvalues and the corresponding eigenvectors are

$$\lambda_- = \frac{-1 - \sqrt{1 + 4b''(0)}}{2} \in (-\infty, -1)$$

$$v_- = \begin{pmatrix}1\\ \lambda_-\end{pmatrix}\frac{1}{\sqrt{1 + \lambda_-^2}}$$

$$\lambda_+ = \frac{-1 + \sqrt{1 + 4b''(0)}}{2} \in (0, +\infty)$$

$$v_+ = \begin{pmatrix}1\\ \lambda_+\end{pmatrix}\frac{1}{\sqrt{1 + \lambda_+^2}}.$$

Hence $(0, 0)$ is a saddle point.

We extend the function b to $\mathbf{R}$ in a function (still denoted by b) which is a nonnegative function of class C^2 with compact support; in the case where $b'(l) = 0$ (and $b''(l) < 0$) we will suppose that b reaches its maximum at $s = l$ and is symmetric with respect to l i.e. $b(2l - s) = b(s)$.

(b) We construct the solution of our problem.

First using a result of Hartman ([Ha2] VIII.3, Theorem 3.4) we know that there exists a solution $X = (Q, P)$ on $(-\infty, 0]$ of (3.19) such that

$$\lim_{t \to -\infty} t^{-1} \log \| X(t) \| = \lambda_+ \tag{3.21}$$

and

$$\lim_{t \to -\infty} \frac{X(t)}{\| X(t) \|} = v_+ . \tag{3.22}$$

In fact X can be extended on $\mathbf{R}$ because the vector field $f\binom{\xi}{\zeta}$ has a linear growth for large values of $\sqrt{\xi^2 + \zeta^2}$.

Let V be the map on $\mathbf{R}$ with

$$V(t) = P(t)^2 - 2b[Q(t)].$$

From (3.19) follows

$$\overbrace{(V(t)e^t)}^{\cdot} = -(P^2(t) + 2b(Q(t)))\,e^t. \tag{3.23}$$

Since P and Q are bounded when t tends to $-\infty$,

$$\lim_{t \to -\infty} (V(t)e^t) = 0 \tag{3.24}$$

From (3.23) and (3.24) we deduce that $V(t)e^t$ is strictly negative on $\mathbf{R}$ or:

$$P(t)^2 < 2b[Q(t)].$$

Hence the curve X does not go out of the bounded open set $\{(\xi, \zeta) \in \mathbf{R}^2 / \zeta^2 < 2b(\xi)\}$.

Furthermore on the segment $(0, l) \times \{0\}$ the vector field is pointing upward. Let K_1 be $\{(\xi, \zeta) \in \mathbf{R}^2/\zeta^2 < 2b(\xi), 0 < \xi < l, \zeta > 0\}$. Then either the curve X goes out of K_1 through the segment $\{l\} \times (0, \sqrt{2b(l)})$, or the curve stays in K_1.

(c) Now we can handle the case $b'(l)$ strictly positive. Let us suppose that X goes never out of K_1, then using (3.21) and (3.22) we know that there exists t_1 in $\mathbf{R}$ such that

$$\forall t \in (-\infty, t_1], \quad Q(t) > 0, \quad P(t) > 0. \tag{3.25}$$

And since $b'(l) > 0$, there exists δ positive such that

$$\forall s \in [Q(t_1), l], \, b'(s) \geqslant \delta.$$

which implies using (3.19)

$$\forall t \in [t_1, +\infty), \quad P(t) \geqslant \delta(1 - e^{t_1 - t}) + P(t_1)e^{t_1 - t}$$

or

$$\forall t \in [t_1, +\infty), \quad P(t) \geqslant \inf(\delta, P(t_1)). \tag{3.26}$$

This together with (3.19) implies that there exists t_2 in $[t_1, +\infty)$ such that X goes out of K_1 at t_1 i.e., $Q(t_1) = l$. Hence X goes out of K_1 in all cases where $b'(l) > 0$.

This gives us a solution to the problem (3.16), (3.17) and (3.18) which is strictly increasing on $(-\infty, 0]$ by taking $X(t)$ and by making a translation on t. Using Theorem 15 this gives the conclusion of the present theorem.
(d) We assume that $b'(l)$ is zero. The curve X stays in $K_2 = \{(\xi, \zeta) \in \mathbf{R}^2/ \zeta^2 < 2b(\zeta), 0 < \xi < 2l\}$ and X has three critical points in $\bar{K}_2$: $(0, 0)$, $(l, 0)$ and $(2l, 0)$.

Using results of Lasalle [La], $X(t)$ tends to one of these critical points when t tends to $+\infty$.

But from (3.21) and (3.22) it comes that $\lim_{t \to \infty} V(t) = 0$ and snce $\dot{V}(t) = -2P(t)^2$ we have

$$V(t) = \int_{-\infty}^{t} -2P(\tau)^2 \, \mathrm{d}r$$

which precludes $\lim_{t \to +\infty} V(t)$ to be positive or zero. This implies that $X(t)$ cannot tend to $(0, 0)$ or $(2l, 0)$ when t tends to $+\infty$ but only to $(l, 0)$.

Hence we study what happens at $(l, 0)$. The linearized system is

$$\begin{pmatrix} (\xi - l) \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b''(l) & -1 \end{pmatrix} \begin{pmatrix} \xi - l \\ \zeta \end{pmatrix} \tag{3.27}$$

First case $b''(l) < -\frac{1}{4}$.

Then the eigenvalues are complex with a strictly negative real part

$$\Lambda_{\pm} = \frac{-1 \pm i\sqrt{-(1 + 4b''(l))}}{2}$$

We use Hartman's theorem ([Ha2], VIII.3): the point $X(t)$ is spiralling around $(l, 0)$ and tends to $(l, 0)$ when t tends to $+\infty$. Hence there exists t_1 in $\mathbf{R}$ with

$$\forall t \in (-\infty, t_1), \quad Q(t) > 0, \quad P(t) > Q(t_1) = l$$

This gives, modulo a translation on t, the solution of the problem.

Second case $-\frac{1}{4} \leqslant b''(l) < 0$.

Then we have two strictly negative real eigenvalues.

$$\Lambda_- = \frac{-1 - \sqrt{1 + 4b''(l)}}{2} \in (-1, -\tfrac{1}{2}]$$

$$\Lambda_+ = \frac{-1 + \sqrt{1 + 4b''(l)}}{2} \in [-\tfrac{1}{2}, 0)$$

We assume that $b'(s) \leqslant b''(l)(s - l)$. Then let K_3 be $\{(\xi, \zeta) \in \mathbf{R}^2 / 0 < \xi < l, \zeta^2 < 2b(\xi), \zeta > 0, \zeta < \Lambda_+(\xi - l)\}$. On the boundary segment $\{(\xi, \Lambda_+(\xi - l))\}$ of K_3, we have

$$f(\xi, \zeta) = \Lambda_+ \begin{pmatrix} \xi - l \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ b'(\xi) - b''(k)(\xi - l) \end{pmatrix} \tag{3.28}$$

because of the relation $\Lambda_+^2 + \Lambda_+ = b''(l)$.

And (3.28) implies that f is pointing inside K_3. This proves that X does not go out of K_3, and we do not find our solution. But $X(t)$ tends to $(l, 0)$ when t tends to $+\infty$, and hence the value $l(1 - \varepsilon)$ is reached by $Q(t)$ in a finite time t_ε for any strictly positive ε. This gives the solution of the following modified problem.

We consider $\mathscr{V}_\varepsilon$ the compact Riemannian submanifold of $\mathscr{V}$ equal to $\overline{L^{-1}[B^3(l(1 - \varepsilon))]}$. Then from Theorem 15 and the above computation $u_\varepsilon(r\omega) = Q(t_\varepsilon + \log r)\omega$ is a minimizing harmonic map from B^3 to $\mathscr{U}_\varepsilon = \mathring{\mathscr{V}}_\varepsilon$ with boundary conditions $u_\varepsilon(x) = l(1 - \varepsilon)(x/|x|)$ on ∂B^3. Hence for every map Y from B^3 to $B^3(l)$ with $Y(x) = lx$ on ∂B^3

$$\int_{B^3} [(1 - \varepsilon)^2 |(\nabla Y)_\|\|^2 + \gamma_\perp((1 - \varepsilon)|Y|)(1 - \varepsilon)^2 |(\nabla Y)_\perp|^2] \mathrm{d}x$$

$$\geqslant \int_{B^3} \left[\left| \frac{\mathrm{d}u_\varepsilon(x)}{\mathrm{d}r} \right|^2 + 2\gamma_\perp(|u_\varepsilon(x)|) \frac{|u_\varepsilon(x)|^2}{r^2} \right] \mathrm{d}x. \tag{3.29}$$

Now if ε tends to zero, t_ε tends to $+\infty$ and hence $u_\varepsilon(r\omega)$ tends to $l\omega$. We pass to the limit in (3.29) using boundedness and continuity of $\gamma_\perp$ and obtain

$$\int_{B^3} [|(\nabla Y)_\parallel|^2 + \gamma_\perp(|Y|)|(\nabla Y)_\perp|^2]\,\mathrm{d}x \geqslant \int_{B^3} \frac{2l^2\gamma_\perp(l)}{r^2}\,\mathrm{d}x. \tag{3.30}$$

This proves that in the case (3.12) $l(x/|x|)$ minimizes the energy among the maps which agree with u_* on ∂B^3. This achieves the proof of Theorem 17.

Q.E.D.

EXAMPLE 3: We consider for a in $(0, +\infty)$ the ellipsoid

$$\mathscr{E}_a = \left\{(x^1, x^2, x^3, x^4) \in \mathbf{R}^4/(x^1)^2 + (x^2)^2 + (x^3)^2 + \frac{(x^4)^2}{a^2} = 1\right\}.$$

Then we consider the maps u of $H^1(B^3, \mathbf{R}^4)$ such that $u(x)$ belongs to $\mathscr{E}_a$ for a.e. x in B^3 and which verify in the trace sense $u(x) = (x, 0)$ on ∂B^3. We try to find a map which minimizes the energy among these maps. Because of the symmetry of $\mathscr{E}_a$ it suffices to consider the map to $\mathscr{E}_a^+ = \{x \in \mathscr{E}_a/x^4 \geqslant 0\}$. The problem was first solved for $a = 1, n \geqslant 7$ in [JK], $a \geqslant 1$ and $n \geqslant 7$ in [B] and $a \leqslant 1$ and $n \geqslant 3$ in [He]. Furthermore Baldes showed in [B] that $u_*(x) = (x/|x|, 0)$ could be minimizing only when $a^2 \geqslant [4(n-1)]/[(n-2)^2]$ for $n \geqslant 3$. It follows from all these papers that one can make the following conjecture for $n \geqslant 3$

$$\begin{cases} \text{if } \; a^2 \geqslant \dfrac{4(n-1)}{(n-2)^2}, \; u_* \text{ is minimizing} \\[2ex] \text{if } \; a^2 < \dfrac{4(n-1)}{(n-2)^2}, \; \text{there exists a } SO(n)\text{-equivariant minimizing map} \\ \qquad\qquad\qquad\qquad \text{of class } C^\infty \text{ on } B^n. \end{cases}$$

The unknown cases were $a > 1$ and $3 \leqslant n \leqslant 6$. Here we solve the case $n = 3$ using Theorem 17

$$\begin{cases} \text{if } \; a^2 \geqslant 8, \; u_* \text{ is minimizing} \\ \text{if } \; a^2 < 8, \; \text{there exists a } S0(3)\text{-equivariant minimizing map of class} \\ \qquad\qquad C^\infty \text{ on } B^3. \end{cases}$$

We use the following parametrization of $\mathscr{E}_a^+$

$$\mathscr{E}_a^+ = \left\{(\omega \sin\theta, a\cos\theta)/\theta \in \left[0, \frac{\pi}{2}\right], \omega \in S^2\right\}.$$

Then the abscissa along a meridian curve is

$$s(\theta) = \int_0^{\theta} \sqrt{a^2 \sin^2 \tau + \cos^2 \tau}\, d\tau$$

And $b(s)$ is defined by $b[s(\theta)] = \sin^2 \theta$.

Hence

$$b'[s(\theta)] = \frac{2 \cos \theta \sin \theta}{\sqrt{\cos^2 \theta + a^2 \sin^2 \theta}}$$

$$b''[s(\theta)] = \frac{2(\cos^4 \theta - a^2 \sin^4 \theta)}{(a^2 \sin^2 \theta + \cos^2 \theta)^2}.$$

Here $l = \int_0^{\pi/2} \sqrt{a^2 \sin^2 \tau + \cos^2 \tau}\, d\tau$, $b'(l) = 0$ and $b''(l) = -2/a^2$.

If $a^2 < 8$ or $b''(l) < -\frac{1}{4}$ then we have a smooth $S0(3)$-equivariant minimizing map.

If $a^2 \geqslant 8$ or $b''(l) \geqslant -\frac{1}{4}$. We must then verify that

$$\forall s \in [0, l], \quad b'(s) \leqslant b''(l)(s - l)$$

to conclude i.e.

$$\forall \theta \in \left[0, \frac{\pi}{2}\right] \frac{2 \cos \theta \sin \theta}{\sqrt{\cos^2 \theta + a^2 \sin^2 \theta}} \leqslant \frac{2}{a^2} \int_0^{\pi/2} \sqrt{\cos^2 \tau + a^2 \sin^2 \tau}\, d\tau$$

By letting

$$H(\theta) = \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + a^2 \sin^2 \theta}} - \frac{1}{a^2} \int_0^{\pi/2} \sqrt{\cos^2 \tau + a^2 \sin^2 \tau}\, d\tau,$$

the required inequality is $H(\theta) \leqslant 0$ which follows from

$$H'(\theta) = \frac{(1 + (1/a^2)) \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta}{(\cos^2 \theta + a^2 \sin^2 \theta)^{3/2}} \geqslant 0 \quad \text{and} \quad H\left(\frac{\pi}{2}\right) = 0.$$

Example 4: This examples shows that Theorem 12 is not optimal in the sense that $a(r) = r^2 g_\perp(r)$ may be decreasing somewhere but the harmonic map considered can be minimizing. Let ε be strictly positive and small enough and let φ be the map from [0, 1] to [0, 1] defined by

$$\begin{cases} \forall t \in [0, 1 - \varepsilon], \ \varphi(t) = 1 \\ \forall t \in [1 - \varepsilon, 1], \ \varphi(t) = 0 \end{cases}$$

Then we take

$$g_{\perp}(r) = \frac{1}{2}\int_0^1 \frac{2 - x^2}{\sqrt{1 - x^2}}\,\varphi(rx)\,x\,\mathrm{d}x$$

$$g_{\parallel}(r) = \int_0^1 \frac{x^2}{\sqrt{1 - x^2}}\,\varphi(rx)\,x\,\mathrm{d}x$$

And one shows easily that $r^2 g_{\perp}(r)$ must be decreasing somewhere in $[1 - \varepsilon, 1]$ if ε is chosen small enough. This property is preserved if we replace φ by a regular approximation of φ. Then we can apply Proposition 7 and the metric coefficients which we obtained define a minimizing harmonic map.

Appendix A

PROPOSITION A: *Let us assume that*

- $\bar{U}$ *is a* C^1*-diffeomorphism between* B^n_* *and* $\mathscr{U}$ *and the* g_{ij} *are of class* C^0 *on* B^n_* *and verify* (1.1), (1.2) *and* (1.3).
  Then, $\forall U \in H^1_*(B^n, \mathscr{U})$, $\forall \varepsilon > 0$, $\exists U_\varepsilon \in C^1_*(B^n, \mathscr{U})$

$$|E(U_\varepsilon) - E(U)| < \varepsilon. \tag{A.1}$$

Proof: For every measurable subset A of B^n, we will write

$$E_A(U) = \frac{1}{2}\int_A g_{ij}[Y(x)]\,\frac{\partial Y^i(x)}{\partial x^\beta}\,\frac{\partial Y^j}{\partial x^\beta}(x)\,\mathrm{d}x.$$

We give the proof in the case where $\{a_1, \ldots, a_k\} = \{0\}$ this simplifies the notations and does not change the sense of the proof. Then the hypotheses (1.1), (1.2) and (1.3) become

$\forall x \in B^n_*$, $\forall y \in \mathbf{R}^n$, if y is orthogonal to x,

$$K_1|y|^2 \leqslant x^2 g_{ij}(x) y^i y^j \leqslant K_2|y|^2 \tag{A.2}$$

$\forall x \in B^n_*$, $\forall y \in \mathbf{R}^n$, if y is parallel to x,

$$K_1 h(|x|)^2|y|^2 \leqslant g_{ij}(x) y^i y^j \leqslant K_2 h(|x|)^2|y|^2 \tag{A.3}$$

$$\int_{B^n} g_{ii}(x)\,\mathrm{d}x < +\infty. \tag{A.4}$$

where K_1 and K_2 are positive constants, and h is a continuous map from $(0, 1]$ to $[\delta, +\infty)$ $(\delta > 0)$ and belongs to $L^1([0, 1], \mathbf{R})$.

(a) **Change of chart**. It is easy to show that there exists real numbers r_0 and ε_0 in $(0, \frac{1}{3})$ and a map τ in $C^0([0, 1], [0, 1]) \cap C^1((0, 1], [0, 1])$ with

$$\begin{cases} \tau(0) = r_0 \\ \forall t \in (0, \varepsilon_0],\ \tau'(t) = h(t) \\ \forall t \in [\frac{2}{3}, 1],\ \tau(t) = t \\ \forall t \in (0, 1],\ \tau'(t) > 0 \end{cases}$$

Then we consider the map T from $B^n \backslash \{0\}$ to $B^n \backslash \overline{B^n(0, r_0)}$ defined by

$$T(x) = \tau(|x|) \frac{x}{|x|}.$$

We can construct a new chart on $\mathcal{U}$ by taking $T \circ \bar{U}^{-1}$. Let γ_{ij} be the metric coefficients of $\mathcal{U}$ in this chart.

If $\xi = T(x)$, $\zeta = T'(x) z$

$$\gamma_{ij}(\xi) \zeta^i \zeta^j = g_{ij}(x) z^i z^j.$$

Let x be in $B^n(0, \varepsilon_0) \backslash \{0\}$,

- If z is orthogonal to x, $\zeta = (\tau(|x|)/|x|) z$ and

$$\gamma_{ij}(\xi) \zeta^i \zeta^j = \frac{\tau(|x|)^2}{|x|^2} \gamma_{ij}(\xi) z^i z^j = g_{ij}(x) z^i z^j,$$

hence using (A.2)

$$\frac{K_1}{r_0^2} |z|^2 \leqslant \gamma_{ij}(\xi) z^i x^j \leqslant \frac{K_2}{r_0^2} |z|^2; \tag{A.5}$$

- If z is parallel to x, $\zeta = \tau'(|x|) z$ and

$$\gamma_{ij}(\xi) \zeta^i \zeta^j = \tau'(|x|)^2 \gamma_{ij}(\xi) z^i z^j = g_{ij}(x) z^i z^j.$$

hence using (A.3)

$$K_1 |z|^2 \leqslant \gamma_{ij}(\xi) z^i z^j \leqslant K_2 |z|^2. \tag{A.6}$$

And (A.5) and (A.6) together with similar estimates on γ_{ij} using (1.1) for x in $B^n \backslash B^n(0, \varepsilon_0)$ imply that everywhere

$$K_3 |\zeta|^2 \leqslant \gamma_{ij}(\xi)\zeta^i\zeta^j \leqslant K_4 |\zeta|^2 \tag{A.7}$$

where K_3 and K_4 are strictly positive constants.

(b) Now we work on $B^n \backslash \overline{B^n(0, r_0)}$. We set

$$\psi = T \circ \bar{U}^{-1} \circ U = T \circ Y;$$

ψ is defined on $B^n \backslash Y^{-1}(\{0\})$. Let us remark that it follows from (A.7) that ψ belongs to $H^1(B^n, B^n)$. For ε_1 in $(0, r_0)$ we write

$$\psi_{\varepsilon_1} = (1 - \varepsilon_1)\psi + \varepsilon_1 \frac{\psi}{|\psi|}$$

We have

$$\frac{\partial \psi_{\varepsilon_1}}{\partial x^\beta} = \frac{\partial \psi}{\partial x^\beta} + \varepsilon_1 \left[\left[\frac{1}{|\psi|} - 1 \right] \frac{\partial \psi}{\partial x^\beta} - \frac{\psi}{|\psi|^3} \left[\sum_p \frac{\psi^p}{|\psi|} \frac{\partial \psi^p}{\partial x^\beta} \right] \right], \tag{A.8}$$

and, with abuse of notation

$$|E(\psi_{\varepsilon_1}) - E(\psi)| \leqslant \left| \int_{B^n} \gamma_{ij}(\psi_{\varepsilon_1}) \left[\frac{\partial \psi^i_{\varepsilon_1}}{\partial x^\beta} \frac{\partial \psi^j_{\varepsilon_1}}{\partial x^\beta} - \frac{\partial \psi^i}{\partial x^\beta} \frac{\partial \psi^j}{\partial x^\beta} \right] \mathrm{d}x \right|$$

$$+ \left| \int_{B^n} [\gamma_{ij}(\psi_{\varepsilon_1}) - \gamma_{ij}(\psi)] \frac{\partial \psi^i}{\partial x^\beta} \frac{\partial \psi^j}{\partial x^\beta} \mathrm{d}x \right|.$$

Using (A.8) and (A.7) and the fact that $|\psi| \geqslant r_0$ a.e., one shows that the first term on the right hand is bounded by a constant time $\varepsilon_1 \|\psi\|^2_{H^1}$. The second term tends to zero; it suffices to apply Lebesgue's theorem.

Hence for any strictly positive ε there exists ε_1 in $(0, r_0)$ such that

$$|E(\psi_{\varepsilon_1}) - E(\psi)| \leqslant \frac{\varepsilon}{2} \tag{A.9}$$

Let us denote $r_2 = (1 - \varepsilon_1)r_0 + \varepsilon_1$ and $r_1 = (r_0 + r_2/2)$. Hence $r_0 < r_1 < r_2$ and the image of ψ_{ε_1} belongs to $B^n \backslash B^n(0, r_2)$.

(c) We consider a sequence ψ_N of maps from B^n to B^n of class C^1 on $\overline{B^n}$ such that $\psi_{N|\partial B^n} = \psi_{|\partial B^n}$ and such that

$$\psi_N \to \psi_{\varepsilon_1} H^1(B^n, B^n) \tag{A.10}$$

$$\psi_N \to \psi_{\varepsilon_1} \quad \text{a.e.} \tag{A.11}$$

We will modify the maps ψ_N. We apply a method inspired by the proof of Theorem 4 of Bethuel and Zheng in [BZ]. First let us introduce the technical objects which are required.

Let η be a function in $C^1(B^n, [0, 1])$ with

$$\begin{cases} \eta(x) \text{ depends uniquely on } |x| \\ \eta(x) = 0 \text{ if } x \in B^n \backslash B^n(0, (2r_0/3)) \\ \eta(x) = 1 \text{ if } x \in B^n(0, (r_0/3)) \\ |\nabla\eta| < (4/r_0). \end{cases} \tag{A.12}$$

For y in $B^n(0, (r_0/8))$, we let $\psi_{N,y} = \psi_N + y\eta \circ \psi_N$.

Let ε_2 be in $(0, \inf(5r_0/24, \varepsilon_0))$ and let v be in $C^1((0, 1], [0, 1])$ with

$$\begin{cases} v(0) = r_0 \\ \text{if } t \text{ belongs to } [r_1, 1],\ v(t) = t \\ \text{if } t \text{ belongs to } [0, \varepsilon_2],\ v'(t) = h(t) \\ \text{if } t \text{ belongs to } [\varepsilon_2, 1],\ 0 < v'(t) \leqslant K_3 h(t) \end{cases} \tag{A.13}$$

Then π will be the map of $C^1(B^n\backslash\{0\}, B^n\backslash\overline{B^n(0, r_1)})$ defined by

$$\pi(z) = v(|z|)\frac{z}{|z|}.$$

We let then $\Phi_{N,y} = \pi \circ \psi_{N,y}$.

Since $\psi_{N,y}$ is of class C^1, we can apply Sard's theorem to show that for a.e. y in $B^n(0, r_0/8)$, $\Phi_{N,y}$ is of class C^1 on B^n except on a finite set where $\nabla\psi_{N,y}$ is invertible, and that $\Phi_{N,y}$ belongs to $H^1(B^n, B^n)$.

Let us evaluate $|\nabla\Phi_{N,y}|^2$

$$\nabla\Phi_{N,y} = \nabla\pi[\psi_N + y\eta \circ \psi_N][\nabla\psi_N + y(\nabla\eta)(\psi_N)\nabla\psi_N].$$

We have

$$\nabla\pi(z) = \frac{v(|z|)}{|z|}\left[Id - \frac{z}{|z|}\otimes\frac{t_2}{|z|}\right] + v'(|z|)\frac{z}{|z|}\otimes\frac{t_2}{|z|},$$

and from (A.13) we compute that there exists a strictly positive constant C_1 such that

$$|\nabla\pi(z)|^2 \leqslant C_1\left(\frac{1}{|z|^2} + h(|z|)\right) \quad \text{if} \quad |z| \leqslant \varepsilon_2. \tag{A.14}$$

Hence we find a constant C_2 strictly positive such that

$$|\nabla\Phi_{N,y}|^2 \leqslant C_2|\nabla\psi_N|^2\left[\frac{1}{|\psi_N + y\eta(\psi_N)|^2} + h(|\psi_N + y\eta(\psi_N)|)^2\right]. \tag{A.15}$$

Then as in the paper of Hardt and Lin [HL] (see also [HKL], p. 556), we apply Fubini's theorem denoting $B^n(r) = B^n(0, r)$,

$$\begin{aligned}
&\int_{B^n(r_0/8)}\int_{\psi_N^{-1}[B^n(r_1)]}|\nabla\Phi_{N,y}|^2\,dx\,dy \leqslant \int_{\psi_N^{-1}(B^n(r_0/3))}\int_{B^n(r_0/8)} C_2|\nabla\Phi_N|^2\\
&\quad\times\left[\frac{1}{|\psi_N + y|^2} + h(|\psi_N + y|)^2\right]dy\,dx + \int_{\psi_N^{-1}[B^n(r_1)\setminus B^n(r_0/3)]}\int_{B^n(r_0/8)} C_2|\nabla\psi_N|^2\\
&\quad\times\left[\frac{1}{|\psi_N + y\eta(\psi_N)|^2} + h(|\psi_N + y\eta(\psi_N)|)^2\right]dy\,dx\\
&\leqslant \int_{\psi_N^{-1}(B^n(r_0/3))} C_2|\nabla\psi_N|^2\left[\int_{B^n(11r_0/24)}\left(\frac{1}{y^2} + h(|y|)^2\right)dy\right]dx\\
&\quad+ \int_{\psi_N^{-1}[B^n(r_1)\setminus(B^n(r_0/3))]} C_2|\nabla\psi_N|^2\left[\sup_{[5r_0/24,1]}\left(\frac{1}{t^2} + h(t)^2\right)\right]\left|B^n\left(\frac{r_0}{8}\right)\right|dx\\
&\leqslant C_3\int_{\psi_N^{-1}(B^n(r_1))}|\nabla\psi_N|^2\,dx,
\end{aligned}$$

where C_3 is a strictly positive constant (see (A.3) and (A.4)).

Hence there exists y_0 in $B^n(r_0/8)$ such that Φ_{N,y_0} has a finite number of singularities and such that

$$\int_{\psi_N^{-1}(B^n(r_1))}|\nabla\Phi_{N,y_0}|^2\,dx \leqslant \frac{C_3}{|B^n(r_0/8)|}\int_{\psi_N^{-1}(B^n(r_1))}|\nabla\psi_N|^2\,dx. \tag{A.16}$$

Since $\psi_{\varepsilon_1}(x)$ belongs to $B^n \backslash B^n(0, r_2)$ a.e. it follows using (A.11) that the measure of $\psi_N^{-1}[B^n(0, r_1)]$ tends to zero when N tends to $+\infty$. Now we use (A.10). We can write $\psi_N = \psi_{\varepsilon_1} + r_N$ where r_N tends to zero in the H^1-topology and

$$\int_{\psi_N^{-1}[B^n(r_1)]} |\nabla \psi_N|^2 \, dx = \int_{\psi_N^{-1}[B^n(r_1)]} |\nabla \psi_{\varepsilon_1}|^2 \, dx + \int_{\psi_N^{-1}[B^n(r_1)]} [2 \nabla \psi_{\varepsilon_1} \nabla r_N + |\nabla r_N|^2] \, dx. \tag{A.17}$$

The first term on the right tends to zero because $|\psi_N^{-1}(B^n(r_1))|$ tends to zero and the second one because r_N tends to zero in H^1. And it follows from (A.16) that

$$\begin{cases} \int_{\psi_N^{-1}[B^n(r_1)]} |\nabla \psi_N|^2 \, dx \to 0 \\ \int_{\psi_N^{-1}[B^n(r_1)]} |\nabla \Phi_N|^2 \, dx \to 0. \end{cases} \tag{A.18}$$

Hence

$$\begin{aligned} |E(\Phi_N) - E(\psi_{\varepsilon_1})| &\leqslant |E_{\psi_N^{-1}[B^n \backslash B^n(r_1)]}(\Phi_N) - E_{\psi_N^{-1}[B^n \backslash B^n(r_1)]}(\psi_{\varepsilon_1})| \\ &\quad + |E_{\psi_N^{-1}[B^n(r_1)]}(\Phi_N) - E_{\psi_N^{-1}[B^n(r_1)]}(\psi_{\varepsilon_1})| \\ &\leqslant |E(\psi_N) - E(\psi_{\varepsilon_1})| \\ &\quad + 2E_{\psi_N^{-1}[B^n(r_1)]}(\Phi_N) + 2E_{\psi_N^{-1}[B^n(r_1)]}(\psi_{\varepsilon_1}) \end{aligned}$$

And this inequality together with (A.10), (A.18) and (A.7) imply using Lebesgue's theorem that $|E(\Phi_N) - E(\psi_{\varepsilon_1})|$ tends to zero when N tends to $+\infty$. Then we choose N large enough to have

$$|E(\Phi_N) - E(\psi_{\varepsilon_1})| \leqslant \frac{\varepsilon}{2}. \tag{A.19}$$

This gives with (A.9)

$$|E(\omega) - E(\Phi_N)| \leqslant \varepsilon.$$

Moreover if we set $Y_N = T^{-1}(\Phi_N) = [(\tau^{-1} \circ v)(|\psi_{N, y_0}|)/|\psi_{N, y_0}|] \psi_{N, y_0}$ it is easy to verify that Y_N is of class C^1 even on $Y_N^{-1}(\{0\})$ and that $Y_N^{-1}(\{0\})$ is finite. Furthermore, because of the definition of τ and v, $\tau^{-1} \circ v$ agrees with the identity map on a neighbourhood of $\{0\}$. And since Sard's theorem implies that we can choose ψ_{N, y_0} such that $\nabla\psi_{N, y_0}$ is invertible on $\psi_{N, y_0}^{-1}\{0\}$, we can obtain hence Y_N such that ∇Y_N is invertible on $Y_N^{-1}\{0\}$.

This achieves the proof of Proposition A. □

Appendix B

This appendix is devoted to the Gauss curvature of $\bar{g}$. We use the notations of Theorem 2 and we denote $\partial a/\partial x^1$ by α, $\partial b/\partial x^1$ by β. We have the Cauchy relations

$$\frac{\partial a}{\partial x^1} = \frac{\partial b}{\partial x^2} = \alpha$$

$$\frac{\partial a}{\partial x^2} = \frac{\partial b}{\partial x^1} = -\beta.$$

Let μ be a map of class C^2 from B^2 into $\mathbf{R}$ such that

$$\mu(x) > |\varphi(x)|, \quad \forall x \in B^2.$$

Let g^* be the metric on B^2 defined by

$$g^*(x) = \mu(x)(\mathrm{d}x)^2 + \operatorname{Re}\varphi(x)[(\mathrm{d}x^2)^2 - (\mathrm{d}x^1)^2] + 2\operatorname{Im}\varphi(x)\,\mathrm{d}x^1\,\mathrm{d}x^2.$$

Finally $\mu^2 - a^2 - b^2$ is denoted d.

The Christoffel symbols of g^* are, with standard notations,

$$\Gamma_{11}^1 = \frac{1}{2d}\left[(\mu + a)\left(\frac{\partial\mu}{\partial x^1} - \alpha\right) + b\left(\frac{\partial\mu}{\partial x^2} - \beta\right)\right]$$

$$\Gamma_{12}^1 = \frac{1}{2d}\left[(\mu + a)\left(\frac{\partial\mu}{\partial x^2} + \beta\right) - b\left(\frac{\partial\mu}{\partial x^1} + \alpha\right)\right]$$

$$\Gamma_{22}^1 = \frac{1}{2d}\left[(\mu + a)\left(-\frac{\partial\mu}{\partial x^1} + \alpha\right) + b\left(-\frac{\partial\mu}{\partial x^2} + \beta\right)\right]$$

$$\Gamma_{11}^2 = \frac{1}{2d}\left[(\mu - a)\left(-\frac{\partial\mu}{\partial x^2} + \beta\right) + b\left(-\frac{\partial\mu}{\partial x^1} + \alpha\right)\right]$$

$$\Gamma^2_{12} = \frac{1}{2d}\left[(\mu - a)\left(\frac{\partial \mu}{\partial x^1} + \alpha\right) - b\left(\frac{\partial \mu}{\partial x^2} + \beta\right)\right]$$

$$\Gamma^2_{22} = \frac{1}{2d}\left[(\mu - a)\left(\frac{\partial \mu}{\partial x^2} - \beta\right) + b\left(\frac{\partial \mu}{\partial x^1} - \alpha\right)\right].$$

Still with standard notations, one finds after computations (noting that $\partial\alpha/\partial x^1 = \partial\beta/\partial x^2$, $\partial\alpha/\partial x^2 = -\partial\beta/\partial x^1$)

$$R^1_{212} = \frac{\mu + a}{2d^2}\left\{-2a\left(\alpha\frac{\partial \mu}{\partial x^1} - \beta\frac{\partial \mu}{\partial x^2}\right) - 2b\left(\alpha\frac{\partial \mu}{\partial x^2} + \beta\frac{\partial \mu}{\partial x^1}\right)\right.$$
$$\left. + \mu\left(\left(\frac{\partial \mu}{\partial x^2}\right)^2 + \left(\frac{\partial \mu}{\partial x^2}\right)^2 + \alpha^2 + \beta^2\right) - \mathrm{d}\Delta\mu\right\}$$

and similarly

$$R^2_{212} = \frac{b}{2d^2}\left\{2a\left(\alpha\frac{\partial \mu}{\partial x^1} - \beta\frac{\partial \mu}{\partial x^2}\right) + 2b\left(\alpha\frac{\partial \mu}{\partial x^2} + \beta\frac{\partial \mu}{\partial x^1}\right)\right.$$
$$\left. - \mu\left(\left(\frac{\partial \mu}{\partial x^1}\right)^2 + \left(\frac{\partial \mu}{\partial x^2}\right)^2 + \alpha^2 + \beta^2\right) + \mathrm{d}\Delta\mu\right\}.$$

Finally if K^* denotes the Gauss curvature of g^* we find

$$K^* = \frac{1}{d^2}\left\{-2a\left(\alpha\frac{\partial \mu}{\partial x^1} - \beta\frac{\partial \mu}{\partial x^2}\right) - 2b\left(\alpha\frac{\partial \mu}{\partial x^2} + \beta\frac{\partial \mu}{\partial x^1}\right)\right.$$
$$\left. + \mu\left(\left(\frac{\partial \mu}{\partial x^1}\right)^2 + \left(\frac{\partial \mu}{\partial x^2}\right)^2 + \alpha^2 + \beta^2\right) - \mathrm{d}\Delta\mu\right\}.$$

In particular if $\mu = (|\varphi|^2 + \theta)^{1/2}$ with $\theta \in C^2(B^2, (0, \infty))$ then

$$K^* = \frac{1}{4\theta^2\mu}\left\{|\nabla\theta|^2 - \theta|\psi|^2 - 2\theta\Delta\theta + \frac{4\theta}{\mu^2}\left[|\varphi|^2|\psi|^2 + \frac{\partial\theta}{\partial x^1}(a\alpha + b\beta)\right.\right.$$
$$\left.\left. + \frac{\partial\theta}{\partial x^2}(-a\beta + b\alpha) + \frac{|\nabla\theta|^2}{4}\right]\right\} \tag{B.1}$$

with $\psi = \alpha + i\beta$. We have

$$\left| \frac{\partial\theta}{\partial x^1} (a\alpha + b\beta) + \frac{\partial\theta}{\partial x^2} (-a\alpha + b\alpha) \right| \leqslant |\nabla\theta| \, |\varphi| \, |\psi|. \tag{B.2}$$

From (B.1) and (B.2) we get

$$K^* \leqslant -\frac{1}{2\theta\mu^3} \{(\Delta\theta)|\varphi|^2 + \theta\Delta\theta - |\nabla\theta|^2\}.$$

Finally we recall that the Gauss curvature of the metric $\theta^{1/2}((\mathrm{d}x)^2 + (\mathrm{d}y)^2)$ is

$$K_0 = \frac{(\nabla\theta)^2 - \theta\Delta\theta}{2\theta^{5/2}}.$$

In particular, if K_0 is negative, K^* is also negative. If we take $\theta = (2 - r^2)^{-4}$ we get

$$K_0 = -16.$$

Appendix C

In this Appendix all the Riemannian surfaces and the functions considered are of class C^∞.

Let (N, g) be a compact Riemannian surface of negative Gauss curvature with boundary. We prove

LEMMA C: *There exists a complete non compact Riemannian surface* (Σ, σ) *of negative Gauss curvature such that*

$$N \subset \Sigma \tag{C.1}$$

$$g = \sigma \quad \text{on} \quad N. \tag{C.2}$$

Proof: The boundary of N is the union of a finite number of disjoint closed curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. We consider one of this curve Γ_i and geodesic orthogonal coordinates along this curve:

$$g = (\mathrm{d}t_i)^2 + f_i^2(t_i, \theta_i)(\mathrm{d}\theta_i)^2, \quad f_i > 0$$

with $\theta_i \in S^1$ and $t_i \in (-\delta_i, 0]$ for some positive real number δ_i. Thje curve Γ_i is $t_i^{-1}(\{0\})$. The Gauss curvature of (N, g) on $t_i^{-1}((-\delta_i, 0])$ is

$$K = -\frac{\partial^2 f_i}{\partial t_i^2} \Big/ f_i.$$

Since K is negative, one can easily construct, for C large enough, a function $\tilde{f}_i$ of class C^∞ from $(-\delta_i, +\infty) \times S^1$ into $(0, +\infty)$ such that

$$\tilde{f}_i = f_i \quad \text{on} \quad (-\delta_i, 0] \times S^1, \tag{C.3}$$

$$\frac{\partial^2 \tilde{f}_i}{\partial t_i^2} > 0 \quad \text{on} \quad (-\delta_i, +\infty) \times S^1 \tag{C.4}$$

and

$$\tilde{f}_i(t, \theta) = C(t+1)^2 \quad \text{on} \quad [1, +\infty) \times S^1. \tag{C.5}$$

Let Σ be the surface obtained by gluing, for $i = 1, \ldots, n$, a cylinder $\Sigma_i = [0, +\infty) \times S^1$ along Γ_i. We provide Σ with the metric σ defined by

$$\sigma = g \quad \text{on} \quad N$$

$$\sigma = (\mathrm{d}t)^2 + \tilde{f}_i^2(t, \theta)(\mathrm{d}\theta)^2 \quad \text{on} \quad \Sigma_i.$$

Clearly the metric σ is smooth and (Σ, σ) is a complete non compact Riemannian surface which satisfies (C.1) and (C.2). Moreover, by (C.4), the Gauss curvature of (Σ, σ) is negative. □

Remark:

a. The surface (Σ, σ) satisfies Morrey's uniformity condition [M]: there are two positive constants α, A such that any point of Σ is in the domain of a coordinate chart $\pi\colon V \to \mathbb{R}^2$ whose image is the unit ball and

$$\alpha |\pi'(y) \cdot Y|^2_{\mathbb{R}^2} \leqslant |Y|^2_\sigma \leqslant A |\pi'(y) \cdot Y|^2_{\mathbb{R}^2}$$

for any y in V and any Y in $T_y(\Sigma)$.

b. If, for π in $[0, +\infty)$, we define

$$\Sigma_\tau = N \cup \left(\bigcup_{i=1}^{n} \{(t, \theta) \in \Sigma_i \,|\, 0 \leqslant t \leqslant \tau\} \right),$$

then for $\tau \geqslant 1$ Σ_τ is convex; see [Ham] p. 6-7 for a definition of convex. This allows to use in the proofs of theorem 3' (when $\partial M \neq \emptyset$) and theorem 3 the results of [Ham] instead of [M], [SU], [L1], [L2] or [S2].

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Note added in proof

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