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Arrangements of lines and monodromy of plane curves

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Introduction

Let $Y = \mathbf{C}^2 \setminus \bigcup_i M_i$ be the complement to an arrangement of real hyperplanes in $\mathbf{C}^N$. The study of the topology of Y is important both in the theory of hypergeometric functions (see [7] and subsequent papers by the same author; [6] and subsequent papers by the second author) and in the singularity theory ([1], [2], [4]; see also [3]). Moreover it plays a role in some problems in algebraic geometry ([10]). In [12] the homotopy type of Y was described by an explicit construction of a regular N-cell complex $X \subset Y$ and of a homotopy equivalence between X and Y. Such a complex was also used for finding presentations of the fundamental group of Y.

In this paper we consider the two dimensional case in which $Y = \mathbf{C}^2 \setminus C$ and C is a finite union of real lines. We found that the method of braid monodromy (introduced in [9] for any algebraic plane curve) is useful to the study of the associated complex X.

Let $C = \{f(x, y) = 0\} \subset \mathbf{C}^2$ be a plane curve, $\pi: \mathbf{C}^2 \to \mathbf{C}_x$ the projection onto the x-axis (where C is generic with respect to π) and set $S(C, \pi) = \{p \in C: \partial f(p)/\partial y = 0\}$, $D(C, \pi) = \pi(S(C, \pi))$. The braid monodromy of C is the homomorphism $\theta: \pi_1(\mathbf{C}_x \setminus D(C, \pi), M) \to B[\pi^{-1}(M), C \cap \pi^{-1}(M)]$, where $M \notin D(C, \pi)$ is a basepoint. The braid $\theta(\gamma)$ is obtained by following the fiber of π over $\gamma: (I, \partial I) \to (\mathbf{C}_x, M)$ by means of a trivialization of $\gamma^*(\pi)$ taking $C \cap \pi^{-1}(M)$ into $C \cap \pi^{-1}(\gamma(t))$, $0 \leqslant t \leqslant 1$.

Call the image $\mathscr{M}(C)$ of θ the monodromy group of C. The main result of this paper could be considered the determination of the monodromy group of C for C an arrangement of r real lines (corollary to Theorem 2). All such groups can be obtained as follows (when an appropriate coordinate system is chosen). Let $\hat{\mathscr{G}}_r$ be the set of subsets $\hat{G}$ of $\mathbf{R}^2$, endowed with two finite sets $V\hat{G} = \{V_1, \ldots, V_r\}$ and $E\hat{G}$ such that: (i) $\hat{G} = \bigcup_{i,j}[V_i, V_j]$, where $[V_i, V_j]$ is the segment determined by V_i and V_j; (ii) $E\hat{G} = \{[V_i, V_j]: i < j$ and $\nexists (h, k) \neq (i, j)$ with $[V_i, V_j] \subset [V_h, V_k]\}$. For $\hat{G} \in \hat{\mathscr{G}}_r$ let $\mathscr{M}(\hat{G})$ be the subgroup of $B[\mathbf{R}^2, V\hat{G}]$ generated by the twists around the segments in $E\hat{G}$.

Then $\mathscr{M}(\hat{G})$ can be identified with the monodromy group of the arrangement given by the lines dual to the V_i's.

The preceding result is derived by a result holding in more general cases: under certain conditions for an ordinary singular point α of a curve C a method for determining *a priori* the braids $\theta(\gamma)$ associated to a class of horizontal circuits γ encircling $\pi(\alpha)$ is found (Theorem 1). These braids are twists around good paths in $\pi^{-1}(M)$. If K is a finite set in a complex plane $\mathbf{C}$ with some points on the real axis $\mathbf{R}$ we call an embedding ϕ_0: $(I, \partial I) \to (\mathbf{C}, K \cap \mathbf{R})$ *good* if an isotopy ϕ_t: $I \to \mathbf{C}$, $0 \leqslant t \leqslant 1$, exists such that $\phi_1(I) \subset \mathbf{R}$ and $\phi_t(I) \cap K = \phi_0(I) \cap K$ for $0 \leqslant t < 1$.

The method of Theorem 1 works when admissible orientations (see 2.1) exist for the real arcs of C intersecting the "angle" spanned by the branches of C in α. By varying admissible orientations one obtains different horizontal paths and different good braids. So Theorem 1 can be used for finding "very good" generators for the above monodromy groups. Also is applies (by the Van Kampen method) to the study of $\pi_1(\mathbf{C}^2 \backslash C)$. Let $g_1, \ldots, g_r \subset \pi^{-1}(M)$ be a well ordered set of generators of $\pi_1(\mathbf{C}^2 \backslash C)$ ($r = \deg C$). Then Theorem 1 gives sufficient conditions in order that a double point gives a commutation between two g_i's, or in general in order that an n-ple point produce global relations which coincide with the local ones (i.e. such relations are obtained by substituting to each local generator one of the g_i's; see 2.2 below). As an example, a weaker form of [5], [11] is derived.

In Section 4 the preceding ideas are applied to the study of the higher homotopy groups of the complement of an arrangement C. In fact, the construction of the braid monodromy can be translated into a study of the associated complex X. For arrangements satisfying certain conditions (which are effectively verifiable in a finite number of steps) we show that there is a subcomplex of X contracting onto a complex which is not a $k(\pi, 1)$ (so X is not a $k(\pi, 1)$ either).

1.1. Good braids

Braid groups can be defined in different ways. Here it is convenient to think of $B[P; K]$ as the group of compact supported homeomorphisms of a 2-plane P which preserve a fixed finite set K, modulo compact supported isotopies (which at every instant preserve K). Recall ([9]) that to each smooth embedding ϕ: $I = [0, 1] \to P$ such that $\phi(\partial I) \subset K$ an anticlockwise half-twist is associated, whose support is an (arbitrarely small) neighborhood of $\phi(I)$ homeomorphic to a 2-disk. Denote by $b(\phi(I)) \in B[P; K]$ the class of such homeomorphism.

Consider now the group $B[\mathbf{C}; K]$, where $\mathbf{C}$ is a complex plane and K is a finite subset of $\mathbf{C}$ containing some points in the real axis $\mathbf{R}$. Let $p: \mathbf{C} \to \mathbf{R}$ be the projection. We call an embedding $\phi: (I, \partial I) \to (\mathbf{C}, K \cap \mathbf{R})$ a *good embedding* (relatively to p) if for every t in $[0, 1]$ the line $\{\mathrm{Re}\, z = p(\phi(t))\}$ intersects $\phi(I)$ in at most one point and the segment $(p(\phi(t)), \phi(t)]$ of $\mathbf{C}$ does not intersect K (up to homotopy, this definition is the same as that in the introduction).

Half-twists associated to homotopic (rel K) embeddings give identical braids and it is not hard to see that b induces an injective map over the set of homotopy classes of embedding (look at the induced permutation of K and at the supports). So one could also define good homotopy classes of embeddings (those having a good representative) and good braids (the images of good classes).

1.2. Braid monodromy for a plane curve

We briefly recall from [9] the construction of the braid monodromy for a plane curve.

Let $\bar{C} \subset \mathbf{CP}^2$ be an algebraic curve of degree r, $\mathbf{C}^2 \subset \mathbf{CP}^2$ an affine chart (with coordinate (x, y)) which is generic for $\bar{C}$, $C = \bar{C} \cap \mathbf{C}^2 = \{f(x, y) = 0\}$. Let $\pi: \mathbf{C}^2 \to \mathbf{C}_x$ be the projection onto the complex plane of the x-coordinate and denote by $S(C, \pi) = \{\alpha \in C: \partial f(\alpha)/\partial y = 0\}$, $D(C, \pi) = \pi(S(C, \pi)$. By genericity $|S(C, \pi)| = |D(C, \pi)|$.

Set $\tilde{\mathbf{C}}^1 = \{x = M\}$, $M \in \mathbf{C}_x \backslash D(C, \pi)$, as a standard fiber of π. Let $\gamma: I \to \mathbf{C}_x$ be a path not intersecting $D(C, \pi)$: the bundle $\gamma^*(\pi)$ and its sub-bundle $\gamma^*(\pi|C)$ are trivial. By a trivialization of the pair $(\pi, \pi|C)$ over γ we mean a trivialization of $\gamma^*(\pi)$ inducing a trivialization of $\gamma^*(\pi|C)$. Following the fiber of $(\pi, \pi|C)$ over γ will mean that one is considering the homeomorphisms $(\pi^{-1}(\gamma(0)), C \cap \pi^{-1}(\gamma(0))) \to (\pi^{-1}(\gamma(t)), C \cap \pi^{-1}(\gamma(t)))$, $t \in [0, 1]$, induced by a given trivialization of $(\pi, \pi|C)$. In particular, if γ is an M-based loop, then one obtains a homeomorphism of $(\tilde{\mathbf{C}}^1, C \cap \tilde{\mathbf{C}}^1)$ with itself. Since π is trivial over the image of γ one can take such homeomorphism with compact support. So we get a homomorphism $\theta: \pi_1(\mathbf{C}_x \backslash D(C, \pi), M) \to B[\tilde{\mathbf{C}}^1, C \cap \tilde{\mathbf{C}}^1]$. Such homomorphism is called the braid monodromy of C.

In practice, the construction of θ is carried out by two steps, a local one and a global one, as we now describe.

Let $\alpha \in S(C, \pi)$, $L = \{x = x(\alpha) + \varepsilon\}$, $0 < \varepsilon \ll 1$, a vertical line near to the fiber of π containing α. In L (which is a complex plane) there is a Lefshetz relative cycle a: we can take a as a non autointersecting broken-line with vertices the points of $C \cap L$ going into α for ε going to 0. Using a trivialization of $(\pi, \pi|C)$ over the circle $c_\alpha = \{x = x(\alpha) + \varepsilon\, e^{i\theta} : 0 \leqslant \theta \leqslant 2\pi\}$ and following the fiber over c_α one obtains a homeomorphism of L which preserves $C \cap L$, with support a small neighborhood of a. The class of such homeomorphism is an element of $B[L; C \cap a]$ which we can call local braid monodromy (relative to α). Next one must transport these local braid monodromies into the standard fiber $\tilde{\mathbf{C}}^1$ by a system of "horizontal" paths (i.e., paths in the x-plane) which are generators of $\pi_1(\mathbf{C}_x \backslash D(C, \pi), M)$. Let Γ join $x(\alpha) + \varepsilon$ and M in $\mathbf{C}_x \backslash D(C, \pi)$, $\gamma = \Gamma^{-1} \cup c_\alpha \cup \Gamma$. Let $\theta(\Gamma) : L \to \tilde{\mathbf{C}}^1$ be the homeomorphism induced by a trivialization of $(\pi, \pi|C)$ over Γ, so $\theta(\Gamma)\,(a)$ is a Lefschetz relative cycle in $\tilde{\mathbf{C}}^1$. Then $\theta(\gamma)$ is represented by a homeomorphism with support a small neighborhood of $\theta(\Gamma)\,(a)$, and it is completely determined by the knowledge of the local braid monodromy and of $\theta(\Gamma)\,(a)$.

In general one takes a well-ordered set of generators for $\pi_1(\mathbf{C}_x \backslash D(C, \pi), M)$, meaning that each generator is an elementary circuit (constructed as above) and two generators intersect only in the basepoint. The ordering starts from one generator and follows the anticlockwise sense around M.

Let $\gamma_1, \ldots, \gamma_\nu$ be a well-ordered set of generators for $\pi_1(\mathbf{C}_x \backslash D(C, \pi), M)$. Then it is possible to prove (see [9]) that the braid $\theta(\gamma_1 \ldots \gamma_\nu)$ corresponding to the ordered product of the γ_i's is the element $\Delta^2 \in B[\tilde{\mathbf{C}}^1, C \cap \tilde{\mathbf{C}}^1]$ generating the center. So one can formally write a formula

$$\Delta^2 = \prod_{i=1}^{\nu} \theta(\gamma_i)$$

which determines θ. Note that such a formula gives also immediately a presentation for $\pi_1(\mathbf{C}^2 \backslash C)$ in some fixed set of (well-ordered) generators contained in $\tilde{\mathbf{C}}^1 \backslash C \cap \tilde{\mathbf{C}}^1$: it suffices to use the Van Kampen method (to this extent the ordering of the γ_i's is not important).

Notice also that $\mathbf{C}^2 \backslash C$ is homotopy equivalent to the 2-complex associated to a presentation of $\pi_1(\mathbf{C}^2 \backslash C)$ coming from the braid monodromy ([8]).

2.1. Construction of braid monodromies

Indicate by $\mathbf{R}^2 \subset \mathbf{C}^2$ the set of points with real coordinates. We will say that something is real if it is contained in $\mathbf{R}^2$.

Let $\mathbf{R}_x \subset \mathbf{C}_x$ be the real axis of $\mathbf{C}_x$; take $M \gg 0$ in $\mathbf{R}_x$ and let $\tilde{\mathbf{R}}^1 \subset \tilde{\mathbf{C}}^1$ be the real axis of $\tilde{\mathbf{C}}^1$ (with y coordinate). Let $\alpha \in S(C, \pi)$ be an ordinary singular point with real coordinates, and suppose that: (a) all the branches $C_1(x), \ldots, C_n(x)$, $x \in \mathbf{R}_x$, of C in α are real; (b) in $C_i \cap \pi^{-1}([x(\alpha), M])$

there are at most ordinary singular points and no branch points, $i = 1, \ldots, n$ (so every C_i extends in $\mathbf{R}^2$ til it intersects $\tilde{\mathbf{R}}^1$); (c) two C_i's do not intersect in $\pi^{-1}([x(\alpha), M])$. Set

$$S(\alpha) = \{(x, y) \in \mathbf{R}^2 \colon x \in [x(\alpha), M], y \in [C_1(x), \ldots, C_n(x)]\}$$

where $[C_1(x), \ldots, C_n(x)]$ is the convex hull (a real segment in $\pi^{-1}(x)$) of the C_i's. Set also $T(\alpha) = \bigcup_{i=1}^n C_i$, and assume that C has only ordinary singularities or branch points in $S(\alpha)$, with all branches real.

Let U be a small neighborhood of $S(\alpha)$ and consider $C \cap U$: besides the arcs C_i's there will be other connected arcs, say $C'_1, \ldots, C'_k$, such that: (1) $C \cap U = \bigcup_i C_i \cup \bigcup_j C'_j$; (2) if $p \in C'_j$ is not a branch point of C, then in a small neighborhood of p C'_j is union of (real) branches of C; (3) if $p \in C'_j$ is a branch point of C, then in a small neighborhood of p C'_j is union of the two real branches of C in p. So each C'_j is obtained by (real) analytic continuation within U, starting from some point p in $C \cap S(\alpha)$ (when a branch point q is encountered, one proceeds on the other branch of q).

Let us give to C_i the orientation from α to $C_i \cap \tilde{\mathbf{R}}^1$, $i = 1, \ldots, n$. Let also σ_α be an orientation for $C'_\alpha = \{C'_1, \ldots, C'_k\}$ (that is an orientation $\sigma_\alpha(C'_j)$ for every curve C'_j). We shall say that σ_α is an admissible orientation of C'_α if the following conditions are satisfied:

(i) for each singular point β in $T(\alpha)$, $x(\beta) > x(\alpha)$, contained in C_i, it holds $(C'_j \cdot C_i)_\beta = (C'_h \cdot C_i)_\beta$ when $C'_j, C'_h \ni \beta$ (brackets means intersection number);

(ii) for each ordinary singular point $\beta \in S(\alpha) \backslash T(\alpha)$, let $\mathbf{R}_\beta = \pi^{-1}(x(\beta)) \cap \mathbf{R}^2$, with the natural orientation of the real axis in the y-plane, $\mathbf{RP}^1(\beta)$ the pencil of real lines in β. Then $\mathbf{RP}^1(\beta) \backslash \{\mathbf{R}_\beta\}$ (a real line) can be divided into two connected components $c(1)(\beta)$, $c(-1)(\beta)$ such that $c(1)(\beta) \supset \{(TC'')_\beta \colon C''$ is a real branch of C in β and $(C'' \cdot \mathbf{R}_\beta)_\beta = 1\}$ and $c(-1)(\beta) \supset \{(TC'')_\beta \colon C''$ is a real branch of C in β and $(C'' \cdot \mathbf{R}_\beta)_\beta = -1\}$ where $(TC'')_\beta$ is the real tangent to C'' in β and C'' has the induced orientation from C'_α.

Set $\operatorname{Sing}(>\alpha) = S(C, \pi) \cap S(\alpha) \cap T(\alpha) \backslash \{\alpha\}$. By condition (i) an admissible orientation σ_α on C'_α induces a map $\sigma_\alpha \colon \operatorname{Sing}(>\alpha) \to \mathbf{Z}_2 = \{-1, 1\}$, given by setting $\sigma_\alpha(\beta) = 1$ $[= -1]$ if $\beta \in C_i$ and $(C'_j \cdot C_i)_\beta = 1$ $[= -1]$ for every $C'_j \ni \beta$.

THEOREM 1. *Let C, α, C'_α be as above, and suppose that σ_α is an admissible orientation for C'_α. Let Γ be a path in $\mathbf{C}_x \backslash D(C, \pi)$ connecting $x(\alpha) + \varepsilon$ with M ($\varepsilon \ll 1$), which coincides with the segment $\Gamma' = [x(\alpha) + \varepsilon, M] \subset \mathbf{R}_x$ except for small semi-circles of radius ε centered at the points $x(\beta) \in \Gamma'$, $\beta \in S(C, \pi)$. Let the semi-circle centered in $x(\beta)$ lie below [above] $\mathbf{R}_x$ when $\beta \in S(\alpha)$ and one of the following conditions holds:*

(1) *$\beta \in T(\alpha)$ (so $\beta \in \mathrm{Sing}(>\alpha)$) and $\sigma_\alpha(\beta) = 1\ [= -1]$;*

(2) *$\beta \notin T(\alpha)$ is an ordinary singular point and, if $\underline{C}''$, $\bar{C}''$ are respectively the branches through β whose tangents have lowest and highest slopes in β, then*

$$(\underline{C}'' \cdot \mathbf{R}_\beta) = 1\ [= -1] \quad \textit{and}$$

$$(\bar{C}'' \cdot \mathbf{R}_\beta) = -1\ [=1];$$

(3) *$\beta(\notin T(\alpha))$ is a branch point contained in C_j', and the orientation of C_j' in β is that induced by the opposite orientation of $\mathbf{R}_\beta$ [by the orientation of $\mathbf{R}_\beta$].*

For the remaining projections let the semi-circle be indifferently below or above $\mathbf{R}_x$.

Let a be a relative Lefshetz cycle in $L = \{x = x(\alpha) + \varepsilon\}$ given by a segment in the real aixs of L.

Then $\theta(\Gamma)(a)$ is good (with respect to the projection of $\tilde{\mathbf{C}}^1$ onto its real axis). It joins the points of $T(\alpha) \cap \tilde{\mathbf{R}}^1$ passing below [above] the points $Q \in C_j' \cap S(\alpha) \cap \tilde{\mathbf{R}}^1$ such that $(C_j' \cdot \tilde{\mathbf{R}}^1)_Q = 1\ [= -1]$.

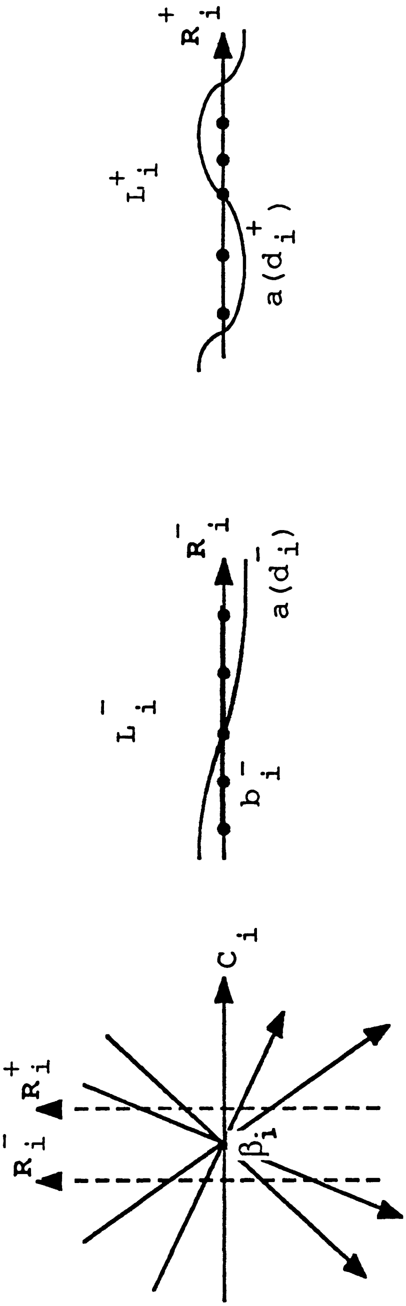
Proof. Let $\beta_1, \ldots, \beta_m$ be the points in $S(C, \pi)$ such that $x(\beta_i) \in \Gamma'$; assume they are ordered by increasing abscissas. Let $d_0^+ = x(\alpha) + \varepsilon$, $d_i^- = x(\beta_i) - \varepsilon$, $d_i^+ = x(\beta_i) + \varepsilon$, $i = 1, \ldots, m$, $d_{m+1}^- = M$. Set $L_i^- = \pi^{-1}(d_i^-)$, $\mathbf{R}_i^-$ its real axis (with natural orientation) $i = 1, \ldots, m+1$, $L_i^+ = \pi^{-1}(d_i^+)$, $\mathbf{R}_i^+$ its real axis, $i = 0, \ldots, m$.

If $p \in \Gamma$, set $\Gamma_p \subset \Gamma$ as the part of Γ between d_0^+ and p. By following the fiber of $(\pi, \pi|C)$ over Γ_p we get a homeomorphism $\theta(\Gamma_p)\colon L \to L_p := \pi^{-1}(p)$ taking a into $a(p) := \theta(\Gamma_p)(a)$. Set $\mathbf{R}_p$ as the real axis of L_p, endowed with the natural orientation.

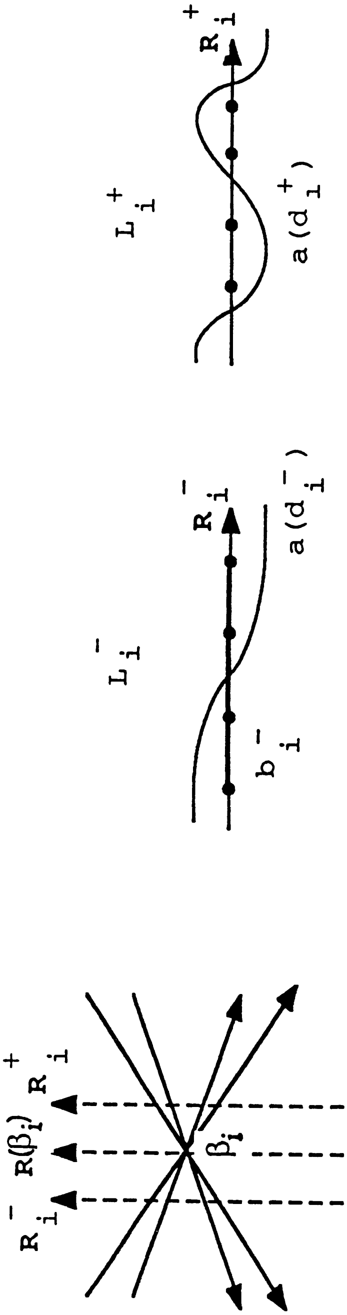
Suppose by induction that for each p in the segment $[d_i^+, d_{i+1}^-]$ of $\mathbf{R}_x$ $a(p)$ is good (with respect to the projection onto $\mathbf{R}_p$) and it joins the points in $T(\alpha) \cap \mathbf{R}_p$ passing below [above] the points $Q \in C_j' \cap \mathbf{R}_p \cap S(\alpha)$ such that $(C_j' \cdot \mathbf{R}_p)_Q = 1\ [= -1]$ $(0 \leqslant i < m)$; we want to show that the same is true when $p \in [d_{i+1}^+, d_{i+2}^-]$. Clearly it will suffice to prove the thesis for $p = d_{i+1}^+$.

Let b_i^- be a Lefschetz cycle in L_i^- relative to β_i. if $\beta_i \notin S(\alpha)$ then we can take $b_i^- \cap S(\alpha) = \varnothing$. Since following the fiber of $(\pi, \pi|C)$ over the semi-circle around $x(\beta_i)$ involves only a small neighborhood of b_i^-, the assumption $a(d_i^-)$ good implies trivially the thesis.

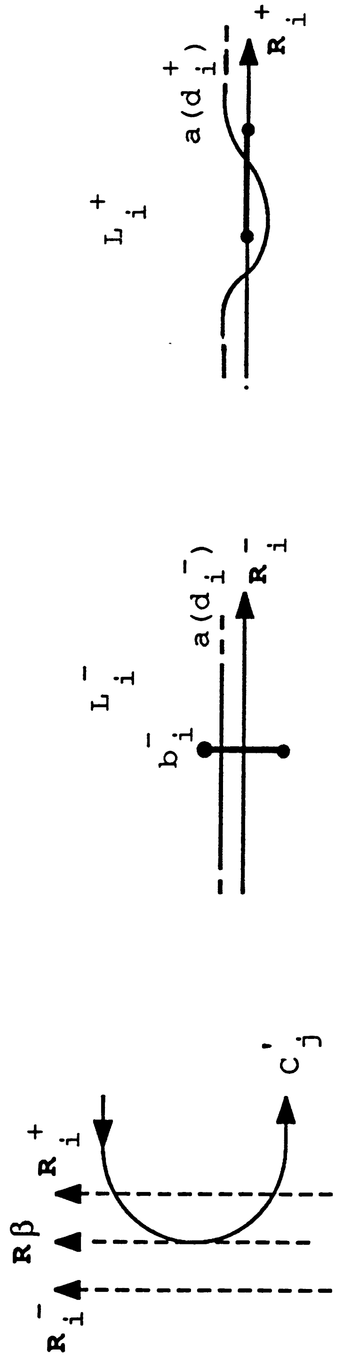
Suppose now $\beta_i \in S(\alpha) \cap T(\alpha)$, so by the above assumptions β_i is an ordinary singular point with real branches C_j, C_l'', $l = 1, \ldots, h-1$ $(\mathrm{ord}(\beta_i) = h)$. Choose b_i^- as a segment in L_i^-. By condition (i) for σ_α $(C_l'' \cdot C_j)_\beta$ is constant over l. In Picture 1 the case $\sigma_\alpha(\beta_i) = 1$ is illustrated: the changing of $a(d_i^-)$ after following $(\pi, \pi|C)$ over the semi-circle around $x(\beta_i)$ (which is now below $\mathbf{R}_x$) is produced by an anticlockwise half-twist around b_i^-. The thesis is easily verified for $i+1$.



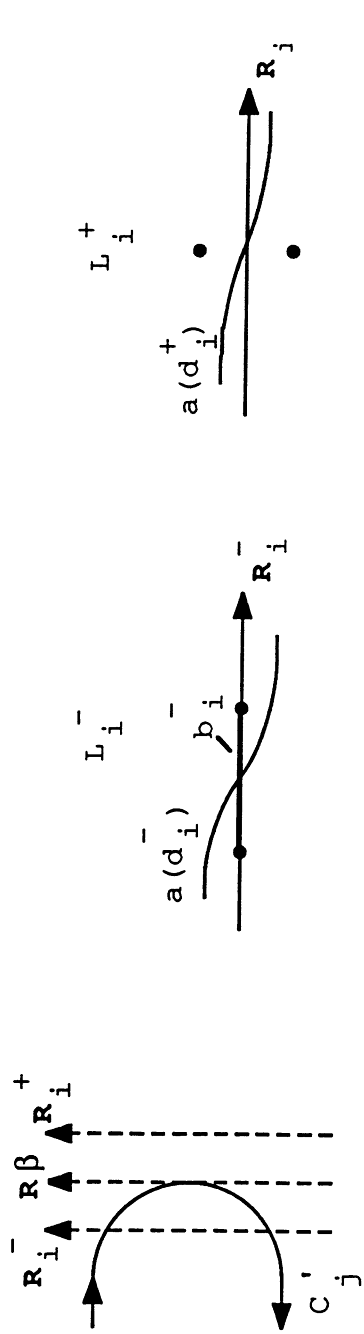
Picture 1



Picture 2



Picture 3



Picture 4

The case $\sigma_\alpha(\beta_i) = -1$ is similar (with the semi-circle above $\mathbf{R}_x$ and a clockwise half-twist around b_i^-).

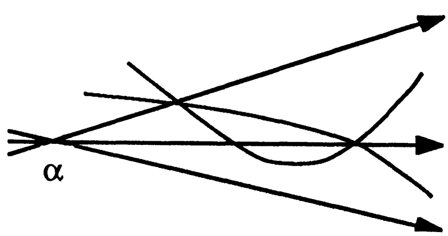
Now let $\beta_i \in S(\alpha)\backslash T(\alpha)$ be an ordinary singular point, with branches C_l'', $l = 1, \ldots, h$, ordered by the slopes of the C_l'' in β_i. Again let b_i^- be a segment in L_i^-. Assumption (ii) about σ_α means that there is a h', $0 \leqslant h' \leqslant h$, such that $(C_l'' \cdot \mathbf{R}_i^-) = 1$ $[= -1]$ for $1 \leqslant l \leqslant h'$, and $(C_l'' \cdot \mathbf{R}_i^-) = -1\,[=1]$ for $h' + 1 \leqslant l \leqslant h$. Picture 2 shows the changing of $a(d_i^-)$ in one of the cases (the semi-circle is below $\mathbf{R}_x$, the half-twist is in the anticlockwise sense). The thesis is verified for $i + 1$. Other cases are similar.

Suppose now β_i a branch point of C, contained in C_j'. Assume first that β_i is a birth-point in $\mathbf{R}^2$ and let b_i^- be a segment in L_i^-. The assumption $a(d_i^-)$ good implies that it intersects b_i^- in exactly one point. The situation is shown in Picture 3 for one of the possible orientations of C_j' (the semi-circle is below $\mathbf{R}_x$; the local braid monodromy is a "half half-twist" in the anticlockwise sense).

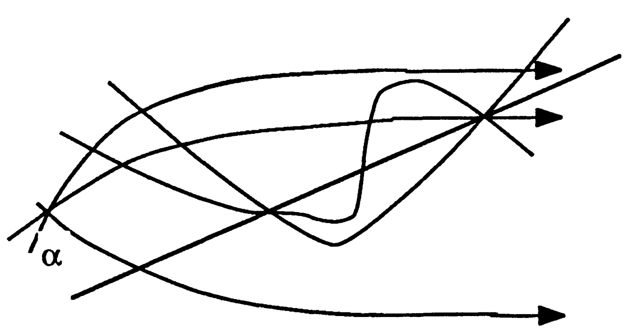
For β_i a death-point the situation is shown in Picture 4.

The cases with different orientations for C_j' are similar. This proves the theorem. Q.E.D.

Admissible orientations do not always exist for given C and α (satisfying above conditions). Picture 5a shows a configuration for which the above condition (i) cannot be satisfied; Picture 5b one in which (i) and (ii) cannot be contemporaly satisfied.



Picture 5a



Picture 5b

Nevertheless there are many cases in which many admissible orientations exist for C'_α. For instance when C is a union of real lines one obtains an admissible orientation by orienting each line L intersecting $S(\alpha)$, $L \not\ni \alpha$, so that $(L \cdot L') = 1$ for all $L' \ni \alpha$. Also, if C has only double points, then whatever orientation for C'_α is admissible.

2.2. Applications to the study of the fundamental group of $\mathbf{C}^2\backslash\mathbf{C}$

Set $K = C \cap \tilde{\mathbf{C}}^1$ and let $B \in \tilde{\mathbf{C}}^1\backslash K$ be a basepoint. Let $g_1, \ldots, g_d$ ($d = \deg C = |K|$) be a well-ordered set of B-based elementary circuits which generate $\pi_1(\tilde{\mathbf{C}}^1\backslash K, B)$. If $\alpha \in S(C, \pi)$ and γ is an M-based elementary circuit in $\mathbf{C}_x\backslash D(C, \pi)$ encircling $\pi(\alpha)$, then there are relations $g_i = \theta(\gamma)(g_i)$, $i = 1, \ldots, d$. By varying α and considering all the possible γ's one obtain a complete (infinite) presentation for $\pi_1(\mathbf{C}^2\backslash C, B)$ (by the Van Kampen method only one γ for each $\pi(\alpha)$ is sufficient, provided such γ's are well-ordered). Recall also that if α is an ordinary singularity of order n, then for each γ as above one obtains relations of the kind:

$$\tilde{g}_{i_n} \ldots \tilde{g}_{i_1} = \tilde{g}_{i_{n-1}} \ldots \tilde{g}_{i_1}\tilde{g}_{i_n} = \cdots = \tilde{g}_{i_1}\tilde{g}_{i_n} \ldots \tilde{g}_{i_2}. \qquad (\Delta)$$

We call an ordinary singular point $\alpha \in C$ *very good* (relative to $\{g_i\}$) if there exists an horizontal path γ as above such that relations (Δ) hold with $\tilde{g}_{i_j} = g_{i_j}$.

Consider the inclusion $\mathbf{C}^2 \subset \mathbf{CP}^2$ and the completion $\bar{C}$ of C ($\bar{C}$ is transversal to the infinite line) and let M big enough so that the ball $B(0; M) \subset \mathbf{C}_x$ contains $D(C; \pi)$. The fibres $(\pi^{-1}(a), \bar{C} \cap \pi^{-1}(a))$, $a \in (\mathbf{C}_x \cup \infty)\backslash B(0; M)$, can be canonically identified to $(\tilde{\mathbf{C}}^1, K)$ since $(\mathbf{C}_x \cup \infty)\backslash B(0; M)$ is contractible. So the monodromy representation factors through $\pi_1(\mathbf{CP}^2\backslash\bar{C})$. In particular given $\alpha \in S(C, \pi)$ as before Theorem 1, one can consider hypotheses analog to (a), (b) and (c) in the interval $[x(\alpha), M]^- = [x(\alpha), -\infty] \cup [+\infty, M]$ $(\subset \mathbf{RP}^1)$ and call $S(\alpha)^- = \{(x, y) \in \mathbf{R}^2\colon\ x \in [x(\alpha), M]^-,\ y \in [C_1(x), \ldots, C_n(x)]\}$ (the identification of the fibers of π outside $B(0; M)$ will be such that $[C_1(x), \ldots, C_n(x)]$ is bounded when $x \to -\infty$). We shall say that such hypotheses hold on the right or on the left of α if we are considering respectively the interval $[x(\alpha), M]$ or $[x(\alpha), M]^-$. It is easy to see that if conditions (a), (b) and (c) are true at the left of α, and an admissible orientation exists for the arcs of $C \cap S(\alpha)^-$, then a theorem completely analog to Theorem 1 is true (except that, in the affine picture for Γ, one has to substitute Γ in Theorem 1 with its complex conjugate in $\mathbf{C}_x$).

As a possible utilization of Theorem 1 above we give the following proposition.

PROPOSITION 1 *Assume that C has only real double points as singularities and for every one of them the hypotheses* (*a*) *and* (*b*) *before Theorem 1 hold at the right or at the left of α. Suppose also that if α is a double point for which the hypotheses hold at right* [*at left*] *then every $C_j' \in C_\alpha'$ intersects $\tilde{\mathbf{R}}^1 \cap S(\alpha)$ in at most one point.*

Let $B \gg 0$ lie on $\tilde{\mathbf{R}}^1$, far from $C \cap \tilde{\mathbf{C}}^1$, and let $\{q_i\}$ be a well-ordered set of generators for $\pi_1(\tilde{\mathbf{C}}^1 \backslash C, B)$ such that the generators corresponding to the points in $C \cap \tilde{\mathbf{R}}^1$ are associated to paths lying in the strip

$$\{-\varepsilon < \operatorname{Im} y \leqslant 0\} \quad \textit{where } \varepsilon < |\operatorname{Im} Q|, \quad \forall Q \in C \cap (\tilde{\mathbf{C}}^1 \backslash \tilde{\mathbf{R}}^1).$$

Then every double point is very good (*with respect to $\{g_i\}$*) *and therefore $\pi_1(\mathbf{C}^2 \backslash C)$ is abelian.*

Proof. Let α be a double point, contained in the branches C_i, C_j, for which (a) and (b) hold at the right. Orient C_α' so that if $P_j = C_j' \cap \tilde{\mathbf{R}}^1 \neq \varnothing$ then $(C_j' \cdot \tilde{\mathbf{R}}^1)_P = 1$. The obtained orientation is admissible and one can apply Theorem 1 (when α is a double point Theorem 1 holds with the same proof without condition (c)). Using notations from the theorem, $\theta(\Gamma)$ (*a*) is a good path connecting $Q_i = C_i \cap \tilde{\mathbf{R}}^1$ and $Q_j = C_j \cap \tilde{\mathbf{R}}^1$ passing below every other point in $C \cap \tilde{\mathbf{R}}^1 \cap S(\alpha)$. Then the twist associated to $\theta(\Gamma)$ (*a*) gives a commutation between the two corresponding generators.

Analogous argument can be used if conditions (a) and (b) hold at the left of α. Therefore every double point produces a commutation between two of the chosen generators, and one deduces that $\pi_1(\mathbf{C}^2 \backslash C)$ is abelian. Q.E.D.

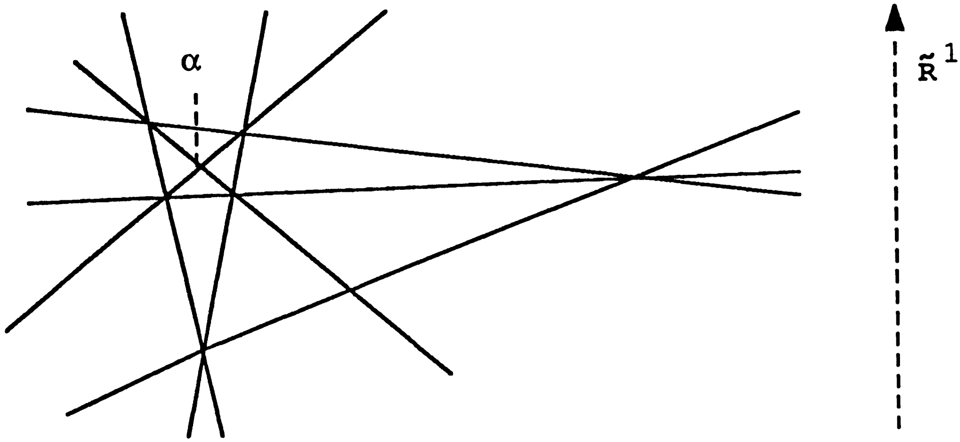
Naturally this is a particular case of [5], [11]. Nevertheless, Theorem 1 gives explicitly also the horizontal paths which produce the relations. In particular this is the case when C is union of lines in general position: here one has a direct proof (not inductive) of the abelianity of $\pi_1(\mathbf{C}^2 \backslash C)$.

Regarding arrangements of lines, following proposition holds.

PROPOSITION 2. *Suppose α is a singularity of $C = L_1 \cup \cdots \cup L_r$, where L_i is a real line. Let $\{g_i\}$ be a system of generators of $\pi_1(\tilde{\mathbf{C}}^1 \backslash C, B)$ chosen as above. Orient each L_j such that $L_j \cap \tilde{\mathbf{R}}^1 \cap S(\alpha) \neq \varnothing$ so that $(L_j \cdot \tilde{\mathbf{R}}^1) = 1$. Then if this orientation can be extended to an admissible orientation of C_α', α is very good* (*with respect to $\{g_i\}$*).

Proof. As in the proof of Proposition 1, $\theta(\Gamma)$ (*a*) is below $\tilde{\mathbf{R}}^1$, so one obtains relations like (Δ) with $\tilde{g}_{i_j} = g_{i_j}$. Q.E.D.

For α a double point of the arrangement condition expressed in Proposition 2 seems to be particularly significant (when applied both at right and at



Picture 6

left of α). One can conjecture that the two generators (among the above constructed $\{g_i\}$) associated to the lines containing α commute if and only if there is an admissible orientation for the lines L_j (at right or at left of α) such that $L_j \cap S(\alpha) \cap \tilde{\mathbf{R}}^1 \neq \emptyset \Rightarrow (L_j \cdot \tilde{\mathbf{R}}^1) = 1$. For instance, in the following configuration such condition is not verified both at the right and at the left of α and it is possible to show that the associated generators do not commute.

At last, note that when admissible orientations exist relatively to some singularity α, then to different orientations for C'_α there correspond different horizontal paths Γ and different $\theta(\Gamma)$ (a), so one can produce a priori many relations relatively to a same singularity.

3. Braid monodromy for arrangements of lines

Let $C = \bigcup_{i=1}^r L_i$ an arrangement of real lines. One can suppose that the coordinate system is chosen so that L_i has equation $a_i x + b_i y = 1$, a_i, $b_i \in \mathbf{R}$, $b_i > 0$, i $= 1, \ldots, r$. Let $Q_i = L_i \cap \tilde{\mathbf{C}}^1 = (M, q_i)$, where the $q_i = (-a_i M + 1)/b_i$ are ordered so that $q_1 < \cdots < q_r$, and set $Q = C \cap \tilde{\mathbf{C}}^1 = \{Q_1, \ldots, Q_r\}$.

Let $V_i = (-a_i, -b_i) \in (\mathbf{R}^2)^*$ be the dual point to the line L_i, $i = 1, \ldots, r$, $V = \{V_1, \ldots, V_r\}$. The ordering of the lines induces the following one on V: $i < j$ iff either the vector OV_j follows OV_i in the anticlockwise sense, starting from the negative direction of the x-axis of $(\mathbf{R}^2)^*$, or OV_i and OV_j have the same direction but $\|OV_i\| > \|OV_j\|$ (by the assumptions, V is contained in the half-plane $\{y < 0\}$). It immediately derives that the broken-line $\langle V_1, \ldots, V_r\rangle$ does not autointersects.

Let $S(C, \pi) = \{\alpha_1, \ldots, \alpha_v\}$, the indices corresponding to increasing abscissas of $\pi(\alpha_i)$ in $\mathbf{R}_x$. For each $\alpha_i \in S(C, \pi)$ denote by $\underline{L}_{\alpha_i}$, $\bar{L}_{\alpha_i}$ the lines through α_i which have respectively lowest and highest slope among the lines containing α_i, and set $\underline{V}_{\alpha_i}$, $\bar{V}_{\alpha_i}$ as their dual points in $(\mathbf{R}^2)^*$. Then $S(\alpha_i)$ is the "angle" between $\underline{L}_{\alpha_i}$, $\bar{L}_{\alpha_i}$ and its dual $S(\alpha_i)^* = [\underline{V}_{\alpha_i}, \bar{V}_{\alpha_i}]$ is a segment in $(\mathbf{R}^2)^*$ belonging to the line α_i^* dual to α_i. We can call the *dual graph* of C the figure $\hat{G} \subset (\mathbf{R}^2)^*$ given by $\hat{G} = \bigcup_{i=1}^{v} S(\alpha_i)^*$; $\hat{G}$ will have as vertex-set $V\hat{G} = V$ and as edge-set $E\hat{G} = \{S(\alpha_i)^*: i = 1, \ldots, v\}$.

THEOREM 2 *Identify* $(\mathbf{R}^2)^*$ *and* $\tilde{\mathbf{C}}^1$ *by a diffeomorphism* J: $(\mathbf{R}^2)^* \to \tilde{\mathbf{C}}^1$ *taking the segment* $[V_i, V_{i+1}]$ *into* $[Q_i, Q_{i+1}]$, $i = 1, \ldots, r - 1$. *Let us connect each* $\pi(\alpha_i) + \varepsilon$, $0 \leqslant \varepsilon \leqslant 1$, *to the basepoint* M *by a path* Γ_i *in* $\mathbf{C}_x \backslash D(C, \pi)$ *which is contained into the lower half-plane of* $\mathbf{C}_x$, $i = 1, \ldots, v$ ($\Gamma_i \cap \Gamma_j = M$ *for* $i \neq j$) *and let* γ_i *be the corresponding elementary circuit (constructed similar to 1.2). Let* a_i *be a Lefshetz relative cycle for* α_i, *given by a real segment in* $\pi^{-1}(\pi(\alpha_i) + \varepsilon)$, $i = 1, \ldots, r$. *Then (up to homotopy)*

$$\theta(\Gamma_i)(a_i) = J(S(\alpha_i)^*), \quad i = 1, \ldots, r,$$

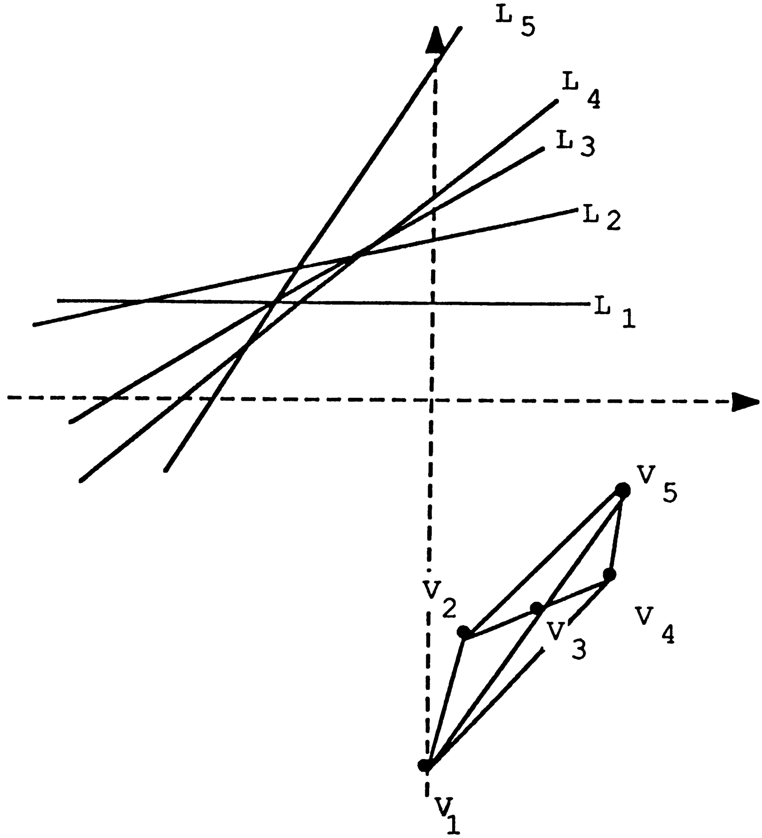
so that the braid monodromy of C *(relatively to* $\{\gamma_i\}$*) is given by the dual graph* $\hat{G}$, *by means of the formula*:

$$\Delta^2 = \prod_{i=v}^{1} [b(S(\alpha_i)^*)]^2$$

(*recall from* 1.1 *that* $b(s)$ *is the half-twist associated to a path* s).

Proof. Let $\alpha_i \in S(C, \pi)$. Orient a line $L_j \not\ni \alpha_i$ which does not separate α_i from O so that $(L_j \cdot \tilde{\mathbf{R}}^1) = 1$; if L_j separates α_i from O then orient it so that $(L_j \cdot \tilde{\mathbf{R}}^1) = -1$. It is easy to see that this orientation for C'_α is admissible (and it is the same as that indicated at the end of 2.1) so one can use Theorem 1. It is also easy to verify that conditions in Theorem 1 requiring for the horizontal paths to contain an upper half-circle are never verified; so Theorem 1 is true for paths Γ_i which are below the real axis of $\mathbf{C}_x$.

Now observe that $J(S(\alpha_i)^*)$ is good (up to homotopy), $i = 1, \ldots, v$. Moreover, the fact that L_j does not separate [separates] α_i from O [or it contains α_i] translates into the fact that V_j is not separated [is separated] from the origin by the line α_i^* in $(\mathbf{R}^2)^*$ [or it lies on that line] (Picture 7). But then $J(S(\alpha_i)^*)$ is (up to homotopy) the same as the path $\theta(\Gamma_i)(a_i)$ deduced by Theorem 1. This proves the theorem. Q.E.D.



Picture 7

This theorem shows that a presentation of $\pi_1(\mathbf{C}^2\backslash C)$ can be deduced simply by the combinatorical of the given arrangement. One can take the origin of $(\mathbf{R}^2)^*$ as a basepoint and elementary circuits which generates $\pi_1((\mathbf{R}^2)^*\backslash V, O)$. For instance, such circuits can be those associated to the segments $[O, V_i]$ (or to small deformations of those segments which contain some other point of V; this happens if there are parallel lines in C). Then the dual graph to C indicates the relations: that is, for each singularity, what conjugates of the generators are to be put in (Δ). One can verify that the conjugates obtained in this way coincide with those obtained by [12; Corollary 12], in the case where the tree there used is taken "following" the line $\tilde{\mathbf{R}}^1$.

Note that the graph $\hat{G}$ is completely determined by its vertices since, as a subset of $(\mathbf{R}^2)^*$, $\hat{G} = \bigcup_{i,j} [V_i, V_j]$.

Let us denote by $\hat{\mathcal{G}}_r$ the set of subsets $\hat{G}$ of $(\mathbf{R}^2)^*$, equipped with two sets $V\hat{G}$ and $E\hat{G}$, such that: (i) $V\hat{G} = \{V_1, \ldots, V_r\}$, where $V_i \in \{y < 0\}$; the ordering of $V\hat{G}$ is given by: $i < j$ iff either OV_j follows OV_i in the anti-clockwise sense, starting from the negative direction of the real axis of $(\mathbf{R}^2)^*$, or OV_i and OV_j have the same direction and $\|OV_i\| > \|OV_j\|$; (ii) $\hat{G} = \bigcup_{i,j}[V_i, V_j]$; (iii) $E\hat{G} = \{[V_i, V_j],\ i < j\colon \nexists (h, k) \neq (i, j),\ h < k$, and $[V_i, V_j] \subset [V_h, V_k]\}$.

Every graph $\hat{G} \in \hat{\mathscr{G}}_r$ determines an arrangement of real lines $C(\hat{G}) = \bigcup_{i=1}^r V_i^*$ in $\mathbf{C}^2$, satisfying the hypotheses of Theorem 2. Denote the image of $\theta\colon \pi_1(\mathbf{C}_x \backslash D(C, \pi)) \to B[\tilde{\mathbf{C}}^1, C \cap \tilde{\mathbf{C}}^1]$ by $\mathscr{M}(C)$ and call it the monodromy group of C. Then (up to the diffeomorphism J) we can think of $\mathscr{M}(C)$ as of a subgroup of the pure braid group $P[(\mathbf{R}^2)^*; V\hat{G}]$ (= compact supported homeomorphisms fixing pointwise $V\hat{G}$, modulo isotopy). Also, to each $\hat{G} \in \hat{\mathscr{G}}_r$ we associate the subgroup $\mathscr{M}(\hat{G}) \subset P[(\mathbf{R}^2)^*; V\hat{G}]$ generated by the full twists $(b[V_i, V_j])^2$ around the edges $[V_i, V_j] \in E\hat{G}$. Then by Theorem 2 $\mathscr{M}(C(\hat{G})) = \mathscr{M}(\hat{G})$, so we have the following corollary.

COROLLARY. *The monodromy groups of arrangements of r real lines (which intersect the y-axis in* $\{y > 0\}$*) are the groups* $\mathscr{M}(\hat{G})$*, for* $\hat{G} \in \hat{\mathscr{G}}_r$.

Up to a diffeomorphism of $\tilde{\mathbf{C}}^1$ preserving $\tilde{\mathbf{R}}^1$ we can see all groups $\mathscr{M}(C)$ as subgroups of a same pure braid group. Thus two main questions remain: when is $\mathscr{M}(\hat{G}) = \mathscr{M}(\hat{G}')$, for $\hat{G}, \hat{G}' \in \mathscr{G}_r$? When is $\mathscr{M}(\hat{G})$ isomorphic to $\mathscr{M}(\hat{G}')$?

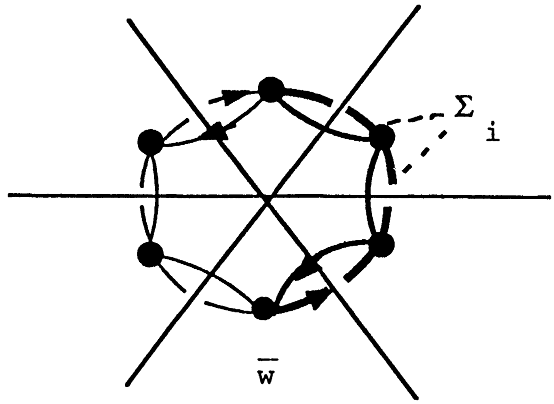
The first question should be affordable and answered by means of some combinatorial properties of the graphs in $\hat{\mathscr{G}}_r$ (the second one seems to be more difficult).

For instance when $\hat{G}$ is associated to r lines in general position, so three vertices of $\hat{G}$ are never in a line, then $\mathscr{M}(\hat{G})$ is the entire pure braid group (this can be derived by an argument similar to that in Proposition 2).

Theorem 1 can be seen as a way of studying the groups $\mathscr{M}(\hat{G})$, finding "very good" generators.

4. Applications to the study of the homotopy type of the complement to an arrangement of lines

Let $C = \bigcup_{i=1}^r L_i$ be an arrangement of real lines. The study of the homotopy type of $\mathbf{C}^2 \backslash C$ is the two dimensional case of the problem (see [1], [2], [4]; see also [3]) of determining the topology of the complement Y to an arrangement of hyperplanes in $\mathbf{C}^N$. Recall from [12] that to every such arrangement a regular N-cell complex is associated, which is homotopy equivalent to Y. So set $Y = \mathbf{C}^2 \backslash C$ and let $X \subset Y$ be the associated 2-complex.



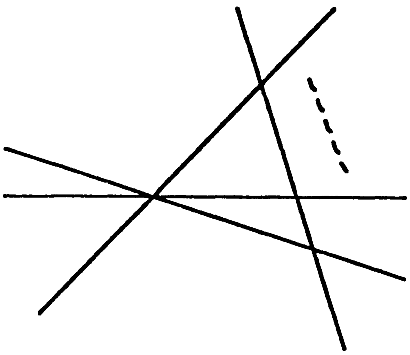
Picture 8

If $S(C, \pi) = \{\alpha_1, \ldots, \alpha_v\}$, where the α_i's are ordered by their x-coordinates, then $X = \bigcup_{i=1}^{v} X(\alpha_i)$, where $X(\alpha_i)$ is the "block" associated to α_i. Using notations from [12] $X(\alpha_i) = \cup D^j_{F^j,w}$, the union being taken over all the facets F^j such that $(F^j)^- \ni \alpha_i$ and all vertices w which are in a chamber having α_i in their closure, $j = 0, 1, 2$ (here the codimensional 2 facets coincide with the α_i's). So, if $\mathrm{ord}(\alpha_i) = n_i$, $X(\alpha_i)$ has $2\,n_i$ vertices, $4\,n_i$ edges and $2\,n_i$ 2-cells. Recall also that $\partial D^2_{\alpha_i,\bar{w}}$ is the union of the two minimal positive paths (in the 1-skeleton) going from $\bar{w}$ to the vertex opposite to $\bar{w}$ with respect to α_i (see picture 8).

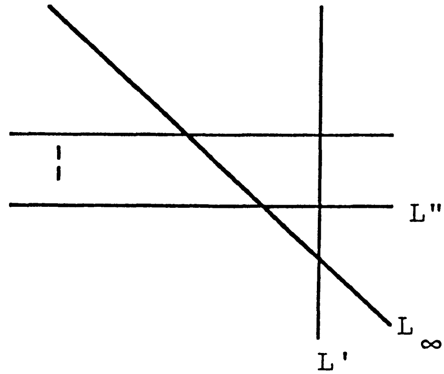
Notice that it is possible to translate (in a homotopy context) the construction of the braid monodromy into a study of X. What one has to consider is, instead of a Lefshetz cycle a relative to α_i, the piece of the 1-skeleton of $X(\alpha_i)$ whose real part is contained in the union of the chambers intersecting $\{x > x(\alpha_i)\}$: call it Σ_i (dark in Picture 8). Then following the fibers of $(\pi, \pi|C)$ over the path Γ correspond to substituting Σ_i by another subset of X_1 which is homotopic to Σ_i. In fact, passing near a singularity α_j means that one is using the "local topology" around α_j: this corresponds exactly to using the block $X(\alpha_j)$ of X. Moreover, from the path $a(p) = \theta(\Gamma_p)(a)$, $p \in \Gamma$, used in the proof of Theorem 1, one easily reconstructs a subset $\Sigma(p) \subset X_1$ homotopic to Σ_i ($\Sigma(p)$ will be homotopic in Y to the subset of $\pi^{-1}(p)$ obtained from $a(p)$ by removing all the points in $a(p) \cap C$ and adding small circles centered in the same points).

We now apply these remarks to the study of the higher homotopy of Y, deducing that for certain classes of arrangements it is not a $k(\pi, 1)$.

LEMMA 1. *The complex $X(n, 1)$ associated to the configuration in $\mathbf{C}^2$ having n lines containing a point O, $n \geqslant 2$, and another line not containing O and transversal to the others is not a $k(\pi, 1)$* (Picture 9a).



Picture 9a



Picture 9b

Proof. Consider a projectivity taking some line containing O into the line at infinity. One obtains a new configuration (Picture 9b) in which the union of the three blocks associated to the double points of $L' \cup L'' \cup L_\infty$ is exactly the associated complex to 3 lines in general position, which is not a $k(\pi, 1)$ ([12; Theorem 2]). Q.E.D.

Set L_∞ as the infinity line of $\mathbf{C}^2$, so $\mathbf{C}^2 \backslash C = \mathbf{CP}^2 \backslash \bar{C}$, where $\bar{C} = C \cup L_\infty$.

THEOREM 3. *Assume that after a projectivity P of $\mathbf{RP}^2$ taking some line $L' \in \bar{C}$ into L_∞ (possibly $L' = L_\infty$) the new arrangement C_P in $\mathbf{C}^2$ has no vertical lines, and the following conditions are true. There is a line $L = \{x = M + \delta y\}$ in C_P, $0 < \delta \ll 1$, such that if we set $\tilde{\mathbf{C}}^1 = \{x = M\}$ then $S(C_P, \pi) \cap L \subset \{y > 0\}$ and $S(C_P, \pi) \cap \{x \geqslant M\} \cap \{x < M + \delta y\} = \varnothing$, (i.e., there are no singularities between $\tilde{\mathbf{R}}^1$ and L). Moreover, there is $\alpha \in S(C_P, \pi)$, $x(\alpha) < M$, such that:*

(i) *every line containing α intersects L in a double point having finite distance;*

(ii) *there is an admissible orientation σ_α for the lines which intersect $S(\alpha)$ such that*

$$L_h \cap L_k \in L \cap S(\alpha) \Rightarrow (L_h \cdot \tilde{\mathbf{R}}^1) = (L_k \cdot \tilde{\mathbf{R}}^1).$$

Then Y is not a $k(\pi, 1)$.

Proof. Let $S(C_P, \pi) \cap S(\alpha) \cap L = \{\beta_1, \ldots, \beta_h\}$, the β_l's being ordered by their y-coordinate. Set $X(L) = \bigcup_{l=1}^{h} X(\beta_l)$, and $\tilde{\Sigma} \subset X(L)_1$ as the union of the cells of $X(L)$ whose real part is contained in $\{x < M + \delta y\}$. We can choose the 1-skeleton of X so that $\tilde{\Sigma} \subset \tilde{\mathbf{C}}^1$ and the set Σ of $X(\alpha)$ (similar to that in Picture 8) is contained into the line $\pi^{-1}(x(\alpha) + \varepsilon)$, $0 < \varepsilon \ll 1$.

Let $\Gamma\colon I \to \mathbf{C}_x \backslash D(C_P, \pi)$ be a path constructed as in Theorem 1, $\Gamma(0) = x(\alpha) + \varepsilon$, $\Gamma(1) = M$. Set $\Gamma_t = \Gamma|[0, t]$, and $\Sigma(t) = \theta(\Gamma_t)(\Sigma) \subset \pi^{-1}(\Gamma(t))$, $\Sigma(I) = \bigcup_{t \in I} \Sigma(t)$. Then $\Sigma(I) \approx \Sigma \times I$. By using an induction similar to that (in the proof) of Theorem 1 it is not hard to see that we can assume $\Sigma(1) \subset \tilde{\Sigma}$ and there is a subcomplex $\Sigma'(I) \subset X$, with same boundary $\Sigma \cup \Sigma(1)$ as $\Sigma(I)$, which is homotopy equivalent to $\Sigma(I)$ in Y (relative to the boundary). For each line L_j in $T(\alpha)$, $\Sigma(1)$ contains the two edges of $\tilde{\Sigma}$ whose real part crosses L_j; and for each line L_j intersecting $(\tilde{\mathbf{R}}^1 \cap S(\alpha)) \backslash T(\alpha)$ it contains the edge whose real part crosses L_j and which is below or above L_j according to $(L_j \cdot \tilde{\mathbf{R}}^1)$ is 1 or -1. One can verify that a positive edge of $\tilde{\Sigma}$, according to the definition given in [9], is an edge traversed in the anticlockwise sense in $\tilde{\mathbf{C}}^1$ (endowed with the induced orientation from $\mathbf{C}^2$).

Let β_j be as above, $\beta_j \notin T(\alpha)$. By (ii), $\Sigma(1) \cap X(\beta_j)$ is either a positive or a negative path in the 1-skeleton of $X(\beta_j)$. Since it contains $m_j - 1$ edges, $m_j = \operatorname{ord}(\beta_j)$, it is contained in the boundary of two of the 2-cells of $X(\beta_j)$, say $D^2_{\beta_j, w_j}$, $D^2_{\beta_j, w_j(L)}$. Here w_j, $w_j(L) \in X(\beta_j)$ are vertices lying in adjacent chambers separated by L. Call $Z(\beta_j) = D^2_{\beta_j, w_j} \cup D^2_{\beta_j, w_j(L)}$ which is topologically a cylinder with boundary two S^1 encircling L.

Take now the subcomplex of X given by

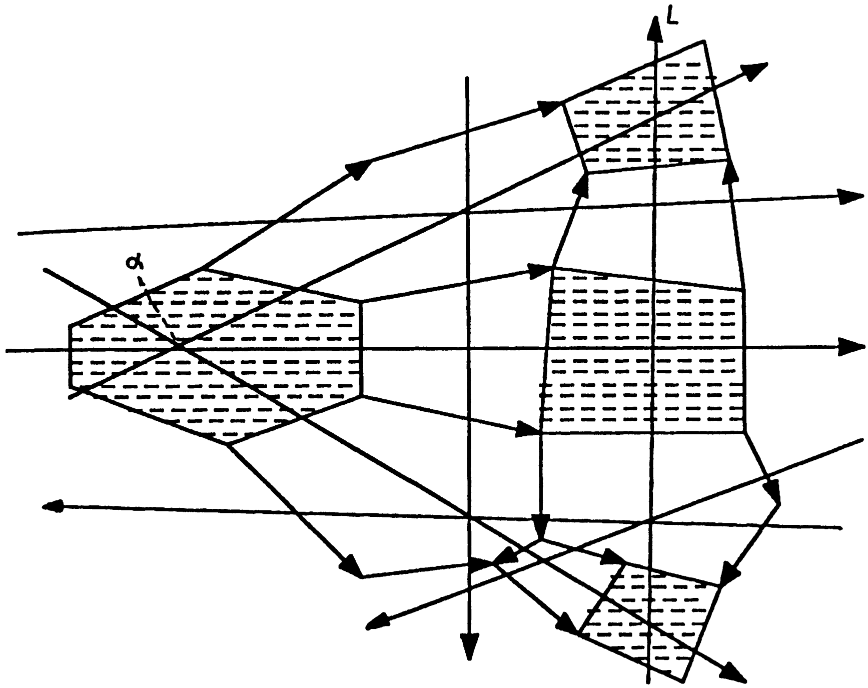
$$K = X(\alpha) \cup \Sigma'(I) \cup \bigcup_{\beta_j \in T(\alpha)} X(\beta_j) \cup \bigcup_{\beta_j \notin T(\alpha)} Z(\beta_j).$$

Since $\Sigma'(I)$ contracts over $\Sigma(1)$ and each $Z(\beta_j)$ contracts, glueing its boundary components, K is homotopically equivalent to the complex in Lemma 1, so it is not a $k(\pi, 1)$ (see Picture 10). Q.E.D.

In Picture 10 the complex K is given by all the dark blocks, and all the 2-cells of the remaining blocks of the picture whose boundaries are consistent with the arrows. Also, an admissible orientation for the lines is shown.

To check the hypotheses of Theorem 3 one has to consider all the possible pairs (L, α), where $L \in \bar{C}$, α is a singularity of $\bar{C} \backslash L$ such that each line through α intersects L in a double point, and all the admissible orientations (a bound for the number of admissible orientations is 2^{r-n-1}, $n = \operatorname{ord}(\alpha)$).

Remarks. (i) Theorems which give an a priori construction of the Lefshetz relative cycles should be found for more general configurations, for instances allowing also tangency points between different branches. In general one will not find good paths in $\tilde{\mathbf{C}}^1$, in the sense of 1.1, but still there will be a bound, depending on the nature of the singularities of C, for the number of intersections of a line $\{\operatorname{Re} y = t\}$ with a Lefshetz cycle.



Picture 10

(ii) A theorem relating the braid monodromy of a plane curve C to that of its dual curve could be proved for many curves C, by using different methods. Nevertheless, unless one also has a theorem 1-like result, such a theorem will not give an *a priori* construction of the braid monodromy of C.

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