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On the space of asymptotically Euclidean metrics

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Abstract. A slice theorem for the action of the group of asymptotically Euclidean diffeomorphisms on the space of asymptotically Euclidean metrics on $\mathbb{R}^n$ is proved.

1. Introduction

Let M be a smooth n-dimensional manifold and let $\mathbf{M}$ denote the space of smooth Riemannian metrices on M. The group $\mathbf{D}$ of all diffeomorphisms on M acts on $\mathbf{M}$ by pullback and the orbit-space $\mathbf{M}/\mathbf{D}$ describes the coordinate independent properties of $\mathbf{M}$. This space has been studied in the case when M is compact by Ebin [9] and Bourgignon [2].

The space $\mathbf{M}/\mathbf{D}$ is of interest in Riemannian geometry as well as in General Relativity, where in the case $n = 3$ it is known as superspace, see for example [10].

There is an analogous problem in the case where $n = 4$ and $\mathbf{M}$ is the space of solutions to the Einstein Equations on M (i.e. a subset of the space $L(M)$ of Lorentz metrics on M). In this case $\mathbf{M}/\mathbf{D}$ is the space of true dynamical degrees of freedom of the gravitational field. This case has been studied by Marsden and Isenberg [11].

The space $\mathbf{M}/\mathbf{D}$ has in general the structure of a stratified ILH Frechet manifold [2], with lower dimensional strata consisting of the metrics with nontrivial isotropy groups for the action of the diffeomorphism group, i.e. those metrics which admit nontrivial Killing fields.

In the above studies the assumption of compactness of M played an essential role in that it allowed the powerful results about Fredholm properties of elliptic operators on compact manifolds to be used. The Fredholm property fails in general if the manifold is non-compact and special assumptions have to be made in order to extend the results to this case. In the paper

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[4] some results in this direction were proved but, due partly to the incomplete state of the theory of elliptic operators on noncompact manifolds at that time, no complete results were obtained.

The aim of this paper is to generalize the results known for M compact to the case where $\mathbf{M}$ is the space of C^∞ asymptotically Euclidean metrics (appropriately defined) on a manifold M diffeomorphic to $\mathbb{R}^n$ and $\mathbf{D}$ is the group of C^∞ asymptotically Euclidean diffeomorphisms of M. It should be possible to extend these results to a wide variety of noncompact settings: M may have more complicated topology and several ends; the metric on the ends of M may be asymptotic to a conical, cylindrical, constant curvature or other metric, but in the interest of simplicity we will only consider the asymptotically Euclidean case here. This case is also one of interest in the study of asymptotically flat solutions of Einstein equations and, in fact, the present paper may be seen as a step toward extending the results in [11] to this case.

1.1. The asymptotically Euclidean case

In the asymptotically Euclidean case, an appropriate setting for the theory of elliptic operators is, instead of the ordinary Sobolev spaces, the class of weighted Sobolev spaces $H^p_{s,\delta}$ introduced by Cantor [3]. Let B be a scalar constant coefficient elliptic operator on $\mathbb{R}^n$ of order m. Then B is a continuous mapping from $H^p_{s,\delta}$ to $H^p_{s-m,\delta+m}$ and is Fredholm (i.e. has finite dimensional kernel and cokernel) if and only if $-\delta - n/p \notin \mathbb{N}$ for $\delta \leqslant -n/p$ and $\delta + m - n/p' \notin \mathbb{N}$ for $\delta > -n/p$. The reason B fails to be Fredholm in the case where either of these conditions fails to hold is that the index $\operatorname{ind}(B) = \dim \ker(B) - \dim \operatorname{coker}(B)$ is a function of δ and changes as δ passes such a point. This is the most important difference from the compact case where a scalar operator always has index 0, which by the above remark no longer is true in the noncompact case.

Here we will, for simplicity, restrict our attention to the case of L^2-type spaces H^s_δ. These spaces were studied in detail by Choquet-Bruhat and Christodoulou [7], where also some results about isomorphism properties of operators were derived. The case of systems of operators with variable coefficients over $\mathbb{R}^n$ was studied in sufficient generality for the present purposes by Lockhart and McOwen [14]. The Fredholm property holds for an elliptic operator with variable coefficients if its symbol tends to that of a constant coefficient operator rapidly enough and it is in fact possible to give a sharp characterization of this.

For fixed $\delta \in \mathbb{R}$, $s \in \mathbb{N}$, let $\mathbf{M}^s$ denote the space of Riemannian metrics on M such that for $g \in \mathbf{M}$, $g - e \in H^s_\delta(S^2T^*M)$, where e is the Euclidean metric on $\mathbb{R}^n$. $\mathbf{M}^s$ is called the space of asymptotically Euclidean metrics on M of order (s, δ).

The group of diffeomorphisms of class H^s_δ will be denoted by D^s_δ. Let $n > 2$. The space of all diffeomorphisms which leave $\mathbf{M}^s$ invariant will be denoted by $\mathbf{D}^{s+1}$. This turns out to be $D^{s+1}_{\delta-1} \circledS G(n)$ where the asymptotic group $G(n)$ is either the orthogonal group $O(n)$ or the Euclidean group $E(n) = O(n) \circledS \mathbb{R}^n$ depending on the value of δ, see §2.4.

It turns out that the Fredholm property of the operator $\Delta_{K,g} = \delta_g \circ K_g$: $H^{s'}_{\delta'}(TM) \to H^{s'-2}_{\delta'+2}$ is crucial and that the hypotheses in [14] require $g - e \in H^s_\delta$ with $s \geqslant 3 + n/2$ and $\delta > -n/2$ for this to hold. While the condition on s is not sharp (due to the fact that we are dealing only with integer values of s), the condition on δ can probably not be weakened (cf. the discussion in [1, p. 245]). We will therefore in the following consider only s, δ satisfying these conditions. See §2.2 for further discussion of the assumptions used in this paper.

1.2. ILH structures

The H^s_δ spaces have the advantage of being Hilbert spaces ([7, p. 130]) which makes it possible to make use of the theory of ILH structures introduced by Omori [18] in studying the quotient $\mathbf{M}/\mathbf{D}$.

Let A: $\mathbf{D}^{s+1} \times \mathbf{M}^s \to \mathbf{M}^s$ denote the right action by pull-back of $\mathbf{D}^{s+1}$ on $\mathbf{M}^s$ and let $\mathbf{O}^s(g)$ denote the orbit $A(\mathbf{D}^{s+1}, g)$ of $g \in \mathbf{M}^{s+1}$ under the action of $\mathbf{D}$. We let $\mathbf{M}$, $\mathbf{D}$ and D denote the corresponding spaces in the limit $s \to \infty$.

To construct the quotient space $\mathbf{M}/\mathbf{D}$ and its ILH structure (following Bourgignon [2]) we need essentially two types of technical results in addition to results about the ILH structure of $\mathbf{M}$, $\mathbf{D}$ and D:

a) A slice for the action (a slice at g is roughly speaking an immersed cell transverse to the orbit $\mathbf{O}^s(g)$). This means in particular that $\mathbf{O}^s(g)$ is an immersed submanifold of $\mathbf{M}^s$ at g.
b) A proof that the orbits $\mathbf{O}^s(g)$ are closed embedded submanifolds.

The proof given in [9] makes use of the compactness of M but it turns out to be possible to use essentially the same method of proof also in the asymptotically Euclidean case by making certain additional estimates near infinity. A result similar to b) was stated in [4, Theorem 5.5 (2)], but the proof given there is incomplete and shows only that the orbit is an immersed manifold.

A detailed discussion of the ILH structure of $\mathbf{M}/\mathbf{D}$ will be postponed to a future paper.

1.3. Overview of this paper

In §2 some of the necessary background material on asymptotically Euclidean manifolds and diffeomorphisms is presented. A detailed analysis of the group $\mathbf{D}^s$ in terms of H^s_δ-spaces does not seem to exist in the literature and is therefore included here.

In §3 the structure of the orbits $\mathbf{O}^s(g)$ is studied and in Theorem 3.4, $\mathbf{O}^s(g)$ is shown to be an embedded submanifold of $\mathbf{M}^s$. The method of proof is a generalization of that in [9].

In §4 the slice theorem is stated and proved following the ideas of Ebin and in §5 some remarks are made about the conclusions that can be drawn from the slice theorem, extending the work of Ebin and Bourgignon. The details of this will be postponed to a future paper.

2. Technical preliminaries

2.1. The weighted Sobolev spaces H^s_δ

In this section we will define the weighted Sobolev spaces H^s_δ and state some of their basic properties.

DEFINITION 2.1: Let $s \in \mathbb{N}$ and $\delta \in \mathbb{R}$. The space $H^s_\delta(\mathbb{R}^n)$ is defined to be the completion of $C^\infty_0(\mathbb{R}^n)$ w.r.t. the norm $\|f\|_{s,\delta}$ given by

$$\|f\|^2_{s,\delta} = \sum_{0 \leqslant |l| \leqslant s} \int_{\mathbb{R}^n} (d_e(x, x_0)^{|l|+\delta} |D^l f(x)|_e)^2 \, \mathrm{d}x^n,$$

where $d_e(x, x_0) = |x - x_0|_e$ is the distance function given by the Euclidean metric e on R^n w.r.t. some basepoint x_0. ■

The definition is a special case of that given by Cantor [3] following the work of Nirenberg and Walker [16]. The above norm makes the spaces $H^s_\delta(\mathbb{R}^n)$ into Hilbert spaces [7, p. 130] and the inclusion $H^s_\delta \subset H^{s'}_{\delta'}$ holds if $s \geqslant s'$ and $\delta \geqslant \delta'$. For ease of reference, we record the following useful results.

LEMMA 2.1 (Holder inclusion property [7, Lemma 2.4]): *If $s > n/2$ then the inclusion $H^s_\delta \in C^{s'}_{\delta'}$ is continuous for $\delta' < \delta + n/2$ and $s' < s - n/2$.* ■

LEMMA 2.2 (Multiplication property [7, Lemma 2.5]): *The multiplication map $(f_1, f_2) \to f_1 \otimes f_2$ of*

$$H^{s_1}_{\delta_1} \times H^{s_2}_{\delta_2} \to H^s_\delta$$

is continuous if $s_1, s_2 \geqslant s$, $s < s_1 + s_2 - n/2$, $\delta < \delta_1 + \delta_2 + n/2$. ■

The following special case of the multiplication property will be useful.

COROLLARY 2.3: *Let $f_1 \in H^{s_1}_{\delta_1}$ for $s_1 > n/2$ and $\delta_1 > -n/2$ and let $f_2 \in H^{s'}_{\delta'}$ with $s' \leqslant s_1$ and $\delta' \in \mathbb{R}$. Then $f_2 \to f = f_1 \otimes f_2$ is a continuous map*

$$H^{s'}_{\delta'} \to H^{s'}_{\delta'+\varepsilon}$$

where $\varepsilon < \delta_1 + n/2$.

Proof: Since $s_1 > n/2$ and $s' \leqslant s$, multiplication by f_1 does not decrease the smoothness of f_2. By the multiplication property, $f_1 \otimes f_2 \in H^{s'}_{\delta_2}$ with $\delta_2 < \delta_1 + \delta' + n/2$. The result follows. ■

2.2. Asymptotically Euclidean metrics on $\mathbb{R}^n$

Let M be a C^∞ manifold, diffeomorphic to $\mathbb{R}^n$.

DEFINITION 2.2: Let δ and s satisfy conditions

$$\delta > -n/2 \tag{2.1-1}$$

and

$$s \geqslant n/2 + 3. \tag{2.1-2}$$

and let e be a given Euclidean metric on M. Let r: $M \to \mathbb{R}$ be defined by $r(x) = |x - x_0|$ for some $x_0 \in M$ and for any $R \in \mathbb{R}$, let ϕ_1, ϕ_2 be a partition of unity such that $\phi_1(x) = 1$ if $r(x) \leqslant R$ and $\phi_2(x) = 1$ if $r(x) \geqslant R$. A Riemannian metric g on M is said to be asymptotically Euclidean of order (s, δ) if there is an $R \in \mathbb{R}$ such that

1) $\phi_1 g \in H^s(S^2T^*M)$.
2) $\phi_2(g - e) \in H^s_\delta(S^2T^*M)$.

The space of asymptotically Euclidean metrics of order (s, δ) on M is denoted by $\mathbf{M}^s_\delta$. ■

The space $\mathbf{M}^s_\delta$ is a smooth Hilbert manifold, see [4]. In this paper we will consider the space $\mathbf{M}^s_\delta$ for the following two cases:

Case 1: $-n/2 < \delta < 1 - n/2$,
Case 2: $n > 3$ and $1 - n/2 < \delta < -1 + n/2$.

The point is that in case 1, the asymptotic group G is the Euclidean group $E(n)$ while in case 2, $G = O(n)$, the Orthogonal group, see §2.4.

Further, we will restrict our attention to values of $n > 2$, since the behaviour of the fundamental solution of a second order elliptic systems is exceptional in the case $n = 2$, namely, it contains logarithmic terms. In particular, this has the consequence that there is no range of δ such that $\Delta_{K,g}: H^{s+1}_{\delta-1} \to H^{s-1}_{\delta+1}$ is an isomorphism, and the analysis of this situation requires methods different from those used here.

For the same reason, we will not consider the range of $\delta > -1 + n/2$ in which case $\Delta_{K,g}: H^{s+1}_{\delta-1} \to H^{s-1}_{\delta+1}$ is an injection but has nontrivial cokernel. In this case, the characterization of $\mathbf{D}^{s+1}$ becomes considerably more complicated. Finally, the case $\delta = 1 - n/2$ is exceptional, since $\Delta_{K,g}: H^{s+1}_{-n-2} \to H^{s-1}_{2-n/2}$ is *not* Fredholm and therefore requires special attention, which is beyond the range of the present paper.

In the following we will consider $\delta \in \mathbb{R}$ to be fixed and when convenient suppress reference to it in our notation.

2.3. Some facts about operators on asymptotically Euclidean manifolds

The following facts are well known (see for example [9]). Let $\eta_t: [-1, 1] \to Diff(M)$ be a curve of diffeomorphisms of M with $\eta_0 = id$ and let $g \in Riem(M)$ be a Riemannian metric on M. Then

$$\frac{\partial}{\partial t}\bigg|_{t=0} \eta_t^* g = \mathbf{L}_X g,$$

where the vector field $X \in \Gamma(TM)$ is given by $X = (\partial/\partial t)|_{t=0}\eta_t$ and $\mathbf{L}_X g$ denotes the Lie derivative of g w.r.t. X. We will denote the mapping $X \to \mathbf{L}_X g$ by $K_g(X)$. Thus, for a given $g \in Riem(M)$, $K_g: \Gamma(TM) \to \Gamma(S^2T^*M)$ is a 1:st. order linear partial differential operator. The formal adjoint of K_g is given by the divergence $\delta_g: \Gamma(S^2T^*M) \to \Gamma(TM)$. The symbol of K_g is injective and therefore the "vector-Laplacian" $\Delta_{K,g}: \Gamma(TM) \to \Gamma(TM)$ defined by $\Delta_{K,g} = \delta_g \circ K_g$ is 2:nd order elliptic.

The next result follows from the results in [8] and [14].

PROPOSITION 2.4: *With (s, δ) satisfying conditions* (2.1), *for any $g \in \mathbf{M}^s$ the following mappings are continuous for $s' \leqslant s$ and any δ'.*

(1) $K_g: H^{s'}_{\delta'}TM \to H^{s'-1}_{\delta'+1}S^2(T^*M)$,
(2) $\delta_g: H^{s'}_{\delta'}S^2(T^*M) \to H^{s'-1}_{\delta'+1}TM$,
(3) $\Delta_{K,g}: H^{s'}_{\delta'}TM \to H^{s'-2}_{\delta'+2}TM$.

Similarly, with Δ_g denoting the scalar Laplacian w.r.t. $g \in \mathbf{M}^s$, we have

(4) $\Delta_g: H^{s'}_{\delta'} \to H^{s'-2}_{\delta'+2}$.

Further, $\Delta_{K,g}$h and Δ_g are Fredholm operators between the above spaces if and only if δ' satisfies the conditions

$$-\delta' - n/2 \notin \mathbb{N} \quad \text{if} \quad \delta' \leqslant -n/2, \tag{2.2-1}$$

$$\delta' + 2 - n/2 \notin \mathbb{N} \quad \text{if} \quad \delta' > -n/2. \tag{2.2-2}$$

■

Next we consider the kernel of the operator $\Delta_{K,g}$ in the various cases of interest.

PROPOSITION 2.5: *Let δ and s satisfy condition* (2.1) *and let $n > 2$. Assume that $g \in \mathbf{M}^s_\delta$ and that $s' \leqslant s$. For $\delta' \in \mathbb{R}$ consider the mapping*

$$\Delta_{K,g}\colon H^{s'}_{\delta'}(TM) \to H^{s'-2}_{\delta'+2}(TM).$$

The following statements are true.

(1) *If $-n/2 < \delta' < -2 + n/2$ then $\Delta_{K,g}$ is an isomorphism.*

(2) *If $-n/2 - 1 < \delta' < -n/2$ then $\Delta_{K,g}$ is a surjection with n-dimensional kernel consisting of asymptotically constant vectorfields: if $X \in \ker \Delta_{K,g} \cap H^{s'}_{\delta'}$, then*

$$X = X_0 + X_1,$$

where $X_0 \in \ker \Delta_{K,e} \cap H^{s'}_{\delta'}$ (i.e. constant) and $X_1 \in H^s_\delta$.

(3) *If $-n/2 - 2 < \delta' < -n/2 - 1$ then $\Delta_{K,g}$ is a surjection with kernel consisting of asymptotically 1:st order vectorfields: if $X \in \ker \Delta_{K,g} \cap H^{s'}_{\delta'}$ then if $-n/2 < \delta < 1 - n/2$,*

$$X = X_0 + X_1,$$

where X_0 is a homogeneous first order vectorfield in $\ker \Delta_{K,e} \cap H^{s'}_{\delta'}$ and $X_1 \in H^s_{\delta-1}$. Further, for $n > 3$ we have the following additional cases: If $\delta = 1 - n/2$ then we get $X_1 \in H^s_{-n/2-\varepsilon}$ for some $\varepsilon > 0$ and if $\delta > 1 - n/2$, then $X = X_0 + X_1$, with X_0 a general first order vectorfield and $X_1 \in H^s_{\delta-1}$, which in this case implies that $X_1 \to 0$ as $x \to \infty$.

If $g \in C^\infty$, then in the above cases, $\ker \Delta_{K,g} \subset C^\infty_{loc}$.

Proof: Part (1) is easily proved along the lines of [8, Lemma 3.1].

Proof of (2): To prove part (2) and (3), we write $\Delta_{K,g}$ as $\Delta_{K,g} = \Delta_{K,e} + A_g$, where the variable coefficient part A_g is given by

$$A_g = \delta_g \circ K_g - \delta_e \circ K_e = \delta_e \circ \Gamma + \Gamma \circ K_e + \Gamma \circ \Gamma,$$

where we have used Γ to denote the operation of pointwise multiplication by elements of Γ^k_{ij}, the Christoffel symbols of g. By assumptions, $g - e \in H^s_\delta$ so $\Gamma^k_{ij} \in H^{s-1}_{\delta+1}$. Thus by Corollary 2.3 we see that for s' and δ' in the present range, the mapping

$$\Gamma\colon H^{s'}_{\delta'} \to H^{s_1}_{\delta'+1+\varepsilon}$$

where $s_1 = \min(s', s - 1)$ and $\varepsilon < \delta + n/2$. In particular, we can take $\varepsilon > 0$. It follows, using Proposition 2.4, that A_g defines a continuous mapping

$$A_g\colon H^{s'}_{\delta'} \to H^{s_1}_{\delta'+2+\varepsilon}.$$

for some $\varepsilon > 0$. Here $s_1 = \min(s' - 1, s - 2)$.

Assume that δ' satisfies the assumptions of (2). By [14, Theorem 3], if $X_0 \in F = \ker \Delta_{K,e} \cap H^{s'}_{\delta'}$ then X_0 is a constant vectorfield when expressed in terms of a Cartesian coordinate system for e. It follows that $X_0 \in H^\infty_\delta$ for any $\delta < -n/2$, multiplication by elements of X_0 defines a continuous map $H^s_\delta \to H^s_\delta$ for *any* s, δ and $A_d X_0 \in H^{s-2}_{\delta+2}$.

Recall that by Proposition 2.4 (3), we have $\Delta_{K,g}\colon H^{s'}_{\delta'} \to H^{s'-2}_{\delta'+2}$. Let $Y \in H^{s'-2}_{\delta'+2}$ be arbitrary and consider the equation $\Delta_{K,g} X = Y$. Writing $X = \Sigma^\infty_{i=0} X_i$, we get the equations

$$\Delta_{K,e} X_0 = -A_g Y \tag{2.3-1}$$

and

$$\Delta_{K,e} X_{i+1} = -A_g X_i, \quad i = 1, \ldots, \infty \tag{2.3-2}$$

By the surjectivity of $\Delta_{K,e}$ in the present range of δ' [14, Theorem 3] we find $X_0 \in F \dotplus H^{s_2}_{\delta'+\varepsilon}$ where $s_2 = \min(s' + 1, s)$. Let $X_{(k)} = \Sigma^\infty_{i=k} X_i$ denote the remainder term. Assume that equations (2.3) have been solved for $i = 1, \ldots, k - 1$. The it remains to solve

$$\Delta_{K,g} X_{(k)} = -A_g X_{k-1}. \tag{2.4}$$

Using the surjectivity of $\Delta_{K,e}$ and estimates similar to the above on equations (2.3), we find there exists a finite k such that part (1) applies to solve (2.4). This proves the surjectivity.

In particular, if $Y = 0$, then we get $X_0 \in F$ and $\Delta_{K,g} X_{(1)} = A_g X_0 \in H^{s-2}_{\delta+2}$. Part (1) applies to show that $X_{(1)} \in H^s_\delta$ exists. Clearly, $X_{(1)}$ is determined uniquely by the choice of X_0. This completes the proof of part (2).

Proof of (3): Let s', δ' be as in part (3), let $Y \in H^{s'-2}_{\delta'+2}$ and consider the equation $\Delta_{K,g} X = Y$. By [14, Theorem 3], $F = \ker \Delta_{K,e} \cap H^{s'}_{\delta'}$ consists of first order vectorfields w.r.t. a Cartesian coordinate system for e. We will denote by F_1 and F_0, the spaces of homogeneous first order and constant vectorfields, respectively. Following the proof of part (2), using the surjectivity of $\Delta_{K,e}$ and estimates on A_g, we first find that there exists a finite k such that part (2) applies to the equation $\Delta_{K,g} X_{(k)} = -A_g X_{k-1}$. Since surjectivity has already been proved in this case, surjectivity follows also in part (3).

Now let $Y = 0$ and write $X = X_0 + X_{(1)}$. Then, assuming that X solves equations (2.3), the equation (2.3-1) implies that $X_0 \in F$ and we get

$$\Delta_{K,g} X_{(1)} = -A_g X_0 \in H^{s-2}_{\delta+1}. \tag{2.5}$$

We now have to consider three different cases depending on the value of δ.

(*i*) If $-n/2 < \delta < 1 - n/2$, then part (2) applies to solve equation (2.5) and we get $X = X_0 + X_1$ with $X_0 \in F_0$ and $X_1 \in H^s_{\delta-1}$ (which includes the constant vectorfields).

(*ii*) If $\delta = 1 - n/2$, then due to the fact that $\Delta_{K,g}$ is not Fredholm as a mapping $\Delta_{K,g}$: $H^s_{-n/2} \to H^{s-2}_{2-n/2}$ [14, Theorem 4], the best estimate we get for X_1 is that $X_1 \in H^s_{-n/2+\varepsilon}$ for any $\varepsilon > 0$ (typically, X_1 may behave like $\log(r)$). Thus we can write $X = X_0 + X_1$ with $X_0 \in F_0$.

(*iii*) For $n > 3$, if $1 - n/2 < \delta < 2 - n/2$, for $X_0 \in F$ and δ in the present range, part (1) applied to solve equation (2.5). In particular, this implies by Lemma 2.1 that $X_1 \to 0$ as $x \to \infty$.

This completes the proof of part (3). ■

Remark:

(1) The reason one does not run into the complications discussed in [15] is that by part (1) of the above Proposition, the variable coefficient operator $\Delta_{K,g}$ is an isomorphism in the *same* range of δ as its constant coefficient part $\Delta_{K,e}$.

(2) The above analysis can be extended to arbitrarily small δ', to give analogous results. It would be interesting to understand the situation for large δ'.

(3) The so-called logarithmic translations which have been studied in General Relativity occurs for $\delta = 1 - n/2$, in which case the present theory does not give any detailed information.

(4) The proof given above is related to that in [17, Theorem 5.2c], which covers the case of the scalar Laplacian corresponding to part (2) of the above Proposition, see Proposition 2.6 below. ■

After analyzing the properties of the operator $\Delta_{K,g}$, it is easy to prove the corresponding properties of the scalar Laplacian Δ_g.

PROPOSITION 2.6: *Let δ and s satisfy condition* (2.1) *and let $n > 2$. Assume that $g \in \mathbf{M}^s_\delta$ and that $s' \leqslant s$. For $\delta' \in \mathbb{R}$ consider the mapping*

$$\Delta_g\colon H^{s'}_{\delta'} \to H^{s'-2}_{\delta'+2}.$$

The following statements are true

(1) *If $-n/2 < \delta' < -2 + n/2$ then Δ_g is an isomorphism.*
(2) *If $-n/2 - 1 < \delta' < -n/2$ then Δ_g is a surjection with one dimensional kernel consisting of constant functions (this follows from the maximum principle.*
(3) *If $-n/2 - 2 < \delta' < -n/2 - 1$ then Δ_g is a surjection with kernel consisting of asymptotically 1:st order functions: if $f \in \ker \Delta_g \cap H^{s'}_{\delta'}$ then if $-n/2 < \delta < 1 - n/2$,*

$$f = f_0 + f_1,$$

where f_0 is a homogeneous first order function in $\ker \Delta_e \cap H^{s'}_{\delta'}$ and $f_1 \in H^s_{\delta-1}$. Further, for $n > 3$ we have the following additional cases: If $\delta = 1 - n/2$ then we get $f_1 \in H^s_{-n/2-\varepsilon}$ for some $\varepsilon > 0$ and if $\delta > 1 - n/2$, then $f = f_0 + f_1$, with f_0 a general first order function and $f_1 \in H^s_{\delta-1}$, which in this case implies that $f_1 \to 0$ as $x \to \infty$.

If $g \in C^\infty$, then in the above cases, $\ker \Delta_g \subset C^\infty_{loc}$. ■

2.4. The structure of the group of asymptotically Euclidean diffeomorphisms

DEFINITION 2.3: For $s > n/2$ and $\delta \in \mathbb{R}$, let D^s_δ denote the group of those diffeomorphisms η such that $\eta - id$ and $\eta^{-1} - id$ are H^s_δ. Denote by R, L: $D^s_\delta \times D^s_\delta \to D^s_\delta$ the right and left composition maps, respectively. ■

PROPOSITION 2.7: *Let $s > n/2$ and let $\delta > -n/2$.*

(1) *The space $D^{s+1}_{\delta-1}$ is a smooth Sobolev manifold of maps.*
(2) *$R_\eta\colon D^{s+1}_{\delta-1} \to D^{s+1}_{\delta-1}$ is smooth for all $\eta \in D^{s+1}_{\delta-1}$ and $L_\eta\colon D^{s+1}_{\delta-1} \to D^{s+1}_{\delta-1}$ is C^l for $\eta \in D^{s+l}_\delta$. In particular, $D^{s+1}_{\delta-1}$ is a topological group w.r.t. the induced H^s_δ topology.* ■

Remark: Part (1) is straightforward, but part (2) is a nontrivial result for the interesting range $-n/2 < \delta < -n/2 + 1$ due to [8, Corollary, p. 277]. They prove the $l = 0$ part but the rest is a straightforward generalization of the corresponding result for M compact. The result was erroneously stated for arbitrary $\delta \in \mathbb{R}$ in [6]. ■

In the following, we will always assume that $n > 2$. Let $s > n/2 + 3$ and let δ be in Case 1 or Case 2 as in §2.2. Let $\mathbf{D}^{s+1}$ denote the group of all diffeomorphisms of M which leave $\mathbf{M}^s$ invariant (we suppress reference to δ in our notation where no confusion can arise). The topology on $D^{s+1}_{\delta-1}$ is the one that is naturally induced from the $H^{s+1}_{\delta-1}$-topology. This topology is not appropriate for $\mathbf{D}^{s+1}$, however. To analyze the space $\mathbf{D}^{s+1}$, we define, for $g \in \mathbf{M}^{s+1}$

$$\mathbf{X}^{s+1} = \{Z \in C^0_{loc}(TM) \mid L_Z g \in T_g\mathbf{M}^s\}. \tag{2.5}$$

$\mathbf{X}^{s+1} = T_{id}\mathbf{D}^{s+1}$ is the "Lie-algebra" of $\mathbf{D}^{s+1}$. Let $X \in \mathbf{X}^{s+1}$. Then, by definition, for $g \in \mathbf{M}^{s+1}$, we have $K_g(X) = L_X g \in T_g\mathbf{M} = H^s_\delta(S^2T^*M)$, so by Proposition 2.4 (2), we find that

$$\Delta_{K,g} X \in H^{s-1}_{\delta+1}(S^2T^*M).$$

By Proposition 2.5, $\Delta_{K,g}\colon H^{s+1}_{\delta-1} \to H^{s-1}_{\delta+1}$ is a surjection with finite dimensional kernel. Hence, elements in $\mathbf{X}^{s+1}$ not in $H^{s+1}_{\delta-1}$ must be in $\ker \Delta_{K,g} \cap H^{s+1}_{-n/2-1-\varepsilon}$ for some $\varepsilon > 0$. It is easy to see that $H^{s+1}_{-n/2-1-\varepsilon}$ is the largest space we have to consider, since it contains all the first order vectorfields. Thus, using Proposition 2.5, we can write $X \in \mathbf{X}^{s+1}$ as $X = Y + \dot{X}$, where $Y \in H^{s+1}_{\delta-1}(TM)$ and $\dot{X} \in H^{s+1}_{-n/2-1-\varepsilon}(TM) \cap \ker \Delta_{K,e}$.

We will consider the cases 1 and 2 defined in §2.2. The point is that in Case 1, $D^{s+1}_{\delta-1}$ contains the translations while in Case 2 it does not. We will not consider the exceptional case $\delta = 1 - n/2$. Using (2.5) we see by expressing $\dot{X}$ in terms of a Cartesian coordinate system for e, that in Case 1, $\dot{X}$ can be chosen to be a pure infinitesimal rotation, while in Case 2, $\dot{X}$ is an infinitesimal Euclidean transformation. It is important to note that in either case we can take $\dot{X} \in H^\infty_{loc}$ since e is C^∞.

Definition 2.4: Let X, Y, $\dot{X}$ be as above, then we define the norm $\|X\|_{\mathbf{X}^{s+1}}$ by

$$\|X\|_{\mathbf{X}^{s+1}} = \|Y\|^{s+1}_{\delta-1} + |\dot{X}|.$$

where the norm $|\ |$ in Case 1 denotes the norm on $\mathbf{o}(n)$, the Lie algebra of $O(n)$ and in Case 2 denotes the norm on $\mathbf{e}(n)$, the Lie algebra of $E(n)$, where $E(n) = \mathbb{R}^n \circledS O(n)$ denotes the Euclidean group in n dimensions. ■

The following Lemma is a straightforward generalization of Proposition 2.7 to the case of $\mathbf{D}^{s+1}$.

LEMMA 2.8: *Let s, δ satisfy conditions* (2.1) *and* (2.2). *The space $\mathbf{D}^{s+1}$ is a smooth manifold modelled on $H^{s+1}_{\delta-1} \times \mathbf{o}(n)$* or *$H^{s+1}_{\delta-1} \times \mathbf{e}(n)$ in Case 1 and Case 2 respectively. The topology induced on $\mathbf{D}^{s+1}$ by the $\|\ \|^{s+1}_{\mathbf{X}}$-norm via the exponential mapping is independent of the choice of $g \in \mathbf{M}^{s+1}$. R_η: $\mathbf{D}^{s+1} \to \mathbf{D}^{s+1}$ is smooth for all $\eta \in \mathbf{D}^{s+1}$ and L_η: $\mathbf{D}^{s+1} \to \mathbf{D}^{s+1}$ is C^l for $\eta \in \mathbf{D}^{s+1+l}$. In particular, $\mathbf{D}^{s+1}$ is a topological group w.r.t. the above topology.* ■

PROPOSITION 2.9: *Let $G(n)$ denote $O(n)$ and $E(n)$ in Case 1 and Case 2 respectively. The following statements are true.*

(1) $\mathbf{D}^{s+1} \cong D^{s+1}_{\delta-1} \circledS G(n)$.
(2) *$D^{s+1}_{\delta-1}$ is a normal subgroup of $\mathbf{D}^{s+1}$ and the quotient $\mathbf{D}^{s+1}/D^{s+1}_{\delta-1}$ is isomorphic to $G(n)$.*

Remark: Part (2) of the Proposition corrects a statement in [17, Corollary, p. 10] to the effect that in general $\mathbf{D}/D^{s+1}_{\delta-1} = E(n)$, where $E(n)$ is the Euclidean group in $\mathbb{R}^n$. This situation may be changed by working with a more restrictive set of asymptotic conditions modelled on the quasi-isotropic gauge introduced in General Relativity by York [19]. ■

2.5. A splitting theorem

This is the basic result needed to show that the orbits $\mathbf{O}^s(g)$ are immersed submanifolds of $\mathbf{M}^s$. The main Lemma that we will use is the following.

LEMMA 2.10: *Let S_i, $i = 1, 2, 3$ be Banach spaces and let A: $S_1 \to S_2$, B: $S_2 \to S_3$ be continuous linear operators and assume that $C = B \circ A$: $S_1 \to S_3$ is Fredholm. Then* ker *B splits and the range of A is closed and splits in S_2.*

Proof: First note that by the Fredholm property of C, $\ker(A)$ and $\operatorname{coker}(B)$ are both finite dimensional. Further, the continuity of B implies that the range of A, $R(A)$ is closed.

Now let $\tilde{S}_1 = S_1/\ker(A)$ and $\tilde{S}_3/\operatorname{coker}(B)$ and define

$$\tilde{C} = \tilde{B} \circ \tilde{A}: \tilde{S}_1 \to \tilde{S}_3$$

in the natural way. Then $\tilde{C}$ is again Fredholm and we can compute that

$$\ker \tilde{C} \cong R(A) \cap \ker B$$

$$\operatorname{coker} \tilde{C} \cong S_2 \ominus [R(A) + \ker (B)]$$

which are finite dimensional.

By the definition of splitting, $X \subset Y$ splits if and only if $Y = X \dotplus Z$ where $\dotplus$ denotes direct algebraic sum and $Z \subset Y$ is some closed subspace. In particular, any finite dimensional subspace splits. Hence, there is a closed subspace $Y \subset \ker (B)$ so that $\ker (B)$ splits into

$$\ker (B) = Y \dotplus R(A) \cap \ker (B)$$

and S_2 splits into

$$S_2 = Z \dotplus [R(A) + \ker (B)].$$

Thus, we can write S_2 as a split sum $S_2 = R(A) \dotplus Z \dotplus Y$. Hence, if we write $R(A) = X \dotplus [R(A) \cap \ker (B)]$, then we have the split sum

$$S_2 = \ker (B) \dotplus Z \dotplus X$$

which completes the proof. ■

Let $g \in \mathbf{M}^{s'}$ for some $s' \geqslant s + 1$ and let $\mathbf{g}(n)$ denote the Lie algebra of the asymptotic group $G(n)$, see §2.4. By Proposition 2.4, $\Delta_{K,g} = \delta_g \circ K_g$ is Fredholm as a map from $H^{s+1}_{\delta-1}$ to $H^{s-1}_{\delta+1}$ and by the finite dimensionality of $\mathbf{g}(n)$ it is clearly also Fredholm as a mapping from $\mathbf{X}^{s+1}$ to $H^{s-1}_{\delta+1}$ so we can apply Lemma 2.9 with $A = K_g$ and $B = \delta_g$ to find that $K_g(\mathbf{X}^{s+1}) \subset T_g\mathbf{M}^s$ is a closed splitting subspace. We record these findings in the following

THEOREM 2.11: *The range of* K_g: $\mathbf{X} \to T_g\mathbf{M}^s$ *splits in* $T_g\mathbf{M}^s$. ■

Remark: We are working in this paper with the L^2-type Sobolev spaces H^s_δ which are Hilbert spaces, so any closed subspace splits, and the Lemma 2.10 is therefore not strictly necessary. However, the Lemma seems to be of independent interest as a generalization of the result of Cantor [5, Lemma 2.2] and may be useful in applications where the L^2-type spaces are not appropriate. ■

3. The structure of orbits

A central point in the construction of the quotient space $\mathbf{M}/\mathbf{D}$ is the analysis of the structure of the group of isometries of a metric g and the orbit $\mathbf{O}^s(g) \subset \mathbf{M}^s$. In this section we provide the necessary extensions of the results of [9, §5].

3.1. Killing fields and isotropy groups

Let $g \in \mathbf{M}$ and let I_g be its isotropy group w.r.t. A. It is well known that I_g is a Lie group of dimension at most $\frac{1}{2}n(n+1)$ and the Lie algebra of I_g is given by the Killing fields of g. For more information about isometries, see [12, Chapter I]. It follows from [12, Theorem I.3.1], that if I_g is of maximal dimension and $g \in \mathbf{M}$, then (M, g) is isometric to $(\mathbb{R}^n, e)$. In order to study the structure of orbits, we need to construct the quotient space $\mathbf{D}^s/I_g$. This was done for the case where M is compact by Ebin [9, §5]. The technical results in §2 and the proofs in [9, §5] make it clear that in the present case, the only important difference from the case treated in [9] is that the isotropy group I_g may fail to be compact.

LEMMA 3.1: *Let $g \in \mathbf{M}$. The map $i\colon I_g \to \mathbf{D}^s$ is an imbedding.*

Remark: The fact that i is a homeomorphism onto its image is used crucially in the proof of [9, Lemma 5.9].

Proof: The smoothness of i and the injectivity of Di are proved in the same way as in [9, §5] using the results of §2. Let $\mathbf{i}$ denote the Lie algebra of I_g. It is clear that I_g is isomorphic to a subgroup of the Euclidean group $E(n) = O(n) \circledS \mathbb{R}^n$. Further, the subspace L of $\mathbf{i}$ such that $Di(\xi) \in \ker \Delta_{K,g}$ is asymptotically first order (Proposition 2.5 (3)) for $\xi \in L$ is a subalgebra of $\mathbf{i}$ which is isomorphic to a subalgebra of $O(n)$.

First, assume that g is not flat. Let $X \in H^\infty_{\delta'}(TM)$ for some $\delta' > -n/2 - 1$ be a Killing field for g. Then $X \in \ker \Delta_{K,g}$ and hence, by Proposition 2.5 (2), $X = X_0 + X_1$ where X_0 is a constant vectorfield w.r.t. Cartesian coordinates for the Euclidean metric e and $X_1 \in H^\infty_\delta$ by Proposition 2.5 (2) (recall that we are assuming that $g \in \mathbf{M}$). This implies that X tends to the constant vectorfield X_0 at infinity. Hence, the flow of X is complete.

For $x, y \in M$, let $d(x, y)$ be the distance function w.r.t. the metric e. Let $x \in M$ be fixed and choose $y \in M$ such that

(1) $\Phi_{X_0,t}(x) = y$ for some t, where Φ_{X_0} denotes the flow of X_0.

and

(2) $d(x, y)$ is large enough so that

$$\| X - X_0 \|_e < \tfrac{1}{2} \| X_0 \|_e,$$

for all $z \in M$ such that $d(x, z) \geqslant d(x, y)$. Let $\Phi_{X,t}$ be the flow of X. It is now easy to see that

$$\frac{\partial}{\partial t} d(x, \Phi_{X,t}(y)) > \varepsilon,$$

for some constant $\varepsilon > 0$. Hence, $d(x, \Phi_{X,t}(y) \to \infty$ as $t \to \infty$. But then, since $X \in I_g$, g is curved at ∞, a contradiction to the assumption that $g \in \mathbf{M}$.

Thus, unless g is flat, I_g is a compact Lie group and the arguments in [9, §5] show that the inclusion $I_g \subset \mathbf{D}$ is an embedding.

It remains to consider the case where g is flat. Then $I_g = E(n)$, the Euclidean group in n dimensions. Consider $i(I_g) \subset \mathbf{D}^s$. Convergence in $\mathbf{D}^s$ implies pointwise convergence, by the Holder inclusion Lemma (§2.1). We can represent I_g and $i(I_g)$ explictly in terms of an Euclidean coordinate system. The result is now obvious in view of the fact that the Holder inclusion property of the H^s_δ spaces implies that the topology on $\mathbf{D}^s$ is stronger than pointwise convergence. ∎

Using Lemma 3.1, it is straightforward to prove the following result, which corresponds to [9, Proposion 5.10].

PROPOSITION 3.2: *Let $g \in \mathbf{M}$. Then the inclusion $I_g \subset \mathbf{D}^s$ is a smooth embedding, the quotient $\mathbf{D}^s/I_g$ is a manifold and the map π: $\mathbf{D}^s \to I_g$ admits smooth cross sections.* ∎

3.2. Proof that orbits are embedded

Let $g \in \mathbf{M}^\infty$ and let $\mathbf{O}^s(g)$ and $\mathbf{O}^s_D(g)$ denote the orbit of g in $\mathbf{M}^s$ under $\mathbf{D}^{s+1}$ and $D^{s+1}_{\delta-1}$ respectively. Theorem 2.11 shows that the basic results on differential topology of infinite dimensional manifolds apply to the present situation and the following result is easily proved along the lines in [9, §6].

PROPOSITION 3.3: *Let s, δ satisfy conditions* (2.1) *and* (2.2) *in* §2. *Then $\mathbf{O}^s(g)$ and $\mathbf{O}^s_D(g)$ are smooth immersed submanifolds of $\mathbf{M}^s$.* ∎

We will now prove that $\mathbf{O}^s(g)$ and $\mathbf{O}^s_D(g)$ are also *embedded* submanifolds of $\mathbf{M}^s$, which is what is necessary to construct the quotient spaces $\mathbf{M}/\mathbf{D}$ and $\mathbf{M}/D$. Let $g \in \mathbf{M}^s$ and let $\kappa_x(\sigma)$ denote the sectional curvature of g w.r.t. the 2-plane σ at $x \in M$ and let

$$I(g) = \sup_{\substack{\|\sigma\|=1 \\ x\in M}} |\kappa(\sigma)|.$$

The following result is easily proved using the results of §2.1 and the properties of κ.

LEMMA 3.4: *With $I(g)$ as above, then $I(g)$ is bounded for $g \in \mathbf{M}^s$, $I(g) = 0$ if and only if g is flat and for $\eta \in \mathbf{D}^{s+1}$ or $\eta \in D^{s+1}_{\delta-1}$, $I(\eta^*g) = I(g)$. Further, the mapping I: $\mathbf{M}^s \to \mathbb{R}$ is continuous.* ■

Thus we have defined a function I: $\mathbf{M}^s \to \mathbb{R}$ which is invariant under the action of $\mathbf{D}^{s+1}$ and $D^{s+1}_{\delta-1}$ and takes the value 0 only at the flat metric. This makes it possible to extend the method used by Ebin to the noncompact case.

THEOREM 3.5: *Let s, δ satisfy conditons* (2.1) *and* (2.2). *Let $g \in \mathbf{M}^\infty$. Then $\mathbf{O}^s(g)$ and $\mathbf{O}^s_D(g)$ are closed embedded submanifolds of $\mathbf{M}^s$.*

Proof: We will consider only the case $\mathbf{O}^s(g)$, the case $\mathbf{O}^s_D(g)$ being similar and easier. We know by Proposition 3.2 that $\mathbf{O}^s(g)$ is an immersed submanifold, so all we need to show is that it is closed. Let $\{\eta_m\}_{m=1}^\infty \subset \mathbf{D}^{s+1}$ be a sequence of diffeomorphisms and let $g_m = \eta_m^* g$. Assume that $g_m \to g_\infty$ in the topology of $\mathbf{M}^s$. The Theorem is proved if we can construct a diffeomorphism $\eta_\infty \in \mathbf{D}^{s+1}$ such that $\eta_\infty^* g = g_\infty$.

First assume that g is not flat. By Lemma 3.4, $I(g) \neq 0$ and $I(\eta_m^* g) = I(g)$, so it follows that $I(g_\infty) = I(g) \neq 0$. Let $p \in M$ and consider the sequence $p_m = \eta_m(p)$. First we will show that this sequence has a convergent subsequence. To get a contradiction, assume that $\{p_m\}$ has a divergent subsequence $\{p_k\}$. By assumption, $s \geqslant n/2 + 3$ so by the Holder estimate [7, Lemma 2.4], g and its derivatives to 2:nd order converge uniformly to e as $p \to \infty$ in M and so we find neighborhoods U_k of geodesic radius ϱ_k of p_k which are covered by exponential coordinates and $\varrho_k \to \infty$ as $k \to \infty$.

This means that on $\eta_k^{-1}(U_k)$, $g_k = \eta_k^* g$ tends to a flat metric. By assumption, $g_k \to g_\infty$ which means that g_∞ is flat. This implies that $0 = I(g_\infty) \neq I(g)$, a contradiction. Thus there is a compact subset $K \subset M$ such that $\eta_m(p)$ remains in K and it follows that there is a subsequence η_k such that $\eta_k(p)$ has a limit $q \in M$.

Now consider a covering $\{U_i\}_{i=1}^{\infty}$ of M where the U_i are open sets of geodesic radius ϱ_i spanned by exponential normal coordinates centered at points $\{p_i\}_{i=1}^{\infty}$. Using the fact that the sequence $\{\eta_m(p_i)\}_{m=1}^{\infty}$ has an accumulation point for every i we can, by using a diagonal procedure, select a subsequence which we again call η_k such that for all i, the sequence $p_{i,k} = \eta_k(p_i)$ has a limit which we call q_i.

Now we are able to use exactly the same method as in the paper by Ebin [9, pp. 29–30]. Choose an ON-frame $\{V_i^s\}_{s=1}^{n}$ at each point p_i and denote its image under η_k by $V_{i,k}^s$. By, if necessary, selecting a further subsequence of $\{\eta_k\}$ which we will continue to index by k, we can make sure that the sequence $\{(V_{i,k}^s, p_{i,k})\}_{k=1}^{\infty}$ converges to (W_i^s, q_i), where $\{W_i^s\}_{i=1}^{n}$ is an ON-frame at q_i.

Using the fact that U_i are covered by exponential coordinates, we can use the knowledge of V_i and W_i to construct a diffeomorphism η_∞ such that $\eta_\infty(p_i) = q_i$ and $\eta_\infty^* g = g_\infty$.

We now turn to the case where g is flat. Then (M, g) is isometric to $(\mathbb{R}^n, e)$. The problem is here simplified by the fact that M is covered by a single exponential chart. Let $g \in \mathbf{M}^\infty$ be flat and let $g_m = \eta_m^* g \to g_\infty$ as above. The assumption that g is flat implies that g_m is flat for all m and so, we can construct η_∞ as follows.

Choose $p \in M$. Using the fact that I_g is transitive, we can find for each η_m a diffeomorphism $\chi_m \in I_g$ such that $\eta_m \circ \chi_m(p) = p$. We can now argue as above, using exponential charts centered at p to find a subsequence η_k and a diffeomorphism η_∞ such that $\eta_\infty(p_i) = q_i$ and $\eta_\infty^* g = g_\infty$.

The proof in [9] shows that $\eta_k \to \eta_\infty$ considered as C_{loc}^1 mappings and that this convergence is actually in H_{loc}^{s+1}, which settles the local question. We will here show that this convergence takes place in $\mathbf{X}^{s+1}$.

We first consider the behaviour of the differentials $\partial\eta_k$ and $\partial\eta_\infty$. Recall that η_k and η_∞ are defined in terms of the exponential mappings Exp_k and Exp_∞ of g_k and g_∞. The tangential derivative $T\ Exp_\gamma$ of the exponential mapping may be computed by solving the Jacobi equation. By the Holder estimate and the multiplication properties of the H_δ^s spaces, the mapping $R(\gamma)$: $\mathbf{M}^s \to C_{\delta'}^0$, where $R(\gamma)$ denotes the Riemann tensor of γ, is continuous. This implies, by the construction of η_∞, that

$$\lim_{x\to\infty} \partial\eta_k(x) - \partial\eta_\infty(x) = 0, \tag{3.1}$$

uniformly in k. This implies in particular, that $\partial\eta_\infty(x) \to A \in O(n)$ as $x \to \infty$. By the arguments in [9], η_∞ is a local diffeomorphism, which together with the above estimate implies that $|\det(\partial\eta_\infty)(x)| > \varepsilon$ for all $x \in M$, where $\varepsilon > 0$ is some constant.

We are now ready to show that the convergence of η_k to η_∞ takes place in $\mathbf{D}^{s+1}$. Let Γ, ${}_k\Gamma$ and ${}_\infty\Gamma$ denote the Christoffel symbols of g, g_k and g_∞, respectively and set ${}_k\varepsilon = {}_k\Gamma - {}_\infty\Gamma$. We know by (3.1) that $\partial f_k/\partial x - \partial f/\partial x \to 0$ in $C^0_{\delta'}$ but we have not investigated the behaviour of the higher derivatives.

Letting $f_k^\infty(x_1, \ldots, x_n)$ and $f_\infty^i(x_1, \ldots, x_n)$ be the expressions for η_k and η_∞ in a coordinate system $(x_1, \ldots, x_n)$, following the argument in [9] shows that

$$\lim_{k\to\infty} \partial^2 f_k^l/\partial x^s\,\partial x^r = \Gamma^t_{rs}\frac{\partial f^l}{\partial x^t} - {}_\infty\Gamma^l_{ij}\left(\frac{\partial f^i}{\partial x^s}\right)\left(\frac{\partial f^j}{\partial x^r}\right) \tag{3.2}$$

in $H^{s-1}_{loc} \cap H^0_{\delta+1}$. Hence f_k converges in H^{s+1}_{loc} which settles the question of smoothness of η_∞. It remains to show that the convergence actually takes place in $\mathbf{D}^{s+1}$.

Now assume that $\eta_k \to \eta_\infty$ in $\mathbf{D}^t \cap H^{s+1}_{loc}Diff(M)$ for some $t \leqslant s$ (where we give $\mathbf{D}^t$ the obvious topology, following §2.4). Let $(x_1, \ldots, x_n)$ be a Cartesian coordinate system for e. Taking traces of both sides of (3.2) w.r.t. s, r and using the multiplication rules for the H^s_δ spaces we find that

$$\lim_{k\to\infty} \|\Delta_e f_k^l - F(\Gamma, {}_\infty\Gamma, f_\infty)\|_{H^{t-1}_{\delta+1}} = 0, \tag{3.3}$$

where Δ_e denotes the scalar Laplace operator w.r.t. e and F is the function determined by taking the trace of the r.h.s. of (3.2).

By the results of §2.4, under the above assumptions, we can write $\eta_k = A_k \circ z_k$, where $A_k \in O(n)$ in Case 1 and $A_k \in E(n)$ in Case 2 and $z_k \in D^t_{\delta-1}$. Let the coordinate expression for A_k be A_k^l. Applying Proposition 2.6 to (3.3) and using the fact that $\Delta_e A_k^l(x) = 0$, we find that $z_k \to z_\infty$ in $D^{t+1}_{\delta-1}$. The spaces $O(n)$ and $E(n)$ are locally compact, so the pointwise convergence of $\eta_k \to \eta_\infty$ is enough to show that $A_k \to A_\infty$ and hence that $\eta_k \to \eta_\infty$ in $\mathbf{D}^{t+1} \cap H^{s+1}_{loc}Diff(M)$. Thus, by induction we find that $\eta_k \to \eta_\infty$ in $\mathbf{D}^{s+1}$. That $I_g\eta_k \to I_g\eta_\infty$ in $\mathbf{D}^{s+1}/I_g$ can be proved exactly as in [9]. This completes the proof of Theorem 3.4. ∎

4. The slice theorem

In this section we will state and prove the slice theorem for the action of $\mathbf{D}^{s+1}$ on $\mathbf{M}^s$.

In the asymptotically Euclidean case, the weak L^2-metric used in [9] to construct a slice for the action of $\mathbf{D}^s$ as an image under the exponential mapping of $\mathbf{M}^s$ of a ball in the normal bundle of $\mathbf{O}^s$, considered as a submanifold of $\mathbf{M}^s$, is not available.

However, the norm defined on H^s_δ by

$$\|h\| = \sum_{|\alpha|=t} \|\nabla^\alpha h\|_{L^2}$$

for some integer t such that $\delta \leqslant t \leqslant s$, can be shown to give a weakly nondegenerate Riemanian structure on $\mathbf{M}$ which is also invariant under the action of D and $\mathbf{D}$. Hence, we could apply a technique similar to that used in [9] for the construction of a slice.

Instead of doing this, we will use a more elementary technique similar to that used by Cantor [4].

The definition of a slice for the action A is as follows.

DEFINITION 4.1: Let (s, δ) be as in §2 and let A be the action as above. Assume that for each $g \in \mathbf{M}$ there exists a closed submanifold $S^s \subset \mathbf{M}^s$ containing g, such that:

(1) If $\eta \in I_g$, $A(\eta, S^s) = S^s$.
(2) If $\eta \in \mathbf{D}^s$, such that $A(\eta, S^s) \cap S^s \neq \varnothing$, then $\eta \in I_g$.
(3) There exists a local cross section χ: $\mathbf{D}^{s+1}/I_g \to \mathbf{D}^{s+1}$ defined in a neighborhood U^{s+1} of the identity coset such that if F: $U^{s+1} \times S^s \to \mathbf{M}^s$ is defined by $(u, t) \to A(\chi(u), t)$, then F is a homeomorphism onto a neighborhood of g.

Then we say that there exists a slice for the action A. ■

Remark: Note that g is assumed to lie in $\mathbf{M}^\infty$. The reason for this is that we wish to construct a C^∞ structure for $\mathbf{M}^\infty/\mathbf{D}^\infty$ and $\mathbf{M}^\infty/D^\infty$ in §6. If we only assume that $g \in \mathbf{M}^{s'}$ for some finite s', then the mappings in part (3) of Definition 4.1 would be of class C^k for some finite k depending on $s' - s$. ■

We are now ready to state the slice theorem.

THEOREM 4.1: *Let (s, δ) be as in §2 and let A be the action of D or $\mathbf{D}$ as above. Then there is a slice for A satisfying the requirements of Definition 4.1.*

Proof: For $g \in \mathbf{M}^\infty$, let ϕ_g: $\mathbf{D}^{s+1}/I_g \to \mathbf{O}^s(g) \subset \mathbf{M}^s$ be the map induced by the action A as in §3.1. Let π: $\mathbf{D} \to \mathbf{D}/I_g$ denote the natural projection and denote the space $H^s_\delta(S_2T^*M)$ by W^s. Note that the inclusion $\mathbf{M}^s \subset W^s$ is open, so we can identify $\mathbf{M}^s$ with W^s locally.

By Theorem 3.4, $\mathbf{O}^s(g) \subset \mathbf{M}^s$ is an embedded submanifold and in particular, with the above notation, *range* $(T_{\pi(I)}\phi_g) \subset W^s$ splits. Now, by working in local coordinates on $\mathbf{D}^{s+1}/I_g$, we can apply [13, Corollary 1s, p. 14] to this situation. This gives a mapping G: $W^s \to W_1^s \times W_2^s$, which is a local isomorphism at g, such that

$$G \circ \phi_g \colon \mathbf{D}^{s+1}/I_g \to W_1^s$$

is an isomorphism from a neighbourhood $U_{\pi(I)}^{s+1} \subset \mathbf{D}^{s+1}/I_g$ to a neighbourhood U_1^s of 0 in W_1^s. From the properties of G and π it is easily seen that for if we define S by

$$S = G^{-1}(0, U_2^s),$$

where U_2^s is a sufficiently small neighbourhood of $0 \in W_2^s$, it will satisfy part (1) and part (2) of the Theorem. Let U_I^{s+1} be a neighbourhood of $I \in \mathbf{D}^{s+1}$. Part (3) follows by Lemma 3.1 and inspection of the mapping

$$G^{-1}(G \circ \phi \circ \pi(\eta), x) \colon U_1^{s+1} \times U_2^s \to U_g^s,$$

where U_g^s is a neighborhood of g. This completes the proof of Theorem 4.1. ■

5. Concluding remarks

The results in this paper lay the foundation for an analysis of the structure of the spaces $\mathbf{M}/\mathbf{D}$ and $\mathbf{M}/D$. In particular, the following results are fairly straightforward applications of the methods of [9] and [2].

(1) $\mathbf{M}$ is an ILH manifold (this is straightforward from the fact that the M_δ^s are smooth Hilbert manifolds and the usual inclusion properties for Sobolev spaces). Further, $\mathbf{D}$ and D are ILH Lie groups. The case of D is straightforward, using the results in §2.4. For the case $\mathbf{D}$ one uses Proposition 2.9 (1).

(2) The actions of $\mathbf{D}$ and D on $\mathbf{M}$ are C^∞ ILH actions and there is a slice for these actions. That the action of D on $\mathbf{M}$ is ILH is straightforward from [4, Proposition 5.3], and the case of $\mathbf{D}$ is similar. That there is a slice for the actions follows from Theorem 4.1 and a straightforward generalization of the techniques used in the proof of [9, Theorem 7.4].

(3) For $g \in \mathbf{M}$, there is a neighborhood U of G such that for $g' \in U$, the isotropy group $I_{g'}$ of g' is conjugate to a subgroup of I_g, the isotropy group of g. This is a standard result in the theory of transformation groups and implies immediately that the set $\mathbf{M}_0$ of metrics with trivial

isotropy group is open in **M** and also, using the techniques of [9, Proposition 8.3] that $\mathbf{M}_0$ is dense in **M**.

(4) **M/D** is a stratified ILH variety but *not* an ILH manifold, while $\mathbf{M}/D$ is an ILH manifold in Case 2, but in Case 1 it has one singular point corresponding to flat space.

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