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On the unitary dual of some classical Lie groups

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§1. Introduction

Let G be a connected real semisimple Lie group, with real Lie algebra $\mathscr{g}_0$ and complexified Lie algebra $\mathscr{g}$. In what follows we will denote a Lie group with roman upper case letters and a Lie algebra by script lower case letters and will use analogous notation to distinguish the real Lie algebra from its complexification. Let $K \subseteq G$ be a maximal compact subgroup and fix a Cartan involution θ so that $\mathscr{g}_0 = \mathscr{k}_0 \oplus \mathscr{p}_0$ is the Cartan decomposition of $\mathscr{g}_0$. Set

$$\hat{G}_u = \{\text{(equivalence classes of) unitary irreducible representations of } G\}.$$

An interesting problem in representation theory is the classification of $\hat{G}_u$. Although the set

$$\hat{G} = \{\text{(equivalence classes of) irreducible admissible representations of } G\}$$

has been parametrized by Langlands (1973) and Vogan (1979) independently, it is not clear yet which subsets of these sets of parameters will classify the unitary dual.

For example, fix a K-type $\mu \in \hat{K}$, Vogan's parametrization consists on attaching to μ (a) a certain parabolic subalgebra $\mathscr{q}_V \subseteq \mathscr{g}$, with quasisplit Levi subgroup L_V; (b) an $(L_V \cap K)$-type μ_V, which is fine, so that we have the following (see §3). There is a bijection $X_{L_V} \to X$, from irreducible $(\ell_V, L_V \cap K)$-modules with lowest $(L_V \cap K)$-type μ_V onto irreducible $(\mathscr{g}, K)$-modules with lowest K-type μ, such that

(a) X is the unique irreducible quotient of the Zuckerman module $\mathscr{R}_{\mathscr{q}_V}(X_{L_V})$,
(b) X_{L_V} is the Harish-Chandra module of a standard principal series representation of L_V.

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This classifies all irreducible Harish-Chandra modules of G in terms of the classification of Harish-Chandra modules of a quasisplit subgroup, which in turn are parametrized by ordinary parabolic induction.

As I said before, it is not clear which subset of parameters $(\mathfrak{q}_V, \mu_V)$ will determine the unitary representations. Part of the problem is that neither ordinary induction nor cohomological parabolic induction preserve unitarity unless we assume some hypothesis on the modules we are inducing from. However, many examples suggest that looking for a (possible different) parabolic subalgebra to attach to the lowest K-type of a unity representation could lead to the solution of the classification problem.

Some progress has been made in this direction. For $G = GL(n)$ Vogan (1986) gave a complete parametrization of $\hat{G}_u$ in terms of unitary almost spherical representations of certain Levi subgroup L of a parabolic subalgebra. These are representations $\sigma \in \hat{L}$ such that there is a unitary character $j \in \hat{L}$ with the property that $j^{-1} \otimes \sigma$ is a spherical representation. To a fixed K-type μ he assigns a parabolic $\mathfrak{q}$ containing $\mathfrak{q}_V$ and an $(L \cap K)$-type μ^L such that if X is unitary with lowest K-type μ, then there is a unitary almost spherical representation Y of L with lowest $(L \cap K)$-type μ^L and such that X is a subquotient of a module $\mathscr{I}Y$, obtained from Y by composition of ordinary and cohomological induction. Also, any unitary almost spherical representation is in turn obtained by ordinary induction from unitary characters and Stein complementary series representations.

For complex classical groups Barbasch (1987) gives a similar parametrization in terms of representations containing a fundamental K-type. These representations are either unipotent, complementary series or edges of complementary series.

For real groups the answer is not clear yet. Let us simplify the problem. Suppose X is a unitary representation of a real reductive Lie group with integral infinitesimal character γ. Barbasch–Vogan (1985) (def. 1.17, 5.23) gave a definition of special unipotent representations with integral infinitesimal character for complex groups. The same definition applies for real groups. Vogan conjectured, following some ideas of Arthur (1984), that X can be obtained by cohomological parabolic induction from a special unipotent representation of a subgroup. If we further assume that γ is regular, then necessarily the special unipotent representations involved are one dimensional and the above conjecture becomes.

CONJECTURE 1.1. Suppose X is an irreducible unitary Harish-Chandra module such that γ is regular integral. Then there are a θ-stable parabolic subalgebra $\mathfrak{q}$ and a unitary one-dimensional character λ of the Levi subgroup L of $\mathfrak{q}$

such that

$$X \cong \mathscr{R}_{\mathfrak{q}}(\mathbb{C}_\lambda) \;=\; A_{\mathfrak{q}}(\lambda).$$

This is an extension of Zuckerman's conjecture which says that the modules $A_{\mathfrak{q}}(\lambda)$ exhaust the set of unitary representations with non-vanishing relative Lie algebra cohomology and which was proved in Vogan–Zuckerman (1984).

Conjecture 1.1 was proved by Enright (1979) in the case when G is complex, by Speh (1981) when G is $SL(n, \mathbb{R})$ and by Baldoni Silva–Barbasch (1983) in the real rank one case.

In this paper we give a proof of this conjecture when G is $SL(n, \mathbb{R})$, $Sp(n, \mathbb{R})$ and $SU(p, q)$. The proof for $SL(n, \mathbb{R})$ is new and different from Speh's original proof and we will need it for the general case.

While this paper was being considered for publication, the author studied the case $G = SO(p, q)$. This case is analogous to the one of $G = Sp(n, \mathbb{R})$ and is done following the algorithm suggested by the proof of this result in this last case. See the comment at the end of this introduction.

The result that we prove is

THEOREM 1.2. *Let $G = SL(n, \mathbb{R})$, $SU(p, q)$ or $Sp(n, \mathbb{R})$. Suppose X is an irreducible Harish-Chandra module with regular integral infinitesimal character and equipped with a non-zero Hermitian form $\langle\ ,\ \rangle$. Then, either*

(a) *$X \cong A_{\mathfrak{q}}(\lambda)$ for some $\mathfrak{q}$ and λ as above;*

or

(b) *X is not unitary. More precisely, there are a lowest K-type V_{δ_1} and a K-type $V_{\delta_2} \subseteq V_{\delta_1} \otimes \mathfrak{p}$, such that*

$$\mathrm{Hom}_K(V_{\delta_i}, X) \neq 0, \qquad i = 1, 2$$

and the restriction of $\langle\ ,\ \rangle$ to the sum $V_{\delta_1} \oplus V_{\delta_2}$ is indefinite. □

The proof is by induction on the dimension of G. Assuming that X cannot be realized as an $A_{\mathfrak{q}}(\lambda)$ module and with the help of Vogan's embedding result we find an appropriate subgroup $L \subset G$ and exhibit X as a Langlands submodule of some derived functor module induced from an $(\ell, L \cap K)$-module X_L, proving non unitarity for X_L and reducing the problem to L. The reduction step is made precise in §5. The main result there is Theorem 5.7, and §§7–9 are devoted to a case-by-case proof of this. In §5 we use Theorem 5.7 to prove Theorem 1.2 for G.

§§2–4 are devoted to notation and the results that will be needed for the proof. §3 deals with Vogan's classification of Harish-Chandra modules. In §4 we define our modules $A_{\mathfrak{q}}(\lambda)$ in question and give some properties needed later. §6 gives some techniques to detect non-unitarity used to prove 1.2b).

The methods of this paper should extend in several directions. If $\langle \gamma, \alpha \rangle \geqslant 1$ for all simple roots α then, the same methods should be applicable to give the same conclusion as in Conjecture 1.1. Likewise, if γ is integral, Vogan's conjecture mentioned above should also be proved this way.

Also, the proof in the case of $Sl(n, \mathbb{R})$ and $Sp(n, \mathbb{R})$ suggest an algorithm for all simple real groups. Namely, a reduction to a special case of a proper subgroup of the same type in Cartan's classification and a real form of $GL(m, \mathbb{C})$.

This paper contains most of the author's doctoral thesis, completed at M.I.T. in 1986. She wishes to thank her advisor David Vogan for much invaluable advice, as well as Dan Barbasch and Jeff Adams for helpful discussions and suggestions.

§2. In this section we set up notation. For undefined terms in this section see, for example, Vogan (1981) Chapter 0.

Let G, $\mathfrak{g}_0$, $\mathfrak{g}$, K and θ as in §1. Let $U(\mathfrak{g})$ = universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ = center of $U(\mathfrak{g})$.

Although we will eventually study connected real simple linear Lie groups, we will consider connected real reductive linear Lie groups. These are Lie groups satisfying:

(a) G is connected

(b) $\mathfrak{g}_0$ is a real reductive Lie algebra

(c) G has a faithful finite dimensional representation.

Fix once and for all a nondegenerate, invariant symmetric bilinear form on $\mathfrak{g}_0$. We will denote this form and its various complexifications, restrictions and dualizations by $\langle\ ,\ \rangle$. We may choose it so that the Cartan decomposition of $\mathfrak{g}_0$ is orthogonal and

$$\langle\ ,\ \rangle|\mathfrak{p}_0 > 0$$

$$\langle\ ,\ \rangle|\mathfrak{k}_0 < 0.$$

Let H be a Cartan subgroup of G. Denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the roots of $\mathfrak{h}$ in $\mathfrak{g}$.

In general if $\mathfrak{s}$ is an abelian reductive Lie subalgebra of $\mathfrak{g}$ and V is an $ad(\mathfrak{s})$-stable subspace of $\mathfrak{g}$ then $\Delta(V, \mathfrak{s})$ is the set of weights of $\mathfrak{s}$ in V (with multiplicities). For any $B \subset \Delta(V, \mathfrak{s})$ let $\varrho(B) = \frac{1}{2}\sum_{\alpha \in B} \alpha$. When there is no confusion we will use $\Delta(V)$ for $\Delta(V, \mathfrak{s})$.

If H is a θ-stable Cartan subgroup, then

$$H = TA; \quad \text{with } T = H \cap K, \quad A = H \cap (\exp \mathfrak{p}_0) = \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0)$$

and $\Delta(\mathfrak{g}, \mathfrak{h})$ is θ-stable.

Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ and

$$W(G, H) = N_G(H)/H \cong N_K(H)/H \cap K \subseteq W.$$

Let $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ be a set of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, the corresponding Borel subalgebra and $\varrho = \varrho_{\mathfrak{g}} = \varrho(\mathfrak{n})$.

Let $\mathfrak{t}_0^c \subseteq \mathfrak{k}_0^c$ be a Cartan subalgebra. Define $\mathfrak{h}^c$ (resp. H^c) to be the centralizer in $\mathfrak{g}$ (resp. G) of $\mathfrak{t}_0^c$. H^c is θ-stable, so we can write

$$H^c = T^c A^c, \quad \text{with } T^c = H^c \cap K$$

a Cartan subgroup of K.

H^c is called the fundamental or maximally compact Cartan subgroup of G.

On the other extreme, if $\mathfrak{a}_0^s \subseteq \mathfrak{p}_0$ is a maximal abelian subalgebra and $\mathfrak{h}_0^s = \mathfrak{t}_0^s + \mathfrak{a}_0^s$ is maximal abelian then $\mathfrak{h}_0^s$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Its centralizer H^s in G is a Cartan subgroup of G, the maximally split one.

Let $(\pi, \mathscr{H})$ be a continuous complex Hilbert space representation and $\mathscr{H}_K$ the subset of $\mathscr{H}$ of K-finite vectors. $\mathscr{H}_K$ is a $(\mathfrak{g}, K)$ module. We call $\mathscr{H}_K$ the Harish-Chandra module of $(\pi, \mathscr{H})$ [cfr. Harish-Chandra (1953)]. Denote by $\mathscr{M}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$-modules.

DEFINITION 2.1. Fix a Cartan subalgebra $\mathfrak{t}_0^c \subseteq \mathfrak{k}_0$ and $x \in i(\mathfrak{t}_0^c)^*$.

We define a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(x) = \mathfrak{l}(x) + \mathfrak{u}(x)$ as follows.

Let

$$\Delta(\mathfrak{l}) = \Delta(\mathfrak{l}, \mathfrak{t}^c) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c) \mid \langle \alpha, x \rangle = 0\}$$

$$\Delta(\mathfrak{u}) = \Delta(\mathfrak{u}, \mathfrak{t}^c) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c) \mid \langle \alpha, x \rangle > 0\}$$

$$\mathfrak{l} = \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathbb{C}X_\alpha + \mathfrak{t}^c,$$

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathbb{C}X_\alpha,$$

Then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is θ-stable and

$$\theta\mathfrak{l} = \mathfrak{l}, \qquad \theta\mathfrak{u} = \mathfrak{u}$$

and $\bar{\mathfrak{l}} = \mathfrak{l}$, with $^{-}$ denoting complex conjugation, $\bar{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}$.

Let L be the normalizer of $\mathfrak{q}$ in G. We call L the Levi subgroup of $\mathfrak{q}$.

§3. In this section we consider the classification of Harish-Chandra modules which consists of exhibiting each irreducible $(\mathfrak{g}, K)$-module as a submodule of a derived functor module.

We will first consider a particular set of irreducible $(\mathfrak{g}, K)$ modules when G is quasisplit.

Let $\mathfrak{a}_0^s \subseteq \mathfrak{p}_0$ be a maximal abelian subalgebra and A^s the corresponding connected subgroup of G. Let $M = K^{A^s}$ and $P^s = MA^sN \subseteq G$, a parabolic subgroup.

Definition 3.1. For a fixed representation (δ, V_δ) of M and $\nu \in \hat{A}^s$, set

$$I^G(\delta \otimes \nu) = \operatorname{Ind}_{P^s}^G(\delta \otimes \nu),$$

the (normalized) induced representation of G.

Let $X^G(\delta \otimes \nu)$ be the Harish-Chandra module of $I^G(\delta \otimes \nu)$.

Now suppose G is any reductive real Lie group. Let X be a $(\mathfrak{g}, K)$ module and μ the highest weight of a lowest K-type of X, Vogan (1981) attaches to μ a set of discrete θ-stable $(\mathfrak{q}_V, H, \delta_V)$, where the Levi subgroup L_V of $\mathfrak{q}_V$ is quasisplit, $H = MA^s$ is a maximally split Cartan subgroup of L_V and $\delta_V \in \hat{M}$. Write $\lambda_V^G(X) = \lambda_V^G(\mu) \in i\mathfrak{t}_0^*$ for the weight attached to μ and used to construct $\mathfrak{q}_V$.

Definition 3.2. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic subalgebra and $L \subseteq G$ its Levi subgroup. Recall from Vogan (1981), (Def. 6.3.1) the cohomological parabolic induction functors (from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules)

$$\mathscr{R}^i_{\mathfrak{q}} = \mathscr{R}^i{}^{(\mathfrak{g}, K)}_{(\mathfrak{q}, L \cap K)} = \Gamma^i{}^{(\mathfrak{g}, K)}_{(\mathfrak{g}, L \cap K)} \circ \operatorname{pro}^{(\mathfrak{g}, L \cap K)}_{(\mathfrak{q}, L \cap K)} (* \otimes \Lambda^{\mathrm{top}}\mathfrak{u})$$

Here is the result of the classification that we are going to use.

PROPOSITION 3.3 (*Vogan* (1981), 6.5.9 (g) *and* 6.5.12 (b)). *Suppose X is an irreducible $(\mathfrak{g}, K)$ module and $(\mathfrak{q}_V, H, \delta_V)$ a set of discrete θ-stable data attached to X. Then there is a character $\nu_V \in \hat{A}^s$ such that, for $S =$* dim $\mathfrak{u} \cap \mathfrak{k}$, *the space*

$$\mathrm{Hom}_{\mathfrak{g},K}(X, \mathscr{R}^S_{\mathfrak{q}_V}(X^{L_V}(\delta_V \otimes \nu_V))$$

is one dimensional.

The 4-tuple $(\mathfrak{q}_V, H, \delta_V, \nu_V)$ is called a set of θ-stable data.

The following is a very technical result which will be needed in the proof of Theorem 5.7.

PROPOSITION 3.4. *Let $\mathfrak{q} = \ell + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic subalgebra and Y an $(\ell, L \cap K)$ module. Write $S =$* dim $\mathfrak{u} \cap \mathfrak{k}$, *$\lambda^L_V = \lambda_V(Y)$, $X = \mathscr{R}^s_{\mathfrak{q}}(Y)$ and $\lambda^G_V = \lambda^G_V(X)$. Assume $\langle \lambda^L_V + \varrho(\mathfrak{u}), \alpha \rangle > 0$; $\alpha \in \Delta(\mathfrak{u})$. Choose $\Delta^+(\mathfrak{k}) = \Delta^+(\ell \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$.*

Suppose μ^L is the highest weight of a lowest K-type for (for $L \cap K$) of Y with respect to the positive system $\Delta^+(\ell \cap \mathfrak{k}, \mathfrak{t}^c)$ and that we choose $\Delta^+(\ell)$ so that $\mu^L + 2\varrho_{\ell \cap \mathfrak{k}}$ is dominant. Let $\mu = \mu^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$.

(a) *If μ is dominant for $\Delta^+(\mathfrak{g})$, then $\lambda^G_V(\mu) = \lambda^L_V + \varrho(\mathfrak{u})$.*

(b) *Let η be the highest weight of a K-type of X. Then*

$$\langle \lambda^G_V(\eta), \lambda^G_V(\eta) \rangle \geqslant \langle \lambda^G_V, \lambda^G_V \rangle.$$

(c) *If equality holds in* (b) *then $\eta = \eta^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ for a highest weight η^L of a lowest $(L \cap K)$-type of Y and V_η is a lowest K-type of X.*

(d) *Conversely, if $\eta = \eta^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$, then V_η is a lowerst K-type of X and equality holds.*

Proof. See Vogan (1981), (a) is similar to 6.5.4 and (b)–(c) is 6.5.9.

§4. The modules $A_{\mathfrak{q}}(\lambda)$

In this section we give a construction of these modules and some properties that we will use later on.

Let G be a connected real reductive linear Lie group, $\mathfrak{q} = \ell + \mathfrak{u} \subseteq \mathfrak{g}$ a θ-stable parabolic subalgebra and L the normalizer of $\mathfrak{q}$ in G. Then $\ell_0 = \mathrm{Lie}(L)$.

Let $\lambda\colon \ell \to \mathbb{C}$ be a one-dimensional representation. Assume that

$$\begin{cases} \text{(a)} & \lambda \text{ is the differential of a unity character of } L \text{ (call it } \lambda \text{ also).} \\ \text{(b)} & \langle \lambda|_{t^c}, \alpha\rangle \geqslant 0 \text{ for all } \alpha \in \Delta(u, t^c). \end{cases} \tag{4.1}$$

We say that λ is an admissible representation of ℓ.

DEFINITION 4.2. With notation as above, we define the Harish-Chandra module $A_q(\lambda)$ by

$$A_q(\lambda) = \mathscr{R}_q^S(\mathbb{C}_\lambda) \quad \text{(Definition 3.2)}$$

with

$$S = \dim u \cap k.$$

Fix positive root systems

$$\Delta^+(\ell \cap k)$$

and

$$\Delta^+(\ell) = \Delta^+(\ell, t), \quad \text{compatible with } \Delta^+(\ell \cap k).$$

Then

$$\Delta^+(k) = \Delta^+(\ell \cap k) \cup \Delta(u \cap k)$$

and

$$\Delta^+(g) = \Delta^+(q) = \Delta^+(\ell) \cup \Delta(u)$$

are positive t-root systems for k and g, respectively. Choose a fundamental Cartan subalgebra $h^c = t^c + a^c$ and a positive root system $\Delta^+(g, h^c)$ so that

$$\Delta^+(g, h)|_{t^c} = \Delta^+(g).$$

Then

$$\varrho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+(g, h^c)} \alpha = \tfrac{1}{2} \sum_{\beta \in \Delta^+(g)} \beta.$$

PROPOSITION 4.3 [Vogan–Zuckerman (1984)], see also Adams (1983) and (1987), Speh–Vogan (1980) and Vogan (1981). *Regard* $\lambda|_{t^c}$ *as a weight in* $(t^c)^*$. *Let*

$$\mu = \lambda|_{t^c} + 2\varrho(\mathfrak{u} \cap \mathfrak{p}) \in (t^c)^*.$$

(a) *The* $(\mathfrak{g}, K)$ *module* $A_{\mathfrak{q}}(\lambda)$ *is the unique irreducible module satisfying:*

(i) *As a K-representation,* $A_{\mathfrak{q}}(\lambda)$ *contains the K-type with highest weight* μ.

(ii) $Z(\mathfrak{g})$ *acts on* $A_{\mathfrak{q}}(\lambda)$ *by the character* $\chi_{\lambda+\varrho}: Z(\mathfrak{g}) \to \mathbb{C}$; *where* $\chi_{\lambda+\varrho}(z) = (\lambda + \varrho)(\zeta(z))$ *and* ζ *is the Harish-Chandra homomorphism.*

(iii) *Any K-type occurring in* $A_{\mathfrak{q}}(\lambda)$ *has a highest weight of the form*

$$\eta = \lambda|_{t^c} + 2\varrho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\substack{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}) \\ n_\beta \in \mathbb{N}}} n_\beta \beta.$$

(b) *Moreover,* μ *is the unique lowest K-type of* $A_{\mathfrak{q}}(\lambda)$.

By 4.1 and Theorem 1.3 in Vogan (1984), we have the following

PROPOSITION 4.4. *In the above setting, the modules* $A_{\mathfrak{q}}(\lambda)$ *are unitarizable.*

PROPOSITION 4.5 (Vogan). (Unpublished). *Fix* $\Delta^+(\mathfrak{k})$. *Let* $\mathfrak{q}_i = \ell_i + \mathfrak{u}_i \subseteq \mathfrak{g}$; $i = 1, 2$ be θ-*stable parabolic subalgebras such that* $\Delta(\mathfrak{q}_i) \supseteq \Delta^+(\mathfrak{k})$ *and* λ_i, *admissible one-dimensional representation of* ℓ_i (Definition 4.1). *Then*

$$A_{\mathfrak{q}_1}(\lambda_1) \cong A_{\mathfrak{q}_2}(\lambda_2).$$

$$\Leftrightarrow \quad \lambda_1 = \lambda_2 \text{ and } \mathfrak{u}_1 \cap \mathfrak{p} = \mathfrak{u}_2 \cap \mathfrak{p}.$$

The proof will follow from two lemmas included here for future reference.

LEMMA 4.6. *Suppose* $\tilde{\mathfrak{q}} = \tilde{\ell} + \tilde{\mathfrak{u}}$, $\mathfrak{q} = \ell + \mathfrak{u} \subseteq \mathfrak{g}$, *are* θ-*stable and* $\lambda: \ell \to \mathbb{C}$; $\tilde{\lambda}: \tilde{\ell} \to \mathbb{C}$, *admissible characters such that*

$$\begin{aligned} &\text{(a) } \tilde{\mathfrak{q}} \supseteq \mathfrak{q}, \text{ that is, } \tilde{\ell} \supseteq \ell \text{ and } \mathfrak{u} \supseteq \tilde{\mathfrak{u}}, \\ &\text{(b) } \lambda \perp \Delta(\tilde{\ell}) \\ &\text{(c) } \mathfrak{u} \cap \mathfrak{p} = \tilde{\mathfrak{u}} \cap \mathfrak{p}. \end{aligned} \tag{4.7}$$

Then $A_{\tilde{\mathfrak{q}}}(\tilde{\lambda}) \cong A_{\mathfrak{q}}(\lambda)$.

Proof. By induction by stages

$$\mathscr{R}_{q}^{S}(\mathbb{C}_\lambda) \cong \mathscr{R}_{\tilde{q}}^{\tilde{S}}(\mathscr{R}_{q \cap \tilde{\ell}}^{\dim \tilde{\ell} \cap (u \cap k)}(\mathbb{C}_\lambda))$$

but $q \cap \tilde{\ell} = \ell + u \cap \tilde{\ell}$ and by (c), $u \cap \tilde{\ell} \subseteq k$, so

$$\mathscr{R}_{q \cap \tilde{\ell}}^{\dim u \cap \tilde{\ell}}(\mathbb{C}_\lambda) \cong \mathbb{C}_\lambda.$$

Hence

$$\mathscr{R}_{q}^{S}(\mathbb{C}_\lambda) \cong \mathscr{R}_{\tilde{q}}^{\tilde{S}}(\mathbb{C}_\lambda) = A_{\tilde{q}}(\lambda).$$

This proves the lemma. Q.E.D.

By this lemma, we may assume that both q_i's in Proposition 4.5 are maximal with respect to conditions (a)–(c).

LEMMA 4.8. *In the above setting*

$$\Delta(\ell_i \cap k) = \{\alpha \in \Delta(g) | \langle \alpha, \lambda_i + 2\varrho(u_i \cap p)\rangle = 0\}.$$

Proof. Suppose $\alpha \in \Delta^+(k, t^c)$ is a simple root so that

(a) $\alpha \notin \Delta(\ell_i \cap k)$.

(b) $\langle \alpha, \mu_i \rangle = 0, \quad \mu_i = \lambda_i + 2\varrho(u_i \cap p)$.

Let

$$\Delta(\bar{\ell}) = \mathrm{Span}(\Delta(\ell_i), \alpha) \cap \Delta(g)$$

$$\Delta(\bar{u}) = \Delta(u_i) \backslash \Delta(\bar{\ell})$$

$$\bar{q} = \bar{\ell} + \bar{u}.$$

We want to contradict the maximality of q_i.
Breaking up $\Delta(u_i \cap p)$ in maximal α strings

$$\{\gamma_0; \gamma_0 + \alpha: \ldots \gamma_0 + r\alpha\},$$

(i.e., $\gamma_0 - \alpha, \gamma_0 + (r + 1)\alpha \notin \Delta(u_i \cap p)$)

and using representation theory of $\mathfrak{sl}(2)$ we can conclude

$$\langle \alpha, 2\varrho(\mathfrak{u}_i \cap \mathfrak{p})\rangle \geqslant 0$$

and we have equality if and only if $\mathfrak{u}_i \cap \mathfrak{p}$ is invariant under the three dimensional subalgebra $\mathfrak{g}^\alpha$ that contains the α-root vector X_α.

But, by definition of λ_i, $\langle \alpha, \lambda_i\rangle \geqslant 0$. So, (a) and (b) imply that $\mathfrak{u}_i \cap \mathfrak{p}$ is invariant under $\mathfrak{g}^\alpha$ and

$$\langle \alpha, \lambda_i\rangle = 0 = \langle \alpha, 2\varrho(\mathfrak{u}_i \cap \mathfrak{p})\rangle.$$

Now we want to prove that

$$\bar{\mathfrak{u}} \cap \mathfrak{p} = \mathfrak{u}_i \cap \mathfrak{p}$$

If $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ and $\beta|_{t^c} = \alpha$ then

$$s_\alpha(\beta|_{t^c}) = -\beta|_{t^c}.$$

If β is complex, then the non-compact root of $-\beta|_{t^c}$ is not in $\Delta(\mathfrak{u}_i \cap \mathfrak{p})$ so it contradicts invariance under $\mathfrak{g}^\alpha$.

Hence α is an imaginary root of $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$. α is also simple for $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$. In fact, since α is simple for $\Delta(\mathfrak{k}, t^c)$, and $\alpha \notin \Delta(\ell_i \cap \mathfrak{k})$ we can assume that if $\gamma, \delta \in \Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ and $\alpha = \gamma + \delta$ then

$$\gamma \in \Delta(\mathfrak{u}_i \cap \mathfrak{p}),$$

say, and $\gamma - \alpha = -\delta \in \Delta(\mathfrak{u}_i \cap \mathfrak{p})$; contradicting invariance again.

Consider a simple factor $\ell_0 \subseteq \bar{\ell}$, not contained in ℓ. Then ℓ_0 is not orthogonal to α. Let $\{\beta_1, \beta_2, \ldots, \beta_l\}$ be a set of simple roots for ℓ_0 containing α.

Say $\alpha = \beta_{i_0}$ and β_{i_0+1} is adjacent to α. Suppose $\ell_0 \cap \mathfrak{p} \neq 0$. Then there is a non-compact root $\beta = \Sigma\, n_i\beta_i$ with some $n_{i_0+1} > 0$ and such that

$$\langle \alpha, \beta\rangle = \sum n_i \langle \alpha, \beta_i\rangle < 0.$$

$\alpha + \beta = \delta$ is a non-compact root, and $\delta \in \Delta(\mathfrak{u}_i \cap \mathfrak{p})$. So the string through δ is not complete.

Hence, ℓ_0 is compact and $\mathfrak{q}_i(\subseteq \bar{\mathfrak{q}})$ is not maximal satisfying (4.7).

This proves Lemma 4.8. Q.E.D.

We are now able to prove Proposition 4.5. By Lemma 4.8

$$\ell_1 \cap k = \ell_2 \cap k$$

$$u_1 \cap k = u_2 \cap k$$

Hence $\lambda_1 + 2\varrho(u_1) = \lambda_2 + 2\varrho(u_2)$. But $\langle \lambda_i, \beta \rangle = \langle 2\varrho(u_i), \beta \rangle = 0$ for all $\beta \in \Delta(\ell_i)$ and $\langle 2\varrho(u_i), \alpha \rangle > 0$, $\langle \lambda_i, \alpha \rangle \geqslant 0$, $\alpha \in \Delta(u_i)$. So

$$\Delta(\ell_i) = \{\beta \in \Delta(g, t^c) | \langle \lambda_i + 2\varrho(u_i), \beta \rangle = 0\}$$

$$\Delta(u_i) = \{\alpha \in \Delta(g, t^c) | \langle \lambda_i + 2\varrho(u_i), \alpha \rangle > 0\}.$$

Hence

$$u_1 \cap p = u_2 \cap p$$

and

$$\lambda_1 = \lambda_2.$$

This proves Proposition 4.5. Q.E.D.

§5. Reduction step for the proof of Theorem 1.2

We are now in a position to prove the main result stated in Chapter 1. We will argue by contradiction and reduction to a proper subgroup $L \subseteq G$.

Suppose $X \in \mathcal{M}(g, K)$ is irreducible and has a Hermitian form $\langle\ ,\ \rangle$. We will assume X cannot be realized as an $A_q(\lambda)$ module, but will exhibit X as a Langlands submodule of some derived functor module induced from an $(\ell, L \cap K)$-module X_L, proving non unitarity for X_L and making sure that this information can be carried over to G and X.

We need to keep track of the existence of Hermitian forms at different steps of induction as well as of their signatures on some finite sets of K-types.

Recall from Vogan (1984) (Definition 2.10) the Hermitian dual of a (g, K) module Y

$$Y^h = \{f: Y \to \mathbb{C} | \dim U(k) \cdot f < \infty; f(\lambda x) = \bar{\lambda} f(x), \lambda \in \mathbb{C}, x \in Y\}.$$

Y^h is a (g, K) module.

It is clear that invariant Hermitian forms on Y are given by $(\mathfrak{g}, K)$ maps $f\colon Y \to Y^h$ such that $f = f^h\colon Y^h \to Y$. Moreover we have

PROPOSITION 5.1. *Suppose $X \in \mathcal{M}(\mathfrak{g}, K)$ is irreducible. Then X admits a non-zero invariant Hermitian form if and only if*

$$X \cong X^h.$$

In this case the Hermitian form is non-degenerate and any two such forms differ by multiplication by a real constant.

PROPOSITION 5.2. *Let $X \in \mathcal{M}(\mathfrak{g}, K)$ be irreducible and $(\mathfrak{q}_V, H, \delta_V, \nu_V)$ a set of θ-stable data attached to X, so that*

$$\dim[\mathrm{Hom}_{\mathfrak{g},K}(X, \mathcal{R}^S_{\mathfrak{q}_V}(X^{L_V}(\delta_V \otimes \nu_V))] = 1$$

(*see* Proposition 3.3). *Let $H = TA$. Then $X \cong X^h$ if and only if there is an element*

$$w \in W(L, A) \quad \text{such that} \quad w\delta_V = \delta_V \quad \text{and} \quad w\nu_V = -\bar{\nu}_V.$$

In this case we get a Hermitian form on X from a form on

$$\mathcal{R}^S_{\mathfrak{q}_V}(X^{L_V}(\delta_V \otimes \nu_V)).$$

This result is essentially due to Knapp and Zuckerman (1976). A formulation close to this one is in Vogan (1984), Corollary 2.15.

COROLLARY 5.3. *Let $X \in \mathcal{M}(\mathfrak{g}, K)$ irreducible, endowed with a non-zero Hermitian form $\langle\ ,\ \rangle$. Write $\mathfrak{q}_V = \mathfrak{q}_V(X)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra such that $\mathfrak{l} \supset \mathfrak{l}_V$, $\mathfrak{u} \subset \mathfrak{u}_V$ and $(\mathfrak{q}_V, H, \delta_V, \nu_V)$, a θ-stable data attached to X. Write*

$$X_L = \mathcal{R}^{\dim \mathfrak{l} \cap \mathfrak{u}_V \cap \mathfrak{k}}_{\mathfrak{q}_V \cap \mathfrak{l}}(X^{L_V}(\delta_V \otimes \nu_V)).$$

Then X^h_L has a Hermitian form $\langle\ ,\ \rangle^L$.

This is a formal consequence of Proposition 5.2.

PROPOSITION 5.4 (Vogan). *Fix $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$, a θ-stable parabolic subalgebra. Suppose $Y \in \mathcal{M}(\mathfrak{l}, L \cap K)$ is equipped with a (possibly degenerate) invariant Hermitian form $\langle\ ,\ \rangle^L$.*

Then there is a natural invariant Hermitian form $\langle\ ,\ \rangle^G$ on

$$[\mathcal{R}_{\mathfrak{q}}^S(Y^h)]^h.$$

Proof. Recall from Vogan (1981) Chapter 6, Definition 6.1.5 the functors

$$\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}: \mathcal{M}(\mathfrak{l}, L \cap K) \to \mathcal{M}(\mathfrak{g}, L \cap K)$$

$$\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Y = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} Y.$$

$$\mathcal{L}^j: \mathcal{M}(\mathfrak{l}, L \cap K) \to \mathcal{M}(\mathfrak{g}, K)$$

$$\mathcal{L}_{\bar{\mathfrak{q}}}^j Y = \mathcal{L}^j Y = \Gamma^j \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}),$$

where $\Gamma^j: \mathcal{M}(\mathfrak{g}, L \cap K) \to \mathcal{M}(\mathfrak{g}, K)$ are the Zuckerman functors (see Vogan (1981) Ch. 6). Set $\tilde{Y} = Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}$. By hypothesis, we have a map

$$\phi^L: Y \to Y^h.$$

This induces maps

$$\phi^{\mathfrak{g}}: \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\tilde{Y}) \to \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\tilde{Y}^h)$$

$$\phi^G: \mathcal{L}_{\bar{\mathfrak{q}}}^S Y \to \mathcal{R}_{\mathfrak{q}}^S Y^h.$$

By Theorem 5.3 [Enright–Wallach (1980)] in Vogan (1984) [see also Duflo–Vergne (1987), Knapp–Vogan (*) and D. Wigner (1987)]

$$\mathcal{R}_{\mathfrak{q}}^{2s-i}(Y^h) \cong (\mathcal{L}_{\bar{\mathfrak{q}}}^i Y)^h.$$

Let $\langle\ ,\ \rangle: \mathcal{L}_{\bar{\mathfrak{q}}}^s Y \times (\mathcal{L}_{\bar{\mathfrak{q}}}^s Y)^h \to \mathbb{C}$ be the natural pairing (see Vogan (1984), Def. 2.10). Define

$$\langle u, v\rangle^G = \langle u, \phi^G v\rangle.$$

This gives an invariant Hermitian form on $\mathcal{L}^s(Y)$ [cfr. the proof of Corollary 5.5, Vogan (1984)]. Q.E.D.

DEFINITION 5.5. If $Z \in \mathcal{M}(\mathfrak{g}, K)$ and $\delta \in \hat{K}$, write

$$Z(\delta) \quad = \quad \mathrm{Hom}_K(V_\delta, Z).$$

Then,

$$Z \cong \bigoplus_{\delta \in \hat{K}} Z(\delta) \otimes V_\delta. \tag{5.6}$$

If we fix a positive definite form on V_δ, $Z(\delta)$ inherits a Hermitian form. Suppose Z is equipped with a non-zero Hermitian form $\langle\ ,\ \rangle$. Write $p(\delta)$ (resp. $q(\delta)$, $z(\delta)$), for the multiplicity of V_δ in the subspace of $Z(\delta)$ where $\langle\ ,\ \rangle$ is positive (resp. negative or zero).

Write the signature of $\langle\ ,\ \rangle$ on $Z(\delta)$ as $\mathrm{sgn}(\langle\ ,\ \rangle|_{Z(\delta)}) = (p(\delta), q(\delta), z(\delta))$. Then write, formally

$$\mathrm{sgn}(\langle\ ,\ \rangle) \quad = \quad \sum_{\delta \in \hat{K}} (p(\delta), q(\delta), z(\delta)).$$

We will prove in the next chapters the following result.

THEOREM 5.7. *Let $G = SL(n, \mathbb{R})$, $SU(p, q)$ or $SP(n, \mathbb{R})$ and $X \in \mathcal{M}(\mathfrak{g}, K)$ irreducible, endowed with a non-zero invariant Hermitian form $\langle\ ,\ \rangle$ and regular integral infinitesimal character.*

If $X \not\cong A_{\mathfrak{q}'}(\lambda')$, for any $\mathfrak{q}'$ and λ'. Then there are a θ-stable parabolic $\mathfrak{q} = \ell + \mathfrak{u}$, an $(\ell, L \cap K)$-module X_L and $(L \cap K)$-types δ_i^L, $i = 1, 2$ such that

(a) *X is the unique irreducible submodule of $\mathcal{R}_{\mathfrak{q}}(X_L)$, and X occurs only once as a composition factor of $\mathcal{R}_{\mathfrak{q}}(X_L)$.*

(b) *X_L^h is endowed with a Hermitian form $\langle\ ,\ \rangle^L \neq 0$.*
Write (p_L, q_L, z_L) for its signature. Then

$$p_L(\delta_1^L) \neq 0 \qquad and \qquad q_L(\delta_2^L) \neq 0.$$

(c) *Choose $\Delta^+(\mathfrak{k}) = \Delta^+(\ell \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$. Then, if δ_i^L has highest weight μ_i^L, $\mu_i = \mu_i^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant.*

Sections 7–9 will be devoted to the proof of this result.

The main ingredient in the proof of 5.7(a) is that Proposition 3.3 gives a group L_V and a module X_{L_V} for which condition (a) is satisfied. Then theorems 6.3.10 and 8.2.15 in Vogan (1981) provide many Levi subgroups to choose from that also satisfy (a) and might satisfy (b) and (c) as well.

Assume this for the moment. Using this result, we want to prove non-unitarity of X. We need to check that the Hermitian form $\langle\ ,\ \rangle^G$ induced on $\mathscr{R}_{\mathfrak{q}}(X_L)^h$ by Proposition 5.4 is a multiple of $\langle\ ,\ \rangle$ on X; that for the $L \cap K$ types satisfying (c) of Theorem 5.7, the corresponding K types occur in X and that the signature of the form on these K-types is the same as that of $\langle\ ,\ \rangle^L$ on the δ_i^L. Here is the result that we need.

THEOREM 5.8 (Vogan). *Suppose $X \in \mathscr{M}(\mathfrak{g}, K)$ is irreducible and has a non-zero Hermitian form $\langle\ ,\ \rangle$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be θ-stable and X_L an $(\mathfrak{l}, L \cap K)$ module such that X is the unique irreducible submodule of $\mathscr{R}_{\mathfrak{q}}^S(X_L)$, X occurs only once as composition factor in $\mathscr{R}_{\mathfrak{q}}^S(X_L)$ and X_L^h has a non-zero Hermitian form $\langle\ ,\ \rangle^L$. If $\delta^L \in (L \cap K)^\frown$ is an $(L \cap K)$-type of X_L with highest weight μ^L such that $\mu = \mu^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ is dominant for $\Delta(\mathfrak{u} \cap \mathfrak{k})$ then if $\delta \in \hat{K}$ has highest weight μ, $X(\delta) \neq 0$ and*

$$\mathrm{Sign}[\langle\ ,\ \rangle|_{X(\delta)}] = \mathrm{Sign}[\langle\ ,\ \rangle^L|_{X_L(\delta^L)}]$$

Proof. Applying the appropriate definitions and results to K and $\mathfrak{q} \cap \mathfrak{k}$ we have maps

$$\mathscr{R}^i_{\mathfrak{q}\cap\mathfrak{k}}\colon \mathscr{M}(\mathfrak{l} \cap \mathfrak{k}, L \cap K) \to \mathscr{M}(\mathfrak{k}, K)$$

$$\mathscr{L}^j_{\bar{\mathfrak{q}}\cap\mathfrak{k}}\colon \mathscr{M}(\mathfrak{l} \cap \mathfrak{k}, L \cap K) \to \mathscr{M}(\mathfrak{k}, K).$$

If $Y \in \mathscr{M}(\mathfrak{l}, L \cap K)$ there are natural maps

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}} \tilde{Y} \twoheadrightarrow \mathrm{pro}_{\mathfrak{q}\cap\mathfrak{k}}^{\mathfrak{k}} \tilde{Y}$$

$$\mathrm{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}}^{\mathfrak{k}} \tilde{Y} \hookrightarrow \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y}.$$

These induce $(\mathfrak{k}, K)$-maps

$$\mathscr{R}^i_{\mathfrak{q}} Y \xrightarrow{r} \mathscr{R}^i_{\mathfrak{q}\cap\mathfrak{k}} Y$$

$$\mathscr{L}^j_{\bar{\mathfrak{q}}\cap\mathfrak{k}} Y \xrightarrow{l} \mathscr{L}^j_{\bar{\mathfrak{q}}} Y.$$

Then, the following diagram is commutative

$$\begin{array}{ccc} [\mathscr{L}^i_{\bar{\mathfrak{q}}} Y]^h & \xrightarrow{\cong} & \mathscr{R}^{2S-i}_{\mathfrak{q}}(Y^h) \\ \Big\downarrow{\scriptstyle l^h} & & \Big\downarrow{\scriptstyle r} \\ [\mathscr{L}^i_{\bar{\mathfrak{q}}\cap\mathfrak{k}} Y]^h & \xrightarrow{\cong} & \mathscr{R}^{2S-i}_{\mathfrak{q}\cap\mathfrak{k}}(Y^h). \end{array}$$

The isomorphisms across are Theorem 5.3 in Vogan (1984) for $(G, \mathfrak{q})$ and $(K, \mathfrak{q} \cap \mathfrak{k})$, respectively.

Arguing as in the Proof of Proposition 5.4 (for K) we have maps

$$\phi^{\mathfrak{q} \cap \mathfrak{k}}: \operatorname{ind}^{\mathfrak{k}}_{\bar{\mathfrak{q}} \cap \mathfrak{k}} \tilde{Y} \longrightarrow \operatorname{pro}^{\mathfrak{k}}_{\mathfrak{q} \cap \mathfrak{k}} \tilde{Y}^h$$

$$\phi^K: \mathscr{L}^s_{\bar{\mathfrak{q}} \cap \mathfrak{k}} Y \longrightarrow \mathscr{R}^s_{\mathfrak{q} \cap \mathfrak{k}} Y^h.$$

and we have the following commutative diagram

$$\begin{array}{ccccc} \mathscr{L}^s_{\bar{\mathfrak{q}}} Y & \xrightarrow{\phi^G} & \mathscr{R}^s_{\mathfrak{q}}(Y^h) & \cong & (\mathscr{L}^s_{\bar{\mathfrak{q}}} Y)^h \\ \uparrow l & & \downarrow r & & \downarrow l^h \\ \mathscr{L}^s_{\bar{\mathfrak{q}} \cap \mathfrak{k}} Y & \xrightarrow{\phi^K} & \mathscr{R}^s_{\mathfrak{q} \cap \mathfrak{k}}(Y^h) & \cong & (\mathscr{L}^s_{\bar{\mathfrak{q}} \cap \mathfrak{k}} Y)^h \end{array} \tag{5.9}$$

And we have a Hermitian form on $\mathscr{L}^s_{\bar{\mathfrak{q}} \cap \mathfrak{k}}(Y)$

$$\langle x,y \rangle^K = \langle x, \phi^K y \rangle.$$

Since $\phi^K = r \circ \phi^G \circ l$, and by Proposition 6.10 in Vogan (1984), l is a unitary map,

$$\langle x, y \rangle^K = \langle lx, ly \rangle^G. \tag{5.10}$$

Write

$$\begin{aligned} \operatorname{sign}(\langle\ ,\ \rangle^K) &= (p_K, q_K, z_K); \qquad p_K, q_K, z_K: \hat{K} \to \mathbb{N} \\ \operatorname{sign}(\langle\ ,\ \rangle^G) &= (p_G, q_G, z_G); \qquad p_G, q_G, z_G: \hat{K} \to \mathbb{N} \end{aligned} \tag{5.11}$$

and again

$$\operatorname{sign}(\langle\ ,\ \rangle^L) = (p_L, q_L, z_L); \qquad p_L, q_L, z_L: (L \cap K)\hat{} \to \mathbb{N}$$

By 5.10

$$\begin{aligned} p_G(\delta) &\geqslant p_K(\delta) \\ q_G(\delta) &\geqslant q_K(\delta) \\ z_G(\delta) &\geqslant z_K(\delta). \end{aligned} \tag{5.12}$$

The main ingredient in the Proof of Proposition 5.8 is the following result due to T. Enright.

PROPOSITION 5.13 [Enright (1984)]. See also Vogan (1984) 6.5–6.8. *Let* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, *θ-stable parabolic.*

Let $\delta^L \in (L \cap K)\hat{}$ *with highest weight* μ^L. *Set* $\mu = \mu^L + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$.

(a) *If* μ *is not* $\Delta(\mathfrak{u} \cap \mathfrak{k})$*-dominant, then*

$$\mathcal{L}^s_{\mathfrak{q} \cap \mathfrak{k}} Y(\delta^L) = 0.$$

(b) *If* μ *is* $\Delta(\mathfrak{u} \cap \mathfrak{k})$ *dominant, write* $\delta \in \hat{K}$ *for the representation of K with highest weight* μ. *Then*

$$p_K(\delta) = p_L(\delta^L)$$

$$q_K(\delta) = q_L(\delta^L)$$

$$z_K(\delta) = z_L(\delta^L).$$

LEMMA 5.14. *Suppose V is a module of finite length and U is irreducible. Assume*

(a) $U \subseteq V$ *occurs exactly once as a composition factor of V.*

(b) *Any non-zero* $W \subseteq V$ *contains U.*

(c) *U is equipped with a Hermitian form.*

Then, up to scalars, V^h *has a unique Hermitian form* $\langle\ ,\ \rangle_1$ *and*

$$U \cong V^h/rad(\langle\ ,\ \rangle_1). \qquad \square$$

The proof of this lemma is standard. We give an outline here.

A non zero Hermitian form on V^h is a non zero map $\varphi: V^h \to V$ where the radical of the form is the kernel of φ. Write R = image of φ. We want to prove that $R = U$.

By (b), $U \subseteq R$. Suppose $R \neq U$. If Q is any irreducible quotient of R/U, then Q is a quotient of V^h. So $Q^h \subseteq V$, and by (b) again, $U \subseteq Q^h$.

But by (c), $U \cong U^h$ and hence $Q \cong U$ since Q is irreducible. (Note that the Hermitian form on U is necessarily non degenerate, since U is irreducible.)

But this means that U occurs twice as a composition factor of V: once as a submodule and once as a quotient of R/U. This contradicts (a).

We can now prove Theorem 5.8. By Proposition 5.13 and 5.12

$$p_G(\delta) \geqslant p_L(\delta^L)$$

$$q_G(\delta) \geqslant q_L(\delta^L) \tag{5.15}$$

$$z_G(\delta) \geqslant z_L(\delta^L).$$

Apply Lemma 5.14 to

$$V = \mathscr{R}^s_{\mathfrak{q}}(X_L) \quad \text{and} \quad U = X.$$

We know that (a)–(c) hold in this Lemma since they are part of our assumptions on X. We also know that $\langle\ ,\ \rangle^G \neq 0$ by 5.15.

Hence, we have the following result:

PROPOSITION 5.16. *In the setting of* Theorem 5.8

$$\langle\ ,\ \rangle^G|_X = c\langle\ ,\ \rangle$$

$$X \cong [\mathscr{R}^s_{\mathfrak{q}}(X_L)]^h/\text{rad}(\langle\ ,\ \rangle^G).$$

So $\langle\ ,\ \rangle^G|_X$ *is nondegenerate and has signature*

$$\text{sgn}(\langle\ ,\ \rangle) = (p_G, q_G).$$

Q.E.D.

It is now straightforward to prove Theorem 1.2. Using Theorem 5.7, proved in sections 7–9 for our groups in question, we have that the hypotheses in Theorem 5.8 are true and by 5.15

$$p_G(\delta^1) > 0 \quad \text{and} \quad q_G(\delta^2) > 0$$

and the form $\langle\ ,\ \rangle$ on X is indefinite too. Q.E.D.

§6. Methods to detect non-unitarity

To prove Theorem 5.7 we will need a few techniques that we will discuss here. Fix a positive root system $\Delta^+(\mathscr{k})$.

LEMMA 6.1 (Parthasarathy's Dirac operator inequality. See Borel–Wallach (1980) II. 6.1.1.). *Let* $(\pi, \mathscr{H})$ *be a unitary representation of G and* $\mathscr{H}_K$ *its Harish-Chandra module.*

Fix a positive $\mathfrak{t}$*-root system* $\Delta^+(\mathfrak{g})$ *compatible with* $\Delta^+(\mathfrak{k})$ *and a* $\mathfrak{k}$*-type* δ *occurring in* $\mathscr{H}_K$ *with highest weight* $\mu \in \mathfrak{t}^{c*}$. *Write*

$$\varrho = \varrho(\Delta^+(\mathfrak{g})) \in (\mathfrak{t}^c)^*$$

$$\varrho_c = \varrho(\Delta^+(\mathfrak{k})) \in (\mathfrak{t}^c)^*$$

$$\varrho_n = \varrho(\Delta^+(\mathfrak{p})) = \varrho - \varrho_c \in (\mathfrak{t}^c)^*.$$

Let c_0 *be the eigenvalue of the Casimir operator of* $\mathfrak{g}$ *acting on* $\mathscr{H}_K$, *and* $w \in W(\mathfrak{k}, \mathfrak{t})$ *making* $w(\mu - \varrho_n)$ *dominant for* $\Delta^+(\mathfrak{k})$. *Then*

$$\langle w(\mu - \varrho_n) + \varrho_c, w(\mu - \varrho_n) + \varrho_c \rangle \geqslant c_0 + \langle \varrho, \varrho \rangle. \tag{6.2}$$

LEMMA 6.3. *Let* $X \in \mathscr{M}(\mathfrak{g}, K)$ *with a non-zero, invariant Hermitian form* $\langle\ ,\ \rangle$. *Suppose the Dirac inequality fails on a K-type* δ, *for some choice of* $\Delta^+(\mathfrak{p})$. *Then*

(a) *There is a* $\mathfrak{k}$*-type* V_η *occurring in* $V_\delta \otimes \mathfrak{p}$ *such that*

$$\langle\ ,\ \rangle|_{V_\delta \oplus V_\eta}$$

is indefinite.

(b) *Suppose G/K is Hermitian symmetric with a one-dimensional compact center, so that we can choose* $z \in i\mathfrak{k}_0$ *with the property that* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ *is the decomposition of* $\mathfrak{g}$ *into the eigenspaces* $0, +1, -1$ *of z, respectively.*

Set $\varrho_n^{\pm} = \varrho(\Delta(\mathfrak{p}^{\pm})$. *Then, if the Diract inequality fails on* δ *for* $\varrho_n^{\pm}$, *there is a* $\mathfrak{k}$*-type* $V_\eta^{\mp}$ *occurring in* $V_\delta \otimes \mathfrak{p}^{\mp}$ *such that*

$$\langle\ ,\ \rangle|_{(V_\delta \oplus V_\eta^{\mp})}$$

is indefinite.

Proof. Recall from Borel–Wallach (1980), II §6, the definition of $(\gamma, S(V))$, the space of spinors of a finite dimensional vector space V defined over $\mathbb{R}$, with a positive definite inner product $\langle\ ,\ \rangle$. Write $\langle\ ,\ \rangle_S$ for the unitary structure on $S(V)$ such that

$$\langle \gamma(v)x, y \rangle_S = -\langle x, \gamma(v)y \rangle_S$$

$$v \in V, \qquad x, y \in S(V).$$

Recall also, the definition of the Dirac operator

$$D: H \otimes S \rightarrow H \otimes S$$

for (π, H) a unitary $(\mathfrak{g}_0, \mathfrak{k}_0)$-module and $S = S(\mathfrak{p}_0)$.

$$D(v \otimes s) = \sum_{\alpha \in \Delta(\mathfrak{p})} \pi(X_\alpha) v \otimes \gamma(X_{-\alpha}) s. \tag{6.4}$$

Since

$$S = \bigoplus_{\Delta^+(\mathfrak{g}) \supseteq \Delta^+(\mathfrak{k})} m \cdot V_{\varrho(\Delta^+(\mathfrak{g})) - \varrho_c}$$

(where $m = 2^{[\dim \alpha^c/2]}$) (cfr. Borel–Wallach (1984) II §6) then $w(\mu - \varrho_n)$ is the highest weight of a $\mathfrak{k}$-representation occurring in $V_\delta \otimes V_{\varrho_n} \subseteq H \otimes S$.

Let $\xi = v \otimes s$ be a weight vector for $w(\mu - \varrho_n)$.

Write also $\langle\ ,\ \rangle_D$ for the tensor product inner product on $H \otimes S$; then the proof of Lemma 6.1 shows that

$$0 > \langle D\xi, D\xi \rangle_D = (\langle w(\mu - \varrho_n) + \varrho_c, w(\mu - \varrho_n) + \varrho_c \rangle$$
$$- c_0 - \langle \varrho, \varrho \rangle) \langle \xi, \xi \rangle_D.$$

So $D\xi \neq 0$ and

$$D\xi = \sum_{\alpha \in \Delta(\mathfrak{p})} \pi(X_\alpha) v \otimes \gamma(X_{-\alpha}) s \in \mathfrak{p} \cdot V_\delta \otimes S \subseteq H \otimes S.$$

This gives a non-zero map

$$\mathfrak{p} \otimes V_\delta \xrightarrow{\sigma} \mathfrak{p} \cdot V_\delta.$$

So $\mathrm{Hom}_{\mathfrak{k}}(\mathfrak{p} \otimes V_\delta, H) \neq 0$. Let $E = \mathrm{Im}\, \sigma$. Since $\langle\ ,\ \rangle_S$ is positive definite this means that $\langle\ ,\ \rangle$ is indefinite on $V_\delta \oplus E$.

This proves (a) of the lemma.

For (b) simply observe that $\mu - \varrho_n^- = \mu + \varrho_n^+$; $\varrho_n^+ = \varrho(\mathfrak{p}^+)$ and $\mathfrak{p}^+$ is a representation of $\mathfrak{k}$. Hence if $\beta \in \Delta(\mathfrak{k})$

$$\langle \varrho_n^+, \beta \rangle = 0.$$

So $V_{\varrho_n^+}$ is one-dimensional. Since $\varrho_n^+ + \alpha$ is not a weight of S, for $\alpha \in \Delta(\not{p}^+)$, $V_{\varrho_n^+}$ is killed by $\gamma(X_\alpha)$ and 6.4 becomes, for $\xi \in V_\delta \otimes V_{\varrho_n^-}$

$$D\xi = \sum_{\alpha \in \Delta(\not{p}^+)} \pi(X_\alpha)v \otimes \gamma(X_{-\alpha})s$$

so $D\xi \in (\not{p}^+) \cdot V_\delta \otimes S \subseteq H \otimes S$. Similarly for ϱ_n^-. Q.E.D.

LEMMA 6.5 (Vogan). *Let G be a connected, reductive linear Lie group. Assume that G is equal rank. Then, any representation with real infinitesimal character has a Hermitian form.*

Proof. By Proposition 5.2 it is enough to prove the lemma for G quasisplit and a Langlands subrepresentation of a principal series $I(\delta \otimes \nu)$ with $\delta \otimes \nu$ a character of a maximally split Cartan subgroup $H^s = T^s A^s$.

Since G is equal rank there is a subset $B = \{\alpha_1, \ldots, \alpha_k\}$ of strongly orthogonal simple real roots such that, since H^s is the maximally split Cartan subgroup of G, then B spans $\mathfrak{a}_0^s = \mathrm{Lie}(A^s)$.

Hence if $w = s_{\alpha_1} \ldots s_{\alpha_k}$ is the product of simple reflections s_{α_i}, w acts by -1 on A^s and by the identity on T_0^s.

Recall from (Vogan (1981) page 172) the maps ϕ_α: $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{m}_0^\alpha$. Consider the exponentiated map

$$\Phi_\alpha\colon SL(2, \mathbb{R}) \to M^\alpha$$

set

$$m_\alpha = \Phi_\alpha \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in M^s.$$

Then, since G is connected, T^s is generated by $T_0^s \cup \{m_\alpha | \alpha \text{ real}\}$. Let $w \in M'/M = W$, then there is $\sigma \in M'$ such that

$$w \cdot m_\alpha = \sigma m_\alpha \sigma^{-1} = m_{w \cdot \alpha}.$$

But $m_{w \cdot \alpha} = m_{-\alpha} = m_\alpha$.

Then

$$(w\delta)(m_\alpha) = \delta(w \cdot m_\alpha) = \delta(m_\alpha)$$

and

$$w \cdot \delta|_{T_0} = \delta.$$

Hence $w\delta = \delta$. Since $I(\delta \otimes \nu)$ is assumed to have real infinitesimal character, ν is real.

Also since $w|_A = -1$ then $w \cdot \nu = -\nu = -\bar{\nu}$.

This is the condition of Proposition 5.2 for the existence of a Hermitian form. Q.E.D.

§7. Proof of Theorem 5.7 for $G = SL(n, \mathbb{R})$

To fix notation consider $G = SL(2n, \mathbb{R})$; the odd case is similar. The maximal compact subgroup K of G is

$$K = SO(2n, \mathbb{R}) = \{g \in G | g^t g = I\}.$$

If θ is the Cartan involution defined by $\theta(X) = -{}^tX$, then

$$\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 | X = {}^tX\}.$$

The maximal compact Cartan subgroup of G is $H^c = T^cA^c$, where

$$T^c = \left\{ g = \begin{bmatrix} r(\theta_1) & & & \\ & r(\theta_2) & & \\ & & \ddots & \\ & & & r(\theta_n) \end{bmatrix} \middle| \theta_i \in \mathbb{R}; r(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \right\}$$

and

$$A^c = \left\{ g = \begin{bmatrix} r_1 & & & & & \\ & r_1 & & & & \\ & & r_2 & & & \\ & & & r_2 & & \\ & & & & \ddots & \\ & & & & & r_n \\ & & & & & & r_n \end{bmatrix} \middle| r_i \in \mathbb{R}; \det g = 1 \right\}$$

Then the roots of t^c in k, p and g are, respectively

$$\Delta(k, t^c) = \{\pm(e_j \pm e_k) | 1 \leqslant j < k \leqslant n\}$$

$$\Delta(p, t^c) = \{\pm 2e_s; \pm(e_j \pm e_k) | 1 \leqslant s \leqslant n; 1 \leqslant j < k \leqslant n\}$$

$$\Delta(g, t^c) = \{\pm 2e_s; \pm(e_j \pm e_k) | 1 \leqslant s \leqslant n; 1 \leqslant j < k \leqslant n\}.$$

The multiplicity of $\pm(e_j \pm e_k)$ as a root in g in 2. $\hat{K}$ can be identified with the set

$$\{\mu = (a_1, a_2, \ldots, a_n) \in \mathbf{Z}^n | a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n-1} \geqslant |a_n|\}.$$

Let $\mu = (a_1, a_2, \ldots, a_n) \in it_0^*$ be the highest weight of a lowest K type of a Harish-Chandra module X. After conjugating by an outer automorphism of K we may assume that $a_n \geqslant 0$.

PROPOSITION 7.1 [see Vogan (1986)]. *Let r be the largest integer such that $a_r \geqslant 2$. Then the subgroup L_V (called L_{c1} in that paper) attached to μ as in* §3 *is isomorphic to $SL(p_1, \mathbb{C}) \times SL(p_2, \mathbb{C}) \times \cdots \times SL(p_s, \mathbb{C}) \times SL(2(n-r), \mathbb{R})$, where $\Pi_r = (p_1, r_2, \ldots, p_s)$ is the coarsest ordered partition of r such that μ is constant on $SL(p_i, \mathbb{C})$. That is*

$$\langle \mu, e_j - e_{j+1} \rangle \begin{cases} > 0 & \textit{for} \quad j = p_1 + p_2 + \cdots + p_k \\ & \qquad k = 1, 2, \ldots, s \\ = 0 & \textit{other } j \leqslant r \\ = 1 \text{ or } 0 & \textit{for} \quad r < j \leqslant n \end{cases}$$

Proof. To obtain $\ell_V(X) = \ell_V(X) = \ell_V(\mu)$, as in Vogan (1981), we need:

$$2\varrho_c = (2n-2, 2n-4, \ldots, 2, 0).$$

Let $\Delta^+(g, h^c)$ be a θ-stable positive system making $\mu + 2\varrho_c$ dominant. The restriction of $\Delta^+(g, h^c)$ to t^c is $\Delta^+(g, t^c)$. Write $\phi(g, t^c)$ for the set of simple roots restricted to t^c. Then

$$\phi(g, t^c) = \{e_1 - e_2; e_2 - e_3; \ldots; e_{n-1} - e_n; 2e_n\}$$

and

$$\varrho = (2n-1, 2n-3, \ldots, 3, 1).$$

We can form an array with the coordinates of $\mu + 2\varrho_c$ by grouping them into maximal blocks of elements decreasing by 2. That is, suppose

$$\mu = (\underbrace{x_1, x_1, \ldots, x_1}_{p_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{p_2 \text{ times}}, \ldots, \underbrace{x_t, \ldots, x_t}_{p_t \text{ times}}, \underbrace{0, \ldots, 0}_{R \text{ times}}) \quad (7.2)$$

where $x_1 > x_2 > \cdots x_t > 0$. (Note that if $x_t = 1$, then $t = s + 1$).

Then, since the coordinates of $2\varrho_c$ decrease by two, the array would look like

$$\boxed{m_1 \quad m_1 - 2 \quad \cdots \quad m_1 - 2p_1 + 2}\ \boxed{m_2 \quad m_2 - 2 \quad \cdots \quad m_2 - 2p_2 + 2}$$

$$\cdots \boxed{2R-2, \quad 2R-4 \quad \cdots \quad 2, \quad 0}$$

Since $2\varrho_c - \varrho = (-1, -1, -1, \ldots, -1)$, then (cf. 3.1) the weight used to build $\ell_V(\mu)$ is

$$\lambda_V(\mu) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{p_1 \text{ times}}, \underbrace{\lambda_2 \ldots \lambda_2}_{p_2 \text{ times}}, \ldots, \underbrace{\lambda_t \ldots \lambda_t}_{p_t \text{ times}}, \underbrace{0, \ldots, 0}_{R})$$

where $\lambda_j = x_j - 1$ (in particular, if $x_t = 1$, then $\lambda_t = 0$). This is easily verified by following the algorithm of Proposition 5.3.3 in Vogan (1981). Simply observe that $\langle \mu, 2e_j \rangle \leqslant 0$ only for $j > r$.

Moreover, the subset of simple roots orthogonal to $\lambda_V(\mu)$ spans the root system (cf. 2.1)

$$(A_{p_1-1} \oplus A_{p_1-1}) \oplus (A_{p_2-1} \oplus A_{p_2-1}) \oplus \cdots \oplus (A_{p_s-1} \oplus A_{p_s-1}) \oplus A_{2(n-r)},$$

since the roots $e_i - e_j$ are restrictions of complex roots and therefore occur twice in $\Delta(\mathfrak{g}, \mathfrak{t}^c)$. Now the proposition is clear. Q.E.D.

We will now obtain some criteria to determine when a representation of K is the lowest K-type of some $(\mathfrak{g}, K)$ module $A_{\mathfrak{q}}(\lambda)$.

Recall from 2.1 that to construct a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ we need a weight $x \in i\mathfrak{t}_0^*$. Suppose

$$x = (\underbrace{x_1, \ldots, x_1}_{r_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{r_2 \text{ times}}, \ldots, \underbrace{x_r, \ldots, x_t}_{r_t \text{ times}}, \underbrace{0, \ldots, 0}_{R \text{ times}})$$

where

$$x_1 > x_2 > \cdots > x_t > 0.$$

Write $\mathfrak{q} = \mathfrak{q}(x) = \mathfrak{l}(x) + \mathfrak{u}(x)$ for the parabolic defined by x as in 2.1. Clearly

$$\left\{\begin{array}{l} 2\varrho(\mathfrak{u} \cap \mathfrak{p}) = (\underbrace{s_1 s_1 \ldots s_1}_{r_1}, \underbrace{s_2 \ldots s_2}_{r_2}, \ldots, \underbrace{s_t \ldots s_t}_{r_t}, \underbrace{0 \ldots 0}_{R}) \\ \quad \text{with } s_j = 2(n - r_1 - \cdots - r_{j-1}) - r_j + 1 \\ \text{and} \\ 2\varrho(\mathfrak{u} \cap \mathfrak{k}) = (\underbrace{u_1, \ldots, u_1}_{r_1}, \underbrace{u_2, \ldots, u_2}_{r_2}, \ldots, \underbrace{u_t, \ldots, u_t}_{r_t}, \underbrace{0 \ldots 0}_{R}) \\ \quad \text{with } u_j = 2(n - r_1 - \cdots - r_{j-1}) - r_j - 1. \end{array}\right. \tag{7.3}$$

PROPOSITION 7.4. *Let μ be as in* (7.2) *and suppose it is the highest weight of a representation of K. Then V_μ is the LKT of a $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$ if and only if*

(a) $x_i - x_{i+1} \geqslant p_i + p_{i+1}$

and

(b) $x_t \geqslant 2R + p_t + 1$.

Proof. Suppose V_μ is the *LKT* of an $A_\mathfrak{q}(\lambda)$. Then $\mu = \lambda + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ and λ is the weight of a one-dimensional character of L satisfying 4.1(a) and (b). Hence λ is orthogonal to the roots of $\mathfrak{t}^c$ in $\mathfrak{l}$ and it is positive in the $\mathfrak{t}^c$-roots in $\mathfrak{u}$. That is,

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{p_1 \text{ times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{p_2 \text{ times}}, \ldots \underbrace{\lambda_t, \ldots, \lambda_t}_{p_t \text{ times}}, \underbrace{0, \ldots, 0}_{R \text{ times}})$$

and

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t \geqslant 0.$$

By 7.3,

$$x_j = \lambda_j + 2(n - p_1 - \cdots - p_{j-1}) - p_j + 1.$$

Then

$$x_t = \lambda_t + 2(n - p_1 - \cdots - p_{t-1}) - p_t + 1 \geqslant 2R + p_t + 1$$

and

$$x_j - x_{j+1} = \lambda_j - \lambda_{j+1} - p_j + 1 + 2p_j + p_{j+1} - 1 \geqslant p_j + p_{j+1},$$

proving (a) and (b).

Conversely, suppose μ is a weight satisfying (a) and (b) then we can define

$$\mathfrak{q} = \mathfrak{q}(\mu) \quad \text{and} \quad \lambda_j = x_j - 2(n - p_1 - \cdots - p_{j-1}) + p_j - 1.$$

Then μ will be the *LKT* of $A_{\mathfrak{q}}(\lambda)$. Q.E.D.

We are now ready to prove Theorem 5.7 for $G = SL(n, \mathbb{R})$.

Suppose X is as in Theorem 5.7 with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$ and $\mu \in (i\mathfrak{t}_0^c)^*$ the highest weight of a lowest K-type of X. Write μ as in 7.2.

Considering what the weights in $V_\mu \otimes \mathfrak{p}$ look like, we will study 2 cases:
1. $2R + p_t + 1 > x_t$.
2. $x_t \geqslant 2R + p_t + 1$.

By the conditions given in 7.4 if V_μ is the lowest K-type of an $A_{\mathfrak{q}}(\lambda)$, then μ is in case 2.

Therefore, the first thing we must do is verify that, in case 1, X is not unitary:

Case 1. We will use the following result.

LEMMA 7.5. *Let μ be as in 7.2 and suppose $x_t < 2R + p_t + 1$. Suppose that $x_i - x_{i+1} = 1$. Then Dirac operator inequality fails on μ for $\varrho_n = (n, n-1, \ldots, 1)$.*

Proof. The hypotheses on μ imply that

$$\mu = (\underbrace{x + t - 1, \ldots, x + t - 1}_{p_1 \text{ times}}, \underbrace{x + t - 2, \ldots, x + t - 2}_{p_2 \text{ times}}$$

$$\ldots \underbrace{x, \ldots, x}_{p_t \text{ times}}, \underbrace{0 \ldots 0}_{R})$$

Note that $1 \leqslant x \leqslant 2R + p_t$ implies that

$$(*) \qquad R + 1 - x \leqslant R \quad \text{and} \quad R + p_t - x \geqslant -R.$$

Now

$$\varrho_n = (n, n-1, \ldots, R+p_t, R+p_t-1, \ldots, R+1, R, R-1, \ldots, 1),$$

so

$$\begin{aligned}\varrho_n - \mu = {} & (n-x-t+1, n-x-t, \ldots, R+p_t-x, \\ & R+p_t-x-1, \ldots, R+1-x, R, R-1, \ldots, 1).\end{aligned}$$

By (*), the sequence of integers

$$R+p_t-x, R+p_t-x-1, \ldots, R+1-x$$

overlaps the sequence $R, R-1, R-2, \ldots, -R+1, -R$.

Clearly, the first $n-R$ coordinates of $\varrho_n - \mu$ decrease by steps of at most one. So if $\omega \in W_K$ is such that $\omega(\mu - \varrho_n)$ is dominant, then the coordinates of $\omega(\mu - \varrho_n)$ will be a sequence of integers decreasing by at most one, ending in 0 or ± 1 and in the latter case, there must be repetitions in the sequence.

Since

$$\varrho_c = (n-1, n-2, \ldots, R+1, R, R-1, \ldots, 2, 1, 0)$$

it follows that

$$\langle \omega(\mu - \varrho_n), \varrho_c) \rangle < \langle \varrho_n, \varrho_c \rangle$$

$$\langle \omega(\mu - \varrho_n), \omega(\mu - \varrho_n) \rangle < \langle \varrho_n, \varrho_n \rangle.$$

Hence

$$\langle \omega(\mu - \varrho_n) + \varrho_c, \omega(\mu - \varrho_n) + \varrho_c \rangle < \langle \varrho_n + \varrho_c, \varrho_n + \varrho_c \rangle = \langle \varrho, \varrho \rangle.$$

Q.E.D.

Now to prove non-unitarity for case 1, take i_0 to be the minimal integer in $\{1, 2, \ldots, t\}$ such that $x_i - x_{i+1} = 1$ for all $i > i_0$.

Let $k_1 = p_1 + p_2 + \cdots + p_{i_0}$, $k_2 = n - k_1$

$$\ell_1 = \mathfrak{gl}(k_1, \mathbb{C}), \qquad \ell_2 = \mathfrak{gl}(2k_2, \mathbb{R})$$

and $\ell = \ell_1 \oplus \ell_2$.

Then $\ell \supseteq \ell_V$ and by Proposition 3.3, X is the Langlands quotient of some module

$$\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}_V}(X^{L_V}(\delta_V \otimes \nu_V))$$

and if we set

$$X_L = \mathscr{R}^{\ell}_{\mathfrak{q}_V \cap \ell}(X^{L_V}(\delta_V \otimes \nu_V)),$$

then, by induction by stages (see Zuckerman (1977) or Vogan (1981), 6.3.10), X is the Langlands quotient of

$$\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L) \cong \mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}_V}(X^{L_V}(\delta_V \otimes \nu_V))$$

and (a) of Theorem 5.7 holds.

Also, by Corollary 5.3, X_L^h has a Hermitian form $\langle\ ,\ \rangle^L$.

Write $\mu^L = \mu - 2\varrho(\mathfrak{u} \cap \mathfrak{p})$. Then μ^L is the highest weight of a lowest $L \cap K$-type of X_L.

Set $\mu^i = \mu^L|_{L_i}$, $i = 1, 2$.

Clearly

$$\mu^2 = \mu|_{SL(2k_2,\mathbb{R})},$$

and, by Lemma 7.5, the Dirac inequality fails on μ^2. By Lemma 6.3(a) there is a K-type V_{η^2} in $V_{\mu^2} \otimes (\ell_2 \cap \mathfrak{p})$ that makes the Hermitian form $\langle\ ,\ \rangle^L$ indefinite.

The roots in $\Delta(\ell_2 \cap \mathfrak{p})$ are

$$\{\underbrace{(0 \ldots \pm 1, 0 \ldots 0 \pm 1\ 0 \ldots 0)}_{k_2}, \underbrace{(0 \ldots 0, \pm 2, 0 \ldots 0)}_{k_2}\}.$$

It is clear that if $\eta^2 = \mu^2 + \beta$ is dominant for some $\beta \in \Delta(\ell_2 \cap \mathfrak{p})$ then, since $a_{i_0} - a_{i_0+1} \geqslant 2$ the K-type $\mu + \beta$ is also dominant for $\Delta(\mathfrak{u} \cap \mathfrak{k})$. Hence by Theorem 5.8 $X(\mu + \beta) \neq 0$, and Theorem 5.7 follows for this case.

For case 2, suppose (7.4)(b) holds and that there is $i_0 < t$ such that (7.4)(a) does not hold for $j = p_1 + p_2 + \cdots + p_{i_0}$. Set

$$p = p_1 + p_2 + \cdots + p_t \qquad R = n - p$$

and

$$\ell_1 = \mathfrak{sl}(p, \mathbb{C}) \qquad \ell_2 = \mathfrak{sl}(2R, \mathbb{R}).$$

Again $\ell = \ell_1 \oplus \ell_2 \supseteq \ell_V$ and arguing as in the preceding case we can find X_L such that (a) in Theorem 5.7 holds.

Write X_L as $X_{L_1} \otimes X_{L_2}$, where X_{L_i} is an $(\ell_i, L_i \cap K)$-module.

By Theorem 6.1 in Enright (1979), and especially its proof (pp. 518–523), if X_{L_1} is not an $A_{\mathfrak{q}}$, (λ') then Dirac inequality fails precisely on the lowest K-type. Write $\mu^L = \mu - 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ and

$$\mu^1 = \mu^L|_{L_1}.$$

By Lemma 6.3 (a) again, there is an $(L_1 \cap K)$-type V_{η^1} with $\eta^1 = \mu^1 + \beta$ for $\beta \in \Delta(\ell_1 \cap \mathfrak{p})$. If for all $i \neq i_0$

$$\langle \mu, e_i - e_{i+1} \rangle \geqslant p_i + p_{i+1} \geqslant 2. \tag{7.6}$$

Then $\mu + \beta$ is dominant.

Otherwise take $k' = \Sigma_{i \in B}\, p_i$ with

$$B = \{i \in \{1, \ldots, t\} \mid (7.6) \text{ holds}\}.$$

Then apply Enright's result to the rest. Q.E.D.

§8. Proof of Theorem 5.7 for $G = SU(p, q)$

Let $n = p + q$. Write I_m for the identity matrix in $GL(m, \mathbb{C})$ and A^* for the conjugate transpose of the matrix A, then

$$G = \left\{ g \in SL(n, \mathbb{C}) \middle| g \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} g^* = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \right\}.$$

Then the maximal contact subgroup K of G is

$$K = \left\{ g \in G \middle| g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}; A \in U(p), B \in U(q) \right\}.$$

If θ is the Cartan involution defined by $\theta(X) = -X^*$, and

$$\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 | \theta(X) = -X\}$$

then

$$\mathfrak{p}_0 = \left\{X = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} | B \text{ arbitrary } p \times q \text{ matrix}\right\}.$$

The compact Cartan subgroup of G is

$$H^c = T^c = \left\{g = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) | \sum_{j=1}^{n} \theta_j = 0\right\}.$$

$\hat{K}$ can be identified with the space

$$\{\mu = (a_1, \ldots, a_p | a_{p+1}, \ldots, a_n) \in \mathbb{R}^n | a_1 \geqslant \cdots \geqslant a_p;$$

$$a_{p+1} \geqslant \cdots \geqslant a_n \Sigma a_j = 0, a_i - a_j \in \mathbb{Z}\}.$$

If we denote by $e_j \in \mathbb{R}^*, j = 1, \ldots, n$, the elements of the dual basis in $\mathbb{R}^n$, then the roots of $\mathfrak{t}$ in $\mathfrak{g}$ correspond to the set

$$\Delta(\mathfrak{g}) = \Delta(\mathfrak{g}, \mathfrak{t}) = \{e_i - e_j | i \neq j; 1 \leqslant i, j \leqslant n\}.$$

Also

$$\Delta(\mathfrak{k}) = \Delta(\mathfrak{k}, \mathfrak{t}) = \{e_i - e_j | 1 \leqslant i, j \leqslant p\} \cup \{e_k - e_m | p < k, m \leqslant n\}$$

the compact imaginary roots of $\mathfrak{t}$ in $\mathfrak{g}$, and

$$\Delta(\mathfrak{p}) = \Delta(\mathfrak{p}, \mathfrak{t}) = \{\pm(e_i - e_{p+j}) | 1 \leqslant i \leqslant p; 1 \leqslant j \leqslant q\}$$

the noncompact imaginary roots of $\mathfrak{t}$ in $\mathfrak{g}$.

Let $\mu = (a_1, a_2, \ldots, a_p | b_1, b_2, \ldots, b_q)$ be the highest weight of a lowest K-type of X. Fix the positive system $\Delta^+(\mathfrak{k})$ so that

$$a_1 \geqslant \cdots \geqslant a_p; \qquad b_1 \geqslant \cdots \geqslant b_q.$$

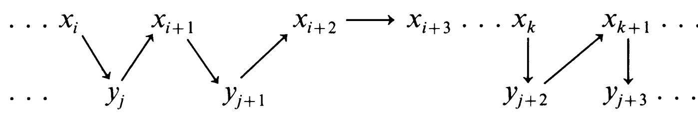
Having in mind the construction of the quasisplit subgroup L_V, write

$$\mu + 2\varrho_c = (x_1, x_2, \ldots, x_p | y_1, y_2, \ldots, y_q).$$

We can form an array of two rows with the coordinates of $\mu + 2\varrho_c$ so that they are aligned in decreasing order from left to right as follows: the x_i are in the first rows; the y_j are in the second; and terms decrease from left to right in the array. For example, if we have

$$x_i > y_j > x_{i+1} > y_{j+1} > x_{i+2} > x_{i+3} \ldots > x_k = y_{j+2} > x_{k+1} = y_{j+3} \ldots$$

the array would look like:



This array gives a choice of positive roots $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$, compatible with $\Delta^+(\mathfrak{k})$. That is, the simple roots are given by the arrows. In the preceding example, they would be

$$\ldots e_i - e_{p+j}; \quad e_{p+j} - e_{i+1};\ e_{i+1} - e_{p+j+1};$$

$$e_{p+j+1} - e_{i+2}; \quad e_{i+2} - e_{i+3}; \quad \ldots$$

$$\ldots e_k - e_{p+j+2}; \quad e_{p+j+2} - e_{k+1}; \quad e_{k+1} - e_{p+j+3}; \quad \ldots$$

Because the terms in each row decrease by at least 2, the entire array is a union of blocks of the following five types.

1.
$$\begin{array}{|cccc|} r & r-2 & \ldots & r-2k \\ r & r-2 & \ldots & r-2k \end{array}$$

2.
$$\begin{array}{cccc} r & r-2 & \ldots & r-2k \\ r-1 & r-3 & \ldots & r-2k-1 \end{array}$$

3.
$$\begin{array}{cccc} r & r-2 & \ldots & r-2k \\ r-1 & r-3 & \ldots & r-2k+1 \end{array}$$

4.
$$\begin{array}{cccc} r & r-2 & \ldots & r-2k \\ r+1 & r-1 & \ldots & r-2k+1 \end{array}$$

5.

r		$r-2$	...			$r-2k$	
$r+1$	$r-1$		...	$r-2k+1$			$r-2k-1$

From now on we will drop the arrows in the pictures, since the ordering of the roots is clear from the arrangement of the coordinates of $\mu + 2\varrho_c$, provided we agree on choosing the order prescribed in block 1.

Using the picture of $\mu + 2\varrho_c$, we can split the coordinates of μ as follows

$$\mu = (\underbrace{g_1 \ldots g_1}_{r_1 \text{ times}} \ldots \underbrace{g_t \ldots g_t}_{r_t \text{ times}} \mid \underbrace{f_1 \ldots f_1}_{s_1 \text{ times}} \ldots \underbrace{f_t \ldots f_t}_{s_t \text{ times}}) \tag{8.1}$$

where r_i is the number of p-coordinates and s_i the number of q-coordinates making up the i-th block of the array of $\mu + 2\varrho_c$, and

$$g_1 \geqslant g_2 \geqslant \cdots \geqslant g_t, \qquad f_1 \geqslant f_2 \geqslant \cdots \geqslant f_t$$

$$r_i \geqslant 0, \quad s_i \geqslant 0, \quad i = 1, \ldots, t.$$

PROPOSITION 8.2. *Let μ be as in* 8.1 *and X a $(\mathfrak{g}, K)$-module with lowest K-type μ. Then*

$$\lambda_V(X) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{r_1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{r_2}, \ldots \underbrace{\lambda_t, \ldots, \lambda_t}_{r_t} \mid \underbrace{\lambda_1, \ldots, \lambda_1}_{s_1},$$

$$\underbrace{\lambda_2, \ldots, \lambda_2}_{s_2} \ldots \underbrace{\lambda_t, \ldots, \lambda_t}_{s_t}) \tag{8.3}$$

$$\mathfrak{l}_V(X) \cong \mathfrak{s}(\mathfrak{u}(r_1, s_1) \oplus \cdots \oplus \mathfrak{u}(r_t, s_t)).$$

(*see* §3).

COROLLARY 8.4. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be θ-stable with*

$$\mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1, q_1) \oplus \mathfrak{u}(p_2, q_2) \oplus \cdots \oplus \mathfrak{u}(p_t, q_t))$$

and

$$\lambda\colon \mathfrak{l} \to \mathbb{C} \quad \textit{an admissible unitary character}$$

Then $\mathfrak{l}_V(A_{\mathfrak{q}}(\lambda)) \subseteq \mathfrak{l}$. *More precisely,*

$$\mathfrak{l}_V(A_{\mathfrak{q}}(\lambda)) \simeq \mathfrak{s}\left[\prod_{i=1}^{t} [(\mathfrak{u}(1))^{di} \oplus \mathfrak{u}(r_i, s_i) \oplus (\mathfrak{u}(1))^{di}]\right]$$

$$\subseteq \mathfrak{s}\left[\prod_{i=1}^{t} (\mathfrak{u}(p_i, q_i))\right]$$

where

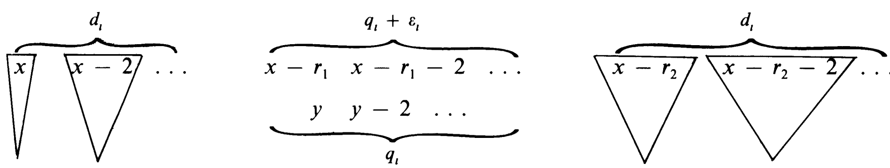
$$r_i = \min(p_i, q_i) + \varepsilon_i$$

$$s_i = \min(p_i, q_i) + \delta_i$$

$$\varepsilon_i = \begin{cases} 1 & p_i \equiv q_i + 1 \pmod 2 \quad \textit{and} \quad p_i > q_i, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\delta_i = \begin{cases} 1 & p_i \equiv q_i + 1 \pmod 2 \quad \textit{and} \quad q_i > p_i, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. This is immediate from Proposition 8.2 and Proposition 3.4(a). We simply need to note that if say $p_i \geqslant q_i$ then the picture for $\mu^L + 2\varrho_{\mathfrak{l} \cap \mathfrak{k}}$ (cfr. 3.4) in the $\mathfrak{u}(p_i, q_i)$ factor looks like



etc. Q.E.D.

We want to obtain now, necessary and sufficient conditions for a representation of K to be the lowest K-type of a $(\mathfrak{g}, K)$ module $A_{\mathfrak{q}}(\lambda)$.

Let $\mu \in i(\mathfrak{t}_0^c)^*$ be the highest weight of a representation of K.

Write

$$\mathfrak{q} \cap \mathfrak{k} = (\mathfrak{q} \cap \mathfrak{k})(\mu), \quad \text{as in 2.1 for } x = \mu,$$

$$\mu' = \mu + 2\varrho(\mathfrak{u} \cap \mathfrak{k}),$$

$$\mathfrak{q} = \mathfrak{q}(\mu) \quad \text{and} \quad \mathfrak{q}' = \mathfrak{q}(\mu').$$

By Proposition 4.4 and Lemmas 4.6, 4.8 we may assume that μ determines $\mathfrak{q}' \cap \mathfrak{k}$ and μ' determines $\mathfrak{q}'$ (note that $\mathfrak{q}(\mu) \neq \mathfrak{q}(\mu')$ but their compact parts coincide).

Write

$$\mathfrak{l}' = \mathfrak{l}(\mu') \cong \mathfrak{s}(\mathfrak{u}(k_1, l_1) \oplus \cdots \oplus \mathfrak{u}(k_t, l_t)), \tag{8.5}$$

where $\Pi_p = (k_1, k_2, \ldots, k_t)$ and $\pi_q = (l_1, l_2, \ldots, l_t)$ are the coarsest partition of p and q, respectively, such that

$$z_i = \mu'|_{\mathfrak{u}(k_i, l_i)} \quad \text{is constant.}$$

Then, an easy argument shows that

Proposition 8.6. *In the above setting, let $n_i = k_i + l_i$, $i = 1, 2, \ldots, t$. Then μ is the lowest K-type of an $A_{\mathfrak{q}}(\lambda)$ iff $z_i - z_{i+1} \geqslant n_i + n_{i+1}$.*

The proof is straightforward if we use the conditions on λ and μ given in 4.1 and 4.3.

We proceed now to the proof of Theorem 5.7. Suppose $X \in \mathcal{M}(\mathfrak{g}, K)$ is as in Theorem 5.7 with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$, and let $\mu \in i(\mathfrak{t}_0^c)^*$ be the highest weight of a lowest K-type of X.

Let us consider a slightly different splitting of the coordinates of μ than that of 8.1. Write

$$\mu = (\underbrace{x_1, \ldots, x_1}_{p_1}, \underbrace{x_2, \ldots, x_2}_{p_2}, \ldots, \underbrace{x_t \ldots x_t}_{p_t} \mid \underbrace{y_1, \ldots, y_1}_{q_1},$$
$$\underbrace{y_2 \ldots y_2}_{q_2} \ldots \underbrace{y_s \ldots y_s}_{q_s}) \tag{8.7}$$

so that

$$x_1 > x_2 > \cdots > x_t$$

$$y_1 > y_2 > \cdots > y_t$$

but here $p_i, q_j > 0$, that is, this splitting is not necessarily compatible with the blocks given by $\mu + 2\varrho_c$.

It is convenient to draw a picture of the coordinates of μ with the same blocks obtained from $\mu + 2\varrho_c$. We are going to study what happens around the first p_1 coordinates of μ.

We may assume that either

$$x_1 + p - 1 > y_1 + q - 1$$

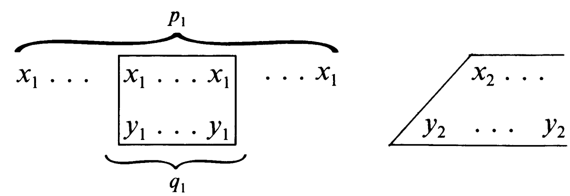
or

$$x_1 + p - 1 = y_1 + q - 1 \quad \text{and} \quad p_1 \geqslant q_1,$$

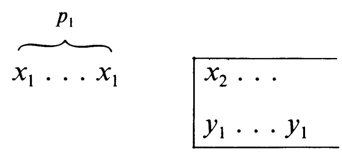
otherwise we can interchange p and q.

If $p_1 < p$ we can have the following configurations for μ

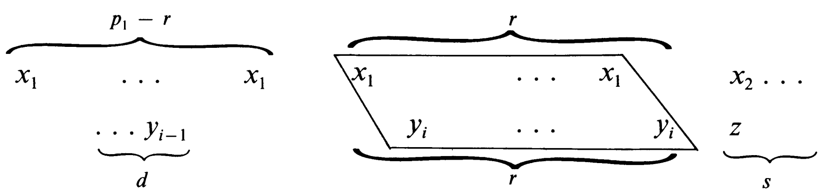
1.



or

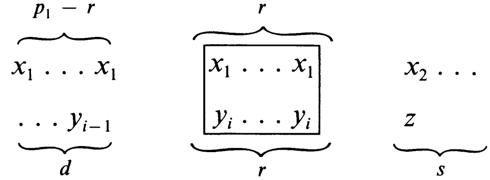


2.



where $y_{i-1} > y_i \geqslant z$.

3.



with $y_{i-1} > y_i \geqslant z$.

4.

$$\begin{array}{cccc} x_1 \dots x_1 \dots & \boxed{\begin{array}{c} x_j \dots x_j \\ y_1 \dots y_1 \end{array}} & z & \dots \\ y_1 \dots y_1 & & y_2 & \end{array}$$

$x_1 > x_j \geqslant z$

5.

$$\begin{array}{cccccc} x_1 \dots x_1 & & \dots & x_j \dots x_j & z & \dots \\ & y_1 \dots y_1 & & y_1 \dots y_1 & y_2 & \end{array}$$

$x_1 > x_j \geqslant z.$

The blocks in these pictures are some of the simple factors of $\ell_V(X)$.

Because of Corollary 8.4, if μ is the lowest K-type of an $A_{\mathfrak{q}}(\lambda)$ module we must be in case one. Therefore, for this case we need to find a reductive subgroup $L \subseteq G$ so that X is a Langlands quotient of a Zuckerman module coming from a representation of L and

(a) in the case when we have an $A_{\mathfrak{q}}(\lambda)$ for L, the derived functor preserves the signature of the form,

(b) otherwise, (a)–(c) of Theorem 5.7 hold.

On the other hand, for cases 2–5 we need to prove non-unitarity. In each case a group L will be found as in 1, making sure that (a)–(c) of Theorem 5.7 hold.

All this will reduce the problem to the case $p_1 = p$.

In this case we have two configurations

6.

$$\begin{array}{ccccc} x_1 & \boxed{\begin{array}{c} x_1 \dots x_1 \\ y_1 \dots y_1 \end{array}} & \dots x_1 \dots & \boxed{\begin{array}{c} x_1 \dots x_1 \\ y_k \dots y_k \end{array}} & z \dots \end{array}$$

$y_1 > y_k \geqslant z.$

7.

$$\begin{array}{ccccc} x_1 & \boxed{\begin{array}{c} \quad \dots x_1 \\ y_1 \; \dots y_1 \end{array}} & \begin{array}{c} \dots \\ \dots \end{array} & \boxed{\begin{array}{c} x_1 \dots \\ y_s \dots y_s \end{array}} & x_1 \dots x_1. \end{array}$$

Case 6 can be included in either 2 or 3 and case 7 will be dealt with in a similar fashion.

Note that as soon as we have shown that (a) of Theorem 5.7 holds, then by Lemma 6.5 the representation of L in question, as well as its Hermitian dual, have a Hermitian form.

For 1, let $\ell = \mathfrak{s}(\mathfrak{u}(p_1, q_a) \oplus \mathfrak{u}(p - p_1, q - q_a))$, here q_a is either q_1 or 0. For 2, choose

$$\ell = \mathfrak{s}(\mathfrak{u}(p_1 - r, d) \oplus \mathfrak{u}(r, r) \oplus \mathfrak{u}(p - p_1, s)).$$

For 3, let

$$\ell = \mathfrak{s}(\mathfrak{u}(p_1 - r - 1, d) \oplus \mathfrak{u}(r + 1, r) \oplus \mathfrak{u}(p - p_1, s)).$$

In cases 1 and 2, $\ell \supseteq \ell_V$. Hence, by induction by stages, arguing as in §7, there is some $(\ell, L \cap K)$-module X_L such that X is the Langlands quotient of

$$\mathscr{R}_{\mathfrak{q}}(X_L), \quad \text{where} \quad \mathfrak{q} = \ell + \mathfrak{u}, \quad \mathfrak{u} \subseteq \mathfrak{u}_V.$$

In case 3, $\ell \not\supseteq \ell_V$. However, Proposition 8.2.15 of Vogan (1981) gives the same result, even if $\mathfrak{q} \not\supseteq \mathfrak{q}_V$.

Cases 4 and 5 are solved the same way as 2 and 3, so we not discuss them in detail.

Now for case 1, assume that $X_L \cong A_{\mathfrak{q}_1}(\lambda)$. Define $\mathfrak{q}_2$ by $\mathfrak{u}_2 = \mathfrak{u} + \mathfrak{u}_1$ and

$$\mathfrak{q}_2 = \mathfrak{q}_1 + \mathfrak{u} = \ell_1 + \mathfrak{u}_1 + \mathfrak{u}.$$

Then

$$\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}}(A_{\mathfrak{q}_1}(\lambda)) \cong \mathscr{R}_{\mathfrak{q}_2}^{\mathfrak{g}}(\mathbb{C}_\lambda)$$

(by induction by stages again).

To see that $\mathscr{R}_{\mathfrak{q}_2}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is a module $A_{\mathfrak{q}_2}(\mathbb{C}_\lambda)$ amounts to checking that $\langle \lambda, \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{u}_2)$. This can be done using 3.4(a).

Set $\mu^L = \mu - 2\varrho(\mathfrak{u} \cap \mathfrak{p})$. By Proposition 3.4 μ^L is the highest weight of a lowest $L \cap K$-type of the module X_L.

Since $p_1 < p$ then $L \neq G$ and $\dim L < \dim G$. Assume μ^L is not the lowest K-type of a module $A_{\mathfrak{q}}(\lambda)$. Hence, by induction, there exists an $L \cap K$-type V_{η^L} in $V_{\mu^L} \otimes (\ell \cap \mathfrak{p})$ such that, on $V_{\mu^L} \oplus V_{\eta^L}$ the Hermitian form is indefinite.

A highest weight of an $L \cap K$-type in $V_{\mu^L} \otimes (\ell \cap \mathfrak{p})$ is then of the form $\mu^L + \beta$ for some $\beta \in \Delta(\ell \cap \mathfrak{p})$.

It is straightforward to check that if $\mu^L + \beta$ is a highest weight, then $\mu + \beta$ is $\mathfrak{k}$-dominant and hence, Theorem 5.8 gives (c) of Theorem 5.7.

For cases 2 and 3 we need the following

LEMMA 8.8. *Let*

$$\mu = (\underbrace{a+1, a+1, \ldots, a+1}_{p} \mid \underbrace{a, a, \ldots, a}_{q})$$

be a weight in $\mathfrak{t}^*$.

If $p = q$ or $p = q + 1$ then Dirac operator inequality fails on μ for

$$\varrho_n^+ = \left(\frac{q}{2}, \frac{q}{2}, \ldots, \frac{q}{2} \middle| \frac{-p}{2}, \frac{-p}{2}, \ldots, \frac{-p}{2}\right)$$

(*see* 6.1).

Proof. Write μ as $\mu_c + \mu_s$ with $\mu_c \in (\text{center } \mathfrak{g})^*$ and $\mu_s \in [\mathfrak{g}, \mathfrak{g}]$. We need to prove that (6.2) does not hold. Note that 6.2 is equal to

$$\langle \mu_c, \mu_c \rangle + \langle w(\mu_s - \varrho_n^+) + \varrho_c, w(\mu_s - \varrho_n^+) + \varrho_c \rangle.$$

If X is a $(\mathfrak{g}, K)$-module with infinitesimal character γ, then

$$\langle \gamma, \gamma \rangle \geqslant \langle \mu_c, \mu_c \rangle + \langle \varrho, \varrho \rangle.$$

Hence it is enough to show that

$$\langle w(\mu_s - \varrho_n^+) + \varrho_c, w(\mu_s - \varrho_n^+) + \varrho_c \rangle < \langle \varrho, \varrho \rangle.$$

This can be computed explicitly. Q.E.D.

We can prove now 5.7(c) for cases 2 and 3. Recall that for 2,

$$\mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1 - r, d) \oplus \mathfrak{u}(r, r) \oplus \mathfrak{u}(p - p_1, s));$$

and for 3,

$$\mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1 - r - 1, d) \oplus \mathfrak{u}(r + 1, r) \oplus \mathfrak{u}(p - p_1, s)).$$

An easy calculation shows that in both cases

$$\mu^L|_{U(r,r)} = (a+1, a+1, \ldots, a+1 | a, a, \ldots, a)$$

or

$$\mu^L|_{U(r+1,r)} = (a+1, a+1, \ldots, a+1 | a, a, \ldots, a).$$

Hence, by Lemma 8.8 and (b) of Lemma 6.3 there is $\beta \in \Delta(\mathfrak{u}(r, r) \cap \mathfrak{p}^-)$ or $\Delta(\mathfrak{u}(r+1, r) \cap \mathfrak{p}^-)$ such that the Hermitian form $\langle\ ,\ \rangle^L$ on $V_{\mu^L} \oplus V_{\mu^L+\beta}$ is indefinite. Now if $\mu^L + \beta$ is dominant, necessarily

$$\beta = (0 \ldots 0 - 1 | 1, 0, \ldots 0).$$

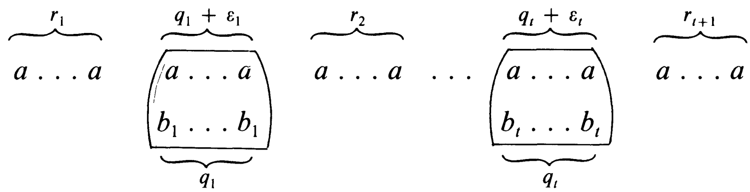
Also $\mu + \beta = (x_1, x_1, \ldots, x_1, x_1 - 1, x_2, \ldots | \ldots y_{i-1}, y_i + 1, y_i, \ldots)$ is $\mathfrak{k}$-dominant and Theorem 5.8 gives again 5.7(c).

4 and 5 are solved in exactly the same way as 2 and 3, using ϱ_n^- and $\mathfrak{p}^+$. So I have reduced the problem to the case

$$p_1 = p.$$

6 can be included in either 2 or 3.

For 7 write $\mu = (a\, a \ldots a | b_1 \ldots b_1, b_2 \ldots b_2 \ldots b_t \ldots b_t)$ the picture for μ is



with $\varepsilon_j = 0, 1$

$$p = \sum_1^{t+1} r_i + \sum_1^{t} q_j + \varepsilon_j.$$

$$q = \sum_1^{t} q_j.$$

Note that if $q_i = 0$ for all $i > 1$ this is case 1. So assume $t \geqslant 2$.

As before, I want to find a group L to which I can apply some reduction argument.

Suppose $r_{t+1} > r_1$ set $s = r_1 + q_1 + \varepsilon_1 + r_2$.

Then let $L = U(s, q_1) \times U(p - s, q - q_1)$. Note that

$$L_V = U(1)^{r_1} \times U(q_1 + \varepsilon_1, q_1) \times U(1)^{r_2} \times \cdots \times U(q_t + \varepsilon_t, q_t) \times U(1)^{r_{t+1}}.$$

So $L \supseteq L_V$, and arguing as in the preceding cases we can verify (a) of Theorem 5.7.

By Proposition 3.4, if $\gamma = (\lambda_V, \nu)$ is the infinitesimal character of $\mathscr{R}_q(X_L)$, then $\gamma^L = (\lambda_V - \varrho(u), \nu)$ is the infinitesimal character of X_L. In fact, by definition of $\Delta(u_V)$,

$$\langle \lambda_V, \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta(u) \subseteq \Delta(u_V).$$

Write $L_1 = U(s, q_1)$.

We want to contradict Theorem 6.1. For some values of r_1, r_2 it could be possible to prove the failure of Dirac inequality as we have done before; that is, by simply using the minimal value of the restriction of ν to the split part of the Cartan of L_1 that makes $\gamma^L|_{L_1}$ regular integral.

However, this is not possible for all values of r_1, r_2. Therefore, we need to involve all of ν instead. This is done by a lengthy and explicit but straightforward calculation. The idea is that it is enough to prove the failure of Dirac operator inequality on

$$\mu^L|_{L_1} \quad \text{and} \quad \varrho_n^-(\ell_1) = \Bigg(\underbrace{\frac{-q_1}{2}, \ldots, \frac{-q_1}{2}}_{s}\Bigg|\underbrace{\frac{s}{2}, \ldots, \frac{s}{2}}_{q_1}\Bigg).$$

That is, if $\gamma^1 = \gamma - \varrho(u)|_{L_1}$ and $w \in W_K$ makes $w\gamma^1$ dominant, it is enough to prove

$$\langle w\gamma_1, w\gamma_1 \rangle - \langle \mu^L|_{L_1} - \varrho_n^-(\ell_1) + \varrho_{\ell_1 \cap k}, \mu^L|_{L_1} - \varrho_n^-(\ell_1) + \varrho_{\ell_1 \cap k} \rangle > 0. \tag{$*$}$$

Now if $r_1 > r_{t+1}$, we choose

$$L = U(r_t + q_t + \varepsilon_t + r_{t+1}, q_t) \times U(p - (r_t + q_t + \varepsilon_t + r_{t+1}), q - q_t)$$

and repeat the same argument for this case.

We next observe that $(*)$ will also hold if $r_1 = r_{t+1}$ and some $\varepsilon_i > 0$ or some $r_j > 0$; $1 < j, i \leqslant t$.

So this reduces to the case

$$\begin{matrix} \overbrace{a \ldots a}^{r} & \overbrace{a \ldots a}^{q_1 + \varepsilon_1} & \overbrace{a \ldots a}^{q_2} & \overbrace{a \ldots a}^{r} \\ & \underbrace{b_1 \ldots b_1}_{q_1} & \underbrace{b_2 \ldots b_2}_{q_2} & \end{matrix} \qquad q_1, q_2 > 0.$$

But by symmetry, using the case $r_1 > r_{t+1}$, we can conclude that $\varepsilon_1 = 0$. But then we have

$$\begin{array}{ccc} \overbrace{a \ldots a}^{r} & \overbrace{a \ldots a}^{q} & \overbrace{a \ldots a}^{r} \\ & \underbrace{b \ldots b}_{q} & \end{array}$$

With which we have dealt before. This is solved in the same way as case 1 for $p_1 < p$.

This proves Theorem 5.7 for $G = SU(p, q)$. Q.E.D.

§9. Proof of Theorem 5.7 for $G = SP(n, \mathbb{R})$

Let I_m be the identity matrix in $GL(m, \mathbb{C})$. We define

$$G = SP(n, \mathbb{R}) = \left\{ g \in SL(2n, \mathbb{R}) \middle| {}^t g \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\}.$$

The maximal compact subgroup K of G is

$$K = SP(n, \mathbb{R}) \cap U(2n) \cong U(n).$$

$\hat{K}$ can be identified with the space

$$\{\mu = (a_1 \ldots a_n) | a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n;\ a_i \in \mathbb{Z}\}.$$

The roots of $\mathfrak{t}$ in $\mathfrak{g}$ are

$$\begin{aligned} \Delta(\mathfrak{g}) &= \Delta(\mathfrak{g}, \mathfrak{t}^c) \\ &= \{\pm(e_j \pm e_k);\ \pm 2e_i | j, k, i = 1, 2, \ldots, n;\ j < k\} \end{aligned}$$

also

$$\Delta(\mathfrak{k}) = \Delta(\mathfrak{k}, \mathfrak{t}^c) = \{\pm(e_j - e_k) | 1 \leqslant j < k \leqslant n\},$$

the compact imaginary roots of t^c in g.

$$\Delta(p) = \Delta(p, t^c) = \{\pm(e_j + e_k);\ \pm 2e_i | 1 \leqslant j < k \leqslant n;\ 1 \leqslant i \leqslant n\},$$

the non-compact imaginary roots of t^c in g.

As for the preceding cases, fix a positive root system $\Delta^+(k)$ so that if

$$\mu = (a_1, a_2, \ldots, a_n) \qquad a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n,$$

then μ is $\Delta^+(k)$-dominant and

$$2\varrho_c = (n-1, n-3, \ldots, -n+3, -n+1).$$

Let $\mu + 2\varrho_c = (x_1, x_2, \ldots, x_n)$.

Choosing a positive Weyl Chamber for $\Delta(g, k^c)$, given by $\mu + 2\varrho_c$ corresponds to forming an array of two rows with the absolute value of the coordinates of $\mu + 2\varrho_c$ so that they are aligned in decreasing order as follows:

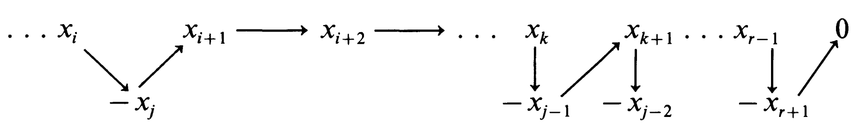
If $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r \geqslant 0 > x_{r+1} \geqslant \cdots \geqslant x_n$ then $x_1, \ldots, x_r$ are in the first row $-x_n, -x_{n-1}, \ldots, -x_{r+1}$ in the second and they all decrease from left to right in the array.

For example, if we have

$$\ldots x_i > -x_j > x_{i+1} > x_{i+2} > \cdots > x_k = -x_{j-1} > x_{k+1}$$

$$= -x_{j-2} > \cdots > -x_{r+1} = x_{r-1} > x_r = 0$$

the array would look like

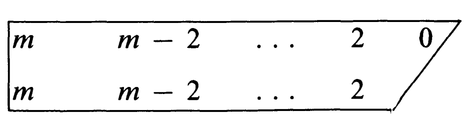


As for the case of $SU(p, q)$, the choice of arrows gives a positive root system $\Delta^+ = \Delta^+(g, t^c)$, compatible with $\Delta^+(k)$.

Again, the entire array is a union of blocks of the following types.

1–5 all blocks of the five types discussed for $SU(p, q)$ not containing 1 or 0.

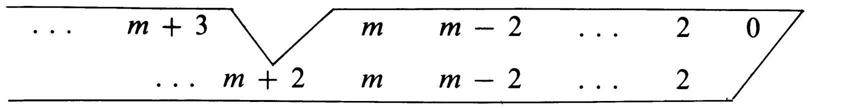
6.



7.

$$\boxed{\begin{array}{cccc} m & m-2 & \dots & 1 \\ m & m-2 & \dots & 1 \end{array}}$$

8.



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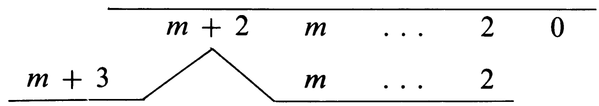
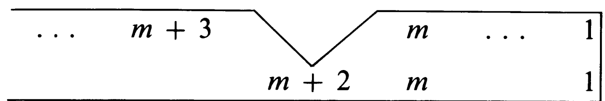


9.



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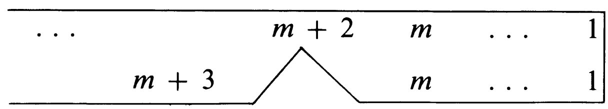
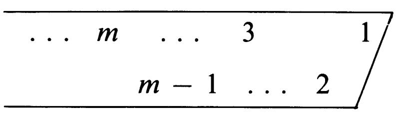
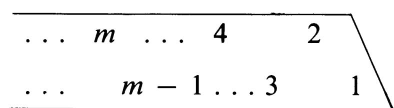


10.



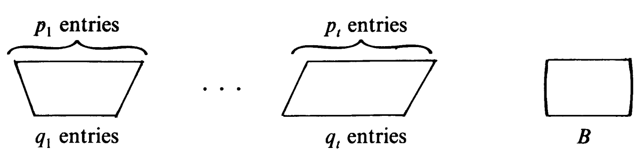
, or



(10 is a particular case of 9).

Again, using the picture, split the coordinates of μ by the blocks that $\mu + 2\varrho_c$ determines as follows.

If $\mu + 2\varrho_c$ gives



(9.1)

with B a block of some type 6–10, set

$$\mu = (\underbrace{a_1 \dots a_1}_{p_1 \text{ times}} \dots \underbrace{a_t \dots a_t}_{p_t \text{ times}} \,|\, \underbrace{c_1 c_2 \dots c_m}_{m \text{ entries}} \,|\, \underbrace{b_t \dots b_t}_{q_t \text{ times}} \dots \underbrace{b_1 \dots b_1}_{q_1 \text{ times}})$$

where m is the total number of coordinates composing the block B.

Write $\lambda_V = \lambda_V(\mu)$ and $\ell_V = \ell_V(\lambda_V(\mu))$ as in §3.

PROPOSITION 9.2. *If* $\mu \in i(\ell_0^c)^*$ *gives figure* 9.1 *then*

$$\lambda_V(\mu) = (\underbrace{\lambda_1 \dots \lambda_1}_{p_1 \text{ times}} \underbrace{\lambda_2 \dots \lambda_2}_{p_2 \text{ times}} \dots \underbrace{\lambda_t \dots \lambda_t}_{p_t \text{ times}} | \underbrace{0 \dots 0}_{m} |$$

$$\underbrace{-\lambda_t \dots -\lambda_t}_{q_t} \dots \underbrace{-\lambda_1 \dots -\lambda_1}_{q_1})$$

$$\ell_V \cong u(p_1, q_1) \oplus \cdots \oplus u(p_t, q_t) \oplus sp(m, \mathbb{R})$$

with

$$\lambda_1 > \lambda_2 > \cdots > \lambda_t > 0.$$

Now suppose μ *is the highest weight of a representation of K. We want an analogue of Proposition* 8.6.

By Proposition 4.5 *we may use* μ *to determine a compact parabolic subalgebra* $q \cap k = \ell \cap k + u \cap k$.

Set $2\varrho(u \cap k) = 2\varrho(\Delta(u \cap k))$. *Suppose that*

$$\mu + 2\varrho(u \cap k) = (\underbrace{a_1 \dots a_1}_{r_1} \dots \underbrace{a_t \dots a_t}_{r_t} | \underbrace{0 \dots 0}_{m} |$$

$$\underbrace{-a_t \dots -a_t}_{s_t} \dots \underbrace{-a_1 \dots -a_1}_{s_1}).$$

PROPOSITION 9.3. *In the above setting, set* $n_i = r_i + s_i$ *then* μ *is the lowest K-type of some* $A_q(\lambda)$

$$\Longleftrightarrow \quad a_i - a_{i+1} > n_i + n_{i+1},$$

and

$$a_t \geqslant n_t + 2m + 1.$$

This also follows from 4.1 *and* 4.3.

We proceed to the proof of Theorem 5.7 for this case. Let X as in Theorem 5.7, with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$, $\mu \in i(\mathfrak{t}^c)^*$, the highest weight of a lowest K-type of X. Suppose X is not a module $A_{\mathfrak{q}}(\lambda)$. Let $\mathfrak{l}_V = \mathfrak{l}_V(X) = (\mathfrak{u}(p_1, q_1) \oplus \mathfrak{u}(p_2, q_2) \oplus \cdots \oplus \mathfrak{u}(p_t, q_t)) \oplus \mathfrak{sp}(m, \mathbb{R})$ (cfr. 9.2) and $p = \Sigma\, p_i$, $q = \Sigma\, q_i$. Set

$$\mathfrak{l}_1 = \mathfrak{u}(p, q), \qquad \mathfrak{l}_2 = \mathfrak{sp}(m, \mathbb{R})$$

then

$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \supseteq \mathfrak{l}_V.$$

Define $\mathfrak{u} \subseteq \mathfrak{u}_V$ by $\mathfrak{u}_V = \mathfrak{u} + (\mathfrak{u}_V \cap \mathfrak{l})$.

Then $\mathfrak{q} \supset \mathfrak{q}_V$ and by Induction by Stages, (a) of Theorem 5.7 holds.

Now let X_L be an $(\mathfrak{l}, L \cap K)$-module such that X occurs only once as composition factor of $\mathscr{R}_{\mathfrak{q}}(X_L)$. We can see X_L as the exterior tensor product $X_L = X_{L_1} \otimes X_{L_2}$ with X_{L_i} an $(\mathfrak{l}_i, L_i \cap K)$-module.

That X_L^h has a Hermitian form $\langle\ ,\ \rangle^L$ follows from Lemma 6.5. Set $\mu^L = \mu - 2\varrho(\mathfrak{u} \cap \mathfrak{p})$, $\mu^i = \mu^L|_{L_i}$.

LEMMA 9.4. $X_{L_1} \cong A_{\mathfrak{q}^0}(\lambda^0)$, *for some* $\mathfrak{q}^0 \subseteq L_1$; $\lambda^0\colon \mathfrak{l}_0 \to \mathbb{C}$.

Proof. By Theorem 5.7(b) and (c) (proved for $SU(p, q)$) and Theorem 5.8 if $X_{L_1} \ncong A_{\mathfrak{q}}(\lambda)$ then there is $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$ such that $\langle\ ,\ \rangle^L$ is indefinite on the sum

$$V_{\mu^1} \oplus V_{\mu^1+\beta}.$$

If $\mu = (x_1, \ldots, x_p | x_{p+1}, \ldots, x_{p+m} | x_{p+m+1}, \ldots, x_m)$ then, since

$$\Delta(\mathfrak{l}_1 \cap \mathfrak{p}) = \pm\{(e_i + e_j) | 1 \leqslant i \leqslant p, p + m \leqslant j \leqslant n\}$$

it is clear that if $\mu^1 + \beta$ is dominant for $\Delta(\mathfrak{l}_1 \cap \mathfrak{k})$, then $\mu + \beta$ is dominant for $\Delta^+(\mathfrak{k})$, unless $x_p = x_{p+1}$ or $x_{p+m} = x_{p+m+1}$.

Suppose then that $x_p = x_{p+1}$.

Note that $\mu^{L_v}|_{L_2} = \mu^2$ and hence μ^2 is fine and X_{L_2} is a principal series.

So $\mu^2 \in \{(0 \ldots 0); (1 \ldots 1, 0 \ldots 0); (0, \ldots, 0, -1, -1 \ldots -1)\}$.

If μ^2 is trivial it is easy to see (looking at the pictures 6–7 given by $\mu + 2\varrho_c$) that $x_p - x_{p+1} > 0$, as well as $x_{p+m} - x_{p+m+1} > 0$.

Note that, by Frobenius reciprocity, both

$$(\underbrace{1, 1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{m-a}) \quad \text{and} \quad (\underbrace{0, \ldots, 0}_{m-a}, \underbrace{-1, -1, \ldots, -1}_{a})$$

should occur in the same principal series. So if μ^2 is a non-trivial fine K-type call η^2 the corresponding other non-trivial fine K-type. Then $\eta = \mu^1 + \eta^2 + 2\varrho(\mathfrak{u} \cap \mathfrak{p})$ is a lowest K-type of X. This implies that

$$\text{if} \quad x_p = x_{p+1} \implies x_{p+m} > x_{p+m+1}$$

$$\text{and} \quad x_{p+m} = x_{p+m+1} \implies x_p > x_{p+1}.$$

So, since

$$\eta + \beta|_{L_1} = \mu + \beta|_{L_1}$$

for any $\beta \in \Delta(\ell_1 \cap \mathfrak{p})$, then either $\mu + \beta$ or $\gamma + \beta$ is $\mathfrak{k}$-dominant proving 5.7(c). Q.E.D.

LEMMA 9.5. *In the above setting, assume that* $X_{L_1} \cong A_{\mathfrak{q}^0}(\lambda^0)$ *for some* $\mathfrak{q}^0 \subseteq \ell_1$ *and* $\lambda^0 \colon \ell^0 \to \mathbb{C}_{\lambda^0}$.

Then, Theorem 5.7 *is true if we assume that*

$$\begin{cases} x_p - x_{p+1} \geqslant 2 \\ \text{and} \quad x_{p+m} - x_{p+m+1} \geqslant 2. \end{cases} \tag{9.6}$$

Proof. Suppose first that

$$\mu^2 = (\underbrace{1, 1, \ldots, 1}_{a}, \underbrace{0 \ldots 0}_{m-a}).$$

Then if

$$\varrho_n^+ = \left[\frac{m+1}{2}, \ldots, \frac{m+1}{2}\right],$$

an easy calculation shows

$$\langle \mu^2 - \varrho_n^+ + \varrho_{\ell_2 \cap \mathfrak{k}}, \mu^2 - \varrho_n^+ + \varrho_{\ell_2 \cap \mathfrak{k}} \rangle < \langle \varrho, \varrho \rangle.$$

By (b) in Lemma 6.3, there is a

$$\beta \in \{(\underbrace{0 \ldots 0\ -1}_{a}\ \underbrace{0 \ldots 0\ -1}_{m-a}), (\underbrace{0, 0 \ldots 0\ -2}_{m})\}$$

making $V_{\mu^2} \oplus V_{\mu^2+\beta}$ into a space on which $\langle\ ,\ \rangle^L$ is indefinite.

Moreover $\mu + \beta$ is $\Delta^+(\not{k})$-dominant, by (9.6). Similarly if

$$\mu^2 = (\underbrace{0 \dots 0}_{m-a} \underbrace{-1, -1, \dots, -1)}_{a}$$

then

$$\beta \in \{(\underbrace{1, 0 \dots 0\, 0}_{m-a}\, \underbrace{1 \dots 0}_{a}); (2, 0 \dots)\}.$$

Now, if $\mu^2 = (0 \dots 0)$ then the Dirac operator inequality fails for any choice of $\varrho_n = \varrho(\Delta^+(\ell_2 \cap \not{p}))$, unless $\gamma|_{\ell_2} = \varrho_{\ell_2}$ in particular, if

$$\varrho_n^+ = \left[\frac{m+1}{2}, \dots, \frac{m+1}{2}\right];$$

and, obviously, $\mu + \beta$ is also dominant for $\beta \in \Delta(\not{p}^- \cap \ell_2)$.

Now if $\gamma|_{\ell_2} = \varrho_{\ell_2}$, then, the Langlands subquotient of X_{L_2} is the trivial representation. (In fact, the representation $X_{L_2} = I(\delta_V^{L_2} \otimes \nu_V^{L_2})$ is a principal series and $\delta_V^{L_2} =$ trivial; $\gamma|_{\ell_2} = \nu_V|_{\ell_2} = \nu_V^{L_2}$.)

Hence the Langlands submodule of

$$\mathscr{R}_{\mathscr{q}}(X_{L_1} \otimes X_{L_2}) = \mathscr{R}_{\mathscr{q}}(X_{L_1}) \otimes \mathscr{R}_{\mathscr{q}}(X_{L_2})$$

is

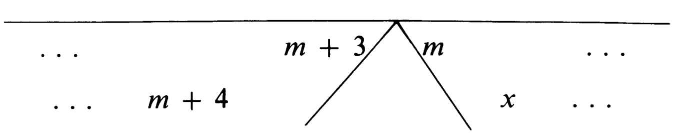
$$X \cong \mathscr{R}_{\mathscr{q}}(A_{\mathscr{q}^0}(\lambda^0)) \otimes \mathscr{R}_{\mathscr{q}} \text{ (trivial representation)}.$$

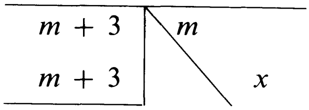
By induction by stages, X is an $A_{\mathscr{q}}(\lambda)$, contradicting our assumptions on X. This proves the lemma. Q.E.D.

To finish the proof of Theorem 5.7, suppose now that $x_p - x_{p+1} \leqslant 1$.

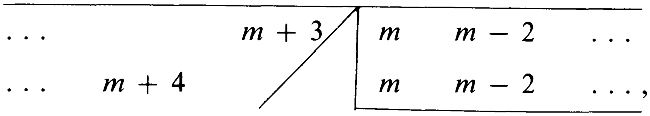
LEMMA 9.7. *Under the hypothesis of Lemma* 9.5 *if* $x_p - x_{p+1} = 1$ *and* $x_{p+m} - x_{p+m+1} \geqslant 2$, *then Theorem* 5.7 *is true.*

Proof. The assumptions on the coordinates of μ imply that the picture of $\mu + 2\varrho_c$ around the coordinates involved is either

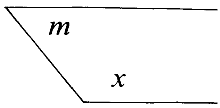




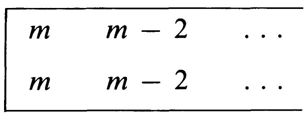
where $x - m = 1$ or 2, or



Observe that $\mu^{L_V} = \mu - 2\varrho(u_V \cap p)$ is fine and that the fine K-type that gives the picture



is $\mu^2 = (1, 1, \ldots, 1, 0 \ldots 0)$; and the fine K-type that gives



is $\mu^2 = (0 \ldots 0)$.

Arguing as in the proof of Lemma 9.5 we can find, in both cases

$$\beta \in \{(0 \ldots 0\ -1, 0 \ldots 0\ -1); (0 \ldots 0\ -2)\}$$

as we want. Q.E.D.

LEMMA 9.8. *Under the hypothesis of Lemma* 9.5, *assume now that*

$$\begin{cases} 0 \leqslant x_p - x_{p+1} \leqslant 1 \\ 0 \leqslant x_{p+m} - x_{p+m+1} \leqslant 1. \end{cases} \tag{9.9}$$

Then the infinitesimal character γ of X is not regular integral.

Proof. We want to contradict the assumption that the infinitesimal character γ is regular and integral. Since we have an $A_q(\lambda)$-module for $L_1 = U(p, q)$, we have some control on γ.

Recall that $L = U(p, q) \times SP(m, \mathbb{R})$ and

$$L \supseteq L_V = \left[\prod_{i=1}^{t} (U(p_i, q_i))\right] \times SP(m, \mathbb{R}).$$

We may assume $p_t \geqslant q_t$. By Corollary 8.4, either

$$\lambda_V|_{U(p_t,q_t)} = (\underbrace{\lambda_t + s, \lambda_t + s - 1 \ldots \lambda_t + 1}_{s} \underbrace{\lambda_t \ldots \lambda_t}_{q_t+1}$$

$$\underbrace{\lambda_t - 1 \ldots \lambda_t - s}_{s} | \underbrace{\lambda_t \ldots \lambda_t}_{q_t})$$

or

$$\lambda_V|_{U(p_t,q_t)} = (\underbrace{\lambda_t + s, \ldots, \lambda_t + \tfrac{1}{2}}_{s} \underbrace{\lambda_t \ldots \lambda_t}_{q_t} \underbrace{\lambda_t - \tfrac{1}{2} \ldots \lambda_t - s}_{s} | \underbrace{\lambda_t \ldots \lambda_t}_{q_t})$$

and

$$v|_{U(p_t,q_t)} = (0 \ldots 0\ v_1 \ldots v_{q_t}\ 0 \ldots 0 | -v_1 \ldots -v_{q_t}).$$

Inside $SP(n, \mathbb{R})$ this gives

$$(\lambda_t + s, \ldots, \lambda_t \ldots \lambda_t \ldots \lambda_t - s | -\lambda_t \ldots -\lambda_t)$$

$$(0 \ldots 0\ v_1 \ldots v_{q_t}\ 0 \ldots 0 | v_1 \ldots v_{q_t}).$$

If γ is regular integral

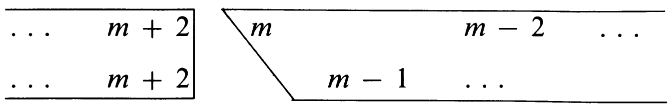
$$\lambda_t + v_{q_t} > \lambda_t + s; \lambda_t - s > 0 > -\lambda_t + v_1 \geqslant -\lambda_t + q_t + v_{q_t} - 1$$

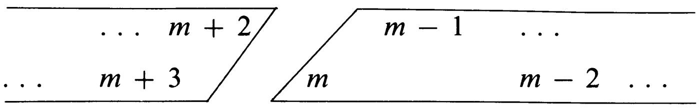
$$\Rightarrow \begin{cases} v_{q_t} > s \\ \lambda_t \geqslant v_{q_t} + q_t \end{cases}$$

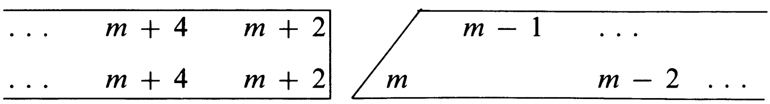
$$\Rightarrow \lambda_t > s + q_t$$

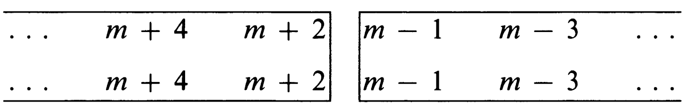
Claim. If μ satisfies (9.9) then $\lambda_t - s \leqslant 1$.

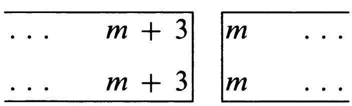
Proof. The picture for $\mu + 2\varrho_c$ around these coordinates can be of the following types.

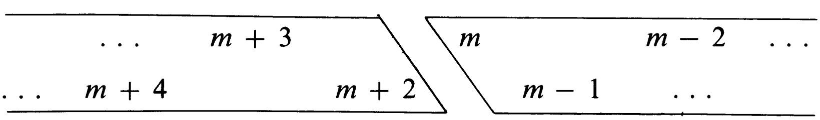
1. 

2. 

3. 

4. 

or 

5. 

So we either have (considering that 5 and 2, and 3 and 1 are symmetric)

$$\mu + 2\varrho_c = (\ldots m + k + 2, m + k|\ldots|-m - k \ldots)$$

and

$$\varrho = (\ldots m + k + 2, m + k|\ldots|-m - k + 1 \ldots)$$

or

$$\mu + 2\varrho_c = (\ldots m + 2|\ldots|-m - 3 \ldots),$$

with

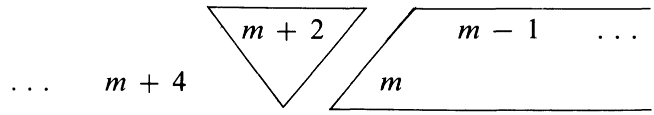
$$\varrho = (\ldots m + 1|\ldots|-m - 2 \ldots)$$

In both cases we get (see Vogan (1981) Proposition 5.3.3)

$$\lambda_V = (\ldots 1\ 1|0 \ldots 0|-1\ -1 \ldots).$$

This proves the claim.

This reduces to the case when $q_t = 0$. But then, $\mu + 2\varrho_c$ gives, at worst,



Because if $q_i = 0$, $p_i = 1$, since $U(p_i, q_i)$ is quasisplit. So, we have $x_{p+m} - x_{p+m+1} \geqslant 2$!

This concludes the proof of Theorem 5.7. Q.E.D.

References

J. Adams: Discrete spectrum of the dual pair $(0(p, q), Sp(2m))$. *Invent. Math.* 74 (1983) 449–475.

J. Adams: Unitary highest weight modules. *Advances in Math.* 63 (1987) 113–137.

J. Arthur: On some problems suggested by the trace formula. *SLN* 1041 (1984) 1–50.

M.W. Baldoni Silva and D. Barbasch: The unitary spectrum for real rank one groups. *Invent. Math.* 72 (1983) 27–55.

D. Barbasch: Unipotent representations for complex semisimple groups II. Preprint (1987).

D. Barbasch and D. Vogan: Unipotent representations for complex semisimple groups. *Ann. Math.* 121 (1985) 41–110.

A. Borel and N. Wallach: Continuous cohomology, discrete subgroups and representations of reductive groups. *Annals of Mathematics Studies* Vol. 94. Princeton University Press (1980).

J. Dixmier: *Enveloping Algebras.* North Holland Publishing Co., Amsterdam, N.Y., Oxford (1977).

M. Duflo and M. Vergne: Sur le foncteur de Zuckerman. *C.R. Acad. Sci. Paris*, t. 304 Serie I, no. 16 (1987) pp. 467–469.

T.J. Enright: Relative Lie algebra cohomology and unitary representations of complex Lie groups. *Duke Math. J.* 46 (1979) 513–525.

T.J. Enright: Unitary representations for two real forms of a semisimple Lie algebra: a theory of comparison. In: *Proc. Special Year in Harmonic Analysis, University of Maryland Lecture Notes in Math.* Vol. 1024 (1984) pp. 1–29. Springer-Verlag, Berlin–Heidelberg–New York–Tokyo.

T.J. Enright and N.R. Wallach: Notes on homological algebra and representations of Lie algebras. *Duke Math. J.* 47 (1980) 1–15.

Harish-Chandra: Representations of semisimple Lie groups I. *Trans. AMS* 75 (1953) 185–243.

A.W. Knapp and D.A. Vogan Jr: Cohomological induction and unitary representations, Preprint.

A. Knapp and G. Zuckerman: Classification theorems for representations of semisimple Lie groups. In: *Non-Commutative Harmonic Analysis. Lecture Notes in Math.* Vol. 587 (1976) pp. 138–159.

R. Langlands: (1973) On the classification of irreducible representations of real algebraic groups. Mimeographed notes, Institute for Advanced Study (1973).

S. Salamanca Riba: On unitary representations with regular infinitesimal character. Thesis dissertation, Massachusetts Institute of Technology, Cambridge, Massachusetts (1986).

W. Schmid: On the characters of discrete series (the Hermitian symmetric case). *Inventiones Math.* 30 (1975) 47–144.

B. Speh: Unitary representations of $SL(n, \mathbb{R})$ and the cohomology of congruence subgroups. In: J. Carmona and M. Vergne (Eds.) *Non-Commutative Harmonic Analysis and Lie groups. Lecture Notes in Math.* Vol. 880 (1981) pp. 483–505.

B. Speh and D. Vogan: Reducibility of generalized principal series representations. *Acta Math.* 145 (1980) 227–229.

D. Vogan: The algebraic structure of the representations of semisimple Lie groups I. *Ann. Math.* 109 (1979) 1–60.

D. Vogan: *Representations of Real Reductive Lie Groups.* Birkhauser, Boston–Basel–Stuttgart (1981).

D. Vogan: Unitarizability of certain series of representations. *Annals Math.* 120 (1984) 141–187.

D. Vogan: The unitary dual of $GL(n)$ over an Archimedian field. *Inv. Math.* 83 (1986) 449–505.

D. Vogan and G. Zuckerman: Unitary representations with non-zero cohomology. *Compositio Mathematica* 53 (1984) 51–90.

D. Wigner: Sur l'homologic relative des algèbres de Lie et une conjecture de Zuckerman. *C.R. Acad. Sci. Paris*, t 305, Série I (1987) pp. 59–62.

G. Zuckerman: On construction of representations by derived factors. Handwritten Notes (1977).