COMPOSITIO MATHEMATICA

V. B. MEHTA
A. RAMANATHAN

Schubert varieties in $G/B \times G/B$

Compositio Mathematica, tome 67, n° 3 (1988), p. 355-358
<http://www.numdam.org/item?id=CM_1988__67_3_355_0>

Compositio Mathematica 67: 355–358 (1988)

Schubert varieties in $G/B \times G/B$

V.B. MEHTA & A. RAMANATHAN
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400 005, India

Received 1 December 1987; accepted 11 February 1988

Introduction

Let G be a semi-simple, simply connected algebraic group defined over an algebraically closed field of characteristic $p > 0$. Let $T \subset G$ be a maximal torus, $B \supset T$ a Borel subgroup and $W = N(T)/T$ the Weyl group. G acts on the homogeneous space G/B and also on $G/B \times G/B$ by the diagonal action: for $g, x_1, x_2, \in G, g(x_1B, x_2B) = (gx_1B, gx_2B)$. By *Schubert Varieties* in $G/B \times G/B$ we mean the closures of the G-orbits in $G/B \times G/B$. It is known ([11, 12]) that these orbit closures are in 1–1 correspondence with the elements of W, the element $w \in W$ corresponding to the closure of the orbit of (eB, wB), where $e \in G$ is the identity element. In particular, taking $w = e$, G/B gets imbedded diagonally in $G/B \times G/B$.

In this paper we prove that these Schubert Varieties are Frobenius-split in the sense of [4, Def. 2]. Our method is as follows: fix $w \in W$ with $l(w) = i$ and denote the Schubert variety in G/B corresponding to w by X_i. Then B acts on X_i on the left and one may form the associated fibre-space $\tilde{X}_i = G \times^B X_i$. The map f: $\tilde{X}_i \to G/B \times G/B$ given by $f(g, x) = (gB, gxB)$ is an isomorphism onto the G-orbit closure of (eB, wB) (cf. [11]). Hence we may work with $\tilde{X}_i$ instead. Express w as a product of reflections associated to the simple roots, $w = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_i}$ and $Z_i \to X_i$ be the corresponding Demazure desingularization of X_i (cf. [2, 3]) and let ψ_i: $Z_i \to X_i$ be the birational map. B acts on Z_i on the left and we may construct the associated fibre-space $\tilde{Z}_i = G \times^B Z_i$. The map ψ_i is B-equivariant and descends to a birational map $\tilde{\psi}_i$: $\tilde{Z}_i \to \tilde{X}_i$. Since X_i is normal [1, 5, 7, 10] and $\tilde{X}_i \to G/B$ is a fibre-space with fibre X_i it follows that $\tilde{X}_i$ is also normal and that $\tilde{\psi}_{i_*}(\mathcal{O}_{\tilde{Z}_i}) = \mathcal{O}_{\tilde{X}_i}$. So to prove that $\tilde{X}_i$ is Frobenius-split, it is sufficient to prove that $\tilde{Z}_i$ is Frobenius-split. We calculate the canonical bundle $K_{\tilde{Z}_i}$ of $\tilde{Z}_i$ (this has been done, without detail, in [11]). From this description of $K_{\tilde{Z}_i}$ it follows from [4 prop. 8] that $\tilde{Z}_i$ is Frobenius-split. It also follows from [6, 8] that $\tilde{X}_i$ is Cohen-Macaulay and has rational singularities. We first recall the basic facts about the standard resolutions of Schubert Varieties in G/B and Frobenius-splitting from [4, 8] and then we prove the main result. Our result should prove useful in the study of the decomposition of the G-module $H^0(G/B, L) \times H^0(G/B, M)$, where L and M are line bundles on G/B, see [11].

Section I

Let G, B and W be as in the introduction, let $w \in W$ with $l(w) = i$ and denote by X_i the Schubert variety in G/B corresponding to w. Then according to [2, 3, 8] there exists a smooth projective variety Z_i, and a map ψ_i: $Z_i \to X_i$ with the following properties:

(1) ψ_i is birational.

(2) There exists i smooth subvarieties of codim 1 in Z_i, denoted by $Z_{i,1} \ldots Z_{i,i}$ intersecting transversally. Further if we denote $\bigcup_{j=1}^{i} Z_{i,j}$ by ∂Z_i, then $\psi_i^{-1}(\partial X_i) = \partial Z_i$, where ∂X_i is the union of the codim 1 Schubert varieties in X_i.

(3) Put $v = ws_{\alpha_i}$ and $X_{i-1} = \overline{BvB/B}$. Then there exists a map f_i: $Z_i \to Z_{i-1}$ such that f_i is a locally trivial $\mathbb{P}^1$ fibration with a section σ_i: $Z_{i-1} \to Z_i$. Further, $\partial Z_i = f_i^{-1}(\partial Z_{i-1}) \cup \sigma_i(Z_{i-1})$.

(4) The canonical bundle K_{Z_i} is given by the formula $K_{Z_i} = \mathcal{O}_{Z_i}(-\partial Z_i) \times \psi_i^* L_\varrho^{-1}$ is the line bundle on $X_i \subset G/B$ associated to half sum of the positive roots.

The varieties Z_i and the morphisms ψ_i are constructed by induction on $l(w)$, see [3, 8] for more details. We recall one proposition from [8].

Proposition 1. *Z_i is Frobenius-split and any sub-intersection of the divisors in ∂Z_i is compatibly split in Z_i.*

Proof. This is [8, Remark 2.5].

Now consider the varieties $\tilde{Z}_i = G \times^B Z_i$ as in the introduction. The maps f_i: $Z_i \to Z_{i-1}$ and σ_i: $Z_{i-1} \to Z_i$ are B-equivariant, hence we get maps $\tilde{f}_i$: $\tilde{Z}_i \to \tilde{Z}_{i-1}$ and $\tilde{\sigma}_i$: $\tilde{Z}_{i-1} \to \tilde{Z}_i$. It follows that there exist i smooth subvarieties of $\tilde{Z}_i$ denoted by $\tilde{Z}_{i,1} \ldots \tilde{Z}_{i,i}$, intersecting transversally, whose union we denote by $\partial\tilde{Z}_i$. Let p_1 and p_2 denote the two projections of $G/B \times G/B$ and for any pair of line bundles L, M on G/B, denote $(p_1^*L \times p_2^*M)$ by (L, M).

Proposition 2. *The canonical bundle $K_{\tilde{Z}_i}$ is given by*

$$K_{\tilde{Z}_i} = \mathcal{O}_{\tilde{Z}_i}(-\partial\tilde{Z}_i) \times \tilde{\psi}_i^*(L_\varrho^{-1}, L_\varrho^{-1}).$$

Proof. (See also [11]). We prove this by induction on $l(w)$. If $l(w) = 0$ then $\tilde{Z}_0 = G/B$ and $\partial\tilde{Z}_0 = \emptyset$. So $\mathcal{O}_{\tilde{Z}_0}(-\partial\tilde{Z}_0) \times \tilde{\psi}_0^*(L_\varrho^{-1}, L_\varrho^{-1})$ is the line-bundle L_ϱ^{-2} on G/B, as $\tilde{\psi}_0$: $G/B \to G/B \times G/B$ is the diagonal imbedding. Assume the result for $l(w) = i - 1$. Now it follows from [8, Lemma 3] that $K_{\tilde{Z}_i/\tilde{Z}_{i-1}} = \mathcal{O}_{\tilde{Z}_i}(-\sigma_i(\tilde{Z}_{i-1})) \times \tilde{\psi}_i^*(1, L_\varrho^{-1}) \times \tilde{f}_i^*\tilde{\sigma}_i^*\tilde{\psi}_i^*(1, L_\varrho)$. Denote this line bundle on $\tilde{Z}_i$ by A. Then $K_{\tilde{Z}_i} = A \times \tilde{f}_i^*(K_{\tilde{Z}_{i-1}}) = A \times \tilde{f}_i^*[\mathcal{O}_{\tilde{Z}_i}(-\partial\tilde{Z}_i) \times \tilde{\psi}_{i-1}^*(L_\varrho^{-1}, L_\varrho^{-1})], = \mathcal{O}_{\tilde{Z}_i}(-\partial\tilde{Z}_i) \times \tilde{\psi}_i^*(1, L_\varrho^{-1}) \times \tilde{f}_i^*\tilde{\sigma}_i^*\tilde{\psi}_i^*(1, L_\varrho) \times \tilde{f}_i^*\tilde{\psi}_{i-1}^*(L_\varrho^{-1}, L_\varrho^{-1})$. But $\tilde{\psi}_{i-1}^*: \tilde{Z}_{i-1} \to G/B \times G/B = \tilde{\psi}_i\tilde{\sigma}_i: \tilde{Z}_{i-1} \to G/B \times G/B$. Hence we get $K_{\tilde{Z}_i} = \mathcal{O}_{\tilde{Z}_i}(-\partial\tilde{Z}_i) \times \tilde{\psi}_i^*(1, L_\varrho^{-1}) \times \tilde{f}_i^*\tilde{\psi}_{i-1}^*(L_\varrho^{-1}, 1)$. But $\tilde{f}_i^*\tilde{\psi}_{i-1}^*(L_\varrho^{-1}, 1) = \tilde{\psi}_i^*(L_\varrho^{-1}, 1)$ as both of them are isomorphic to $q^*(L_\varrho^{-1}, 1)$ where q is the projection $\tilde{Z}_i \to G/B$. Hence we get $K_{\tilde{Z}_i} = \mathcal{O}_{\tilde{Z}_i}(-\partial\tilde{Z}_i) \times \tilde{\psi}_i^*(L_\varrho^{-1}, L_\varrho^{-1})$.

THEOREM 1. *$\tilde{Z}_i$ is Frobenius-split and any sub-intersection of the divisors in $\partial\tilde{Z}_i$ is compatibly split in $\tilde{Z}_i$.*

Proof. From Prop. 2, we know that $K_{\tilde{Z}_i}^{-1} = \mathcal{O}_{\tilde{Z}_i}(\partial\tilde{Z}_i) \times \tilde{\psi}_i^*(L_\varrho, L_\varrho)$. From [8, Remark 2], we know that $\sigma = D + \tilde{D}$ is an element of $H^0(G/B, L_\varrho^2)$ such that σ^{p-1} splits G/B. Consider the section $t = \partial\tilde{Z}_i + \tilde{\psi}_i^*(D, \tilde{D})$ of $K_{\tilde{Z}_i}$. It follows from [4, Prop. 8] that t^{p-1} splits $\tilde{Z}_i$ and that any sub-intersection of the divisors in $\partial\tilde{Z}_i$ is compatibly split in $\tilde{Z}_i$ by t^{p-1}.

COROLLARY 1. *Let N be the length of the maximal element $w_0 \in W$. Then by the above, $\tilde{Z}_0$ is compatibly-split in $\tilde{Z}_N$. So it follows from* [4, *Prop. 4*] that *$\tilde{\psi}_0(\tilde{Z}_0) = G/B$ is compatibly-split in $\tilde{\psi}_N(\tilde{Z}_N) = G/B \times G/B$, where G/B is imbedded diagonally in $G/B \times G/B$.*

This was first proved by the second author by other methods (cf. [9]).

COROLLARY 2. *From Corollary 1 and* [9, *Cor. 2.3*] *it follows that any imbedding of G/B by a complete linear system is projectively normal.*

This was first proved in [7] (see also [9]).

COROLLARY 3. *It follows from* [6, 8] *that the Schubert varieties $\tilde{X}_i$ in $G/B \times G/B$ are Cohen–Macaulay and have rational singularities.*

Remark. It can be proved, using the methods of [9], that these Schubert varieties in $G/B \times G/B$ are scheme-theoretically defined by quadrics. This will be taken up in a later paper. Analogues follow for $G/P_1 \times G/P_2$.

References

1. H.H. Andersen, Schubert Varieties and Demazure's character formula. *Invent. Math.* 79 (1985) 611–618.

2. Demazure, M., Desingularization des Varietes de Schubert generalisees. *Ann. Sc. Ec. Norm. Super.* 7 (1974) 53–88.
3. G. Kempf, Linear systems on homogeneous spaces. *Ann. Math.* 103 (1976) 557–591.
4. V.B. Mehta and A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties. *Ann. Math.* 122 (1985) 27–40.
5. V.B. Mehta and V. Srinivas, Normality of Schubert varieties. *Amer. Jour. Math.* 109 (1987) 987–989.
6. V.B. Mehta and V. Srinivas, A note on Schubert varieties in G/B. Preprint.
7. S. Ramanan and A. Ramanathan, Projective normality of flag varieties and Schubert varieties. *Invent. Math.* 79 (1985) 217–224.
8. A. Ramanathan, Schubert Varieties are arithmetically Cohen–Macaulay. *Invent. Math.* 80 (1985) 283–294.
9. A. Ramanthan, Equations defining Schubert Varieties and Frobenius-splitting of the diagonal. *Publ. I.H.E.S.* 65 (1987) 61–90.
10. C.S. Seshadri, Line bundles on Schubert varieties Vector bundles on Algebraic Varieties, *Proceedings of the Bombay Colloquium* 1984.
11. Kumar, Shrawan, Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture. Preprint.
12. T. Springer, Quelques applications de la cohomologie d'intersection. *Seminaire Bourbaki Expose* 589, Vol. 1981/1982.