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Birational moduli and nonabelian cohomology

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Introduction

Global moduli spaces in algebraic geometry exist only in very special situations. Indeed, to handle the set of isomorphism classes of objects of a given kind, the first thing one generally tries is to decompose it into a (generally infinite) disjoint union of "quotients" of certain algebraic schemes by certain "algebraic" equivalence relations. But even if such a decomposition into "quotients" is available, the corresponding "quotients" generally do not exist as algebraic schemes (or as algebraic spaces); this is the case for instance with the polarized nonsingular projective varieties where the ruled ones spoil the picture [17]. On the other hand it may happen that no such decomposition into "quotients" is available at all: this seems to be the case (at least apriori) with: a) finitely presented algebras, b) complete local algebras, c) linear algebraic groups, a.s.o.

To remedy the lack of global moduli spaces there are at least two changes of viewpoint which can be made: first one can adopt a "local" viewpoint on moduli (in the sense of Kodaira-Spencer, Schlessinger, . . .); secondly one can adopt a "birational" viewpoint on moduli (as suggested by work of Matsusaka, Shimura, Koizumi [16, 17, 24]).

It is the birational viewpoint which we follow in this paper. It consists in associating to each isomorphism class ξ of objects of a given kind a field $k(\xi)$ which should play the role of "residue field at ξ" on a global moduli space. In [17] Matsusaka proved the existence of the fields $k(\xi)$ for nonsingular polarized projective varieties and called them "fields of moduli"; his strategy was geometric, via "quotients", hence does not seem to apply to cases a), b), c) above.

The aim of this paper is to introduce a new method by which we prove the existence of the fields $k(\xi)$ for large classes of objects (possibly equipped with certain additional structures) belonging to classes a), b), c) above. For precise statements, see Section 2. Our method is of purely algebraic nature;

it is of independent interest since it is based on killing nonabelian cocycles of certain nonprofinite groups.

Rather than speaking about "fields of modulii" we will speak in our paper about "coarse representability" of certain functors of fields; as we shall see "coarse representability" is "essentially equivalent" to the existence of "fields of moduli" (cf. assertion (5) in Theorem (1.5)).

The paper is divided into two parts. In part I we prove an abstract criterion of "coarse representability" for functors of fields (assertion (6) in Theorem (1.5)), we consider some basic examples of such functors and state our main Theorem (2.10) which asserts that most our functors are "coarsely representable" in characteristic zero. In part II we use nonabelian cohomology to split algebras over skew group algebras in order to check that our functors satisfy the axioms appearing in our criterion of coarse representability.

As a concluding remark note that there are remarkable cases where the "local moduli theory" is trivial whereas the "birational moduli theory" is not (e.g., the case of smooth affine varieties, which have no nontrivial infinitesimal deformations but lots of global deformations "depending effectively on a certain number of moduli"). On the other hand there are cases when there is no satisfactory "local moduli theory" whereas there is a satisfactory "birational moduli theory" (e.g. the case of nonnecessary isolated singularities). Both cases above will be discussed in our paper.

PART I: FUNCTORS OF FIELDS

1. Abstract theory

(1.1) Let $\mathbf{B}^a \subset \mathbf{B} \subset \mathbf{B}^e$ be categories (here "$\subset$" means "subcategory") and let $\mathbf{C}: \mathbf{B} \to \mathbf{S}$ (= category of sets) be a contravariant functor. For any object $X \in \mathbf{B}^e$ define the functor $h_X: \mathbf{B} \to \mathbf{S}$ by $h_X(Y) = \mathrm{Hom}_{\mathbf{B}^e}(Y, X)$ for all $Y \in \mathbf{B}$.

We say that an object $X \in \mathbf{B}^e$ coarsely represents $\mathbf{C}$ if there is a morphism $\varphi: \mathbf{C} \to h_X$ satisfying the following properties

(m_1) $\varphi(Y): \mathbf{C}(Y) \to h_X(Y)$ is a bijection for all $Y \in \mathbf{B}^a$

(m_2) For any morphism $\varphi': \mathbf{C} \to h_{X'}$ with $X' \in \mathbf{B}^e$

there is a unique morphism $f \in \mathrm{Hom}_{\mathbf{B}^e}(X, X')$ such that $\varphi' = h_f \circ \varphi$ (where $h_f: h_{X'} \to h_X$ is the morphism naturally induced by f).

Clearly (m_2) uniquely determines X up to a "canonical" isomorphism in $\mathbf{B}^e$.

The prototype for the definition above is Mumford's concept of coarse moduli space (in that case the objects of $\mathbf{B}$ are locally noetherian schemes, those of $\mathbf{B}^e$ are also schemes or more generally algebraic spaces while the objects of $\mathbf{B}^a$ are the spectra of algebraically closed fields).

(1.2) From now on we shall fix a ground field k; all objects and maps will be "over k". Throughout the paper we assume that $\mathbf{B}$ is the dual of the category of fields, $\mathbf{B}^a$ is the full subcategory of $\mathbf{B}$ whose objects are the algebraically closed fields while $\mathbf{B}^e$ is the category which we shall now describe. Given a set X, by a birational structure on it we mean a family of fields $\{k(x);\ x \in X\}$; a set together with a birational structure on it will be called a birational set. By a morphism between two birational sets X and Y we mean a map $f: X \to Y$ together with field homomorphisms $f_x^*: k(f(x)) \to k(x)$ for all $x \in X$. Birational sets form a category which we denote by $\mathbf{B}^e \cdot \mathbf{B}$ is viewed as a subcategory of $\mathbf{B}^e$ by letting a field K be identified with the birational set $X = \{x\}$, $k(x) = K$. A birational set X is called of finitely generated type if $k(x)$ is a finitely generated extension of k for all $x \in X$. A birational set X coarsely representing a functor $\mathbf{C}: \mathbf{B} \to \mathbf{S}$ will be called a birational moduli set for $\mathbf{C}$.

Intuitively the field $k(x)$ should be viewed as the "residue field" of X at x. Moreover note that if $X \in \mathbf{B}^e$ and $K \in \mathbf{B}$ then

$$h_X(K) = \{(x, u);\ x \in X,\ u: k(x) \to K \text{ a field homomorphism}\}$$

so h_X is a "birational analogue" of the "functor of points" of a scheme in algbraic geometry.

(1.3) What we do next is to give a criterion for a functor $\mathbf{C}: \mathbf{B} \to \mathbf{S}$ to possess a birational moduli set of finitely generated type; our criterion will involve (and was motivated by) concepts introduced by Matsusaka, Shimura, Koizumi (especially their concepts of "fields of moduli").

First we fix some notations. If K/K_0 is a field extension, $\mathbf{g}(K/K_0)$ will denote the group of K_0-automorphisms of K and we write $\mathbf{g}(K)$ instead of $\mathbf{g}(K/k)$. If $\mathbf{g} \subset \mathbf{g}(K)$ is a sub-group then $K^{\mathbf{g}}$ will denote the subfield of K of elements fixed by $\mathbf{g}$; $\mathbf{g}$ is called Galois-closed if $\mathbf{g} = \mathbf{g}(K/K^{\mathbf{g}})$.

Now given a functor $\mathbf{C}: \mathbf{B} \to \mathbf{S}$, K a field and $\xi \in \mathbf{C}(K)$ we say that a subfield K_0 of K is a field of definition for ξ if $\xi \in \operatorname{Im}(\mathbf{C}(K_0) \to \mathbf{C}(K))$. Denote by $D(\xi) = D(\xi, \mathbf{C})$ the set of those subfields of K which are fields of definition for ξ. Moreover note that $\mathbf{g}(K)$ acts on $\mathbf{C}(K)$ on the right by the formula $\xi^\sigma = \mathbf{C}(\sigma^{-1})(\xi)$ for $\sigma \in \mathbf{g}(K)$, $\xi \in \mathbf{C}(K)$; let

$$\mathbf{g}(\xi, \mathbf{C}) = \mathbf{g}(\xi) = \{\sigma \in \mathbf{g}(K);\ \xi^\sigma = \xi\}$$

be the isotropy of ξ under this action. Finally define

$$K_\xi = \text{intersection of all members of } D(\xi)$$

$$K_\xi^\infty = \text{intersection of all perfect members of } D(\xi)$$

$$K^\xi = K^{\mathbf{g}(\xi)}.$$

If K is algebraically closed then some easy remarks are in order (cf. [16] [24]):

1) If $K_0 \in D(\xi)$ then $\mathbf{g}(K/K_0) \subset \mathbf{g}(\xi)$, in particular we have $K^\xi \subset K_\xi^\infty$.

2) $\mathbf{g}(\xi)$ acts on $D(\xi)$ hence globally invariates K_ξ^∞; so if $\mathbf{g}(\xi)$ is Galois-closed and K/K_ξ^∞ is not algebraic then K_ξ^∞/K^ξ is an algebraic normal extension.

3) It is not reasonable to expect that $K_\xi \in D(\xi)$; this fails in very nice situations (e.g., $k = \mathbb{Q}$, $\mathbf{C}(K)$ = set of isomorphism classes of non-singular projective curves over K [24]).

(1.4) A field will be called universal if it is algebraically closed and has uncoutable transcendence degree over k. We denote by $\mathbf{B}^u$ the full subcategory of $\mathbf{B}$ consisting of universal fields and by $\mathbf{B}^\omega$ the full subcategory of $\mathbf{B}$ consisting of those fields which are countably generated over k; so $\mathbf{B}^u = \mathbf{B}^a \backslash \mathbf{B}^\omega$. A field extension K/K_0 is called regular if K_0 is seperably closed in K.

Let $\mathbf{C}: \mathbf{B} \to \mathbf{S}$ be a functor, $\xi \in \mathbf{C}(K)$ and consider the following properties:

(g_1) $\mathbf{g}(\xi)$ is Galois-closed

(g_2) $K_\xi^\infty = K^\xi$

(g_3) K_ξ^∞/K_ξ is purely inseparable

(d_1) $D(\xi)$ contains an algebraic extension of K^ξ

(d_2) $D(\xi)$ contains a regular extension of K^ξ belonging to $\mathbf{B}^\omega$

(d_3) $D(\xi)$ contains a finitely generated extension of k.

A functor $\mathbf{C}:\mathbf{B} \to \mathbf{S}$ will be said to have property (g_i) or (d_i) for some $1 \leqslant i \leqslant 3$ if for all $K \in \mathbf{B}^u$ and all $\xi \in \mathbf{C}(K)$, ξ has the corresponding property. Consider also the following properties which make sense for any functor $\mathbf{C}:\mathbf{B} \to \mathbf{S}$.

(ω) For any $K \in \mathbf{B}$ we have $\mathbf{C}(K) = \xrightarrow{\lim} \mathbf{C}(E)$ where E runs through the set of all subfields of K belonging to $\mathbf{B}^\omega$.

(s) For any extension K'/K, $K, K' \in \mathbf{B}^a$ the map $\mathbf{C}(K) \to \mathbf{C}(K')$ is injective.

(m) $\mathbf{C}$ has a birational moduli set of finitely generated type.

Note that $(g_1) + (g_2) + (g_3)$ implies the fact that ξ has a "field of moduli" in Koizumi's sense [16] while (g_1) says that ξ has a "field of moduli" in Shimura's sense [24]. Note also that in our applications property (s) will be essentially a "specialisation" property.

It will be convenient to consider the following variations of (d_1) and (d_2) (making sense for any functor $\mathbf{C}:\mathbf{B} \to \mathbf{S}$):

(δ_1) For all $K \in \mathbf{B}^u$ and all $\xi \in \mathbf{C}(K)$ there exists an extension $\tilde{K}/K$ such that $D(\tilde{\xi})$ contains an algebraic extension of K^ξ (where $\tilde{\xi}$ is the image of ξ via $\mathbf{C}(K) \to \mathbf{C}(\tilde{K})$).

(δ_2) For all $K \in \mathbf{B}^a \cap \mathbf{B}^\omega$ and all $\xi \in \mathbf{C}(K)$ there exists an extension $\tilde{K}/K$ with $\tilde{K} \in \mathbf{B}^\omega$ such that $D(\tilde{\xi})$ contains a regular extension of K^ξ.

The following result summarizes the relevant implications between the above properties (the last implication being the key one in our approach):

(1.5) THEOREM. *For a functor* $\mathbf{C}:\mathbf{B} \to \mathbf{S}$ *the following hold*:

1) $(s) + (\delta_1) \Rightarrow (d_1)$

2) $(s) + (d_1) + (d_3) \Rightarrow (g_1)$

3) $(\omega) + (s) + (\delta_1) + (\delta_2) \Rightarrow (d_2)$

4) $(\omega) + (g_1) + (d_2) \Rightarrow (g_2)$

5) $(\omega) + (s) + (d_1) + (d_3) + (g_1) + (g_2) + (g_3) \Longleftrightarrow (\omega) + (d_3) + (m)$

6) $(\omega) + (s) + (\delta_1) + (\delta_2) + (d_3) \Rightarrow (m)$ if $\operatorname{char}(k) = 0$.

REMARKS.

a) Implication 6) is a formal consequence of 1)–5)

b) Implication 4) is contained in [16]

c) The equivalence 5) is a characterisation of coarse representability (under the hypothesis $(\omega) + (d_3)$).

d) Implication 2) plays a key role in our approach.

Proof. 1) is standard and we omit proof.

2) For any subfield L of a field $K \in \mathbf{B}^u$ and for any $\xi \in \mathbf{C}(K)$ put $\mathbf{g}(\xi/L) = \mathbf{g}(\xi) \cap \mathbf{g}(K/L)$.

CLAIM 1. If $L \notin \mathbf{B}^\omega$ then $\mathbf{g}(\xi/L)$ is Galois-closed.

Indeed, by (d_1) $\xi = \mathbf{C}(j)(\xi^0)$ where j is the natural inclusion of K_0, the algebraic closure of $K^{\mathbf{g}(\xi/L)}$, into K and $\xi^0 \in \mathbf{C}(K_0)$. Now $K_0 \in \mathbf{B}^u$ hence by (d_3) there is a finite Galois extension K_1 of $K^{\mathbf{g}(\xi/L)}$ contained in K_0 with $K_1 \in D(\xi)$. We have $\mathbf{g}(K/K_1) \subset \mathbf{g}(\xi/L) \subset \mathbf{g}(K/K^{\mathbf{g}(\xi/L)})$. Upon letting H to be the image of $\mathbf{g}(\xi/L)$ under the projection $g(K/K^{g(\xi/L)}) \to g(K_1/K^{g(\xi/L)})$ we have by Galois theory that $H = \mathbf{g}(K_1/(K_1)^H)$ hence $\mathbf{g}(\xi/L) = \mathbf{g}(K/(K_1)^H)$ which easily implies our claim.

Now let's prove that $\mathbf{g}(\xi) = \mathbf{g}(K/K^{\mathbf{g}(\xi)})$. We will make a "reduction to the uncountable case". Let k' be a purely transcendental extension of k having uncountable transcendence degree over k, let $K' = Q(K \otimes_k k')$ be the quotient field of $K \otimes_k k'$, F an algebraic closure of K' and ξ_F be the image of ξ via $\mathbf{C}(K) \to \mathbf{C}(F)$. By our claim 1 we have $\mathbf{g}(\xi_F/k') = \mathbf{g}(F/F^{\mathbf{g}(\xi_F/k')})$. Now take $\sigma \in \mathbf{g}(K/K^{\mathbf{g}(\xi)})$, let $\sigma' \in \mathbf{g}(K'/k')$ be its unique extension and let $\tilde{\sigma} \in \mathbf{g}(F/k')$ be any extension of σ'.

CLAIM 2. $\tilde{\sigma} \in \mathbf{g}(F/F^{\mathbf{g}(\xi_F/k')})$

Assuming this for a moment we get that $\tilde{\sigma} \in \mathbf{g}(\xi_F/k')$ so $\xi_F = (\xi_F)^{\tilde{\sigma}} = (\xi^\sigma)_F$ and we conclude by (s) that $\xi = \xi^\sigma$ i.e., that $\sigma \in \mathbf{g}(\xi)$. So 2) will be proved if we prove Claim 2. Let

$$\mathbf{g} = \{\tilde{\tau} \in \mathbf{g}(F/k');\quad \tilde{\tau}(K) = K,\ \tilde{\tau}|_K \in \mathbf{g}(\xi)\}$$

Clearly there is a surjective homomorphism $\mathbf{g} \to \mathbf{g}'$ where

$$\mathbf{g}' = \{\tau' = \tau \otimes 1 \in \mathbf{g}(K'/k');\quad \tau \in \mathbf{g}(\xi)\}$$

and the kernel of $\mathbf{g} \to \mathbf{g}'$ is $\mathbf{g}(F/K')$. So we have

$$F^{\mathbf{g}(\xi_F/k')} \subset F^{\mathbf{g}} = (F^{\mathbf{g}(F/K')})^{\mathbf{g}} = (K_i')^{\mathbf{g}} = ((K')^{\mathbf{g}'})_i = Q(K^{\mathbf{g}(\xi)} \otimes_k k')_i$$

where the index "i" means "perfect closure" and the last equality is a consequence of a remark made in [27] pp. 405–406. Now our Claim 2 follows because $\tilde{\sigma}$ is the identity on $Q(K^{\mathbf{g}(\xi)} \otimes_k k')$.

3) Let $K \in \mathbf{B}^u$, $\xi \in \mathbf{C}(K)$ and let K_0 be the algebraic closure of K^ξ in K. By (ω), $K_0 \in \mathbf{B}^\omega$. By (d_1) there exists some $\xi_0 \in \mathbf{C}(K_0)$ whose image in $\mathbf{C}(K)$

is ξ. Applying (δ_2) to K_0 and ξ_0 there is an extension $\tilde{K}/K_0$ such that $\tilde{K} \in \mathbf{B}^\omega$ and $D(\tilde{\xi})$ contains a regular extension of $K_0^{\xi_0}$ ($\tilde{\xi}$ = image of ξ_0 in $\mathbf{C}(\tilde{K})$). Now there exists a K_0-isomorphism u of $\tilde{K}$ onto a subfield K_1 of K. With ξ_1 = image of $\tilde{\xi}$ in $\mathbf{C}(K_1)$ = image of ξ_0 in $\mathbf{C}(K_1)$, clearly we have that $D(\xi_1)$ contains a regular extension of $K_0^{\xi_0}$ contained in K_1. We shall be done if we prove that $K_0^{\xi_0} = K^\xi$. But this is easily checked through property (s).

4) Let $x \in K \backslash K^\xi$; it is sufficient to find a perfect field $E \in D(\xi)$ with $x \notin E$. By (d_2) there exists $F \in D(\xi) \cap \mathbf{B}^\omega$ with F a regular extension of K^ξ. Let F_i be the perfect closure of F in K; since K^ξ is perfect, K^ξ is algebraically closed in F_i. So one can find $\sigma \in \mathbf{g}(K/K^\xi)$ such that $\sigma x \notin F_i$. By (g_1), $\sigma \in \mathbf{g}(\xi)$ hence $x \notin \sigma^{-1} F_i \in D(\xi)$.

5) To prove implication from left to right we need the following definition. Given a subcategory $\mathbf{B}^0$ of $\mathbf{B}$ a morphism $\varphi^0 : \mathbf{C}|_{\mathbf{B}^0} \to h_X|_{\mathbf{B}^0}$ is said to have property (m_1) if it induces isomorphisms on all objects of $\mathbf{B}^a \cap \mathbf{B}^0$. Then we proceed in several steps:

STEP 1. Let $K \in \mathbf{B}^u$ and denote by $\mathbf{B}^K$ the subcategory of $\mathbf{B}$ whose objects are the subfields of K and whose morphisms are defined by

$$\mathrm{Hom}_{\mathbf{B}^K}(E, F) = \{u \in \mathrm{Hom}_{\mathbf{B}}(E, F);\ u \text{ can be extended to some } \tilde{u} \in \mathbf{g}(K)\}$$

Define a birational set as follows. Put $X = \mathbf{C}(K)/\mathbf{g}(K)$, let $s: X \to \mathbf{C}(K)$ be any section of the projection $p : \mathbf{C}(K) \to X$ and put $k(x) = K_{s(x)}$ for all $x \in X$ ($k(x)/k$ is finitely generated by (d_3)). We construct a morphism $\varphi^K : \mathbf{C}|_{\mathbf{B}^K} \to h_X|_{\mathbf{B}^K}$ having property (m_1) as follows. For any subfield E of K and any $\xi_E \in \mathbf{C}(E)$ let $\xi = \xi_K$ be the image of ξ_E in $\mathbf{C}(K)$; we have $s(p(\xi)) = \xi^\sigma$ for some $\sigma \in \mathbf{g}(K)$ and we put $\varphi^K(E)(\xi_E) = (x_\xi, u_\xi)$ where $x_\xi = p(\xi)$ and $u_\xi : k(x_\xi) \to E$ is defined as the composition

$$k(x_\xi) = K_{s(p(\xi))} \xrightarrow{\sigma} K_\xi \subset E$$

Note that by $(g_2) + (g_3) u_\xi$ does not depend on the choice of σ. Moreover if E is algebraically closed $\varphi^K(E)$ is injective due to properties $(s) + (g_1) + (g_2) + (g_3)$ and surjective due to properties $(\omega) + (d_1) + (g_2) + (g_3)$.

STEP 2. Let K and φ^K be as in STEP 1. We can construct a functor $\beta : \mathbf{B}^\omega \to \boldsymbol{\beta}^K$ with the property that for any $E \in \mathbf{B}^\omega$ we have an isomorphism $\beta_E : E \simeq \beta(E)$ in $\mathbf{B}$ and for each arrow $u : E \to E'$ in $\mathbf{B}^\omega$ we have $\beta(u) \circ \beta_E = \beta_{E'} \circ u$. Define a morphism $\varphi^\omega : \mathbf{C}|_{\mathbf{B}^\omega} \to h_X|_{\mathbf{B}^\omega}$ as follows for any $E \in \mathbf{B}^\omega$ let $\varphi^\omega(E)$ be defined by the commutative diagram:

$$\begin{array}{ccc} \mathbf{C}(E) & \xrightarrow{\varphi^{\omega}(E)} & h_X(E) \\ {\scriptstyle \mathbf{C}(\beta_E)}\downarrow & & \searrow {\scriptstyle h_X(\beta_E)} \\ \mathbf{C}(\beta(E)) & \xrightarrow[\varphi^K(\beta(E))]{} & h_X(\beta(E)) \end{array}$$

Clearly φ^{ω} has (m_1).

STEP 3. Using axiom (ω) φ^{ω} can be extended to a morphism $\varphi : \mathbf{C} \to h_X$ which will still have property (m_1).

STEP 4. We claim that φ has also property (m_2). To check this take any morphism $\varphi' : \mathbf{C} \to h_{X'}$ choose $K \in \mathbf{B}^u$ and consider the $\mathbf{g}(K)$-equivariant maps of sets $\varphi(K) : \mathbf{C}(K) \xrightarrow{\sim} h_X(K)$, $\varphi'(K) = \mathbf{C}(K) \to h_{X'}(K)$. Taking orbits we get maps $\tilde{\varphi} : \mathbf{C}(K)/\mathbf{g}(K) \xrightarrow{\sim} h_X(K)/\mathbf{g}(K) \simeq X$ and $\tilde{\varphi}' : \mathbf{C}(K)/\mathbf{g}(K) \to h_{X'}(K)/\mathbf{g}(K)$ and define $f : X \to X'$ by $f = \pi \circ \tilde{\varphi}' \circ \tilde{\varphi}^{-1}$ where $\pi : h_{X'}(K)/\mathbf{g}(K) \to X'$ is the natural projection. Moreover if $\varphi(K)(\xi) = (x_{\xi}, u_{\xi})$ and $\varphi'(K)(\xi) = (x'_{\xi}, u'_{\xi})$ for $\xi \in \mathbf{C}(K)$ then by functoriality of φ' we have $u'_{\xi}(k(x'_{\xi})) \subset K_{\xi}$ while by the very construction of X we haye $u_{\xi}(k(x_{\xi})) = K_{\xi}$ consequently we get field homomorphisms $f_x^* : k(f(x)) \to k(x)$ hence a morphism $f : X \to X'$ unique with the property $h_f \circ \varphi = \varphi'$.

The other implication in 5) is proved along the same lines; it will not be used in the sequel and we omit details.

As already noted 6) follows from the preceeding implications.

(1.6) The following general situation will often occur in what follows. Let's make the following definition: a morphism $\mathbf{C}' \to \mathbf{C}$ between functors from $\mathbf{B}$ to $\mathbf{S}$ will be called a full embedding if the map $\mathbf{C}'(K) \to \mathbf{C}(K)$ is injective for all $K \in \mathbf{B}$ and

$$\mathbf{C}(K) \cap \mathbf{C}(j)^{-1}(\mathbf{C}'(K')) = \mathbf{C}'(K)$$

for any field extension $j : K \to K'$.

Now if $\mathbf{C}' \to \mathbf{C}$ is a full embedding and $\xi \in \mathbf{C}(K)$ for some $K \in \mathbf{B}$ then clearly $D(\xi, \mathbf{C}) = D(\xi, \mathbf{C}')$, $\mathbf{g}(\xi, \mathbf{C}) = \mathbf{g}(\xi, \mathbf{C}')$. Consequently if $\mathbf{C}$ has one of the properties (g_i), (d_i), (δ_i), (ω), (s) the same holds for $\mathbf{C}'$.

(1.7) The following construction will play a role later. Suppose $\mathbf{C} : \mathbf{B} \to \mathbf{S}$ is a functor. An element $\xi \in \mathbf{C}(K)$ is called bounded if there exists a field extension K'/K, a subfield K_0 of K' finitely generated over k and an element $\xi_0 \in \mathbf{C}(K_0)$ such that ξ and ξ_0 have the same image in $\mathbf{C}(K')$. For any $K \in \mathbf{B}$

let $\mathbf{C}^b(K)$ denote the set of all bounded elements in $\mathbf{C}(K)$. Then $K \mapsto \mathbf{C}^b(K)$ defines a functor $\mathbf{C}^b : \mathbf{B} \to \mathbf{S}$ fully embedded into $\mathbf{C}$.

2. Some remarkable functors. Main result

(2.1) The typical examples of functors from $\mathbf{B}$ to $\mathbf{S}$ which we are going to consider are the "moduli functors" associated to suitable fibred categories over $\mathbf{B}$. More precisely let $\mathbf{C}$ be a fibred category over $\mathbf{B}$; by this we mean that for any $K \in \mathbf{B}$ we are given a category $\mathbf{C}_K$, for any field homomorphism $u : K \to K'$ we are given a "base change" functor $\mathbf{C}_u : \mathbf{C}_K \to \mathbf{C}_{K'}$, and for any pair of field homomorphisms $K \xrightarrow{u} K' \xrightarrow{v} K''$ we are given a functorial isomorphism $\mathbf{C}_{u,v} : \mathbf{C}_v \circ \mathbf{C}_u \to \mathbf{C}_{vu}$, all these data being subject to same natural compatibility conditions [10]. Given $\mathbf{C}$ as above one can define the "moduli functor" (still denoted by $\mathbf{C}$) from $\mathbf{B}$ to $\mathbf{S}$ by the formula $\mathbf{C}(K) = \mathbf{C}_K/\text{iso}$ (= set of isomorphism classes of objects in $\mathbf{C}_K$). If $A \in \mathbf{C}_K$ and ξ_A is its image in $\mathbf{C}(K)$ we put $D(A) = D(\xi_A)$ and $\mathbf{g}(A) = \mathbf{g}(\xi_A)$.

(2.2) The functors **PAL**, $\mathbf{PAL}^f$, **HAL**. By a K-algebra (K a field) we understand either an associative unitary (not necessarily commutative) K-algebra or a Lie K-algebra. By a polarization on a K-algebra A we mean a finite dimensional K-linear subspace P of A which generates A as a K-algebra. By a polarized K-algebra we mean a K-algebra A with a given polarization P_A on it. A polarized K-algebra is of course finitely generated; it is called finitely presented (respectively homogeneous) if the kernel of $K\langle P_A \rangle \to A$ is a finitely generated (respectively finitely generated and homogeneous) ideal of $K\langle P_A \rangle$ = free (associative or Lie) K-algebra on P_A. The polarized (respectively polarized finitely presented, respectively homogeneous) K-algebras form a category which we call $\mathbf{PAL}_K$ (respectively $\mathbf{PAL}^f_K$, $\mathbf{HAL}_K$); a morphism is by definition a K-algebra map $f : A \to B$ such that $f(P_A) \subset P_B$. For any field homomorphism $K \to K'$ we define base change functors $\mathbf{PAL}_K \to \mathbf{PAL}_{K'}$ by $A \mapsto A' = K' \otimes_K A$, $P_{A'} = K' \otimes_K P_A$ (and analogously for $\mathbf{PAL}^f_K$, $\mathbf{HAL}_K$); the resulting fibred category and moduli functor are denoted by **PAL** (respectively $\mathbf{PAL}^f$, **HAL**).

Finite dimensional K-algebras A have a natural structure of polarized finitely presented K-algebras via $P_A = A$. Another remarkable example of algebras which carry a natural polarization will be given below (cf. (2.4) and (2.8)).

(2.3) The functors **CLA**, **CLS**. A complete local K-algebra will always be assumed commutative, noetherian, with residue field K. Denote by $\mathbf{CLA}_K$

the category of complete local K-algebras. Define the base change functors $\mathbf{CLA}_K \to \mathbf{CLA}_{K'}$ by $A \mapsto K' \hat{\otimes}_K A$ ($\hat{\otimes}$ = completed tensor product) and denote by **CLA** the resulting fibred category and moduli functor. As we shall see below the functor $\mathbf{CLA}^b$ of bounded complete local algebras (cf. (1.7)) will prove itself to have good moduli theoretic properties. Note for instance that any $A \in \mathbf{CLA}_K$ which is algebraisable in the sense of [1] is bounded. It seems to be an open problem whether any complete local algebra is bounded (cf. the end of [2]).

For technical reasons it is convenient to consider also a "relative" situation. Namely, for a fixed integer $N \geqslant 1$ let $K[[X]] = K[[X_1, \ldots, X_N]]$ be the power series algebra over K and let $\mathbf{CLS}_K$ be the category of complete local K-algebras A equipped with a local algebra homomorphism $K[[X]] \to A$ (the morphisms in $\mathbf{CLS}_K$ being assumed to agree with the maps $K[[X]] \to A$). These $\mathbf{CLS}_K$ define a fibred category and a moduli functor **CLS**.

(2.4) The functors **AFF**, $\mathbf{AFF}^+$. By an affine K-algebra we mean a finitely generated, commutative, geometrically reduced K-algebra. Denote by $\mathbf{AFF}_K$ the category of affine K-algebras (which is antiequivalent to the category of affine K-varieties) and by **AFF** the resulting fibred category and moduli functor. We say that $A \in \mathbf{AFF}_K$ has non-negative Kodaira dimension (compare with [21]) if it is geometrically integral and there exists a smooth completion X of V_{reg} (= regular locus of $V = \mathrm{Spec}(A)$) such that $S_m(X, D) := H^0(X, \omega_X^{\otimes m}((m-1)D)) \neq 0$ for some $m \geqslant 1$ (where D is the reduced divisor $X \backslash V_{\mathrm{reg}}$ assumed to have normal crossings and ω_X is the canonical bundle on X). If $\mathrm{char}(k) = 0$ then by [4] [21] $S_m(X, D)$ (viewed as a subspace of the space of regular m-uple n-forms on V_{reg}, where $n = \dim(A)$) does not depend on the choice of the completion of V_{reg}; if $K = \mathbb{C}$, $S_m(X, D)$ can be interpreted as the space of regular m-uple n-forms on V_{reg} with finite "volume". Denote by $\mathbf{AFF}_K^+$ the full subcategory of $\mathbf{AFF}_K$ of all algebras having a non-negative Kodaira dimension and by $\mathbf{AFF}^+$ the resulting fibred category and moduli functor.

(2.5) The functor **COH**. For any field K let $\mathbf{COH}_K$ denote the category of finitely generated $K[[X]]$-modules ($X = (X_1, \ldots, X_N)$). Define base change functors $\mathbf{COH}_K \to \mathbf{COH}_{K'}$ by $E \mapsto E \otimes_{K[[X]]} K'[[X]] = K' \hat{\otimes}_K E$. We get a fibred category and a moduli functor **COH**.

(2.6) The functor **LFS**. Fix a complete local k-algebra R with $\mathrm{prof}(R) \geqslant 2$ and for any field K let $\mathbf{LFS}(K)$ be the set of isomorphism classes of locally free coherent sheaves on the punctured spectrum $Y_K = \mathrm{Spec}\,(R_K) \backslash \{M(R_K)\}$ where $R_K = K \hat{\otimes}_k R$ and $M(R_K)$ is the maximal ideal of R_K. We defined a functor $\mathbf{LFS}: \mathbf{B} \to \mathbf{S}$.

(2.7) The functors **AHA**, $\mathbf{AHA}^p$, $\mathbf{AHA}^r$. Let $\mathbf{AHA}_K$ denote the category of affine Hopf K-algebras (which is anti-equivalent to the category of linear algebraic K-group [12]). With obvious base change functors we get a fibred category and hence a moduli functor **AHA**.

A reductive algebraic K-group P (char$(K) = 0$) will be called pure if: $\mathrm{Aut}(P)/\mathrm{Int}(P)$ is finite. If K is algebraically closed P is pure if and only if its center has dimension $\leqslant 1$ ([12] p. 218 and [7], p. 409). An affine Hopf K-algebra A(Char$(K) = 0$) will be called pure if, upon letting $L = \mathrm{Spec}(A)$, $U = R_u(L) =$ unipotent radical of L we have that L/U is pure (L/U exists and is reductive by [12] pp. 80 and 117). Let $\mathbf{AHA}^p_K$ be the full subcategory of $\mathbf{AHA}_K$ of pure Hopf algebras and $\mathbf{AHA}^p$ the resulting fibred category and moduli functor.

Suppose $A \in \mathbf{AHA}_K$, char$(K) = 0$. By a rigidification on A (or on $L = \mathrm{Spec}(A)$) we mean the giving of the isomorphism class V of a faithful representation V of L/U (where once again $U = R_u(L)$). Since L/U is reductive the set of all possible rigidifications on a given A is a "discrete" set (i.e., it does not increase by base change $K \to K'$, $K, K' \in \mathbf{B}^a$). By a rigidified affine Hopf K-algebra we mean a pair consisting of an object $A \in \mathbf{AHA}_K$ and a rigidification V on it. Rigidified affine Hopf K-algebras form a groupoid which we call $\mathbf{AHA}^r_K$; we obtain a fibred groupoid and a moduli functor $\mathbf{AHA}^r$.

(2.8) Proposition. *There exist full embeddings*:

a) $\mathbf{AFF}^+ \xrightarrow{\alpha} \mathbf{PAL}^f \longrightarrow \mathbf{PAL}, \quad \mathbf{HAL} \longrightarrow \mathbf{PAL}^f$

b) $\mathbf{LFS}^b \xrightarrow{\beta} \mathbf{COH}^b \xrightarrow{\gamma} \mathbf{CLS}^b \longrightarrow \mathbf{CLS}, \quad \mathbf{CLA}^b \longrightarrow \mathbf{CLS}^b$

c) $\mathbf{AHA}^p \xrightarrow{\delta} \mathbf{AHA}$

where for α, γ, δ *we assume* char$(k) = 0$.

Proof. The only non-obvious arrows are α, β, γ.

To construct α the key point is that any $A \in \mathbf{AFF}^+_K$ has a canonical polarization P_A. It is constructed as follows. Let $V = \mathrm{Spec}(A)$, (X, D) a smooth normal crossing completion of V_{reg} and let $m \geqslant 1$ be the smallest integer such that $S_m(X, D) \neq 0$. Then consider for each integer $n \geqslant 1$ the K-linear subspace of A:

$$A_n = \{f \in A; f \cdot \beta_1 \otimes \cdots \otimes \beta_n \in S_{mn}(X, D)$$
$$\text{for all} \quad \beta_1, \ldots, \beta_n \in S_m(X, D)\}$$

Clearly $\dim_K A_n < \infty$ for all $n \geqslant 1$ and $A = \bigcup_{n\geqslant 1} A_n$. Let N be the smallest integer such that A_N generates A as a K-algebra and define a polarization $P_A = A_N$. Note that if $u: A \to A'$ is a K-isomorphism and $V' = \mathrm{Spec}(A')$, (X', D') a smooth normal crossing completion of $(V')_{\mathrm{reg}}$ then by [4, 21] the map $u^*: (V')_{\mathrm{reg}} \to V_{\mathrm{reg}}$ induces K-isomorphisms $S_m(X, D) \to S_m(X', D')$ hence u induces K-isomorphisms $A_n \to A'_n$ for all $n \geqslant 1$; in particular $u(P_A) = P_{A'}$, so we have a correctly defined morphism $\mathbf{AFF}^+ \to \mathbf{PAL}^f$ which obviously is a full embedding.

To construct β write $R = k[[X]]/I$ and send a bounded locally free sheaf F on Y_K into the $K[[X]]$-module $F' = H^0(Y_K, F)$. By a theorem of Grothendieck [10] F' is finitely generated. Now β is a full embedding by the yoga in [10].

Finally, to construct γ we proceed as follows. To any $K[[X]]$-module E which is bounded and finitely generated we associate the local complete K-algebra $K[[X]] \oplus E$ ($E^2 = 0$) equipped with the obvious inclusion map $K[[X]] \to K[[X]] \oplus E$; this will be an element of $\mathbf{CLS}^b(K)$. To check that γ is a full embedding one has to prove that if L/K is a field extension, A is a bounded complete local $K[[X]]$-algebra and $L \hat{\otimes}_K A \simeq L[[X]] \oplus E$ is an isomorphism of $L[[X]]$-algebras (with E a finitely generated bounded $L[[X]]$-module) then $A \simeq K[[X]] \oplus E_0$ for some bounded finitely generated $K[[X]]$-module E_0. It is sufficient to show that $(\mathrm{nil}(A))^2 = 0$ and the natural map $u: K[[X]] \to A_{\mathrm{red}} = A/\mathrm{nil}(A)$ is an isomorphism. The condition on nil is clear while for the second condition the formula $(L \hat{\otimes}_K A)_{\mathrm{red}} = L \hat{\otimes}_K (A_{\mathrm{red}})$ (which holds by separability of L/K) shows that $1 \hat{\otimes} u: L \hat{\otimes}_K K[[X]] \to L \hat{\otimes}_K (A_{\mathrm{red}})$ is an isomorphism which implies that so is u (look at the associated graded rings).

(2.9) It is an easy exercise to check that the functors $\mathbf{PAL}^f$, $\mathbf{AHA}$, $\mathbf{AHA}^r$, $\mathbf{AFF}$ have properties (ω) (d_3) (s). Clearly $\mathbf{CLS}$ has property (ω). Moreover $\mathbf{CLS}^b$ has property (d_3) (use the fact that if a system of algebraic equations with coefficients in a universal field K involving at most countably many unknowns has a solution in a field extension of K, then it has a solution in K; we would like to stress the following technical point: here the universality of K is essential and this justifies both our definition of property (d_3) and the somewhat boring "reduction to the uncountable case" in the proof of 2) in (1.5)). Finally note that by a result of Seidenberg [22] $\mathbf{CLA}$ has property (s); same arguments as in [22] show that in fact $\mathbf{CLS}$ has (s).

The main effect of our theory will be the following:

(2.10) THEOREM. *Suppose* $\mathrm{char}(k) = 0$. *Then the functors*

a) $\mathbf{HAL}$, $\mathbf{AFF}^+$, $\mathbf{PAL}^f$; b) $\mathbf{CLA}^b$, $\mathbf{COH}^b$, $\mathbf{LFS}^b$; c) $\mathbf{AHA}^p$, $\mathbf{AHA}^r$ *have property* (m).

In view of 6) in Theorem (1.5) together (1.6), (2.8), (2.9) the statement above will be proved if we prove the following.

(2.11) THEOREM. *The functors* **PAL** *and* **CLS** *satisfy* (δ_1) *and* (δ_2). *Moreover if* char(k) $= 0$, **AHA** *satisfies* (δ_1), $\mathbf{AHA}^p$ *satisfies* (δ_2) *and* $\mathbf{AHA}^r$ *satisfies* (δ_1) *and* (δ_2).

As a Corollary of Theorem (2.11) we get that if char(k) $= 0$ then **AHA** has property (g_1).

Theorem (2.11) gives significant information also in characteristic $p > 0$. Indeed together with (1.5), (1.6), (2.8), (2.9) it shows that in arbitrary characteristic $\mathbf{PAL}^f$ and $\mathbf{CLA}^b$ have the properties (g_1), (g_2), (d_1), (d_2). A typical corollary in characteristic $p > 0$ concerns the Frobenius automorphism φ. Indeed if k is the prime field $\mathbb{F}_p$ and A is a bounded local complete K-algebra (or a polarized finitely presented K-algebra) with K universal such that $A \simeq A^{\varphi^n}$ for some $n \geqslant 1$ then A is defined over the algebraic closure of $\mathbb{F}_p$.

An example of remarkable functor having property (m) in arbitrary characteristic (cf. [16]) is the functor $\mathbf{CRV}: \mathbf{B} \to \mathbf{S}$ $\mathbf{CRV}(K) =$ set of isomorphism classes of smooth projective curves over K.

The proof of Theorem (2.11) will be done in Part II of our paper (cf. Corollaries (4.3), (5.3), (6.4), (6.9), (6.10)) using a purely algebraic strategy from our book [5].

(2.12). Note that one could try to prove (δ_1) in a "geometric" way as follows. If **C** is one of the fibred categories under consideration then each object $A \in \mathbf{C}_K$ (K universal) should be viewed intuitively as a "family" over the parameter space Spec(K). Then one could try to replace Spec(K) by a K_0-variety Spec(S) ($K_0 = K^A$, $K_0 \subset S \subset K$) on which $\mathbf{g}(A)$ acts by birational automorphisms and then try to use representability of the functor of isomorphisms between objects in $\tilde{\mathbf{C}}_{\mathrm{Spec}(S)}$ where $\tilde{\mathbf{C}}$ is an extension of **C** to the category **SCH** of schemes (over k). There are serious difficulties with this approach (indeed although one can find by property (d_3) a field definition K_1 for A which is finitely generated over K_0 one cannot find apriori such a K_1 which in addition is stable under $\mathbf{g}(A)$ (in fact if K_1/K_0 is transcendental then K_1 is never stable under $\mathbf{g}(A)$!).

Note also that one could try to prove (δ_1) by extending the method of Chow points due to Matsusaka and Shimura. There are difficulties also with this approach. Indeed in their method it is essential that the moduli functor takes the form

$$K \mapsto \coprod H_i(K)/R_i(K)$$

with H_i certain quasi-projective k-schemes (appearing as locally closed pieces of certain Chow varieties or Hilbert schemes) and R_i certain "algebraic" (nonnecessary closed) equivalence relations on H_i. But it is not clear (at least apriori) that this holds for the functors under consideration: to see this one would probably need a theory of Chow coordinates (or a "Hilbert scheme") for the corresponding fibred categories (e.g., for complete local algebras or for polarized finitely presented algebras!).

(2.13) Let's close by making some remarks on automorphisms in a fibred category $\mathbf{C}$ over $\mathbf{B}$. Let $K \in \mathbf{B}^a$, $A \in \mathbf{C}_K$; there exists an exact sequence

$$1 \to \mathrm{Aut}_K(A) \to G(A) \to \mathbf{g}(A) \to 1$$

where $\mathrm{Aut}_K(A)$ is the automorphism group of A as an object in $\mathbf{C}_K$ and $G(A) = G(A, \mathbf{C})$ is the group defined as follows. Its elements are pairs $s = (\sigma, v)$ with $\sigma \in \mathbf{g}(A)$ and $v\colon A \to A^\sigma := \mathbf{C}_{\sigma-1}(A)$ an isomorphism in $\mathbf{C}_K$; the multiplication is defined by

$$(\sigma, v)(\tau, w) = (\sigma\tau, c_{\sigma,\tau} \circ v^\tau \circ w)$$

where $v^\tau = \mathbf{C}_{\tau-1}(v) \in \mathrm{Hom}\,(A^\tau, (A^\sigma)^\tau)$ and $c_{\sigma,\tau} = \mathbf{C}_{\sigma^{-1},\tau^{-1}}(A) \in \mathrm{Hom}\,((A^\sigma)^\tau, A^{\sigma\tau})$. Note that $G(A)$ acts on K via $\mathbf{g}(A)$. A key point in our method will be to kill cocycles of $G(A)$ with values in general linear groups $GL_n(K)$.

As an example, if $\mathbf{C} = \mathbf{CLA}$ and if we view K as a subset of $A \in \mathbf{CLA}_K$ then $G(A)$ identifies with the group of all k-automorphisms of A sending K onto K.

The following relation between our setting and Weil's Galois descent worths being noted (although won't be used later). Let K_0 be a subfield of K and $A \in \mathbf{C}_K$. If $K_0 \in D(A)$ then one can find a group homomorphism $s\colon \mathbf{g}(K/K_0) \to G(A)$ which composed with the projection $G(A) \to \mathbf{g}(A)$ yields the natural inclusion $\mathbf{g}(K/K_0) \subset \mathbf{g}(A)$. Conversely if such a "section" s exists one can ask whether $K_0 \in D(A)$. Upon letting $s(\sigma) = (\sigma, s_\sigma)$ for $\sigma \in \mathbf{g}(K/K_0)$ with $s_\sigma\colon A \to A^\sigma$ we see that we have

$$s_{\sigma\tau} = c_{\sigma,\tau} \circ s_\sigma^\tau \circ s_\tau$$

for all σ, i.e., the family $\{s_\sigma; \sigma \in \mathbf{g}(K/K_0)\}$ satisfies a condition analogue to Weil's cocycle condition [26]. So if K/K_0 was a Galois extension, we would get for "reasonable" $\mathbf{C}$'s that Weil's discent works and $K_0 \in D(A)$. In our situation; however, K is universal while K_0 is the algebraic closure of K^A so K/K_0 is always transcendental; moreover we do not dispose apriori of a "section" s as above. So Weil's Galois descent cannot be applied to deal with property (d_1)!

PART II: ALGEBRAS OVER SKEW GROUP ALGEBRAS

3. Killing nonabelian cocycles

(3.1) We place ourselves in the setting of [3], section 1. So let G be a group (not assumed to be profinite!); by a G-field (respectively G-group, G-ring, . . .) we will understand a field (respectively group, ring, . . .) together with a G-action on it by field (respectively group, ring, . . .) automorphisms. If K is a G-field and L is a linear algebraic K^G-group then $L(K)$, the group of K-points of L, is a G-group.

Recall that if Γ is a G-group one defines the set $Z^1(G, \Gamma)$ of 1-cocycles as the set of all maps $f: G \to \Gamma$ satisfying $f(st) = f(s)s(f(t))$ for all $s, t \in G$. A cocycle f is called a coboundary if there exists $x \in \Gamma$ such that $f(s) = x^{-1}sx$ for all $s \in G$.

(3.2) We make two definitions. An extension E/K of G-fields will be called constrained if the extension E^G/K^G is algebraic (terminology is inspired from differential algebra [14]); note that if E/K is constrained and K is algebraically closed then $K^G = E^G$.

Moreover a subgroup G_1 of G is called cofinite if there exists a sequence of subgroups $G_1 = G_2 \subset \cdots \subset G_m = G$ such that G_i is normal of finite index in G_{i+1} for $1 \leqslant i \leqslant m - 1$. Clearly the extension K^{G_1}/K^G is then necessarily finite.

(3.3) THEOREM. *Let K be a G-field, L a linear algebraic K^G-group and $f \in Z^1(G, L(K))$ a cocycle. Then:*

a) *There exists a cofinite subgroup G_1 of G and a finitely generated constrained extension of G_1-fields K_1/K such the image of f via $Z^1(G, L(K)) \to Z^1(G_1, L(K_1))$ is a coboundary.*

b) *If L is geometrically irreducible there exists a finitely generated regular extension of G-fields K_1/K such that the image of f via $Z^1(G, L(K)) \to Z^1(G, L(K_1))$ is a coboundary.*

Proof. Embed L into GL_N for some N and suppose L is defined in $K^G[X]_d$ by an ideal I where $X = (X_{ij})$ and $d = \det(X)$. There is a unique G-action on $K[X]$ which agrees with our G-action on K and such that $sX_{ij} = \Sigma X_{ip}(f(s))_{pj}$ where $f(s) \in L(K)$ is viewed as an element in $GL_N(K)$. Since $sd = (\det f(s))d$ the action above extends to a G-action on $K[X]_d$; clearly $J = IK[X]_d$ is globally G-invariant. To prove b) note that the radical

$r(J)$ of J is a prime ideal in $K[X]_d$ and clearly is globally G-invariant. Then we put $K_1 = Q(K[X]_d/r(J))$ and let $x \in L(K_1)$ be the K_1-point of L corresponding to the map $K^G[X]_d/I \to K_1$; clearly $f(s) = x^{-1}sx$ for all $s \in G$ and b) is proved.

To prove a) let S be the set of all ideals J' in $K[X]_d$ satisfying the following properties:

1) J' contains J

2) J' is G'-invariant for some cofinite subgroup G' of G.

Let J_1 be a maximal member in S and let G_1 be the corresponding cofinite group from condition 2). We claim J_1 is a prime ideal. Indeed let $M = \{P_1, \ldots, P_m\}$ be the set of primes in $K[X]_d$ minimal over J_1. Then $G_2 = \text{Ker}(G_1 \to \text{Aut}(M))$ is still cofinite so $P_1 \in S$ hence by maximality $J_1 = P_1$. Let $K_1 = Q(K[X]_d/J_1)$ and $x \in L(K_1)$ as in the proof of b). We are left to prove that $K_1^{G_1}/K^G$ is algebraic. It is sufficient to check that any element $a \in K_1^{G_1}$ is algebraic over K; indeed if $a^n + b_1a^{n-1} + \cdots + b_n = 0$ with $b_i \in K$ is an equation of minimal degree satisfied by a then for any $s \in G_1$ we have $(b_1 - sb_1)a^{n-1} + \cdots + (b_n - sb_n) = 0$, hence by minimality $sb_i = b_i$ for all i and $s \in G_1$; so a is algebraic over K^{G_1} which in its turn is finite over K^G.

Assume there exists $a \in K_1^{G_1}$ transcendental over K and look for a contradiction. By Chevalley's constructibility theorem there exists $g \in K[a]$, $g \neq 0$ such that the image of the map Spec $(R_1[a] \to \text{Spec}(K[a])$ contains Spec $(K[a]_g)$ (where $R_1 = K[X]_d/J_1$ and $R_1[a]$ is the R_1-subalgebra of K_1 generated by a). We claim there exists a cofinite subgroup G_2 of G and a G_2-invariant prime ideal $P \neq 0$ in $K[a]$ not containing g. If K^{G_1} is infinite this is clear. To prove the claim in general note that there exists at least one polynomial $h \in K^{G_1}[a]$ none of whose prime factors $h_1, \ldots, h_m$ in $K[a]$ divides g. Clearly G_1 acts on $K[a]$ and also on the set of ideals $F = \{h_1K[a], \ldots, h_mK[a]\}$. Then the claim follows, by taking $G_2 = \text{Ker}(G_1 \to \text{Aut}(F))$. With P at hand consider the set $E = \{Q_1, \ldots, Q_s\}$ of minimal primes in the fibre of Spec $(R_1[a]) \to$ Spec $(K[a])$ at P; clearly G_2 acts on $R_1[a]$ and also on E. Then if we let $G_3 = \text{Ker}(G_2 \to \text{Aut}(E))$ we get that $Q = Q_1$ is G_3-invariant hence so will be $Q \cap R_1$, hence so will be the inverse image of $Q \cap R_1$ in $K[X]_d$ which we call J_3. Now $Q \neq 0$ hence $Q \cap R_1 \neq 0$ (because $Q(R_1) = Q(R_1([a]))$ so J_3 strictly contains J_1. Since G_3 is cofinite in G this contradicts the maximality of J_1 and the Theorem is proved.

(3.4) Let K be a G-field. Denote by $K[G]$ the skew group K-algebra on G; recall that as a K-linear space, $K[G]$ has a basis consisting of the elements of G, while the multiplication is defined by $(c_1s_1)(c_2s_2) = (c_1s_1(c_2))(s_1s_2)$ for all $c_1, c_2 \in K$, $s_1, s_2 \in G$. We shall be interested in the category of $K[G]$-modules (note that the (G, K)-spaces from [15] are $K[G]$-modules while the converse is not true since we do not assume – and this will be important – that the action map $G \to \mathbf{g}(K)$ is injective). If M is a $K[G]$-module then $s(cx) = (sc)(sx)$ for all $s \in G$, $c \in K$, $x \in M$. When we say a $K[G]$-module is finite dimensional we mean it has finite dimension over K. The field K is a $K[G]$-module in a natural way.

For any $K[G]$-module M put $M^G = \{x \in M; sx = x \text{ for all } s \in G\}$; M^G is a K^G-linear space and we have a natural injective map

$$K \otimes_{K^G} (M^G) \to M, \qquad c \otimes x \mapsto cx$$

We will often identify $K \otimes_{K^G} (M^G)$ with the image of the above map. If this map is surjective we say that M is a split $K[G]$-module; clearly M is split if and only if it has a K-basis contained in M^G. Moreover one easily checks that any sub-$K[G]$-module of a split $K[G]$-module is split (use an argument similar to that in [5] p. 55). If K_1/K is an extension of G-fields and if M is a $K[G]$-module then $K_1 \otimes_K M$ has a natural structure of $K_1[G]$-module defined by $s(c \otimes x) = sc \otimes sx$ for $s \in G$, $c \in K_1$, $x \in M$.

(3.5) A useful remark is that if M is a split $K[G]$-module, G_1 is a subgroup of G and K_1/K is an extension of G_1-fields then the following hold:

1) $K_1 \otimes_K M$ is a split $K_1[G_1]$-module and

2) the natural map

$$f\colon\ (K_1^{G_1}) \otimes_{K^G} (M^G) \to (K_1 \otimes_K M)^{G_1}$$

is an isomorphism.

The first assertion is clear since $K_1 \otimes_K M$ has a K_1-basis consisting of G-invariant elements in M. To prove the second assertion it is sufficient to check that f becomes an isomorphism after tensorization with K_1 over $K_1^{G_1}$. But after tensorization both the source and the target of f naturally identify with $K_1 \otimes_K M$ so we are done.

Now exactly as in [15] our Theorem (3.3) on killing cocycles (applied to $L = GL_N$) leads to "existence of invariant bases" so we get.

(3.6) COROLLARY. *Let K be a G-field, M a $K[G]$-module of finite dimension. Then*:

a) *There exist a cofinite subgroup* G_1 *of* G *and a finitely generated constrained extension* K_1/K *of* G_1*-fields such that* $K_1 \otimes_K M$ *is a split* $K_1[G_1]$*-module.*
b) *There exists a finitely generated regular extension* K_1/K *of* G*-fields such that* $K_1 \otimes_K M$ *is a split* $K_1[G]$*-module.*

4. Polarized $K[G]$-algebras

(4.1) Let K be a G-field. Following [19] p. 952, by a $K[G]$-algebra we mean a K-algebra A which is also a $K[G]$-module such that the multiplication map $A \otimes_K A \to A$ and the unit $K \to A$ (if there is any) are $K[G]$-module maps (here $A \otimes_K A$ is a $K[G]$-module via $s(a_1 \otimes a_2) = sa_1 \otimes sa_2$ for $s \in G_1$, a_1, $a_2 \in A$). By a polarized $K[G]$-algebra we will mean a polarized K-algebra A which is also a $K[G]$-algebra such that P_A is a $K[G]$-submodule of A.

Following [19] p. 957 we say that the (polarized) $K[G]$-algebra A is split if there is an isomorphism of (polarized) $K[G]$-algebras $A \simeq K \otimes_{K^G} (A^0)$ for some (polarized) K^G-algebra A^0 where $K \otimes_{K^G} (A^0)$ is given the structure of $K[G]$-algebra defined by $s(c \otimes x) = sc \otimes x$ for $s \in G$, $c \in K$, $x \in A^0$.

(4.2) THEOREM. *Let* A *be a polarized* $K[G]$*-algebra. Then:*
1) *There exists cofinite subgroup* G_1 *of* G *and a finitely generated constrained extension* K_1/K *of* G_1*-fields such that* $K_1 \otimes_K A$ *is a split polarized* $K_1[G_1]$*-algebra.*
2) *There exists a finitely generated regular extension of* G*-fields* K_1/K *such that* $K_1 \otimes_K A$ *is a split polarized* $K_1[G]$*-algebra.*

Proof. Let $\pi: K\langle P\rangle \to A$ be the natural surjection, $P = P_A$ and $J = \mathrm{Ker}\,(\pi)$. To prove assertion 1), by part a) in (3.6) there exists a cofinite subgroup G_1 of G and a finitely generated constrained extension of G_1-fields K_1/K such that $P_1 := K_1 \otimes_K P$ is a split $K_1[G_1]$-module. Consequently $K_1\langle P_1\rangle$ is a split $K_1[G_1]$-algebra. Since $K_1 \otimes_K J$ is a $K_1[G_1]$-submodule of $K_1\langle P_1\rangle$ it is split; this immediately implies that $K_1 \otimes_K A$ is split polarized $K_1[G_1]$-algebra. The proof of 2) is similar using part b) in (3.3) instead of part a).

(4.3) COROLLARY. *The functor* **PAL** *has properties* (δ_1) *and* (δ_2).

Proof. Any polarized K-algebra A has a structure of polarized $K[G]$-algebra with $G = G(A, \mathbf{PAL})$ (see (2.13)): for any $s = (\sigma, v) \in G$, $\sigma \in \mathbf{g}(A, \mathbf{PAL})$, $v: A \to A^\sigma$ and any $a \in A$ we put $sa = p_\sigma(v(a))$ where $p_\sigma: \sigma \otimes 1_A: A^\sigma = K^\sigma \otimes_K A \to A = K \otimes_K A$ (where K^σ is K itself viewed as a K-algebra via $\sigma^{-1}: K \to K$). We conclude by (4.2).

5. Complete local $K[[X]][G]$-algebras

(5.1) Let K be a G-field. By a complete local $K[G]$-algebra we mean a complete local K-algebra which is also a $K[G]$-algebra. If $X = (X_1, \ldots, X_N)$ by a complete local $K[[X]][G]$-algebra we mean a complete local $K[G]$-algebra together with a local algebra homomorphism $u: K[[X]] \to A$ such that $u(X_i) \in A^G$ for $1 \leqslant i \leqslant N$.

Let A be a complete local $K[[X]][G]$-algebra; it is called split if there is a $K[[X]]$-algebra isomorphism $A \simeq K \hat{\otimes}_{K^G} (A^0)$ with A^0 a complete local $K^G[[X]]$-algebra such that for the induced $K[G]$-algebra structure on $K \hat{\otimes}_{K^G} (A^0)$ we have $s(c \hat{\otimes} x) = sc \hat{\otimes} x$ for all $s \in G$, $c \in K$, $x \in A^0$.

(5.2) THEOREM. *Let A be complete local $K[[X]][G]$-algebra. Then:*

1) *There exists a field extension $\tilde{K}/K$ such that $D(\tilde{K} \hat{\otimes}_K A$, **CLS**$)$ contains an algebraic extension of K^G.*
2) *There exists a countably generated regular extension of G-fields $\tilde{K}/K$ such that $\tilde{K} \hat{\otimes}_K A$ is a split complete local $K[[X]][G]$-algebra.*

Proof. We shall prove 1) and 2) simultaneously refering to them as to case 1) and 2). For all $n \geqslant 2$, $A_n = A/M^n$ is a finite dimensional $K[G]$-module ($M = M(A)$). By (3.6) one can construct inductively a sequence $G = G_1 \supset G_2 \supset G_3 \supset \cdots$ of subgroups of G and a sequence $K = K_1 \subset K_2 \subset K_3 \subset \cdots$ of fields such that for all $n \geqslant 2$ the following conditions are satisfied:

a) K_n is a G_n-field

b) K_n/K_{n-1} is a finitely generated extension of G_n-fields which is constrained in case 1) and regular in case 2).

c) G_n is a cofinite subgroup of G_{n-1} in case 1) and $G_n = G_{n-1}$ in case 2).

d) $K_n \otimes_K A_n$ is a split $K_n[G_n]$-module (call it B_n). Now put

$$k_n = K_n^{G_n}, \quad C_n = B_n^{G_n}, \quad \tilde{K} = \cup K_n, \quad \tilde{k} = \cup k_n, \quad A_n^0 = \tilde{k} \otimes_{k_n} C_n$$

Note that $\tilde{k}/K^G$ is algebraic in case 1) and $\tilde{K}/K$ is regular in case 2); moreover in case 2) $\tilde{K} = \tilde{K}^G$. Clearly A_n^0 is a $\tilde{k}$-subalgebra of $\tilde{K} \otimes_K A_n$ and we have $\tilde{K} \otimes_{\tilde{k}} (A_n^0) = \tilde{K} \otimes_K A_n$. Since the natural maps $f_n: B_{n+1} \to K_{n+1} \otimes_{K_n} B_n$ are maps of $K_{n+1}[G_{n+1}]$-modules we get by (3.5):

$$f_n(C_{n+1}) \subset (K_{n+1} \otimes_{K_n} B_n)^{G_{n+1}} = k_{n+1} \otimes_{k_n} C_n$$

Consequently the maps $\tilde{K} \otimes_K A_{n+1} \to \tilde{K} \otimes_K A_n$ send A_{n+1}^0 onto A_n^0. We claim that with these data one can construct a complete local $\tilde{k}[[X]]$-algebra A^0 and a $\tilde{K}[[X]]$-isomorphism $f = \tilde{K} \hat{\otimes}_{\tilde{k}} A^0 \to \tilde{K} \hat{\otimes}_K A$. Moreover we may assume in case 2) that the G-action induced via f on $\tilde{K} \hat{\otimes}_{\tilde{k}} A^0$ is the "split" action; clearly this will close the proof of the theorem.

Now the claim above can be proved by using an argument from [5] p. 80; we reproduce it for convenience. If s is the embedding dimension of A and $Y = (Y_1, \ldots, Y_s)$ are indeterminates one can find surjective maps p_n: $\tilde{k}[[Y]] \to A_n^0$ which agree with the projections $A_{n+1}^0 \to A_n^0$. Upon letting $J_n = \ker(p_n)$ we have K-isomorphisms

$$\tilde{K}[[Y]]/J_n\tilde{K}[[Y]] \simeq \tilde{K} \otimes_K A_n$$

which are compatible with the projections obtained by "passing from $n + 1$ to n". Put $J_0 = \cap J_n$ and $A^0 = \tilde{k}[[Y]]/J_0$. We have isomorphisms:

$$\tilde{K} \hat{\otimes}_{\tilde{k}} A^0 \simeq \tilde{K}[[Y]]/J_0\tilde{K}[[Y]] \xrightarrow{\alpha} \tilde{K}[[Y]]/\cap(J_n\tilde{K}[[Y]])$$

$$\xrightarrow{\beta} \varprojlim (\tilde{K}[[Y]]/J_n\tilde{K}[[Y]] \simeq \varprojlim (\tilde{K} \otimes_K A_n) = \tilde{K} \hat{\otimes}_K A.$$

Indeed to see that α is an isomorphism we use the fact that $\cap(J_n\tilde{K}[[Y]]) = J_0\tilde{K}[[Y]]$ which is proved as follows. Upon letting $I_n = J_n/J_0 \subset C = \tilde{k}[[Y]]/J_0$ and $B = \tilde{K}[[Y]]/J_0\tilde{K}[[Y]]$ we are reduced to proving that for any extension $C \subset B$ of local noetherian rings with C complete and for any sequence of ideals $(I_n)_{n \geqslant 1}$ in C with $\cap I_n = 0$ we have $\cap(I_nB) = 0$. Now by [18] p. 103 there is a function m: $\mathbb{N} \to \mathbb{N}$ such that $I_{m(n)} \subset (M(C))^n$ for all $n \geqslant 1$ hence $\bigcap_n (I_{m(n)}B) \subset \bigcap_n (M(B))^n = 0$ and we are done. To check that β is an isomorphism one uses the standard fact that any complete local ring is complete in any separated linear topology on it.

On the other hand if we denote by x_{in} the image of X_i in A_n then $x_{in} \in A_n^0$ hence we get $\tilde{k}$-algebra homomorphisms u_n: $\tilde{k}[[X]] \to \tilde{k}[[Y]]/J_n$ which agree with the projections $\tilde{k}[[Y]]/J_{n+1} \to \tilde{k}[[Y]/J_n$. Since $\tilde{k}[[Y]]/J_0 = \varprojlim \tilde{k}[[Y]]/J_n$, u_n yield a $\tilde{k}$-algebra map $\tilde{k}[[X]] \to A^0$. It is easy to see that the $\tilde{K}$-isomorphism $\tilde{K} \hat{\otimes}_{\tilde{k}} A^0 \to \tilde{K} \hat{\otimes}_K A$ constructed above is in fact a $K[[X]]$-algebra map and our Theorem is proved.

(5.3) COROLLARY. *The functor* **CLS** *has properties* (δ_1) *and* (δ_2).

Proof. Any local complete $K[[X]]$-algebra A has a natural structure of $K[[X]][G]$-algebra with $G = G(A, \textbf{CLS})$ (exactly as in (4.3)) and we conclude by (5.2).

6. Hopf $K[G]$-algebras

(6.1) Throughout this section we shall often identify an affine Hopf K-algebra A with the linear algebraic K-group $L = \mathrm{Spec}(A)$ and we write $A = \Gamma(L)$; moreover if K is algebraically closed we shall sometimes use the letter L to denote also the group $L(K)$ of K-points of L.

Following [19] p. 952 by a Hopf $K[G]$-algebra we mean a Hopf K-algebra [13, 15] which is also a $K[G]$-algebra such that the comultiplication $A \to A \otimes_K A$ and the counit $A \to K$ are $K[G]$-module maps. A Hopf $K[G]$-algebra A is called split if there is a Hopf K-algebra isomorphism $A \simeq K \otimes_{K^G} (A^0)$ with A^0 Hopf K^G-algebra such that the induced $K[G]$-module structure on $K \otimes_{K^G} (A^0)$ is given by $s(c \otimes x) = sc \otimes x$ for all $s \in G$, $c \in K$, $x \in A^0$.

(6.2) THEOREM. *Let K be algebraically closed of characteristic zero and let A be an affine Hopf $K[G]$-algebra. then $D(A$, **AHA**$)$ contains an algebraic extension of K^G.*

The key point in proving (6.2) is the following:

(6.3) THEOREM. *Let K/K_0 be an extension of algebraically closed fields of characteristic zero and L a linear algebraic K-group with unipotent radical $U = R_u(L)$. Then $K_0 \in D(L$, **AHA**$)$ if and only if $K_0 \in D(\mathrm{Lie}\,(U)$, **PAL**$)$, where* Lie (U) *is the* Lie *algebra of U viewed as a polarized K-algebra via* $P_{\mathrm{Lie}(U)} = \mathrm{Lie}\,(U)$.

(6.4) COROLLARY. *If* char $(k) = 0$ *the functor* **AHA** *has property* (δ_1).

(6.5). *Proof of Theorem* (6.3). If $L = L^0 \otimes_{K_0} K$ with L^0 a linear algebraic K_0-group then $U = U^0 \otimes_{K_0} K$ where U^0 is the unipotent radical of U hence K_0 is a field of definition for U, in particular for Lie (U). Conversely, if K_0 is a field of definition for Lie (U) then so it will be for U because U is isomorphic as an affine variety with the spectrum of the symmetric algebra on Lie (U), the isomorphism being given by "exp" while the multiplication on U is defined by the Campbell-Hausdorff formula which involves only rational coefficients [12] p. 228. So we may write $U \simeq U^0 \otimes_{K_0} K$ for some unipotent K_0-group U^0. Now by [12], p. 117 L is a semidirect product of U with some linearly reductive subgroup $P \subset L$. P is then reductive and in particular $P = P^0 \otimes_{K_0} K$ for some reductive K_0-group P^0 [7].By [3] the group Aut (U) of algebraic group automorphisms of U is an algebraic K-group; moreover we must have Aut $(U) =$ Aut $(U^0) \otimes_{K_0} K$. Furthermore the group homomorphism $\varrho: P \to \text{Aut}\,(U)$ defined by $\varrho(p)u = p^{-1}up$ ($p \in P$, $u \in U$) is also algebraic. We claim there is a K-point σ of Aut (U) and a morphism of algebraic K_0-groups $\varrho^0: P^0 \to \text{Aut}\,(U^0)$ such that $\varrho^0 \otimes 1_K = \text{Inn}_\sigma \cdot \varrho$ where $\text{Inn}_\sigma \in \text{Aut}\,(\text{Aut}\,(U))$ is defined by $\text{Inn}_\sigma(\tau) = \sigma^{-1} \circ \tau \circ \sigma$. Indeed since P is linearly reductive, by [8] p. 194 we have in particular $H^1(P, \text{Lie}\,(\text{Aut}\,(U)) = 0$ (with P acting on Lie (Aut (U)) via ϱ and the adjoint representation of Aut (U)). By [9] p. 116 the above cohomology group identifies with the space of "first order deformations" of ϱ modulo the "first order deformations arising from infinitesimal inner automorphisms of Aut (U)". Now the existence of ϱ^0 and σ follows for instance from [6] (2.11) plus an obvious specialisation argument. With ϱ^0 and σ at hand we may define an isomorphism of algebraic K-groups

$$\varphi : L = U \times_\varrho P \to U \times_r P$$

by the formula $\varphi(u, p) = (\sigma^{-1}(u), p)$ where $U \times_\varrho P$ is set theoretically $U \times P$ with multiplication given by $(u_1, p_1)(u_2, p_2) = ((\varrho(p_2)u_1)u_2, p_1p_2)$ and $U \times_r P$ is defined similarly with $r = \varrho^0 \otimes 1_K$ instead of ϱ. But $U \times_r P = (U^0 \times_{\varrho^0} P^0) \otimes_{K_0} K$ and Theorem (6.3) is proved.

(6.6) *Proof of Theorem* (6.2). A is the coordinate Hopf algebra of an algebraic K-group L. Let U be the unipotent radical of L and J the defining prime ideal of U in A. We claim that $s(J) = J$ for all $s \in G$. Indeed upon letting σ to be the image of s in $\mathbf{g}(K)$ it is sufficient to prove that the natural map $p_\sigma : L^\sigma \to L$ given in some matrix representation by $(x_{ij}) \mapsto (\sigma x_{ij})$ carries the unipotent radical of L^σ onto the unipotent radical of L (here of course $L^\sigma = \text{Spec}\,(A^\sigma)$. But this follows from the fact that the map p_σ is an abstract group isomorphism (of course not an algebraic K-group isomorphism!), it takes Zariski closed sets into Zariski closed sets and takes unipotent matrices into unipotent matrices, so our claim follows. We deduce that the coordinate Hopf algebra $B = A/J$ of U has an induced structure of Hopf $K[G]$-algebra. Then one easily checks that Lie (U) also has a (naturally induced) structure of Lie $K[G]$-algebra (use for instance the $K[G]$-algebra structure on the convolution algebra $B^* = (\text{Hom}_K(B, K), *)$ and the description of Lie (U) as a Lie subalgebra of the Lie algebra $(B^*, [\,,])$, $[f, g] = f*g - g*f$ cf. [13]). Now if K_0 is the algebraic closure of K^G in K by (4.2) and property (s) we have $K_0 \in D(\text{Lie}\,(U), \mathbf{PAL})$ hence by (6.3) $K_0 \in D(L, \mathbf{AHA})$ and we are done.

(6.7) Let's discuss rigidified and pure affine Hopf algebras. Suppose K is algebraically closed of characteristic zero, A is an affine Hopf $K[G]$-algebra,

$L = \text{Spec}\,(A)$, $U = R_u(L)$. The arguments in (6.6) show that U and hence also L/U have on their coordinate Hopf algebras natural structures of Hopf $K[G]$-algebras. Let now B be an affine Hopf $K[G]$-algebra; by a $K[G]$-representation V of B (or of Spec (B)) we mean a finite dimensional $K[G]$-module V together with a Hopf $K[G]$-algebra map $\Gamma(GL(V)) \to B$; V is called faithfull if the above map is surjective. Here $\Gamma(GL(V))$ has the structure of Hopf $K[G]$-algebra induced by that of V via the following formulae: if $e_1, \ldots, e_n$ is a K-basis of V, e is the column vector with entries $e_1, \ldots, e_n$ and $se = a(s)e$ where $s \in G$, $a(s) \in GL_n(K)$ and if $X = (X_{ij})$ is a matrix of indeterminates which are coordinates on $GL(V)$ then we put $sX = a(s)^{-1}Xa(s)$ (product of matrices).

(6.8) THEOREM. *Let K be algebraically closed of characteristic zero, A an affine Hopf $K[G]$-algebra, L =* Spec (A), *$U = R_u(L)$ and let V be a faithful $K[G]$-representation of L/U. Assume there is a maximal reductive subgroup P of L whose ideal in A is G-globally invariant. Then:*

1) *There exists a cofinite subgroup $\tilde{G}$ of G and a finitely generated constrained extension $\tilde{K}/K$ of $\tilde{G}$-fields such that $\tilde{K} \otimes_K A$ and $\tilde{K} \otimes_K \Gamma(L/U)$ are split Hopf $K[G]$-algebras and $\tilde{K} \otimes_K V$ is a split $K[G]$-module.*

2) *There exists a finitely generated regular extension of G-fields $\tilde{K}/K$ such that $\tilde{K} \otimes_K A$, $\tilde{K} \otimes_K \Gamma(L/U)$ are split Hopf $K[G]$-algebras and $\tilde{K} \otimes_K V$ is a split $K[G]$-module.*

(6.9) COROLLARY. *If* char (k) = 0, $\mathbf{AHA}^r$ *has properties* (δ_1) *and* (δ_2).

Proof. Let $L \in \mathbf{AHA}_K$, $U = R_u(L)$ and $\varrho\colon L/U \to GL(V)$ a rigidification. Let H be the group of all triples $s = (\sigma, u_\sigma, v_\sigma)$ where $\sigma \in \mathbf{g}(K)$, $u_\sigma\colon L \to L^\sigma$ and $v_\sigma\colon V \to V^\sigma$ are isomorphisms and the following diagram is commutative:

$$\begin{array}{ccc} L/U & \xrightarrow{\varrho} & GL(V) \\ {\scriptstyle \bar{u}_\sigma}\downarrow & & \downarrow{\scriptstyle GL(v_\sigma)} \\ (L/U)^\sigma & \xrightarrow[\varrho^\sigma]{} & GL(V^\sigma) \end{array}$$

where $\bar{u}_\sigma$ is deduced from u_σ while $GL(v_\sigma)(x) = v_\sigma^{-1}xv_\sigma$. Write $L = Ux_\alpha P$ for some $\alpha\colon P \to$ Aut (U) and let G be the subgroup of H consisting of all $(\sigma, u_\sigma, v_\sigma)$ for which $u_\sigma(P) = P^\sigma$. By (6.8) we shall be done if we prove that G and H have the same image in $\mathbf{g}(K)$. Now if $(\sigma, u_\sigma, v_\sigma) \in H$, by the conjugacy of maximal reductive groups in L^σ([12] p. 117) there exists a K-point $x \in U$ such that $x^{-1}(u_\sigma(P))x = P^\sigma$. Put $w_\sigma = \text{Int}_x \circ u_\sigma$. Then $\bar{w}_\sigma = \bar{u}_\sigma$ and consequently $(\sigma, w_\sigma, v_\sigma) \in G$ which ends our proof.

(6.10) COROLLARY. If char $(k) = 0$, $\mathbf{AHA}^p$ has property (δ_2).

Proof. Let K be algebraically closed, $A \in \mathbf{AHA}^p_K$, $L = \text{Spec}(A)$, $U = R_u(L)$, $P = L/U$. By (6.9) and the fact that $k \in D(P, \mathbf{AHA})$ [7] it is sufficient to construct a faithful representation ϱ: $P \to GL(V)$ such that for any $\varphi \in \text{Aut}(P)$ there exists $\psi \in \text{GL}(V)$ such that $\varrho \circ \varphi = \text{Int}_\psi \circ \varrho$. Start with any faithful representation ε: $P \to GL(W)$, select a (finite) set $\tau_1, \ldots, \tau_N \in \text{Aut}(P)$ of representatives modulo Int (P) and let W_i be the representations of P defined by the composition $P \xrightarrow{\tau_1} P \xrightarrow{\varrho} GL(W)$. Then we are done by putting

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_N \text{ (as representations)}$$

(6.11). *Proof of* (6.8). We prove 1) and 2) simultaneously. Once again write $L = U \times_\alpha P$ with α: $P \to \text{Aut}(U)$ and let ϱ: $L/U \to GL(V)$ define our representation. Composing with the isomorphism $P \to L \to L/U$ we get a representation ε: $P \to GL(V)$. By (3.6) there exist finitely generated extensions $K \subset K_1 \subset K_2$ and subgroups $G \supset G_1 \supset G_2$ such that K_1/K is a G_1-extension, K_2/K_1 is a G_2-extension, $K_1 \otimes_K V$ is a split $K_1[G_1]$-module, $K_2 \otimes_K \text{Lie}(U)$ is a split $K_2[G_2]$-module (hence a split Lie $K_2[G_2]$-algebra) and moreover

a) in case 1) G_1, G_2 are cofinite in G and K_1/K, K_2/K_1 are constrained,

b) in case 2) $G = G_1 = G_2$ and K_1/K, K_2/K_1 are regular.

We claim that $\Gamma(\text{Aut}(U))$ has a natural structure of Hopf $K[G]$-algebra induced by that of $\Gamma(U)$; this can be seen by taking the embedding $\text{Aut}(U) \xrightarrow{\exp^*} \text{Aut}(\text{Lie}(U)) \to GL(\text{Lie}(U))$. Moreover the splitting of $K_2 \otimes_K \text{Lie}(U)$ as a Lie $K_2[G_2]$-algebra yields (via the exponential map) a splitting

$$K_2 \otimes_K U \simeq K_2 \otimes_{(K_2)^{G_2}} (U^0)$$

of Hopf $K_2[G_2]$-algebras, hence a splitting

$$K_2 \otimes_K \text{Aut}(U) \simeq K_2 \otimes_{(K_2)^{G_2}} (\text{Aut}(U^0))$$

as Hopf $K_2[G_2]$-algebras.

We claim that α: $P \to \text{Aut}(U)$ yields a map of Hopf $K[G]$-algebras between the corresponding coordinate algebras. This can be seen as follows: the action of G on $A = \Gamma(L) = \Gamma(U \times_\alpha P)$ yields for any $s \in G$ a K-isomorphism

$$\varphi = \varphi_s: U \times_\alpha P \to U^\alpha \times_{\alpha^\sigma} P^\sigma$$

where $\sigma = s/K$ such that $\varphi(U) = U^\sigma$, $\varphi(P) = P^\sigma$ (the latter follows from our condition that the ideal of P in A is G-invariant). For any $u \in U$, $p \in P$ we have

$$\varphi((u, 1)(1, p)) = \varphi(u, 1)\varphi(1, p) = (\varphi(u), 1)(1, \varphi(p))$$

$$= (\alpha^\sigma(\varphi(p))\varphi(u), \varphi(p)) = \varphi(\alpha(p)u, p) = (\varphi(\alpha(p)u), \varphi(p))$$

This gives the commutativity of the diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \mathrm{Aut}\,(U) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle C_\varphi} \\ P^\sigma & \xrightarrow[\alpha^\sigma]{} & \mathrm{Aut}\,(U^\sigma) \end{array}$$

where $C_\varphi(f) = \varphi \circ f \circ \varphi^{-1}$ hence our claim is proved.

To conclude note that we have two Hopf $K_2[G_2]$-algebra maps:

$$\varepsilon^* : \Gamma(GL(K_2 \otimes_K V)) \to \Gamma(K_2 \otimes_K P)$$

$$\alpha^* : \Gamma(K_2 \otimes_K \mathrm{Aut}\,(U)) \to \Gamma(K_2 \otimes_K P)$$

Since $\Gamma(GL(K_2 \otimes_K V))$ is split, ker (ε^*) is split; since ε^* is surjective, $\Gamma(K_2 \otimes_K P)$ is split. Since ε^* and α^* take G_2-invariants into G_2-invariants we get $\varepsilon = K_2 \otimes_{K_0} (\varepsilon^0)$, $\alpha = K_2 \otimes_{K_0} (\alpha^0)$ where $K_0 = K_2^{G_2}$ and

$$\varepsilon^0 : P^0 \to GL(V^0),\ P^0 \text{ a reductive } K_0\text{-group}$$

$$\alpha^0 : P^0 \to \mathrm{Aut}\,(U^0)$$

which closes our proof.

7. Remarks and open questions

(7.1) Let K be algebraically closed (of characteristic zero, to fix ideas) and A an affine $K[G]$-algebra (respectively an affine Hopf $K[G]$-algebra). One could ask whether there exist a cofinite subgroup G_1 of G and an extension K_1/K of G_1-fields such that $K_1 \otimes_K A$ is a split $K_1[G_1]$-algebra (respectively a split Hopf $K_1[G_1]$-algebra). By out theory this is easily seen to be true if A has non negative Kodaira dimension (respectively if A is pure). But it fails in general. Here is an example. Let $A = k[t_1, t_2, t_1^{-1}, t_2^{-1}] = \Gamma(G_m \times G_m)$ and let G be the infinite cyclic group with generator s act on A via k-automorphisms by the formulae

$$st_1 = t_1 t_2^m, \quad st_2 = t_2$$

where m is a fixed integer. This makes A into an affine Hopf $K[G]$-algebra. Suppose there exists G_1 and K_1 as above. Upon modifying m we may assume $G_1 = G$. Let M be the K_1-linear subspace of $K_1 \otimes_K A$ spanned by $(e_p)_{p \in \mathbb{Z}}$ where $e_p = t_1 t_2^{mp}$. We have $se_p = e_{p+1}$ hence M is a $K_1[G]$-submodule of $K_1 \otimes_K A$ hence it is split. In particular there exists $f \in M^G, f \neq 0$; write $f = \Sigma\, a_p e_p$ with $a_p \in K_1$. We get $f = sf = \Sigma\, (sa_p)e_{p+1}$ hence $sa_p = a_{p+1}$ for all $p \in \Sigma$ hence $a_p \neq 0$ for all $p \in \mathbb{Z}$, contradiction.

(7.2) There are very natural "moduli functors" from **B** to **S** which are not coarsely representable. We give here an example. For any field K let $\mathbf{ALG}_K$ denote the category of K-algebras; for any field extension $K \to K'$, the base change functors $\mathbf{ALG}_K \to \mathbf{ALG}_{K'}$ $A \mapsto K' \otimes_K A$ yield a fibred category and a moduli functor **ALG**; moreover denote by $\mathbf{ALG}^c$ the subfunctor of **ALG** (fully embedded into **ALG**) of commutative associative unitary algebras. Since **ALG**, $\mathbf{ALG}^c$ have no finiteness properties it is not reasonable to expect that they have property (m); but one might still hope that they are coarsely representable (by some birational set not necessarily of finitely generated type). The fact is that neither **ALG** nor $\mathbf{ALG}^c$ are coarsely representable. Indeed coarse representability implies property (d_1); on the other hand we can show that $\mathbf{ALG}^c$ (and hence also **ALG**) does not have this property. Just take k to be arbitrary, $K \in \mathbf{B}^u$ arbitrary, $A = K(T) =$ field rational functions in the indeterminate T and $\xi \in \mathbf{ALG}^c(K)$ be the isomorphism class of A. Clearly K^ξ/k is algebraic. On the other hand if $E \in D(\xi)$ and A_E is an E-algebra such that $A \simeq K \otimes_E A_E$ then $K \otimes_E A_E$ is a field which may happen only if K/E is algebraic hence only if E/K is transcendental; consequently $\mathbf{ALG}^c$ does not have property (d_1).

(7.3) Here are some questions for which we would like to have a positive answer.

1) Do **AFF** or **AHA** have property (m) (at least if char $(k) = 0$)?

2) Do the functors in Theorem (2.10) have property (m) in characteristic $p > 0$?

3) Are **CLA**, **COH**, **LFS** coarsely representable (by a birational set not necessarily of finitely generated type)?

Concerning 1) it would suffice for **AFF** to have properties (δ_1), (δ_2) and for **AHA** to have property (δ_2).

Concerning 2) what would be missing for **PAL**^{f} and **CLA**^{b} is property (g_3).

Concerning 3) note that **CLA** satisfies (d_1) and (d_2) (unlike **ALG** or **ALG**^{c}, for instance).

References

1. M. Artin: Algebraic approximations of structures over complete local rings. *Publ. Math. IHES* 36 (1969) 23–58.
2. S.S. Abhyankar, T.T. Moh and M. Van der Put: Invariants of analytic rings. *Publ. Math. IHES* 36 (1969) 165–194.
3. A. Borel and J.P. Serre: Théorèmes de finitude en cohomologie Galoisienne. *Comm. Math. Helv.* 39 (1964) 111–164.
4. F.A. Bogomolov: Analytic sections in conic bundles. *Proc. of the Steklov Institute* 3 (1985) 17–25.
5. A. Buium: Differential Function Fields and Moduli of Algebraic Varieties. *Lecture Notes in Math.* 1226, Springer (1986).
6. A. Buium and M. Takeuchi: Rigidity of maps from Hopf algebras to group algebras, to appear in *J. Algebra.*
7. M. Demazure: Schémas en groupes réductifs. *Bull. Soc. Math. France* 93 (1965) 369–413.
8. M. Demazure and P. Gabriel: *Groupes Algébriques I.* Masson & Cie, Paris, North-Holland, Amsterdam (1970).
9. M. Demazure and A. Grothendieck: Schémas en Groupes I. *Lecture Notes in Math.* 151, Springer (1970).
10. A. Grothendieck: Séminaire de Géometrie Algébrique. *IHES* (1960–61).
11. F. Grünewald and J.O. Halloran: Nilpotent groups and unipotent algebraic groups. *J. Pure Appl. Algebra* 37(3) (1985) 299–315.
12. G.P. Hochschild: *Basic Theory of Algebraic Groups and Lie Algebras.* Springer (1981).
13. G. Heyneman and M.E. Sweedler: Affine Hopf algebras I. *J. Algebra* 13-2 (1969) 192–242.
14. E.R. Kolchin: *Differential Algebra and Algebraic Groups.* Academic Press, New York (1973).
15. E.R. Kolchin and S. Lang: Existence of invariant bases. *Proc. of Amer. Math. Soc.* 11 (1960) 140–148.
16. S. Koizumi: The fields of moduli of polarized abelian varieties. *Nagoya Math. J.* 31 (1968) 185–245.
17. T. Matsusaka: Algebraic deformations of polarized varieties. *Nagoya Math. J.* 31 (1968) 185–245.
18. M. Nagata: *Local Rings.* Interscience Publishers, New York (1962).
19. W. Nichols and B. Weisfeiler: Differential formal groups of J.F. Ritt. *Amer. J. Math.* 104 (5) (1982) 943–1005.
20. A. Robert: Automorphism groups of transcendental field extensions. *J. Algebra* 16(2) (1970) 252–271.
21. F. Sakai: Kodaira dimension of complements of divisors. In: *Complex Analysis and Algebraic Geometry.* Tokio, Iwanami (1977) pp. 239–257.
22. A. Seidenberg: On analytically equivalent ideals. *Publ. Math. IHES* 36 (1969) 69–74.
23. J.P. Serre: *Cohomologie Galoisienne.* Springer Verlag.
24. G. Shimura: On the field of rationality for abelian varieties. *Nagoya Math. J.* 45 (1972) 167–178.

25. M.E. Sweedler: *Hopf Algebras*. Benjamin, New York (1969).
26. A. Weil: The field of definition of a variety. *Amer. J. Math.* 78 3 (1956) 509–524.
27. M. Rosenlicht: Some basic theorems on algebraic groups. *Amer. J. Math.* 78 (1956) 401–463.