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Non-reflexive curves

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1. Introduction

In recent years the study of curves over fields of positive characteristic has again attracted the attention of people fascinated by its richness.

In this paper we give a systematic account of our results relating the notion of non reflexivity and the geometry of curves.

The central notions we deal with, and of which we give a review in Section 2, are reflexivity and duality. Our main tool is the Hasse differential calculus to which we devote Section 3. In Section 4 we derive some interesting properties of a plane curve defined over a field of positive characteristic from the properties of its parametrization at a general point. In Section 5 we study how non-reflexivity affects the equation of a plane curve. In Section 6 we give necessary and sufficient conditions for a plane curve to have a non reflexive dual and show that the general member of each of the families considered in Section 5 has a reflexive dual. In Section 7 we show that any extremal curve, that is a curve in a projective space such that its degree is equal to the degree of the projection from its conormal variety to its dual variety, is a strange curve.

These results in many ways complete and generalize the works of Pardini [Pa] and Homma [Ho], as well as some unpublished work of Kleiman and the author [He-Kl, 1].

2. Preliminary remarks

Let $Z \subset \mathbb{P}_k^N$ be an irreducible variety defined over an algebraically closed field k.

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Recall that the conormal variety $C(Z)$ of Z is the closure in $\mathbb{P}_k^N \times (\mathbb{P}_k^N)^*$ of the set

$$\{(P, H)/P \in Z^{sm} \text{ and } H \supset T_P Z\}.$$

The dimension of $C(Z)$ is always $N - 1$.

The *dual variety* Z' of Z is the projection on $(\mathbb{P}_k^N)^*$ of $C(Z)$.

We note by π and π' respectively the projections of $C(Z)$ onto Z and onto Z'.

A variety Z is called *reflexive* if $C(Z) = C(Z')$. It follows immediately from this definition that if Z is reflexive, then $(Z')' = Z$, and Z' is also reflexive.

Examples of non reflexive varieties are the strange curves, that is, projective curves such that all the tangent lines to the curve at simple points contain a given point. Indeed, the dual of a strange curve is a hyperplane whose dual is a point, so $(Z')' \neq Z$.

A fundamental result, called the *Segre–Wallace criterion* (see [Wa] or [Kl, 2]), asserts that:

Z is reflexive if, and only if, π' is separable.

In such case, it follows that π' is birational.

If Z is a curve, not a line, then the fibres of π' are zero dimensional, and so Z' is a hypersurface. In this case we have that π' is of finite degree, that is, $K(C(Z))$ is a finite field extension of $K(Z')$.

Define

$$\pi_i' = [K(C(Z)) \colon K(Z')]_i,$$

and

$$\pi_s' = [K(C(Z)) \colon K(Z')]_s.$$

So $\deg \pi' = \pi_i' . \pi_s'$.

In (3.5) of [He-Kl, 2] we proved the following result, called the *Generic Order of Contact Theorem*:

(2.1) *A projective curve Z is not reflexive if, and only if, for a general point P of Z and a general tangent hyperplane H to Z at P, we have*

$$\pi_i' = I(P, Z . H),$$

where $I(P, Z . H)$ is the intersection multiplicity of Z and H at P.

An immediate consequence of (2.1) is that there are no non reflexive curves of degree less than $p = char k$.

If H is a general tangent hyperplane of a non reflexive curve Z, that is, a hyperplane corresponding to a general point of Z', then (2.1) implies that

$$H.Z \geqslant \sum_{j=1}^{\pi'_s} I(P_j, Z.H)P_j = \sum_{j=1}^{\pi'_s} \pi'_i P_j,$$

where $P_1, \ldots, P_{\pi'_s}$ are the points at which H touches Z.

Taking degrees in the above equality, we get

$$\deg \pi' = \pi'_i . \pi'_s \leqslant \deg Z. \tag{2.2}$$

The above inequality is trivially true when Z is reflexive.

A curve Z will be called *extremal* if we have an equality in (2.2). Such curves will be studied in Section 7.

For the rest of this section we will assume that

$$Z\colon G(X_0, X_1, X_2) = 0,$$

is an irreducible plane curve. In this case π is birational hence it induces a rational map

$$\varphi = \pi' \circ \pi^{-1}\colon Z \to Z',\ \varphi(P) = [G_0(P); G_1(P); G_2(P)],$$

where G_i is the partial derivative of G with respect to X_i.

If Z is given in affine coordinates by $f(x, y) = 0$, with x a separating transcendental of $K(Z)$, then it is easy to see that the map φ is given by $\varphi(P) = [(y'x - y)(P); -y'(P); 1]$, where $y' = -f_X(x, y)/f_Y(x, y)$.

When Z is reflexive, it follows that π' is also birational, hence the map φ is birational and its inverse is the map

$$\varphi'\colon Z' \to (Z')' = Z,$$

obtained in the same way as φ.

The converse of the statement that if, Z is reflexive, then $(Z')' = Z$, is false as we may see in the following known example

(2.3) Example. Let $char k = p > 0$ and let q be a power of p. Define Z by

$$X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0.$$

Since Z is smooth, the rational map $\varphi: Z \to Z'$ is defined everywhere on Z and it is the restriction of the Frobenius map of order q of the projective plane:

$$[X_0; X_1; X_2] \to [X_0^q; X_1^q; X_2^q].$$

Denoting by $[Y_0; Y_1; Y_2]$ the coordinates of $(\mathbb{P}_k^2)^*$, it is clear that the equation of Z' is

$$Y_0^{q+1} + Y_1^{q+1} + Y_2^{q+1} = 0.$$

Applying the same procedure to Z', it follows that $(Z')' = Z$.

Now, the image of $[x_0; x_1; x_2] \in Z$ by $\varphi' \circ \varphi$ is the point $[x_0^{q^2}; x_1^{q^2}; x_2^{q^2}]$ which is, in general, different from $[x_0; x_1; x_2]$, hence $\varphi' \circ \varphi \neq Id$, and therefore Z is not reflexive.

Since we are going to work locally on curves we will introduce some notation and establish some technical results about parametrizations.

A *parametrization* of the curve Z centered at the point $[a_0; a_1; a_2] \in Z$ is a point

$$P(t) = [P_0(t); P_1(t); P_2(t)] \in \mathbb{P}^2_{k((t))},$$

not rational over k, such that $P(0) = [a_0; a_1; a_2]$ and $G(P(t)) = 0$.

$P(t)$ will be called *primitive* if it is not rational over $k((t^p))$.

(2.4) LEMMA. *Let $P(t) = [P_0(t); P_1(t); P_2(t)]$ be a point in $\mathbb{P}^2_{k((t))}$. Then we have*

(i) *$P(t)$ is rational over $k((t^p))$ if, and only if,* $\overrightarrow{P(t)}$ and $\overrightarrow{P'(t)}$ *are linearly dependent over $k((t))$.*

(ii) *Suppose that $P(t)$ is not rational over $k((t^p))$ and let $Q(t) \in \mathbb{P}^2_{k((t))}$. Then* $\overrightarrow{P(t)} \cdot \overrightarrow{Q(t)} = 0$ and $\overrightarrow{P'(t)} \cdot \overrightarrow{Q(t)} = 0$ *if, and only if,*

$$Q(t) = [P_1(t)P_2'(t) - P_1'(t)P_2(t); P_0'(t)P_2(t) - P_0(t)P_2'(t); \qquad (2.5)$$
$$P_0(t)P_1'(t) - P_0'(t)P_1(t)].$$

Proof. (i) If $P(t)$ is rational over $k((t^p))$, then

$$\overrightarrow{P(t)} = h(t)(H_0(t^p), H_1(t^p), H_2(t^p))$$

with $h(t) \neq 0$, so

$$\overrightarrow{P'(t)} = h'(t)(H_0(t^p), H_1(t^p), H_2(t^p)),$$

hence $\overrightarrow{P(t)}$ and $\overrightarrow{P'(t)}$ are linearly dependent over $k(t))$.

Conversely, if $\overrightarrow{P(t)}$ and $\overrightarrow{P'(t)}$ are linearly dependent over $k((t))$, then for all $i, j = 0, 1, 2$,

$$P_i(t)P_j'(t) - P_i'(t)P_j(t) = 0.$$

Since some of the $P_i(t)$ is non zero, by symmetry we may assume that $P_2(t) \neq 0$. It follows that for $i = 0, 1$,

$$\left(\frac{P_i(t)}{P_2(t)}\right)' = \frac{P_2(t)P_i'(t) - P_i(t)P_2'(t)}{(P_2(t))^2} = 0,$$

hence, for $i = 0, 1$,

$$\frac{P_i(t)}{P_2(t)} \in k((t^p)),$$

therefore $P(t)$ is rational over $k((t^p))$.

(ii) This follows from (i) and the fact that the right hand side of (2.5) is the point of $\mathbb{P}^2_{k((t))}$ corresponding to $\overrightarrow{P(t)} \times \overrightarrow{P'(t)}$.

(2.6) REMARK. Let $P(t)$ be a primitive parametrization of Z centered at a smooth point, then $[G_0(P(t)); G_1(P(t)); G_2(P(t))]$ makes sense as a point of $\mathbb{P}^2_{k((t))}$ and it is equal to the right hand side of (2.5).

Indeed, differentiating the equality $G(P(t)) = 0$ we get

$$P_0'(t)G_0(P(t)) + P_1'(t)G_1(P(t)) + P_2'(t)G_2(P(t)) = 0.$$

From Euler's identity for homogeneous polynomials we get

$$P_0(t)G_0(P(t)) + P_1(t)G_1(P(t)) + P_2(t)G_2(P(t)) = 0.$$

Since $P(t)$ is primitive, then $P(t)$ and $Q(t) = [G_0(P(t)); G_1(P(t)); G_2(P(t))]$ satisfy the conditions of Lemma 2.4 (ii), from which the result follows.

The number π_i' may be characterized in terms of parametrizations as follows.

(2.7) If $P(t)$ is a primitive parametrization centered at a general point of Z, then $[G_0(P(t)); G_1(P(t)); G_2(P(t))]$ is rational over $k((t^{\pi_i'}))$ but not rational over $k((t^{p\pi_i'}))$.

It follows immediately from this that Z is not reflexive if, and only if, $[G_0(P(t)); G_1(P(t)); G_2(P(t))]$ is rational over $k((t^p))$.

3. Hasse differential calculus

Let A be a unitary commutative k-algebra. A *Hasse family of differential operators* on A or, briefly a *Hasse family* on A, is a family

$$\mathcal{D} = (D_{\underline{n}}, \underline{n} \in \mathbb{N}^r)$$

of k-vector space endomorphisms of A satisfying the following conditions:
(i) $D_{\underline{0}} = Id$
(ii) $D_{\underline{n}}(c.1) = 0$ for all $c \in k$ and all $\underline{n} \neq \underline{0}$
(iii) $D_{\underline{n}} \circ D_{\underline{m}} = \binom{\underline{n}+\underline{m}}{\underline{n}} D_{\underline{n}+\underline{m}}$
(iv) $D_{\underline{n}} a.b = \Sigma_{\underline{i}+\underline{j}=\underline{n}} (D_{\underline{i}} a)(D_{\underline{j}} b)$,
where, for $\underline{m} = (m_1, \ldots, m_r)$ and $\underline{n} = (n_1, \ldots, n_r)$,

$$\binom{\underline{m}}{\underline{n}} = \binom{n_1}{m_1} \cdots \binom{n_r}{m_r}.$$

If $\underline{n} = (0, \ldots, \nu, \ldots, 0)$, where ν is the ith entry, then $D_{\underline{n}}$ will be denoted by D_i^ν.

It is easy to verify that for each i, $(D_i^\nu, \nu \in \mathbb{N})$ is a Hasse family on A.

Conversely, given r Hasse families $(D_i^\nu, \nu \in \mathbb{N})$, $i = 1, \ldots, r$, on A, then

$$D_{\underline{n}} = D_1^{n_1} \circ \cdots \circ D_r^{n_r},$$

define a Hasse family $\mathcal{D} = (D_{\underline{n}}, \underline{n} \in \mathbb{N}^r)$ on A.

(3.1) Example. Let $A = k[\underline{t}] = k[t_1, \ldots, t_r]$. Put $\underline{t}^{\underline{m}} = t_1^{m_1} \cdots t_r^{m_r}$ and

$$D_{\underline{n}} \underline{t}^{\underline{m}} = \binom{\underline{m}}{\underline{n}} \underline{t}^{\underline{m}-\underline{n}}.$$

The family $\mathcal{D} = (D_{\underline{n}}, \underline{n} \in \mathbb{N}^r)$, where $D_{\underline{n}}$ has been extended by linearity on A in a Hasse family on A.

This definition is taken in order that, if $P(t_1, \ldots, t_r) \in A$, then $D_{\underline{n}} P(t_1, \ldots, t_r)$ is the coefficient of $\underline{u}^{\underline{n}}$ is the expansion of $P(t_1 + u_1, \ldots, t_r + u_r)$ as a polynomial in $\underline{u}$. This was what motivated Hasse to introduce such operators (see [Ha]).

Condition (iii) for a Hasse family $\mathscr{D}$ implies that

$$D_{\underline{n}} = D_r^{n_r} \circ \cdots \circ D_1^{n_1}, \tag{3.2}$$

and that

$$v!D_i^n = (D_i^1)^v, \tag{3.3}$$

where

$$(D_i^1)^v = D_i^1 \circ \cdots \circ D_i^1 \quad (v - times).$$

So, if $chark = 0$, we have that

$$D_{\underline{n}} = (1/\underline{n}!)(D_r^1)^{n_r} \circ \cdots \circ (D_1^1)^{n_1},$$

where

$$\underline{n}! = n_1! \cdots n_r!.$$

Therefore, in characteristic zero, the whole family $\mathscr{D}$ is determined by the first order operators $D_r^1, \ldots, D_1^1$.

If $chark = p > 0$, then the picture is quite different as we will see in (3.7).

It is well known and easy to prove that if p is a prime number, n and m are integers with $n \geqslant 0$, then

$$\binom{pm}{pn} \equiv \binom{m}{n} \quad modp. \tag{3.4}$$

From this it follows easily that if $m = \Sigma_{i=0}^s m_i p^i$, $n = \Sigma_{i=0}^s n_i p^i$, where the m_i and n_i are integers, with $n_i \geqslant 0$, $i = 0, \ldots, s$, then

$$\binom{m}{n} \equiv \binom{m_0}{n_0} \cdots \binom{m_s}{n_s} \quad modp. \tag{3.5}$$

This in turn implies that if also $m \geqslant 0$, then

(3.6) $\binom{m}{n} \not\equiv 0\,(modp)$ if, and only if, m is p-adically bigger than or equal to n.

These are classical results in elementary number theory. (3.4) and (3.5) can already be found in [Lu], while (3.6) is usually credited to F.K. Schmidt.

Condition (iii) of the definition of a Hasse family together with (3.5) imply easily the following result due to Dieudonné [D].

Let $n = \Sigma_{j=0}^{s} n_j p^j$ with $0 \leqslant n_j < p, j = 0, \ldots, s$, then we have

$$D_i^n = \frac{1}{n_0! \cdots n_s!} (D_i^{p^s})^{n_s} \circ \cdots \circ (D_i^p)^{n_1} \circ (D_i^1)^{n_0}. \tag{3.7}$$

It follows that for each i, the family $(D_i^n, n \in \mathbb{N})$ is determined by the operators

$$D_i^1, \quad D_i^p, \quad D_i^{p^2}, \ldots$$

The following is an easy consequence of (3.7).

(3.8) If $D_i^j a = 0$ for some $a \in A$ and $j \in \mathbb{N}$, then $D_i^m a = 0$ for all m p-adically bigger than or equal to j.

In particular, we have

(3.8.1) If $D_i^{p^\mu} a = 0$, then $D_i^{p^\mu + 1} a = \cdots = D^{p^{\mu+1} - 1} a = 0$.
(3.8.2) If $p \neq 2$ and $D_i^2 a = 0$, then for all $m \geqslant 0$ we have

$$D_i^{mp+2} a = \cdots = D_i^{mp+p-1} a = 0.$$

(3.9) Lemma. *Let X and T be indeterminates and q be a power of p. If $f(T) \in k[T]$, then*

$$D_X^n f(X^q) = \begin{cases} (D_T^j f)(X^q) & \text{if } n = jq \\ 0 & \text{if } n \not\equiv 0 \, mod q \end{cases}$$

where D_X^n (respectively D_T^n) is the operator D_n of Example (3.1) *defined on $k[X]$ (respectively on $k[T]$).*

Proof. Let $f(T) = \Sigma_{i=0}^m a_i T^i$. Then $D_X^n f(X^q) = \Sigma_{i=0}^m \binom{iq}{n} a_i X^{iq-n}$, from which the result follows in view of (3.4) and (3.6).

The following is a higher order generalization of Euler's identity for homogeneous polynomials.

(3.10) Proposition. *Let $P_j(T_1, \ldots, T_r)$ and $Q_j(T_1, \ldots, T_r)$ be polynomials in $k[T_1, \ldots, T_r]$, $j = 1, \ldots, s$, such that the P_j are homogeneous of degree n and the Q_j have no indeterminate raised to a power bigger than or equal to* *a power of q of p. If*

$$F(X_1, \dots, X_r) = \sum_{j=1}^{s} P_j(X_1^q, \dots, X_r^q)Q_j(X_1, \dots, X_r),$$

then

$$\sum_{i=1}^{r} X_i^q D_{X_i}^q F(X_1, \dots, X_r) = nF(X_1, \dots, X_r).$$

Proof. Using condition (iv) of the definition of a Hasse family together with (3.9), we get that

$$D_{X_i}^q P_j(X_1^q, \dots, X_r^q)Q_j(X_1, \dots, X_r)$$

$$= Q_j(X_1, \dots, X_r)(D_{X_i}^1 P_j)(X_1^q, \dots, X_r^q).$$

Now, this together with Euler's identity for homogeneous polynomials imply the result.

(3.11) Lemma. *Let B be a commutative ring and* $(D_n, n \in \mathbb{N})$ *a family of mappings from B into B satisfying condition (iv) of a Hasse family. Then for any* $b \in B$ and $n, m \geqslant 1$, *we have*

$$D_n b^m = mb^{m-1}D_n b + \sum_{i=2}^{m} \sum_{j=1}^{n-1} b^{m-i}(D_j b)(D_{n-j} b^{i-1}).$$

Proof. By induction on n and m.

(3.12) Proposition. *Let A be a k-algebra with a Hasse family* $\mathscr{D} = (D_n, n \in \mathbb{N})$ *on it. Let* $B = A[y]$, *where y satisfies a polynomial relation* $P(y) = 0$ *with coefficients in A such that* $P'(y)$ *is not nilpotent in B. Then* $\mathscr{D}$ *extends uniquely to a Hasse family on* $B_{P'(y)} = S^{-1}B$, *where* $S = \{1, P'(y), (P'(y))^2, \dots\}$.

Proof. Let $P(y) = a_0 + a_1 y + \cdots + a_r y^r$. The only way to define $D_1 y$ is by using the relation $P(y) = 0$ and (3.11), which imply that

$$D_1 y = -\frac{D_1 a_0 + (D_1 a_1)y + \cdots + (D_1 a_r)y^r}{P'(y)} \in B_{P'(y)}.$$

Using (3.11) to define $D_1 y^m$ for $m > 1$, we get

$$D_1 y^m = m y^{m-1} D_1 y \in B_{P'(y)}.$$

Again the relation $P(y) = 0$ and (3.11) allow us to define $D_2 y$ as an element of $B_{P'(y)}$. So, inductively we have that (3.11) and the previously computed elements determine $D_2 y^m$ for all $m > 1$. Now proceed inductively again to define $D_n y^m$, for $n, m \geqslant 1$, as elements of $B_{P'(y)}$. It is easy to extend the definition of D_n on $B_{P'(y)}$.

It remains to prove that what we get in this way is actually a Hasse family on $B_{P'(y)}$. This will be omitted since it is a long and tedious chain of verifications.

(3.13) COROLLARY. *Let $Z: f(x_1, \ldots, x_{r+1}) = 0$ be an affine irreducible hypersurface defined over k, with $x_1, \ldots, x_r$ separating transcendentals. Let*

$$B = k[X_1, \ldots, X_{r+1}]/(f)$$

be the k-algebra of regular functions on Z. Then there is a Hasse family $\mathscr{D} = (D_{\underline{n}}, \underline{n} \in \mathbb{N}^r)$ on $B_{f_{r+1}}$, where f_{r+1} is the derivative of f with respect to X_{r+1}, uniquely determined by the conditions

$$D_{\underline{n}} x_1^{m_1} \cdots x_r^{m_r} = \binom{\underline{m}}{\underline{n}} x_1^{m_1 - n_1} \cdots x_r^{m_r - n_r}.$$

It is easy to prove that Hasse families behave well under the formation of rings of fractions and under completions. The precise statements are the following:

(3.14) (Hironaka [Hi]). *Let $\mathscr{D} = (D_{\underline{n}}, \underline{n} \in \mathbb{N}^r)$ be a Hasse family on A. Let S be a multiplicative subset of A and I an ideal of A. Then $\mathscr{D}$ extends uniquely to Hasse families on $S^{-1}A$ and on the I-adic completion of A.*

So the Hasse family of Example (3.1) extends uniquely to a Hasse family on $k[[\underline{t}]]$. Also in this case, $D_{\underline{n}} P(t_1, \ldots, t_r)$ is the coefficient of $\underline{u}^{\underline{n}}$ in the expansion of $P(\underline{t} + \underline{u})$ as a power series in $\underline{u}$.

Let the notation be as in (3.13). Suppose that Z is an irreducible hypersurface and let $K(Z)$ be its field of rational functions. By (3.14) $\mathscr{D}$ extends uniquely on $K(Z)$. If $h \in K(Z)$ is regular at $P \in Z^{sm}$, and $x_i - x_i(P)$, $i = 1, \ldots, r$, are local parameters of Z at P, then in $\hat{\mathcal{O}}_{Z,P}$ we have

$$h = \sum_{\underline{n} \in \mathbb{N}^r} (D_{\underline{n}} h)(P)(x_1 - x_1(P))^{n_1} \cdots (x_r - x_r(P))^{n_r}.$$

It follows from (3.13) and (3.14) that we have a Hasse family on the function field of any algebraic variety Z in a number of variables equal to the dimension of Z.

4. Local theory for plane curves

Let

$$Z\colon G(X_0, X_1, X_2) = 0$$

be an irreducible plane curve, and let P be a smooth point of Z. By a convenient choice of the coordinates of $\mathbb{P}_k^2$ we may assume that $P = [1; a_0; b_0]$ and that $t = x - a_0$ is a local parameter of Z at P, where $x = X_1/X_0$.

Hence a primitive parametrization of Z centered at P may be given by

$$P(t) = [1; a_0 + t; b_0 + b_1 t + \cdots],$$

where for $y = X_2/X_0$,

$$b_i = (D_x^i y)(P).$$

From Remark (2.6) we have that

$$[G_0(P(t)); G_1(P(t)); G_2(P(t))] = [a_0 b_1 - b_0 + 2a_0 b_2 t + (3a_0 b_3 + b_2)t^2 + (4a_0 b_4 + 2b_3)t^3 + \cdots; -b_1 - 2b_2 t - \cdots; 1] \qquad (4.1)$$

which are the power series expansions in $\hat{\mathcal{O}}_{Z,P}$ of the components of

$$[xy' - y; -y'; 1] \qquad (4.2)$$

From (4.1) we get immediately the following remarks:
(4.3) No non linear curve in characteristic two is reflexive.
(4.4) Let x be a separating transcendental of $K(Z)$, then the following are equivalent.
(a) Z is reflective (b) *chark* $\neq 2$ and $D_x^2 y \neq 0$ (c) $y'' \neq 0$.

Assertion (4.3) was first observed for any curve by N. Katz in [Ka], while (4.4) is Wallace's Hessian Criterion given in [Wa] (see also [He-Kl, 2]).

Note that (4.4) implies that

(4.5) Z is reflexive if and only if, *chark* $\neq 2$ and for a general point $P \in Z$ we have

$$I(P, Z \cdot T_P Z) = 2.$$

If Z is not reflexive and x is a separating transcendental of $K(Z)$, then *chark* $= 2$ or $D_x^2 y = 0$. Hence from (3.8.1) and (3.8.2), it follows that there exists $q = p^e$ such that for any smooth point P of Z, where $x - x(P)$ is a uniformizing parameter, a primitive parametrization of Z at P may be given by

$$P(t) = [1; a_0 + t; b_0 + b_1 t + b_q t^q + b_{q+1} t^{q+1} + b_{2q} t^{2q} + b_{2q+1} t^{2q+1} + \cdots] \tag{4.6}$$

such that b_q, as a rational function on Z, is not identically zero.

From (4.1) we get

$$[G_0(P(t)); G_1(P(t)); G_2(P(t))] = [a_0 b_1 - b_0 + (a_0 b_{q+1} - b_q)t^q$$
$$+ (a_0 b_{2q+1} - b_{2q})t^{2q} + \cdots; -b_1 - b_{q+1} t^q - b_{2q+1} t^{2q} - \cdots; 1]. \tag{4.7}$$

Since at a general point $P \in Z$, we have $b_q \neq 0$, then either $b_{q+1} \neq 0$ or $a_0 b_{q+1} - b_q \neq 0$, so from (4.6) and (4.7) we get that

$$I(P, Z \cdot T_P Z) = q = \pi_i'.$$

This is the *Generic Order of Contact Theorem* for plane curves.

A point $P \in Z^{sm}$ is called a *flex* if $I(P, Z \cdot T_P Z) > \pi_i'$.

A flex is called *ordinary* if $I(P, Z \cdot T_P Z) = \pi_i' + 1$, otherwise it is called a *higher flex*.

At finite distance, a flex is a smooth point for which

$$b_q(P) = (D_x^q y)(P) = 0,$$

therefore any non linear curve has at most finitely many flexes.

The following result gives a peculiar property of non reflexive curves.

(4.8) Proposition. *Let Z be a non reflexive curve and $P(t)$ a place centered at a point P which is not a higher flex. Then the image of $P(t)$ by φ is a non singular branch of Z' at $\varphi(P)$.*

Proof. Indeed, the condition on P implies that $b_q \neq 0$ or $b_{q+1} \neq 0$ at P, the result now follows immediately from (4.7).

(4.9) COROLLARY. *Let Z be a smooth non reflexive curve of degree $\leqslant 3$, then chark $= 2$ and Z' is smooth.*

Proof. From (2.2) we have that $\pi_i' . \pi_s' \leqslant \deg Z \leqslant 3$. Since Z is not reflexive, then $\pi_i' = 2$ or 3 and $\pi_s' = 1$.

Since Z has no bitangent lines (by Bézout's theorem), it follows that at all points of Z' there is only one branch; and since Z has no higher flexes, it follows from (4.8) that all branches of Z' are non singular, hence Z' is smooth.

Now, since φ is purely inseparable, Z and Z' have the same genus.

If $\deg Z = 2$, and since the degree of a non reflexive curve is at least equal to *chark*, it follows that *chark* $= 2$.

If $\deg Z = 3$, then $\deg Z' = 3$. From Plücker's formula, see [Kl, 1], we have

$$\deg \pi' \deg Z' = \deg Z(\deg Z - 1) = 6.$$

It then follows that $\deg \pi' = 2$ and therefore *chark* $= 2$.

(4.10) COROLLARY. *If Z is a smooth non reflexive curve of degree $q + 1$ such that $\pi_i' = q$, then Z' is smooth of degree $q + 1$ and π' is purely inseparable.*

Proof. The hypotheses, together with the Generic Order of Contact Theorem and Bézout's Theorem imply that Z has no bitangent lines, nor higher flexes. Hence by (4.8) Z' is mooth.

Now, from Plücker's formula,

$$\deg \pi' \deg Z' = \deg Z(\deg Z - 1),$$

and from (2.2), it follows that $\deg Z' \geqslant \deg Z - 1$.

On the other hand, by the Riemann–Hurwitz formula, it follows that $\deg Z' \leqslant \deg Z$.

Therefore, either $\deg Z' = q$, in which case $\deg \pi' = q + 1$ which is impossible because $p | \deg \pi'$, or $\deg Z' = q + 1$.

Hence $\deg Z' = q + 1$ and $\deg \pi' = q = \pi_i'$.

(4.11) REMARK. It is possible to show that the curves in (4.10) are projectively equivalent to the curve of Example (2.3) (cf. Pardini [Pa], where it is proved for $q = p$. When $q > p$ the proof is similar).

(4.12) PROPOSITION. *Let Z: $G(X_0, X_1, X_2) = 0$ be an irreducible plane curve of degree at least two. Z is not reflexive if, and only if, for all $i, j = 0, 1, 2$, the following equalities are satisfied on Z.*

$$G_i^2 G_{jj} + G_j^2 G_{ii} - 2G_i G_j G_{ij} = 0. \tag{4.13}$$

Proof. Since Z is not a line, then $Z \cap \{X_0 \neq 0\}$ is not empty. Let

$$f(x, y) = G(1, X_1/X_0, X_2/X_0).$$

Suppose that x is a separating transcendental of $K(Z)$ (if not, y is and the proof is similar).

By implicit differentiation of the relation $f(x, y) = 0$ we get

$$f_X + y' f_Y = 0$$

$$f_{XX} + y' f_{YX} + y'(f_{XY} + y' f_{YY}) + y'' f_Y = 0,$$

from which we get

$$y'' = -\frac{f_X^2 f_{YY} + f_Y^2 f_{XX} - 2f_X f_Y f_{XY}}{(f_Y)^3}.$$

From (4.4) we have that Z is not reflexive if, and only if $y'' = 0$, hence

$$f_X^2 f_{YY} + f_Y^2 f_{XX} - 2f_X f_Y f_{XY} = 0,$$

which gives one of the equations (4.13). The other relations are obtained in a similar way.

5. Equations for non-reflexive plane curves

In this section Z will be a plane irreducible curve defined over k by an equation

$$Z\colon G(X_0, X_1, X_2) = 0.$$

We will give explicit forms for the polynomial G when Z is non reflexive and satisfies some regularity condition.

Let P be a point of Z. Define e_P as the multiplicity of the jacobian ideal of Z at P, that is,

$$e_p = \min \{I(P, G_i . G), i = 0, 1, 2\}$$

(5.1) THEOREM. *Let* $char k > 2$ *and* Z *be such that* $\Sigma_{P \in Z} e_P < (1/2) \deg Z$. *Then* Z *is not reflexive if, and only if,* $G_{ij} = 0$ *for all* $i, j = 0, 1, 2$.

Proof. Suppose that X is not reflexive, then (4.13) holds for $i, j = 0, 1, 2$. For fixed i, it follows, for all $j = 0, 1, 2$ and all $P \in Z$, that

$$I(P, G_{ii} . G) + 2I(P, G_j . G) \geqslant I(P, G_i . G).$$

This implies that

$$I(P, G_{ii} . G) + 2e_P \geqslant I(P, G_i . G). \tag{5.2}$$

Suppose that $G_{ii} \neq 0$. Then $G_i \neq 0$, $\deg G_{ii} = \deg G - 2$ and $\deg G_i = \deg G - 1$.

From Bézout's theorem it follows after summation of (5.2) over all the points of Z, that

$$\deg G(\deg G - 2) + \sum_{P \in Z} e_P \geqslant \deg G(\deg G - 1).$$

Hence $\Sigma_{P \in Z} e_P \geqslant (1/2) \deg Z$, a contradiction. So $G_{ii} = 0$.

Now, from (4.13) it follows that $2G_i G_j G_{ij} = 0$ on Z. Since $char k \neq 2$ and Z is irreducible, it follows that either $G_i = 0$, $G_j = 0$ or $G_{ij} = 0$ on Z. Any one of these conditions implies immediately that $G_{ij} = 0$.

The converse is trivially true in view of Proposition (4.12).

(5.3) REMARKS:

This theorem, for nonsingular curves, was first obtained by R. Pardini in [Pa].

The hypothesis that $char k > 2$ is essential in the theorem, since every non linear curve in characteristic two is non reflexive.

Some sufficient condition such as $\Sigma_{P \in Z} e_P < (1/2) \deg Z$ is also essential as we see in the following example.

Let $char k = 3$ and Z: $X_0 X_1^2 + X_2^3 = 0$. Z is not reflexive since it is strange. The conclusion of the theorem does not hold because $G_{01} = 2X_1 \neq 0$. In this case, $3 = \Sigma_{P \in Z} e_P \geqslant (1/2) \deg Z = 3/2$.

(4) COROLLARY. *Let* $char k = p > 2$ *and let* Z *be such that* $\Sigma_{P \in Z} e_P <$ (. 2) $\deg Z$. *If* Z *is not reflexive, then* $char k | (\deg Z - 1)$ *and there exist* *homogeneous polynomials* P_i, $i = 0, 1, 2$, *of the same degree such that*

$$G(X_0, X_1, X_2) = \sum_{i=0}^{2} X_i P_i(X_0^p, X_1^p, X_2^p).$$

Proof. From Euler's identity and (5.1) we have that

$$0 = \sum_{j=0}^{2} G_{ij} X_i = (\deg X - 1) G_j.$$

Since not all the G_j are zero, it follows that $char k | (\deg X - 1)$.
Now, using Euler's identity again, we have that

$$G(X_0, X_1, X_2) = \sum_{i=0}^{2} X_i G_i(X_0, X_1, X_2),$$

with $G_{ij} = 0$ for all $i, j = 0, 1, 2$, hence $G_i = P_i(X_0^p, X_1^p, X_2^p)$ for some homogeneous polynomials P_i.

It follows immediately from (5.4) that if $char k > 2$, then the general curve of any fixed degree n is reflexive (In [He-Kl, 2] the reader may find more general results with this flavour).

(5.5) THEOREM. *Let* $char k = p$ *and let* $Z\colon G = 0$ *be a curve such that*

$$\sum_{P \in Z} e_P < \left(1 - \frac{1}{p}\right) (\deg Z)^2,$$

and $G_{ij} = 0$ *for all* $i, j = 0, 1, 2$. *Then* $\pi_i' = p^e$ *with* $e \geqslant 2$ *if, and only if,* $D_i^{p^m} G = 0$ *for all* $i = 0, 1, 2$ *and all* $m = 1, \ldots, e - 1$.
Proof. By induction on e. Let

$$Z^0 = Z^{sm} \cap \{[a_0; a_1; a_2] / a_0 \neq 0\}. \tag{5.6}$$

Suppose that $e = 2$. The condition that $G_{ij} = 0$ implies that the Taylor expansion of $G(1, x_1, x_2)$ at $Q = (1, a_1, a_2) \in Z^0$ is given by

$$G(1, x_1, x_2) = \sum_{i=1}^{2} G_i(Q)(x_i - a_i) + \sum_{i=1}^{2} D_i^p G(Q)(x_i - a_i)^p + \cdots$$

Now, a straightforward computation, using Euler's identity for homogeneous polynomials, shows that

$$\sum_{i=1}^{2} G_i(Q)(x_i - a_i) = \sum_{i=0}^{2} G_i(Q) x_i.$$

Now, since $G_{ij} = 0$ for all $i, j = 0, 1, 2$, then G has the form described in (5.4), so from (3.10) we have that

$$\sum_{i=1}^{2} D_i^p G(Q)(x_i - a_i)^p = \sum_{i=0}^{2} D_i^p G(Q) x_i^p,$$

where $x_0 = 1$.

Since by hypothesis we have that $I(Q, Z \,.\, T_Q Z) \geqslant \pi_i' > p$, it follows that

$$\sum_{i=0}^{2} G_i(Q) X_i \quad \text{divides} \quad \sum_{i=0}^{2} D_i^p G(Q) X_i^p.$$

This implies that

$$G_j^p(Q) D_i^p G(Q) = G_j^p(Q) D_j^p G(Q), \text{ for all } i, j = 0, 1, 2, \text{ and all } Q \in Z^0. \tag{5.7}$$

Hence (5.7) is verified on Z; so for any $P \in Z$ we have, for all $i, j = 0, 1, 2$, that

$$pI(P, G_j \,.\, G) + I(P, D_i^p G \,.\, G) \geqslant pI(P, G_i \,.\, G).$$

This implies that

$$pe_P + I(P, D_i^p G \,.\, G) \geqslant pI(P, G_i \,.\, G). \tag{5.8}$$

Suppose that $D_i^p G \neq 0$.

It follows that $G_i \neq 0$, because otherwise from (5.7) we would have $G_j = 0$ for $j = 0, 1, 2$ which is impossible. So we have

$$\deg D_i^p G = \deg Z - p \quad \text{and} \quad \deg G_i = \deg Z - 1.$$

After summation of (5.8) over Z, we get by Bézout's theorem that

$$p \sum_{P \in Z} e_P + \deg Z(\deg Z - p) \geqslant p \deg Z(\deg Z - 1),$$

hence

$$\sum_{P \in Z} e_P \geqslant \left(1 - \frac{1}{p}\right) (\deg Z)^2,$$

a contradiction. Hence $D_i^p G = 0$ for all $i, j = 0, 1, 2$.

Suppose now that the results hold for $e - 1$ and that $\pi_i' \geqslant p^e$.
By the inductive assumption we have that

$$D_i^{p^m} G = 0, \quad i = 0, 1, 2; m = 1, \ldots, e - 2.$$

From this it follows that

$$D_i^{p^m} G_j = 0 \quad \text{for } i, j = 0, 1, 2, \text{ and } m = 1, \ldots, e - 2. \tag{5.9}$$

So the Taylor expansion of $G(1, x_1, x_2)$ at $Q = (1, a_1, a_2) \in Z^0$ is given by

$$G(1, a_1, a_2) = \sum_{i=1}^{2} G_i(Q)(x_i - a_i) + \sum_{i=1}^{2} D_i^{p^{e-1}} G(Q)(x_i - a_i)^{p^{e-1}} + \cdots.$$

On the other hand, (5.9) also implies that

$$G(X_0, X_1, X_2) = \sum_{i=0}^{2} X_i P_i(X_0^{p^{e-1}}, X_1^{p^{e-1}}, X_2^{p^{e-1}}),$$

hence by (3.10) we have that

$$\sum_{i=1}^{2} D_i^{p^{e-1}} G(Q)(x_i - a_i)^{p^{e-1}} = \sum_{i=0}^{2} D_i^{p^{e-1}} G(Q) x_i^{p^{e-1}}.$$

Now, the same kind of reasoning as in the case $e = 2$ allows us to conclude that $D_i^{p^e} G = 0$ for $i = 0, 1, 2$. So one direction of the theorem is proved.

The converse is clear since the conditions that $D_i^{p^m} G = 0$ and $G_{ij} = 0$ for all $i, j = 0, 1, 2$ and $m = 1, \ldots, e - 1$, imply that the Taylor expansion of $G(1, x_1, x_2)$ at $Q \in Z^0$ starts with the term $\Sigma_{i=1}^2 D_i^{p^e} G(Q)(x_i - a_i)^{p^e}$ after the linear term, so $I(Q, Z \cdot T_Q Z) \geqslant p^e$ for a general point Q, hence by the Generic Order of Contact Theorem we get that $\pi_i' \geqslant p^e$.

REMARK. Some sufficient condition such as $\Sigma_{P \in Z}\, e_P < (1 - 1/p)(\deg Z)^2$ is needed in the theorem as we see in the following example.

Let *chark* $= 3$ and let

$$Z: G = Z_1 X_2^9 + X_1^3 X_0^7 + X_0^{10} = 0.$$

It is easy to check that $G_{ij} = 0$ for all i, j. It is also easy to verify that the relations (5.7) are satisfied for all $Q \in Z^0$; this implies that $\pi_i' > 3$, so it must be 9.

On the other hand we have that

$$D_1^3 G = X_0^7,$$

which is not zero on Z, hence the conclusion of the theorem does not hold. The explanation for this is that

$$81 = \sum_{P \in Z} e_P \geqslant \left(1 - \frac{1}{p}\right)(\deg Z)^2 = \frac{200}{3}.$$

(5.10) COROLLARY. *Let* $char k = p > 2$ *and let* Z *be a curve such that* $\Sigma_{P \in Z} e_p < (1/2) \deg Z$. *If* $\pi_i' > p^e = q$ *with* $e \geqslant 1$, *then the equation of* Z *is of the form*

$$\sum_{i=0}^{2} X_i P_i(X_0^q, X_1^q, X_2^q) = 0,$$

where the P_i *are homogeneous polynomials.*

Proof. This follows from (5.1) and (5.5).

This answers a question posed by Kleiman in [Kl, 3].

It remains to prove a result similar to (5.10) when $char k = 2$.

Any curve in characteristic two falls into one of the following types according to its degree:

a) If deg Z is odd, then the equation of Z is of type

$$G(X_0, X_1, X_2) = \sum_{i=0}^{2} X_i P_i(X_0^2, X_1^2, X_2^2) + X_0 X_1 X_2 P_3(X_0^2, X_1^2, X_2^2),$$

b) If deg Z is even, then the equation of Z is of type

$$G(X_0, X_1, X_2) = \sum_{i,j=0}^{2} X_i X_j P_{ij}(X_0^2, X_1^2, X_2^2)$$

(5.11) THEOREM. *Let* $char k = 2$ *and let* Z: $G = 0$ *be a curve such that* $\Sigma_{P \in Z} e_P < (1/2) \deg Z$. *Then* $\pi_i' \geqslant 2^e$ *with* $e \geqslant 2$ *if, and only if,* $G_{ij} = 0$ *and* $D_i^{2^m} G = 0$ *for all* $i, j = 0, 1, 2$ *and* $m = 1, \ldots, e - 1$.

Proof. If $G_{ij} = 0$ and $D_i^{2^m} G = 0$ for all $i, j = 0, 1, 2$ and $m = 1, \ldots, e - 1$, then clearly $\pi_i' \geqslant 2$ because the Taylor expansion of $G(1, x_1, x_2)$ at $Q \in Z^0$ will start after the linear term with the term

$$\sum_{i=1}^{2} D_i^{2^e} G(Q)(x_i - a_i)^{2^e}.$$

To prove the converse, in view of Theorem 5.5, we have only to prove that $G_{ij} = 0$ for all $i, j = 0, 1, 2$.

The second order term of the Taylor expansion of $G(1, x_1, x_2)$ at $Q \in Z^0$ is

$$D_1^2 G(Q)(x_1 - a_1)^2 - D_2^2 G(Q)(x_2 - a_2)^2 + G_{12}(Q)(x_1 - a_1)(x_2 - a_2). \tag{5.12}$$

If deg Z is odd we cannot apply (3.10) directly to G since the degree of P_3 is not equal to the degree of P_i, $i = 0, 1, 2$. But if we use (3.10) separately on each summand, it is easy to check that

$$a_1^2 D_1^2 G(Q) + a_2^2 D_2^2 G(Q) = a_0^2 D_0^2 G(Q) + a_1 a_2 G_{12}(Q).$$

Using this relation and playing with Euler's identity, we get that (5.12) is equal to

$$\sum_{i=0}^{2} x_i^2 D_i^2 G(Q) + \sum_{0 \leqslant i < j \leqslant 2} x_i x_j G_{ij}(Q), \tag{5.13}$$

where $x_0 = 1$.

Now, the condition that $\pi_i' \geqslant 2^e$ with $e \geqslant 2$ implies that the polynomial $\Sigma_{i=1}^2 G_i(Q) x_i$ divides the polynomial in (5.13), so we get, for $Q \in Z^0$ and for $i, j = 0, 1, 2$, the following relations

$$G_j^2(Q) D_i^2 G(Q) + G_i^2 D_j^2 G(Q) = G_i G_j G_{ij}. \tag{5.14}$$

These relations, as in the proof of Theorem (5.1), imply that $D_i^2 G(Q) = 0$ for $i = 0, 1, 2$, and consequently that $G_{ij} = 0$ for all $i, j = 0, 1, 2$.

If deg Z is even, then by (3.10), Euler's identity, and the relation

$$a_1 a_2 G_{12}(Q) = a_0 a_1 G_{01}(Q) + a_0 a_2 G_{02}(Q) + \sum_{i=0}^{2} a_i^2 P_{ii}(1, a_1^2, a_2^2),$$

which comes from the equation type of Z, we get that (5.12) is equal to

$$\sum_{i=0}^{2} x_i^2 D_i^2 G(Q) + \sum_{0 \leqslant i < j \leqslant 2} x_i x_j G_{ij}(Q) + \sum_{i=0}^{2} x_i G_i(Q). \tag{5.15}$$

Since the above polynomial is divisible by $\Sigma_{i=0}^2 x_i G_i(Q)$, we get the same relations as in (5.14) and the proof proceeds as above.

(5.16) COROLLARY. *Let* $char\, k = 2$ *and let* Z *be such that* $\Sigma_{P \in Z}\, e_P < (1/2) \deg Z$. If $\pi_i' \geqslant 2^e$ with $e \geqslant 2$, *then the equation of* Z *is of the form*

$$\sum_{i=0}^{2} X_i P_i(X_0^{2^e}, X_1^{2^e}, X_2^{2^e}) = 0$$

where the P_i's *are homogeneous polynomials.*

REMARK. It follows from (5.16) that every smooth curve of even degree in characteristic two is such that $\pi_i' = 2$.

So far we have only met two types of non reflexive plane curves in characteristic $p > 2$, namely the strange curves and curves of type $P_{n,q}$, that is curves of degree n with equations

$$\sum_{i=0}^{2} X_i P_i(X_0^q, X_1^q, X_2^q) = 0,$$

where q is a power of p.

The curves of type $P_{n,q}$ only occur in degree $n = mq + 1$ and all smooth non reflexive curves, when $p > 2$, are of this type. These curves are parametrized by a linear space in the projective space $\mathbb{P}^N$, $N = n(n + 3)/2$, which parametrizes all plane curves of degree n.

Now we give an example of a family of non reflexive curves which are not strange nor of type $P_{n,q}$.

Let $char\, k = p$ and $n = rp$, with $r > 1$. Let F_n be the family of curves of degree n of the following type:

$$y + \sum_{m=0}^{r} a_m x^{mp} + \sum_{m=0}^{r-1} b_m x^{mp+1} = 0.$$

where $a_m, b_m \in k$, $a_r b_{r-1} \neq 0$.

It is easy to check that these curves are not strange and since $y'' = 0$, they are non-reflexive.

It is still an open problem to describe all non reflective curves.

6. Plane curves with non-reflexive duals

In this section we are going to study curves with non reflexive duals. It has been observed in Section 2 that such curves are necessarily non reflexive. We give here necessary and sufficient conditions for the non reflexivity of the dual of a curve in terms of the vanishing of certain rational functions on the curve.

Since we are interested here in curves with no reflexive duals, we exclude the lines and the strange curves from our considerations.

Let Z be a non reflexive curve with $\pi_i' = q$. Then, as we saw in Section 4, at a general point P, Z may be parametrized by

$$P(t) = [1; a_0 + t; b_0 + b_1 t + b_q t^q + b_{q+1} t^{q+1} + b_{2q} t^{2q} + b_{2q+1} t^{2q+1} + \cdots]$$

with b_q not identically zero on Z. So from (4.7) we have that a primitive image of the parametrization $P(t)$ by φ is given by

$$[Q_0; Q_1; Q_2] = [a_0 b_1 - b_0 + (a_0 b_{q+1} - b_1)t + (a_0 b_{2q+1} - b_q)t^2 + \cdots;$$

$$-b_1 - b_{q+1}t - b_{2q+1}t^2 - \cdots; 1]. \tag{6.1}$$

Now, applying Remark 2.6, we get the following parametrization, not necessarily primitive, of Z''.

$$[R_0; R_1; R_2] = [Q_1 Q_2' - Q_1' Q_2; Q_0' Q_2 - Q_0 Q_2'; Q_0 Q_1' - Q_0' Q_1]. \tag{6.2}$$

Now, by (2.7) we have that Z' is non reflexive if, and only if, $[R_0; R_1; R_2]$ is rational over $k((t^p))$. This, in view of (2.4, i), is equivalent to the conditions $R_i' R_j - R_i R_j' = 0$ for $i, j = 0, 1, 2$.

To prove the main result of this section we will need the following lemmas

(6.3) LEMMA. *The power series* $R_0(t^q)R_1'(t^q) - R_0'(t^q)R_1(t^q)$ *and* $R_0'(t^q)R_2(t^q) - R_0(t^q)R_2'(t^q)$ *are the power series expansions at* P *of rational functions on* Z.

Proof. It is enough to prove that individually each of the power series $R_i(t^q)$ and $R_i'(t^q)$, $i = 0, 1, 2$, is the power series expansion at P of a rational function on Z.

Recall from Section 2 that $Q_0(t^q)$, $Q_1(t^q)$, $Q_2(t^q)$ are, respectively, the power series expansions at P of the rational functions $xy' - y$, $-y'$ and 1.

Now using the relation $D_t(f(t^q)) = f'(t^q)$ which was established in (3.9), we get that

$$R_0(t^q) = (Q_1 Q_2' - Q_1' Q_2)(t^q) = Q_1(t^q) D_t^q(Q_2(t^q)) - (D_t^q(Q_1(t^q)) Q_2(t^q),$$

which is the power series expansion at P of the rational function $D_x^q y'$.

Since $R_0'(t^q) = D_x^q R_0(t^q)$, then it is the power series expansion at P of the rational function $D_x^q D_x^q y' = 2D_x^{2q} y'$.

In a similar way we obtain that $R_1(t^q)$, $R_1'(t^q)$, $R_2(t^q)$, $R_2'(t^q)$ are the power series expansions at P of the rational functions $D_x^q(xy' - y)$, $2D_x^{2q}(xy' - y)$, $y'D_x^q(xy' - y) - (xy' - y)D_x^q y'$ and $D_x^q[y'D_x^q(xy' - y) - (xy' - y)D_x^q y']$.

(6.4) LEMMA. *The power series $(R_0R_1' - R_0'R_1)(t)$ is zero at a general point P of Z if, and only if, $2(b_{2q+1}b_q - b_{2q}b_{q+1}) = 0$ on Z.*

Proof. By a straightforward computation we find that the coefficient of t^0 of the power series $R_0R_1' - R_0'R_1$ is

$$2(b_{2q+1}b_q - b_{2q}b_{q+1}). \tag{6.5}$$

Now, the vanishing of the above power series implies the vanishing of (6.5).

Conversely, the vanishing of (6.5) at a general point P implies that the rational function of which $R_0R_1' - R_0'R_1$ is the power series expansion at P has value zero on Z, so it is the zero function and consequently the power series in zero at every general point P of Z.

(6.6) PROPOSITION. *Z' is not reflexive if, and only if, we have on Z*

$$2(b_{2q+1}b_q - b_{2q}b_{q+1}) = 0.$$

Proof. One direction, namely ($\Rightarrow$), was already proved in (6.4).

For the converse, note that the coefficient of t^0 in the power series $R_0R_2' - R_0'R_2$ is

$$2b_1(b_{q+1}b_{2q} - b_qb_{2q+1}).$$

Since by hypothesis this is zero at a general point of Z, it follows that the rational function of which the above power series, after replacing t by t^q, is the power series expansion at P, is identically zero. So at a general point P of Z, the above power series is zero.

Now since by (6.4) we have that $R_0R_1' - R_0'R_1 = 0$ and we proved that $R_0R_2' - R_0'R_2 = 0$, it follows that $R_1R_2' - R_1'R_2 = 0$. Hence $\overrightarrow{R(t)}$ and $\overrightarrow{R'(t)}$ are linearly dependent over $k((t))$, and therefore Z' is not reflexive.

(6.7) REMARKS:

(6.7.1) It is possible to prove the following result:

Let Z be such that $\pi_i'(Z) = q$. Then $\pi_i'(Z') \geqslant p^r$ if, and only if, we have for $i = 0, \ldots, r - 1$, on Z

$$(p^i + 1)(b_qb_{(p^i+1)q+1} - b_{q+1}b_{(p^i+1)q}) = 0.$$

(6.7.2) These results show that the set of points Z of $\mathbb{P}_k^N$, $N = n(n+3)/2$, representing curves such that $\pi_i'(Z') \geqslant p^r$ is a closed set.

(6.7.3.) If *chark* $= p > 2$ and q is a power of p, then the general curve of the family $P_{n,q}$ with $n = mq + 1$ and $m \geqslant 2$ has a reflexive dual.

To prove this it is enough to show that for each $m \geqslant 2$, there is a curve in the set $P_{n,q}$ such that $b_{2q+1}b_q - b_{2q}b_{q+1} \neq 0$.

In affine coordinates, let

$$y = xf(x^q) + g(x^q),$$

with $\deg f = m$ and $\deg g \leqslant m$. From (3.9) and the property (iv) of Hasse families, we get that

$$D_x^q y = xf'(x^q) + g'(x^q),$$

$$D_x^{q+1} y = f'(x^q),$$

$$D_x^{2q} y = (1/2)xf''(x^q) + (1/2)g''(x^q),$$

and

$$D_x^{2q+1} y = (1/2)f''(x^q).$$

In this case,

$$b_{2q+1}b_q - b_{2q}b_{q+1} = (1/2)g'(x^q)f''(x^q) - (1/2)g''(x^q)f'(x^q)$$

Choosing $g(x)$ of degree 1 and $f(x)$ of degree m such that $f''(x) \neq 0$, we get the curve we wanted.

(6.7.4) It follows from the above remark that a general curve Z of type $\Sigma_{i=0}^2 X_i P_i(X_0^q, X_1^q, X_2^q) = 0$, with $\deg P_i \geqslant 2$, is such that $(Z')' \neq Z$. This is because the general curve of the above type is such that Z' is reflexive, hence $(Z')'$ is reflexive, therefore different from Z which is not reflexive.

The only smooth non reflexive curves Z such that $(Z')' = Z$ we know, are curves of type $\Sigma_{i=0}^2 X_i L_i(X_0^q, X_1^q, X_2^q) = 0$ with $\deg L_i = 1$. Are there other such curves?

(6.7.5) If *chark* $= p > 2$, then the general member Z of the family F_n has a reflexive dual.

In fact, let $n = mp$, and let

$$Z: y - x^{p+1} + x^{2p} + x^{mp} = 0.$$

We have

$$D_x^p y D_x^{2p+1} y - D_x^{p+1} y D_x^{2p} y = 1 + [m(m-1)/2]x^{(m-2)p},$$

which is not identically zero on Z.

Here again, the general curve Z of F_n is such that $(Z')' \neq Z$.

7. Extremal curves

Our goal in this section is to characterize all extremal curvves, that is curves Z in any projective space, with $\deg \pi' = \deg Z$.

The problem is easily reduced by projection to plane curves, since:
a) A curve is strange if, and only if, a generic projection of it is strange.
b) π'_i, π'_s, $\deg \pi'$ and the degree of the curve are invariant by generic projections.

Let Z be a plane curve given in affine coordinates by the equation

$$Z: f(x, y) = 0,$$

where we assume that x is a separating transcendental. We saw ın Section 2 that the rational map φ is given in affine coordinates by $[-\beta; -\alpha; 1]$, where

$$\alpha = D_x^1 y = -f_X(x, y)/f_Y(x, y),$$

and

$$\beta = y - \alpha x.$$

So we have that

$$K(C(Z)) = K(Z) = k(x, y) = k(\alpha, \beta, x),$$

and that

$$K(Z') = k(\alpha, \beta).$$

Now, the equation of the curve and the relation $y = \alpha x + \beta$, determine a polynomial

$$\psi(T) = f(T, \alpha T + \beta) \in k(\alpha, \beta)[T], \tag{7.1}$$

such that $\psi(x) = 0$. So $\psi(T)$ is a multiple of the minimal polynomial of x over $K(Z')$. In a moment we are going to see that this polynomial, in the relevant cases, is not identically zero.

The following statements are clearly satisfied,

(7.2.1) Z is a strange curve if, and only if, 1, α, and β are linearly dependent over k.

(7.2.2) If α or β is constant, then Z is a strange curve.

(7.2.3) α and β are constant if, and only if, Z is a line.

(7.3) LEMMA. *Let $g(X, Y) = X^n h_n(Y) + X^{n-1} h_{n-1}(Y) + \cdots + h_0(Y)$ be a polynomial in $k[X, Y]$. If α and β are indeterminates, then the following identity holds:*

$$g(X, \alpha X + \beta) = \sum_{i \geq 0} \left(\sum_{j=0}^{i} D_Y^{i-j} h_j(\beta) \alpha^{i-j} \right) X^i. \tag{7.4}$$

Proof. The result follows by using the Taylor expansions:

$$h_j(\alpha X + \beta) = \sum_{m \geq 0} D_Y^m h_j(\beta)(\alpha X)^m.$$

(7.5) PROPOSITION. *Z is such that $\psi(T)$ is identically zero if, and only if, Z is a line.*

Proof. Write

$$f(X, Y) = X^n h_n(Y) + X^{n-1} h_{n-1}(Y) + \cdots + h_0(Y).$$

Since $f(X, Y)$ is irreducible, we must have $h_0(Y) \neq 0$. By (7.4) the vanishing of $\psi(T)$ implies the following equalities:

$$h_0(\beta) = 0,$$

$$D_Y^1 h_0(\beta) + h_1(\beta) = 0,$$

$$D_Y^2 h_0(\beta)\alpha^2 + D_Y^1 h_1(\beta)\alpha + h_2(\beta) = 0, \qquad \text{etc.}$$

The first equality implies that $\beta \in k$. Suppose that $\alpha \notin k$. Hence $h_i(\beta) = 0$, $i = 0, 1, \ldots, n$, so $Y - \beta$ divides $h_i(Y)$, $i = 0, \ldots, n$. This contradicts the fact that $f(X, Y)$ is irreducible. Therefore, $\alpha \in k$, so by (7.2.3) Z is a line.

The converse is trivially satisfied.

Since $f_X(x, y) + y'f_Y(x, y) = 0$, it follows that x is a root of the polynomial.

$$\phi(T) = f_X(T, \alpha T + \beta) + \alpha f_Y(T, \alpha T + \beta). \tag{7.6}$$

It is clear that $\psi'(T) = \phi(T)$. Our task now is to characterize those curves for which $\phi(T)$ is identically zero.

(7.7) LEMMA. *Let $g(X, Y) = \Sigma_{m=0}^{r} g_m(X, Y)$, where $g_m(X, Y)$, for $m = 0, 1, \ldots, r$, is a homogeneous polynomial of degree m in $k[X, Y]$. If α and β are indeterminates, then*

$$g(X, \alpha X + \beta) = \sum_{m=0}^{r} \left(\sum_{i=0}^{m} D_Y^{m-i} g_{r-i}(1, \alpha)\beta^{m-i} \right) X^{r-m}.$$

Proof. Let $g_m(X, Y) = \Sigma_{i=0}^{m} a_i X^{m-i} Y^i$. By the binomial expansion we get that the coefficient of X^{m-i} in $g_m(X, \alpha X + \beta)$ is

$$a_i\beta^i + \binom{i+1}{i} a_{i+1}\alpha\beta^i + \binom{i+2}{i} a_{i+2}\alpha^2\beta^i + \cdots + \binom{m}{i} a_m\alpha^{m-i}\beta^i$$
$$= D_Y^i g_m(1, \alpha)\beta^i.$$

From this the result follows easily.

Write

$$f(X, Y) = f_1(X^p, Y^p) + g_r(X, Y) + g_{r-1}(X, Y) + \cdots + g_0(X, Y),$$

where $g_i(X, Y)$, $i = 0, \ldots, r$, is a homogeneous polynomial of degree i. From (7.7) we have that

$$\psi(T) = f_1(T^p, \alpha^p T^p + \beta^p) + \sum_{m=0}^{r} \left(\sum_{i=0}^{m} D_Y^{m-i} g_{r-i}(1, \alpha)\beta^{m-i} \right) T^{r-m},$$

so,

$$\phi(T) = \psi'(T) = \sum_{m=0}^{r-1} \left((r - m) \sum_{i=0}^{m} D_Y^{m-i} g_{r-i}(1, \alpha)\beta^{m-i} \right) T^{r-m-1}. \tag{7.8}$$

(7.9) LEMMA. *Let* $f(X, Y) = f_1(X^p, Y^p) + g(X, Y) \in k[X, Y]$, *with* $g(X, Y) \neq 0$ *and no monomial of* $g(X, Y)$ *in* $k[X^p, Y^p] - \{0\}$. *Let* T *be an indeterminate,* α *a transcendental over* k *and* $\beta = (a\alpha + b)/(c\alpha + d)$ *with* $a, b, c, d \in k$ *and* $c \neq 0$. If $\phi(T) = f_X(T, \alpha T + \beta) + \alpha f_Y(T, \alpha T + \beta)$ *is the zero polynomial in* $k(\alpha)[T]$, *then* p *divides* $\deg(g(X, Y))$ *and* X *divides* $g(X, Y)$.

Proof. Write $g(X, Y) = g_r(X, Y) + \cdots + g_1(X, Y)$, where each $g_i(X, Y)$ is a homogeneous polynomial of degree i with no monomials in $k[X^p, Y^p] - \{0\}$, and $g_r(X, Y) \neq 0$.

The vanishing of $\phi(T)$ implies by (7.8) that for $j = 0, \ldots, r - 1$,

$$(r - j) \sum_{i=0}^{j} D^{j-i} g_{r-i}(1, \alpha)\beta^{j-i} = 0. \tag{7.10}$$

When $j = 0$, we get

$$r g_r(1, \alpha) = 0.$$

Since $g_r(X, Y) \neq 0$ and α is transcendental over k, it follows that $p \mid r$.

If $p \mid j$, then the coefficient of Y^{r-j} in $g_{r-j}(X, Y)$ is zero because no one of its monomials is in $k[X^p, Y^p] - \{0\}$ and $p \mid r$. This holds in particular for $j = 0$.

To conclude the proof we have to show that the coefficient of Y^{r-j} in $g_{r-j}(X, Y)$ is zero for $j = 1, \ldots, r - 1$. This will be proved by induction on j. Our inductive assumption is the following:

The coefficient of Y^{r-i} in $g_{r-i}(X, Y)$ is zero for $i = 0, \ldots, j - 1$.

If $p \mid j$, the result follows from the above remark.

Suppose that $p \nmid j$. From (7.10) we have that

$$\sum_{i=0}^{j} D_Y^{j-i} g_{r-i}(1, \alpha)\beta^{j-i} = 0. \tag{7.11}$$

From our inductive assumption, it follows that each $D_Y^{j-i} g_{r-i}(1, \alpha)$, for $i = 0, \ldots, j - 1$, is a polynomial in α of degree at most $r - j - 1$.

Now, replace β by $(a\alpha + b)/(c\alpha + d)$ in (7.11) and clear denominators. Since $c \neq 0$, we get a polynomial in α with leading coefficients c^j times the coefficient of Y^{r-j} in $g_{r-j}(X, Y)$, which is zero, hence the coefficient of Y^{r-j} in $g_{r-j}(X, Y)$ is zero.

(7.12) REMARK. The first part of the above proof shows that if α is not constant and $\deg \phi(T) < \deg Z - 1$, then $p \mid \deg Z$. Hence we have the following result:

If $p \nmid \deg Z$, then either $\deg \phi(T) = \deg Z - 1$, or Z is a strange curve with its center at infinity.

The following is the main result of this section.

(7.13) THEOREM. *Let $Z: f(x, y) = 0$ be an irreducible plane curve with x a separating transcendental over k. If the polynomial $\phi(T)$ of (7.6) is zero in $k(\alpha, \beta)[T]$, then Z is a strange curve.*

Proof. If either α or β belong to k, then by (7.2.2) Z is strange, and there is nothing to prove. So we may assume that α and β are transcendental over k.

Write

$$f(X, Y) = f_1(X^p, Y^p) + g_r(X, Y) + \cdots + g_1(X, Y),$$

where each $g_i(X, Y)$ is either zero or homogeneous of degree i with no monomials in $k[X^p, Y^p] - \{0\}$ and $g_r(X, Y) \neq 0$.

If $\phi(T) = 0$, then from the first part of the proof of (7.9) we have that $p \mid r$, and (7.10) holds.

From (7.10) for $j = 1$, we have that

$$D_Y^1 g_r(1, \alpha)\beta + g_{r-1}(1, \alpha) = 0.$$

We claim that $D_Y^1 g_r(1, \alpha) \neq 0$, because otherwise, writing

$$g_r(X, Y) = a_1 X^{r-1} Y + \cdots + a_{r-1} X Y^{r-1},$$

we would have

$$D_Y^1 g_r(1, \alpha) = a_1 + 2a_2\alpha + \cdots + (r-1)a_{r-1}\alpha^{r-2} = 0.$$

This implies that either, $a_i = 0$ whenever $p \nmid i$, this is not admissible since $g_r(X, Y) \notin k[X^p, Y^p]$; or α is algebraic over k, which we have already excluded.

So

$$\beta = -g_{r-1}(1, \alpha)/D_Y^1 g_r(1, \alpha) \in k(\alpha). \tag{7.14}$$

Now, write

$$f(X, Y) = f_1(X^p, Y^p) + X^s h_s(Y) + X^{s+1} h_{s+1}(Y) + \cdots,$$

where $s \geqslant 0$, $h_s(Y) \neq 0$ and each $h_i(Y)$ is such that no monomial of $X^i h_i(Y)$ is in $k[X^p, Y^p] - \{0\}$.

If $s = 0$, then by taking the derivative of (7.4) with respect to X, we have that the coefficient of T^0 in $\phi(T)$ is

$$D_Y^1 h_0(\beta)\alpha + h_1(\beta).$$

The vanishing of this coefficient implies that either

$$\alpha \in k(\beta),$$

or $D_Y^1 h_0(\beta) = 0$. This second possibility is ruled out, because otherwise, β being transcendental over k, we would have $D_Y^1 h_0(Y) = 0$ so $X^0 h_0(Y)$ would lie in $k[X^p, Y^p]$, and this has been excluded.

Suppose now that $s \geqslant 1$. The coefficient of T^{s-1} in $\phi(T)$ is, by (7.4), $sh_s(\beta)$.

Since $h_s(Y) \neq 0$ and β is transcendental over k, the vanishing of this coefficient implies that $p|s$.

Now, the coefficient of T^s in $\phi(T)$ is, again by (7.4),

$$D_Y^1 h_s(\beta)\alpha + h_{s+1}(\beta).$$

The vanishing of this coefficient implies that either

$$\alpha \in k(\beta),$$

or $D_Y^1 h_s(\beta) = 0$. This second possibility is ruled out for the same reason as above.

In conclusion we have that in any case, $p|s$ and $\alpha \in k(\beta)$.

This together with (7.14) yield

$$k(\alpha) = k(\beta).$$

Hence there exist $a, b, c, d \in k$ with $ad - bc \neq 0$ such that

$$\beta = (a\alpha + b)/(c\alpha + d).$$

We are now going to prove that $c = 0$; this in view of (7.2.1) will imply that Z is strange.

Suppose that $c \neq 0$. Then from (7.9) we show that X divides

$$g_r(X, Y) + \cdots + g_1(X, Y) = X^s h_s(Y) + X^{s+1} h_{s+1}(Y) + \cdots,$$

hence $s \geqslant 1$. Therefore,

$$f(X, Y) = f_1(X^p, Y^p) + x^s h(X, Y), \tag{7.15}$$

with $p|s$, $X \nmid h(X, Y)$ and no monomial of $h(X, Y)$ is in $k[X^p, Y^p] - \{0\}$.
Now, from (7.15) we have that

$$\phi(T) = T^s(h_X(T, \alpha T + \beta) + \alpha h_Y(Y, \alpha T + \beta)),$$

so,

$$h_X(T, \alpha T + \beta) + \alpha h_Y(T, \alpha T + \beta) = 0.$$

Applying Lemma (7.9) to $h(X, Y)$ in place of $f(X, Y)$, it follows that X divides $h(X, Y)$, contradiction. Therefore, $c = 0$.

(7.16) COROLLARY. *If Z is an extremal projective curve, then Z is strange.*
Proof. We have already observed that the question may be reduced to plane curves, so we assume that Z is a plane curve.
If $\deg \pi' = \deg Z$, then the minimal polynomial of $K(Z)$ over $K(Z')$ has degree equal to $\deg Z$ and since it divides $\phi(T)$, which is of degree less than or equal to $\deg Z - 1$, it follows that $\phi(T)$ is zero. The result now follows from (7.13).

In particular it follows that if, $\deg Z = p$ and Z is not reflexive, then Z is a strange curve. More generally if the curve has degree equal to π_i', then it is an extremal and therefore, it is strange. It is possible to derive from this Theorem 3.4 of Homma's paper [Ho], we will do this elsewhere since it requires some extra knowledge about strange curves.

(7.17) COROLLARY (Homma [Ho]). *Any smooth and extremal curve is a conic in characteristic two.*

(7.18) COROLLARY ([He-Kl, 1] and [Ho]). *Let Z be a smooth plane curve such that Z' is smooth.*
(i) *If Z is reflexive, then it is a conic.*
(ii) *If Z is not reflexive, then $\varphi: Z \to Z'$ is purely inseparable.*
Proof.
(i) If Z is reflexive, then $\varphi: Z \to Z'$ is birational, hence $Z \simeq Z'$, so $\deg Z = \deg Z'$. By Plücker's formula we have,

$$\deg Z = \deg Z' = \deg Z(\deg Z - 1).$$

This implies that $\deg Z = 2$.

(ii) Let Z be non reflexive. If Z is a conic, then *chark* $= 2$ and φ is purely inseparable.

Suppose that Z is not a conic. From Hurwitz's formula we have that

$$2g(Z) - 2 \geqslant \pi'_s(2g(Z') - 2),$$

hence

$$n(n - 3) \geqslant \pi'_s n'(n' - 3), \tag{7.19}$$

where $n = \deg Z$ and $n' = \deg Z'$.

Since no smooth curve of degree bigger than 2 is extremal, it follows from Plücker's formula that

$$n(n - 1) = \deg \varphi . n' \leqslant (n - 1)n',$$

so $n' \geqslant n$. From (7.19) it follows that if $n \neq 3$, then $\pi'_s = 1$.

Now, for $n = 3$, we have from the inequality $\deg \varphi \leqslant n - 1 = 2$, that φ must be purely inseparable.

(7.20) COROLLARY (Homma [Ho]). *Let Z be a nonreflexive smooth plane curve.*

(i) *Let* $\deg Z \geqslant 4$. *Then Z' is smooth if, and only if, Z is projectively equivalent to the curve $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$, where $q = \pi'_i$.*

(ii) *Let* $\deg Z \leqslant 3$. *Then chark $= 2$ and Z' is smooth.*

Proof. (i) Let $\deg Z \geqslant 4$. If Z' is smooth, then from (7.18), φ is purely inseparable, hence $g(Z) = g(Z')$, and therefore $\deg Z = \deg Z'$.

From Plücker's formula we have that

$$\deg \varphi . \deg Z' = \deg Z(\deg Z - 1),$$

hence

$$\pi'_i = \deg \varphi = \deg Z - 1,$$

so

$$\deg Z = \pi'_i + 1.$$

Now, from (5.10) and (5.16) it follows that Z' has an equation of the form

$$\sum_{i=0}^{2} X_i L_i(X_0^q, X_1^q, X_2^q) = 0,$$

where $q = \pi_i'$ and the L_i are linear homogeneous polynomials. By a straightforward generalization of a result of Pardini, [Pa] Proposition 3.7, such curves are all projectively equivalent to each other. From this, the result follows.

The converse of this statement follows from Example (2.3).

(ii) This was already proved in (4.9). We give here another proof.

If deg $Z \leqslant 3$, then deg $\varphi \leqslant 3$. From (7.17) deg $\varphi < 3$, hence deg $\varphi = 2$, so *chark* $= 2$ and φ are purely inseparable.

If deg $Z = 3$, then from Plücker's formula deg $Z' = 3$. Since $g(Z') = g(Z) = 1$, it follows that Z' is smooth.

If deg $Z = 2$, then Z' is a line, hence smooth.

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