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1. Introduction

Let $\mathfrak{a}$ be a n-dimensional Euclidean space with inner product $(.,.)$ and R a rank n root system, not necessarily reduced, contained in $\mathfrak{a}^*$. We use the notation P for the weight lattice of R and H for the complex torus with character lattice P and Lie algebra $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a} = \mathfrak{a} \oplus i\mathfrak{a} = \mathfrak{a} \oplus \mathfrak{t}$. So we can write $H = \exp \mathfrak{h}$. H can be decomposed in a split part and a compact part: $H = A \cdot T$ with $A = \exp \mathfrak{a}$ and $T = \exp \mathfrak{t}$. Choose an orthonormal basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{a}$ and let $\{x_1, \ldots, x_n\}$ be the coordinates on $\mathfrak{a}$ relative to this basis. A multiplicity function $\mathscr{g}: R \to \mathbb{R}$ (or $\mathbb{C}$) is by definition a Weyl group invariant function. For a fixed multiplicity function $\mathscr{g}$ we consider three closely related quantum mechanical one particle systems on $\mathfrak{a}$, described by the following Schrödinger operators: (we take $\hbar = 1$)

$$S_A(X) = -\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{\alpha \in R_+} \frac{\mathscr{g}_\alpha}{4\sinh^2(\alpha(X)/2)} \tag{1.1}$$

$$S_T(X) = -\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{\alpha \in R_+} \frac{\mathscr{g}_\alpha}{4\sin^2(\alpha(X)/2)} \tag{1.2}$$

$$S_{\mathfrak{a}}(X) = -\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{\alpha \in R_+} \frac{\mathscr{g}_\alpha}{(\alpha(X))^2} \tag{1.3}$$

To comprehend the relation between S_A, S_T and $S_{\mathfrak{a}}$ we start with the operator

$$S(h) = -\frac{1}{2}\sum_{j=1}^{n}\partial(X_j)^2 + \sum_{\alpha \in R_+} \frac{\mathscr{g}_\alpha}{(h^{\alpha/2} - h^{-\alpha/2})^2} \tag{1.4}$$

on H. Now we can interpret S_A and $-S_T$ as the restrictions of S to A and T respectively, and $S_{\mathfrak{a}}$ as the "lowest homogeneous part" of S (see [Op], Section 4).

The main object of this paper is to prove that the commutant of S is a polynomial algebra of differential operators on H with n generators. Physically this means that we find a complete set of preserved quantities for the systems described by S_A, S_T and $S_\mathfrak{a}$. If we descend to the level of classical mechanics this implies that the Hamiltonian systems given by

$$H_A(p, X) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{\alpha\in R_+} \frac{g_\alpha}{4\sinh^2(\alpha(X)/2)} \tag{1.5}$$

$$H_T(p, X) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{\alpha\in R_+} \frac{g_\alpha}{4\sin^2(\alpha(X)/2)} \tag{1.6}$$

$$H_\mathfrak{a}(p, X) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{\alpha\in R_+} \frac{g_\alpha}{\alpha^2(X)} \tag{1.7}$$

respectively are completely integrable.

In case $R = A_2$ the complete integrability of the system (1.7) was proved already in 1866 by Jacobi ([J]). Marchioro (in 1970!) rediscovered this fact and treated both the classical and the quantum mechanical scattering problem (see [Mar]). The results of Marchioro on the scattering behaviour were generalized in the quantum case to an aribtrary number of particles (i.e., $R = A_n$ where $n + 1$ is the number of particles) by Calogero ([C]). Moser proved integrability for the Hamiltonians (1.5), (1.6) and (1.7) (with $R = A_n$) by giving a Lax representation for the equations of motion (see [Mo]). By extending the methods of Moser, partial results on the integrability of these systems for the classical root systems were achieved by Olshanetsky and Perelemov ([OP]). As for the quantum mechanical case we mention Harish Chandra's result on the structure of the space of invariant differential operators on a (non compact) Riemannian symmetric space $X = G/K$ (see [HC]), which can be interpreted physically as a proof of the integrability for the systems (1.1), (1.2) and (1.3) if we take R equal to the restricted root system of X and $g_\alpha = (1/2)(\alpha, \alpha)k_\alpha \cdot (k_\alpha + 2k_{2\alpha} - 1)$ where $2k_\alpha$ is equal to the multiplicity of the root α in G (see Section 4). In [K] Koornwinder shows that the integrability of these systems for $R = A_2$ or BC_2 is not perturbed if one takes for the multiplicity g an arbitrary, complex valued Weyl group invariant function. Sekiguchi and later independently Macdonald (see [se], [Mac]) proved this remarkable fact for $R = A_n$. Macdonald's proof consists of a direct calculation, but uses in an essential way the special features of the root system A_n.

In contrast with the above mentioned proof of Macdonald for the root system A_n the general proof presented in this paper is very indirect. It is based on Heckman's construction of hypergeometric functions (see [H]), which in turn is based on Deligne's solution of the Riemann monodromy problem. The construction in [H] is carried through only for "generic parameters" (see [H], Definition 7.1 and Theorem 7.5). However, in Section 2 we show that the hypergeometric function can be continued analytically to the full parameter space (thereby filling in the last remaining gap in the definition of hypergeometric functions). This continuation theorem implies some combinatorical information which is used in Section 3 to prove a conjecture (see [HO], Conj. 2.10) on the structure of the space of hypergeometric differential operators. In fact we prove a stronger result that includes the existence of so called shift operators of hypergeometric differential operators (see [Op], Cor 3.12). In Section 4 we translate the results of Section 3 in physical terms and we prove the theorems on complete integrability of the systems discussed here.

2. Analytic continuation of the hypergeometric function in the parameter space

For notions and notations which are used in this paper without proper introduction we refer the reader to the papers [HO], [H] and [Op].

Fix a rank n root system R. As in ([HO], Section 2) we consider, for a fixed complex valued multiplicity function k on R, the partial differential operator

$$L(k) = \sum_{j=1}^{n} \partial(X_j)^2 - \sum_{\alpha \in R_+} k_\alpha (1 + h^\alpha)(1 - h^\alpha)^{-1} \partial(X_\alpha)$$

on the complex torus H. This operator has an asymptotic expansion on A_-:

$$L(k) = \sum_{j=1}^{n} \partial(X_j)^2 - 2\partial(X_\varrho) - 2 \sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^{\infty} h^{j\alpha} \partial(X_\alpha).$$

Substitution of a formal series on A_- of the form

$$\psi(\lambda, k; a) = \sum_{\kappa \in Q_+} \Delta_\kappa(\lambda, k) a^{\lambda + \varrho(k) + \kappa}$$

in the eigenfunction equation

$$L(k)\psi = (\lambda - \varrho(k), \lambda + \varrho(k))\psi \quad (\lambda \in \mathfrak{h}^*)$$

for $L(k)$ leads to the following recurrence relations for the coefficients Δ_κ (see also [HO], Section 3):

$$(2\lambda + \kappa, \kappa)\Delta_\kappa(\lambda, k) = 2 \sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^{\infty} (\lambda + \varrho(k) + \kappa - j\alpha, \alpha)\Delta_{\kappa - j\alpha}(\lambda, k) \tag{2.1}$$

These relations determine the coefficients Δ_κ uniquely as elements of $\mathbb{C}(\mathfrak{h}^*) \otimes \mathbb{C}[K]$ ($\mathbb{C}(\mathfrak{h}^*)$ is the quotient field of $\mathbb{C}[\mathfrak{h}^*]$ and K is the vector space of all multiplicity functions of R) once $\Delta_0 \in \mathbb{C}(\mathfrak{h}^*) \otimes \mathbb{C}[K]$ is chosen. In comparison with ([HO], Section 3): take $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \kappa^\vee) + 1 \neq 0$, $\forall \kappa \in Q_+\backslash\{0\}$ then

$$\psi(\lambda, k; a) \quad = \quad \Delta_0(\lambda, k)\phi(\lambda + \varrho(k), k; a)$$

or equivalently:

$$\Delta_\kappa(\lambda, k) \quad = \quad \Delta_0(\lambda, k)\Gamma_{\kappa + \lambda + \varrho(k)}(\lambda + \varrho(k), k).$$

The following lemma is a mild extension of ([Hel], Ch IV, Lemma 5.3).

LEMMA 2.1. *Let $U \subset \mathfrak{h}^* \times K$ be open, connected and bounded. Choose Δ_0 in such a way that Δ_κ is analytic on $\bar{U}$, $\forall \kappa \in Q_+$. Let $a = \exp X \in A_-$. Then there exists a constant $d = d_{U,a}$ such that:*

$$|\Delta_\kappa(\lambda, k)| \leqslant d \cdot a^{-\kappa} \quad \forall \kappa \in Q_+, (\lambda, k) \in \bar{U}. \tag{2.2}$$

Proof. Introduce the following function on Q_+: $m(\tau) = m(\Sigma_{i=1}^n t_i\alpha_i) = \Sigma_{i=1}^n t_i$. This height function extends to a norm on $\mathfrak{a}^* = \mathbb{R}R$. By equivalence of norms on $\mathfrak{a}^*$ we have:

$$a|\tau| \leqslant m(\tau) \leqslant b|\tau|$$

for certain constants $a, b \in \mathbb{R}_{>0}$. here $|\tau| = (\tau, \tau)^{1/2}$. Select $c_1 \in \mathbb{R}_{>0}$ such that

$$|(\lambda + \varrho + \tau, \alpha)| \leqslant c_1(1 + m(\tau)) \ \forall (\lambda, k) \in \bar{U}, \tau \in Q_+, \alpha \in R_+.$$

Furthermore we can choose a constant $N_0 \in \mathbb{N}$, and $c_2 \in \mathbb{R}_{>0}$ such that

$$|(2\lambda + \kappa, \kappa)| \geqslant c_2(m(\kappa))^2 \ \forall \lambda \in \mathfrak{h}^*: \exists k \in K: (\lambda, k) \in \bar{U}$$

$$\forall \kappa \in Q_+: m(\kappa) \geqslant N_0.$$

So if $\kappa \in Q_+$, $m(\kappa) \geqslant N_0$ then (see formula (2.1))

$$|\Delta_\kappa(\lambda, k)| \leqslant c(m(\kappa))^{-1} \sum_{\alpha \in R_+} |k_\alpha| \sum_{j \geqslant 1} |\Delta_{\kappa - j\alpha}(\lambda, k)|, \forall(\lambda, k) \in \bar{U} \tag{2.3}$$

with $c = 2c_1/c_2$.

Choose $N_1 \in \mathbb{N}$ such that

$$c \sum_{\alpha \in R_+} |k_\alpha| \sum_{j \geqslant 1} a^{j\alpha} \leqslant N_1 \ \forall k \in K: \exists \lambda \in \mathfrak{h}^*: (\lambda, k) \in \bar{U}$$

and let $N = \max(N_0, N_1)$. Select $d = d_{U,a}$ such that

$$|\Delta_\kappa(\lambda, k)| \leqslant d \,.\, a^{-\kappa}, \forall(\lambda, k) \in \bar{U}, \forall \kappa \in Q_+ : m(\kappa) \leqslant N.$$

Then we prove (2.2) with induction on $m(\kappa)$. Let $\kappa \in Q_+$ with $m(\kappa) > N$ and suppose (2.2) is true for all $\tau \in Q_+$, $m(\tau) < m(\kappa)$. In combination with (2.3) we obtain:

$$|\Delta_\kappa(\lambda, k)| \leqslant c(m(\kappa))^{-1} \sum_{\alpha \in R_+} |k_\alpha| \sum_{j \geqslant 1} d \,.\, a^{j\alpha} \,.\, a^{-\kappa} \leqslant \frac{d \,.\, N}{m(\kappa)} a^{-\kappa} \leqslant d \,.\, a^{-\kappa}.$$

□

COROLLARY 2.2. *Let U and Δ_0 be as in Lemma 2.1. Let V be open, bounded with $\bar{V} \subset A_-$. Then $\psi = \Sigma_{\kappa \in Q_+} \Delta_\kappa a^{\lambda + \varrho + \kappa}$ converges uniformly on $\bar{U} \times \bar{V}$. Consequently, ψ is analytic on $U \times A_-$.* □

COROLLARY 2.3. *The function* ϕ: $\mathfrak{h}^* \times K \times A_- \to \mathbb{C}$

$$(\lambda, k, a) \to \phi(\lambda + \varrho(k), k; a)$$

as introduced in ([HO], *formula* (3.11)) *is meromorphic with simple poles along hyperplanes of the form $H_\kappa \times K \times A_-$, $\kappa \in Q_+ \backslash \{0\}$. Here $H_\kappa = \{\lambda \in \mathfrak{h}^* | (\lambda, \kappa^\vee) + 1 = 0\}$. Moreover, for $\lambda_0 \in \mathfrak{h}^*$ with $\lambda_0 \in H_\kappa$ for precisely one $\kappa = \kappa_0 \in Q_+ \backslash \{0\}$ we have*:

$$\begin{aligned} &[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))\phi(\lambda + \varrho(k), k; a)]_{\lambda = \lambda_0} \\ &\quad = [(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))\Gamma_{\kappa_0 + \lambda + \varrho(k)}(\lambda + \varrho(k), k)]_{\lambda = \lambda_0} \\ &\qquad \times \phi(\lambda_0 + \kappa_0 + \varrho(k), k; a) \end{aligned} \tag{2.4}$$

Proof. From the recurrence relations (2.1) we see that poles of the coefficients Δ_τ can occur only along hyperplanes of the form $H_\kappa \times K$ with $\kappa \in Q_+$, $\kappa \leqslant \tau$, and that the hyperplane H_{κ_0} does not occur if we start with $\Delta_0(\lambda, k) = 2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)$. The set of hyperplanes $\{H_\kappa, \kappa \in Q_+\}$ is locally finite, so we can take a neighbourhood U of (λ_0, k) in $\mathfrak{h}^* \times K$ such that $\bar{U}$ does not intersect the set $\{H_\kappa \times K, \kappa \in Q_+, \kappa \neq \kappa_0\}$. As a direct consequence of Corollary 2.2 we find that $\psi(\lambda, k; a) = (2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))\phi(\lambda + \varrho(k), k; a)$ is analytic on $U \times A_-$. Formula (2.4) is a consequence of the observation that, with $\Delta_0(\lambda, k) = 2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)$, $\Delta_\kappa(\lambda_0, k) = 0$ if $\kappa \ngeqslant \kappa_0$ and the fact that ψ converges uniformly in a neighbourhood of (λ_0, k). □

In ([HO], formula (6.4)) the c-function c: $\mathfrak{h}^* \times K \to \mathbb{C}$ is defined as

$$c(\lambda, k) = \prod_{\beta \in R_{0,+}} \frac{c_\beta(\lambda, k)}{c_\beta(-\varrho(k), k)}$$

with

$$c_\beta(\lambda, k) = \frac{2^{(\lambda, \beta^\vee)}\Gamma(-(\lambda, \beta^\vee))}{\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta + k_{2\beta})\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta + \frac{1}{2})}.$$

The following proposition is clear from this definition:

PROPOSITION 2.4. *c is meromorphic on $\mathfrak{h}^* \times K$ with poles along sets of hyperplanes S_1 and S_2 with:*

$$S_1 = \{(\lambda, k) | (\lambda, \beta^\vee) \in \mathbb{Z}_+ \text{ for some } \beta \in R_{0,+}\}$$

$$S_2 \subset \{(\lambda, k) | \tfrac{1}{2}(\varrho(k), \beta^\vee) + \tfrac{1}{2}k_\beta + k_{2\beta} \in \mathbb{Z}_- \text{ for some } \beta \in R_{0,+}, \text{ or } \tfrac{1}{2}(\varrho(k), \beta^\vee) + \tfrac{1}{2}k_\beta + \tfrac{1}{2} \in \mathbb{Z}_- \text{ for some } \beta \in R_{0,+}\}.$$

The poles along S_1 are simple. □

REMARK 2.5. The set S_2 of hyperplanes in K along which the c-function has poles is always strictly contained in the set given in Proposition 2.4, but is not so easy to describe explicitly in general. However, if we take a root system R with only one root length (so $R = A_n$, D_n or E_n) then it is not hard to show that S_2 coincides (with multiplicity) with the set of poles of the function $\prod_{i=1}^n \prod_{j=1}^{m_i} \Gamma(k + j/(m_i + 1))$ (where m_i $(i = 1, \ldots, n)$ are the exponents of R).

For rank$(R) \leqslant 2$ we obtain the following list: S_2 coincides (with multiplicity) with the set of poles of:

BC_1: $\Gamma(k_1 + k_2 + 1/2)$ $(k_1 = k_\alpha, k_2 = k_{2\alpha})$

A_2: $\Gamma(k + 1/2)\Gamma(k + 1/3)\Gamma(k + 2/3)$

BC_2: $\Gamma(k_1 + 1/2)\Gamma(k_2 + k_3 + 1/2)\Gamma(k_1 + k_2 + k_3 + 1/2)$

$(k_1 = k_\alpha, k_2 = k_{2\beta}, k_3 = k_\beta)$

G_2: $\Gamma(k_1 + 1/2)\Gamma(k_2 + 1/2)\Gamma(k_1 + k_2 + 1/3)\Gamma(k_1 + k_2 + 2/3)$. □

Heckman showed ([H], Theorem 7.5) that if k is admissible $(\frac{1}{2} + k_\alpha + k_{2\alpha} \notin \mathbb{Z})$ and $\lambda \in \mathfrak{h}^*$ satisfies the condition $(\lambda + \tau_1, \lambda + \tau_1) \neq (\lambda + \tau_2, \lambda + \tau_2)$ if $\tau_1 \neq \tau_2$, $\tau_i \in P$ and $\tau_1 - \tau_2 \in Q$ then span$\{\phi(w\lambda + \varrho, k; h)\}_{w \in W}$ is a system of Nilsson class functions on $W \backslash H^{\text{reg}}$ of monodromy type $M(\lambda, k)$. In addition, if $(\lambda, k) \notin S_2$ then $F(\lambda, k; a) = \Sigma_{w \in W}\, c(w\lambda, k)\phi(w\lambda + \varrho(k), k; a)$ $(a \in A_-)$ has an analytic continuation to a W-invariant tubular neighbourh[illegible] $A \subset H$ which is W-invariant. A first result on meromorph[illegible]ce of F on the parameters (λ, k) will be achieved by means of the fol[illegible] lemma.

Lemma 2.6. *Let X be a connected complex manifold and $B \subset X$ an open subset. Let $\Omega \subset \mathbb{C}^m$ be a domain and f a holomorphic function on $\Omega \times B$. Suppose that for almost all $\omega \in \Omega$ the function $f(\omega, .)$ can be extended to an analytic function on X. Then f can be continued analytically to $\Omega \times X$.*

Proof. It is enough to show the lemma for the special case where $X = \{|x_i| < r, i = 1, \ldots, n\}$, $B = \{|x_i| < r' < r, i = 1, \ldots, n\}$ and $\Omega = \{|\omega_i| < r, i = 1, \ldots, m\}$. In this situation we may assume $m = 1$. Furthermore we assume that f is bounded on $\Omega \times B$ by replacing r and r' by $r - \varepsilon$ and $r' - \varepsilon$ respectively ($\varepsilon > 0$ arbitrary small). Now Lemma 2.2.11 of [Hö] or, more accurately, the proof of this lemma gives the desired result. □

Corollary 2.7. *Let V be a W-invariant tubular neighbourhood of $A \subset H$. The function F: $\mathfrak{h}^* \times K \times V \to \mathbb{C}$*

$$(\lambda, k, h) \to F(\lambda, k; h)$$

can be extended to a meromorphic function with simple poles along the following hyperplanes:
(*a*) $\{(\lambda, k, h) | (\lambda, \alpha) = 0$ for some $\alpha \in R\}$
(*b*) $H_\kappa \times K \times V$, $\kappa \in Q\backslash\{0\}$
Furthermore it has poles (*not simple in general*) *along*:
(*c*) S_2.

Proof. Define $V_- = V \cap A_- \cdot T$. From Corollary 2.3 and Proposition 2.4 we conclude that the restriction of F to $\mathfrak{h}^* \times K \times V_-$ is meromorphic with poles along the hyperplanes (a), (b) and (c), where those along (a) and (b) are simple. Now ([H], Theorem 7.5) states that for almost all (λ, k) the function $F(\lambda, k; .)$ has an analytic continuation to V. So if we apply Lemma 2.6 we obtain Corollary 2.7. □

Formula (2.4) enables us to calculate the residues of F along the hyperplanes $H_\kappa \times K \times V$, $\kappa \in Q\backslash\{0\}$. Suppose that $\lambda_0 \in \mathfrak{a}^*_- + \mathfrak{t}^*$ with $\lambda_0 \in H_{\kappa_0}$ but $\lambda_0 \notin H_\kappa$ if $\kappa \neq \kappa_0$ for a fixed $\kappa_0 \in Q\backslash\{0\}$. Suppose $k \in K$ such that $(\lambda_0, k) \notin S_2$. There are two cases we have to deal with:

First case. Assume that $\kappa_0 \neq n \cdot \alpha\ \forall n \in \mathbb{Z}_+$, $\alpha \in R_0$ (the indivisible roots of R). Then $c(w\lambda, k)$ is holomorphic in a neighbourhood of (λ_0, k) $(\forall w \in W)$. We obtain in this case (take $k \in K$ and $a \in A_-$ fixed):

$$[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)) . F(\lambda, k; a)]|_{\lambda=\lambda_0}$$

$$= \sum_{\substack{w \in W \\ w\kappa_0 > 0}} c(w\lambda_0, k)[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)\Gamma_{w\kappa_0}(w\lambda, k)]|_{\lambda=\lambda_0}$$

$$. \phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; a).$$

Second case. Now let $\kappa_0 = n \cdot \alpha$ for some $n \in \mathbb{Z}_+$, $\alpha \in R_0$. Then $c(w\lambda, k)$ will have a simple pole in a neighbourhood of (λ_0, k) for those $w \in W$ for which $w\kappa_0 < 0$, along the hyperplane $H_{\kappa_0} \times K$ (see Proposition 2.4). Thus:

$$[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)) \cdot F(\lambda, k; a)]|_{\lambda=\lambda_0}$$

$$= \sum_{\substack{w \in W \\ w\kappa_0 > 0}} \{[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))c(r_{w\alpha} w\lambda, k)]|_{\lambda=\lambda_0} + c(w\lambda_0, k)[(2(\lambda, \kappa_0)$$

$$+ (\kappa_0, \kappa_0))\Gamma_{nw\alpha}(w\lambda, k)]|_{\lambda=\lambda_0}\}\phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; a).$$

So in both cases we obtain the formula:

$$[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0)) \cdot F(\lambda, k; a)]|_{\lambda=\lambda_0}$$

$$= \sum_{\substack{w \in W \\ w\kappa_0 > 0}} d(w, \kappa_0; \lambda_0, k) . \phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; a)$$

where (i) $d(w, \kappa_0; \lambda_0, k) = c(w\lambda_0, k)[(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))\Gamma_{w\kappa_0}(w\lambda, k)]|_{\lambda=\lambda_0}$

if $\kappa_0 \neq n \cdot \alpha\ \forall n \in \mathbb{Z}_+,\ \alpha \in R_0$

(ii) $d(w, \kappa_0; \lambda_0, k) = [(2(\lambda, \kappa_0) + (\kappa_0, \kappa_0))(c(r_{w\alpha} w\lambda, k) + c(w\lambda_0, k)\Gamma_{w\kappa_0}(w\lambda, k)]|_{\lambda=\lambda_0}$

if $\kappa_0 = n \cdot \alpha$ for some $n \in \mathbb{Z}_+,\ \alpha \in R_0$.

In both cases $d(w, \kappa_0; \lambda, k)$ is holomorphic in a neighbourhood of (λ_0, k) on the hyperplane $H_{\kappa_0} \times K$.

THEOREM 2.8. *The poles of F along the hyperplanes*
(*a*) $\{(\lambda, k, h)|(\lambda, \alpha) = $ for some $\alpha \in R\}$
(*b*) $H_\kappa \times K \times V$ for some $\kappa \in Q\backslash\{0\}$
are all removable. In other words: F: $\mathfrak{h}^* \times K \times V \to \mathbb{C}$ *is a meromorphic function with poles only along the set of hyperplanes* S_2 (*see Proposition 2.4 and Remark 2.5*).

Proof. Because F is W invariant as function of the parameter $\lambda \in \mathfrak{h}^*$ it is obvious that F cannot have simple poles along the set of hyperplanes (a). So let us restrict our attention to (b). Take λ_0, κ_0 as we did in formula (2.5). The expression

$$\operatorname{res}(\lambda_0, k, a) = \sum_{\substack{w \in W \\ w\kappa_0 > 0}} d(w, \kappa_0; \lambda_0, k) . \phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; a),$$

being the residue of F along $H_{\kappa_0} \times K \times V$ at (λ_0, k, a), has analytic continuation on $(\lambda_0, k) \times V$. In particular: $\operatorname{res}(\lambda_0, k, .)$ has trivial monodromy with respect to the curves s_j ($j = 1, \ldots, n$) (see [HO], Section 5). Observe that the functions $\phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; .)$ have analytic continuations as multivalued functions on H^{reg} (apply Lemma 2.6). Put, for convenience, $d(w, \kappa_0; \lambda_0, k) = 0$ if $w\kappa_0 \not> 0$. If we choose λ_0 outside some subset of codimension $\geqslant 1$ in H_{κ_0} then, for every $j \in \{1, \ldots, n\}$, the subsums

$$\Sigma_{w,j}(a) = d(w, \kappa_0; \lambda_0, k) \cdot \phi(w(\lambda_0 + \kappa_0) + \varrho(k), k; a)$$

$$+ d(r_j w, \kappa_0; \lambda_0, k) \cdot \phi(r_j w(\lambda_0 + \kappa_0) + \varrho(k), k; a)$$

of $\operatorname{res}(\lambda_0, k, a)$ can be separated with the monodromy action of the loops t_Z (see [HO], Section 5) that satisfy $t_Z s_j = s_j t_Z$. A rigorous proof of this is given in lemma 2.9. Hence: $\forall w \in W$, $\forall j \in \{1, \ldots, n\}$ the sum $\Sigma_{w,j}$ has trivial monodrony with respect to s_j. If we apply rank one reduction (with respect to the simple root α_j) to $\Sigma_{w,j}$ we obtain an asymptotic expansion for the ordinary hypergeometric function associated with R_{α_j}. Take the coordinate $z = \frac{1}{2} - \frac{1}{4}(h^{\alpha_j/2} + h^{-\alpha_j/2})$ as in [HO], Section 4. The ordinary hypergeometric function we have obtained has exponents: 0 and $\frac{1}{2} - k_{\alpha_j/2} - k_{\alpha_j}$ at $z = 0$, 0 and $\frac{1}{2} - k_{\alpha_j}$ at $z = 1$, $\frac{1}{2}k_{\alpha_j/2} + k_{\alpha_j} \pm (w(\lambda_0 + \kappa_0), \alpha_j^\vee)$ at $z = \infty$. Suppose that one of the coefficients $d(w, \kappa_0; \lambda_0, k)$ or $d(r_j w, \kappa_0; \lambda_0, k)$ is equal to zero, for instance $d(w, \kappa_0; \lambda_0, k) = 0$. If $d(r_j w, \kappa_0; \lambda_0, k) \neq 0$ one of the following relations between the exponents at 1 and ∞ has to hold:

(1) $\frac{1}{2}k_{\alpha_j/2} + k_{\alpha_j} - (w(\lambda_0 + \kappa_0), \alpha_j^\vee) \equiv 0(\operatorname{mod} \mathbb{Z})$ or

(2) $\frac{1}{2}k_{\alpha_j/2} + k_{\alpha_j} - (w(\lambda_0 + \kappa_0), \alpha_j^\vee) + (\frac{1}{2} - k_{\alpha_j})$

$$= \tfrac{1}{2}k_{\alpha_j/2} - (w(\lambda_0 + \kappa_0), \alpha_j^\vee) + \tfrac{1}{2} \equiv 0 \ (\operatorname{mod} \mathbb{Z}).$$

However, for almost all $k \in K$ and $\lambda_0 \in H_{\kappa_0}$ these relations are violated. Consequently: for general $k \in K$ and $\lambda_0 \in H_{\kappa_0}$ both $d(w, \kappa_0; \lambda_0, k)$ and $d(r_j w, \kappa_0; \lambda_0, k)$ have to be zero if one of them is. Thus $d(w, \kappa_0; \lambda_0, k) = 0$ $\forall w \in W$. We have shown now that, except for S_2, the set of singularities of F has codimension $\geqslant 2$ and thus that F has analytic continuation to $(\mathfrak{h}^* \times K \times V)\backslash S_2$. □

To complete the proof of Theorem 2.8 we will prove:

LEMMA 2.9. *Let p_i denote the orthogonal projection on the hyperplane $\alpha_i^\perp$ ($\alpha_i \in R$ a simple root). Let $w, w' \in W$ with $w\kappa, w'\kappa > 0$. Suppose that the map $H_\kappa \to \alpha_i^\perp$*

$\lambda \to p_i((w - w')\lambda)$ is constant. Then $w = w'$ or $w = r_i w'$.

Proof. $\lambda \to p_i((w - w')\lambda)$ is constant on H_κ if and only if $(w - w')(\kappa^\perp) \subset \mathbb{C}\alpha_i$ or equivalently $(w(w')^{-1} - 1)(w'\kappa^\perp) \subset \mathbb{C}\alpha_i$. Let us study the set $\mathscr{A} = \{x \in W | (x - 1)(w'\kappa^\perp) \subset \mathbb{C}\alpha_i\}$. Observe that $r_i\mathscr{A} = \mathscr{A}$. Furthermore: if $x \in \mathscr{A}$ then x or $r_i x$ has to be a reflection. To see this we assume that x itself is not a reflection and that $x \neq 1$. Then stab(x) has codimension 2 and so $x = r_\alpha r_\beta$ for some $\alpha, \beta \in R$. Also: $\mathrm{Im}(x - 1) = \mathbb{C}\alpha + \mathbb{C}\beta \supset \mathbb{C}\alpha_i$ and thus $r_i x$ is a reflection. We are left with two cases to treat:
(a) $\kappa \neq n\alpha\ \forall n \in \mathbb{Z}, \alpha \in R$. Then $x \in \mathscr{A}$ and x a reflection imply that $x = r_i$ and so $\mathscr{A} = \{1, r_i\}$
(b) $\kappa = n\alpha$ for some $\alpha \in R$, $n \in \mathbb{Z}_+$. Suppose $x \in \mathscr{A}$ a reflection. Then $x = r_i$ or $r_{w'\alpha}$. So we have $\mathscr{A} = \{1, r_i, r_{w'\alpha}, r_i r_{w'\alpha}\}$. But $w = r_{w'\alpha}w'$ is in contradiction with $w\alpha$, $w'\alpha > 0$ and $w = r_i r_{\omega'\alpha} w' \Rightarrow w\alpha = r_i(-w'\alpha) > 0 \Rightarrow w'\alpha = \alpha_i \Rightarrow w = w'$.

We resume the combinatorical implications of theorem 2.8:

COROLLARY 2.10. *Let the rational functions $\Delta_\kappa(\lambda, k)$ on $\mathfrak{h}^* \times K$ be defined by the recurrence relations (2.1) and $\Delta_0(\lambda, k) = 1$. Then the only poles that can occur in $\Delta_\kappa(\lambda, k)$ are simple poles along the hyperplanes $H_\tau \times K$ $(=\{(\lambda, k)|2(\lambda, \tau) + (\tau, \tau) = 0\})$ where τ satisfies:*
(1) $\tau \leqslant \kappa$
(2) $\tau = n\alpha$ for some $\alpha \in R_+$, $n \in \mathbb{Z}_+$
Furthermore, if $\kappa = n\alpha$ for some $\alpha \in R_+$, $n \in \mathbb{Z}_+$ then we have: $c(\lambda + \kappa, k) + c(\lambda, k) . \Delta_\kappa(\lambda, k)$ has a removable pole along $H_\kappa \times K$. □

3. Existence of hypergeometric differential operators and their shift operators

Let $\mathscr{V} = (\mathbb{C}(\mathfrak{h}^*))^{Q_+}$ (where $\mathbb{C}(\mathfrak{h}^*)$ is the field of rational functions on $\mathfrak{h}^*$) be the $\mathbb{C}(\mathfrak{h}^*)$-vectorspace of functions from Q_+ to $\mathbb{C}(\mathfrak{h}^*)$. If we choose $k \in K$ fixed we can identify $\mathscr{V}$ with the space of formal series on A_- of the form $r = \sum_{\kappa \in Q_+} r_\kappa(\lambda) h^{\lambda + \varrho(k) + \kappa}$, $r_\kappa(\lambda) \in \mathbb{C}(\mathfrak{h}^*)$ by means of $r \to (r_\kappa(\lambda))_{\kappa \in Q_+}$. A formal differential operator on A_- with asymptotic expansion $D = h^{\varrho(\ell)} \sum_{\kappa \in Q_+} h^\kappa \partial(p_\kappa(\lambda))$ with $\ell \in K$, $p_\kappa(\lambda) \in \mathbb{C}[\mathfrak{h}^*]$ can be viewed in this way as an element $M(k; D)$ of $\mathfrak{gl}(\mathscr{V})$, with matrix:

$$(M(k; D))_{\kappa,\nu}(\lambda) = p_{\kappa-\nu}(\lambda + \varrho(k) + \nu).$$

PROPOSITION 3.1. (*see* [HO], *Lemma 2.7 and* [Op], *Proposition 3.2*). *Let* $q_\kappa \in \mathbb{C}(\mathfrak{h}^*)$ *with* $q_\kappa = 0$ *if* $\kappa \notin Q_+$. *Let* $N \in \mathfrak{gl}(\mathscr{V})$ *be defined by* $N_{\kappa,\nu}(\lambda) = q_{\kappa-\nu}(\lambda + \varrho(k) + \nu)$. *Then*

$$M(k + \ell; L(k + \ell) + (\varrho(k + \ell), \varrho(k + \ell))) \circ N$$

$$= N \circ M(k; L(k) + (\varrho(k), \varrho(k))) \tag{3.1}$$

if and only if the q_k *satisfy the following recurrence relations*:

$$(2(\lambda - \varrho(k)) + \kappa, \kappa)q_\kappa(\lambda) = 2 . \sum_{\substack{\alpha \in R_+ \\ j \geqslant 1}} \{(k_\alpha + \ell_\alpha)(\alpha, \lambda + \varrho(\ell)$$

$$+ \kappa - j\alpha)q_{\kappa - j\alpha}(\lambda) - k_\alpha(\alpha, \lambda)q_{\kappa - j\alpha}(\lambda + j\alpha)\} \tag{3.2}$$

Proof. This is an easy computation in $\mathfrak{gl}(\mathscr{V})$ and left to the reader. □

COROLLARY 3.2. *For a given* $q_0 \in \mathbb{C}(\mathfrak{h}^*)$ *there exists precisely one* $N \in \mathfrak{gl}(\mathscr{V})$ *of the form described in Proposition 3.1 that satisfies (3.1).*

PROPOSITION 3.3. *The equation* $M(k, L(k)) . r = p . r$ $(p \in \mathbb{C}(\mathfrak{h}^*), r \in \mathscr{V})$ *has a non trivial solution space if and only if* $p(\lambda) = (\lambda + \varrho(k), \lambda - \varrho(k))$. *The* $(\lambda + \varrho(k), \lambda - \varrho(k))$*-eigenspace is one dimensional and spanned by the vector* $\phi(k) \in \mathscr{V}$ *with* $\phi(k) = (\Delta_\kappa(k))_{\kappa \in Q_+}$ *where* $\Delta_0(k) = 1$ *and* $\Delta_\kappa(k)$ *defined by the recurrence relations (2.1).*

Proof. See [HO], formulas (3.11)–(3.14).

In [HO] we formulated a conjecture on the structure of $\mathbb{D}(k) = (\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}))^{W,L(k)}$ ([HO], Conjecture 2.10): let $\gamma(k)$: $\mathbb{D}(k) \to \mathbb{C}[\mathfrak{h}^*]$ be the map defined by $\gamma(k)(\Sigma_{\kappa \in Q_+} h^\kappa \partial(p_\kappa(\lambda))) = \{\lambda \to p_0(\lambda + \varrho(k))\}$. Then $\gamma(k)$ is an isomorphism $(\forall k \in K)$. In [Op] we extended this to shift operators (and changed it in an inessential way: the ring $\mathscr{R}$ was replaced by the ring $\mathscr{S}$). Recall the following notations from [Op] (Corollary 3.11): $R = \amalg_{i=1}^m C_i$ is the decomposition of R in conjugacy classes of roots, $B = \{b_i\}_{i=1}^m$ is a basis for K consisting of the vectors $b_i = e_i$ if $2C_i \cap R = \emptyset$ (where $e_i \in K$ is the function defined by $(e_i)_\alpha = 0$ if $\alpha \notin C_i$ and $(e_i)_\alpha = 1$ if $\alpha \in C_i$) and $b_i = (2e_i - e_j)$ if $2C_i = C_j$. The next lemma is crucial for the proof of ([HO], Conjecture 2.10) and for the proof of the existence of the shift operators $G(\ell)$ $(\ell \in \mathbb{Z} . B)$ (see [Op], Corollary 3.12).

LEMMA 3.4. *let $N \in \mathfrak{gl}(\mathscr{V})$ be as in Proposition 3.1 and suppose that*
(a) $\ell \in \mathbb{Z}_- . B$, and q_0 is a polynomial such that

$$(b)\ \frac{q_0(\lambda + \varrho(k)) . c(\lambda, k)}{c(\lambda, k + \ell)} \in \mathbb{C}[\mathfrak{h}^*]^W.$$

Then $q_\kappa \in \mathbb{C}[\mathfrak{h}^*]$, $\forall \kappa \in Q_+$.

Proof. With induction on the ordering $<$ on Q_+. Take $\mu \in Q_+$ and suppose that $q_\tau \in \mathbb{C}[\mathfrak{h}^*]$ $\forall \tau \in Q_+$, $\tau < \mu$. From (3.2) we see that the only possible pole that can occur in $q_\mu(\lambda)$ is along the hyperplane $H_\mu + \varrho(k)$. However, from (3.1) and Proposition 3.3 it is clear that

$$N . \phi(k) = q_0(\lambda + \varrho(k)) . \phi(k + \ell)$$

This is equivalent to:

$$\sum_{\substack{\kappa + \tau = \mu \\ \kappa, \tau \in Q_+}} q_\kappa(\lambda + \varrho(k) + \tau) . \Delta_\tau(\lambda, k) = q_0(\lambda + \varrho(k)) . \Delta_\mu(\lambda, k + \ell)$$

or

$$\begin{aligned} q_\mu(\lambda + \varrho(k)) &= (q_0(\lambda + \varrho(k)) . \Delta_\mu(\lambda, k + \ell) \\ &\quad - q_0(\lambda + \varrho(k) + \mu) . \Delta_\mu(\lambda, k)) \\ &\quad - \sum_{\substack{\kappa + \tau = \mu \\ \kappa, \tau \in Q_+ \setminus \{0\}}} q_\kappa(\lambda + \varrho(k) + \tau) . \Delta_\tau(\lambda, k). \end{aligned}$$

The only term in the right hand side where a pole along H_μ can occur is, according to Corollary 2.10, the term

$$q_0(\lambda + \varrho(k)) . \Delta_\mu(\lambda, k + \ell) - q_0(\lambda + \varrho(k) + \mu) . \Delta_\mu(\lambda, k)$$

This expression can, according to the same Corollary 2.10, only have a pole along H_μ if $\mu = n\alpha$ for some $n \in \mathbb{Z}_+$, $\alpha \in R_+$. In this case (take $\lambda_0 \in H_\mu$ a generic point):

$$\begin{aligned} &\lim_{\lambda \to \lambda_0} [2(\lambda, \mu) + (\mu, \mu)][q_0(\lambda + \varrho(k))\Delta_\mu(\lambda, k + \ell) \\ &\quad - q_0(\lambda + \varrho(k) + \mu)\Delta_\mu(\lambda, k)] = \lim_{\lambda \to \lambda_0} [2(\lambda, \mu) + (\mu, \mu)] \\ &\quad \times \left[- q_0(\lambda + \varrho(k)) \frac{c(\lambda + \mu, k + \ell)}{c(\lambda_0, k + \ell)} + q_0(\lambda + \mu + \varrho(k)) \frac{c(\lambda + \mu, k)}{c(\lambda_0, k)} \right] \end{aligned}$$

$$= \lim_{\lambda \to \lambda_0} [2(\lambda, \mu) + (\mu, \mu)] \Bigg[- \left(\frac{q_0(\lambda + \varrho(k))c(\lambda_0, k)}{c(\lambda_0, k + \ell)} \right)$$

$$\times \left(\frac{c(\lambda + \mu, k + \ell)}{c(\lambda_0, k)} \right) + \left(\frac{q_0(\lambda + \mu + \varrho(k))c(\lambda + \mu, k)}{c(\lambda + \mu, k + \ell)} \right)$$

$$\times \left(\frac{c(\lambda + \mu, k + \ell)}{c(\lambda_0, k)} \right) \Bigg] = 0$$

because $\frac{q_0(\lambda + \varrho(k))c(\lambda, k)}{c(\lambda, k + \ell)}$ is a W invariant polynomial and $(\lambda + \mu) \to r_\alpha \lambda_0$ if $\lambda \to \lambda_0$. □

LEMMA 3.5. *Let $k \in K$ be a generic point and let $D = h^{\varrho(\ell)} \sum_{\kappa \in Q_+} h^\kappa \partial(p_\kappa)$ with $p_\kappa \in \mathbb{C}[\mathfrak{h}^*]$ and $\ell \in \mathbb{Z}_-.B$ be a formal differential operator on A_- with the property that (formally) $D \circ (L(k) + (\varrho(k, \varrho(k))) = (L(k + \ell) + (\varrho(k + \ell), \varrho(k + \ell))) \circ D$. Then the sum converges and D is in fact the lifting of an element of $\mathbb{A}_n$ (the algebra of polynomial differential operators on $\mathbb{C}^n$) under the map $H \to W\backslash H \cong \mathbb{C}^n$, $h \to (z_1, \ldots, z_n)$. So in particular (see* [OP], *Proposition 2.5), $D \in (\mathscr{S} \otimes \mathfrak{U}(\mathfrak{h}))^W$.*

Proof. The functions $z_i = \sum_{w \in W/\mathrm{stab}(\lambda_i)} h^{-w\lambda_i} (i = 1, \ldots, n)$ form a system of coordinates on A_-. So on A_- we have

$$\partial(X) = \sum_{i=1}^n \partial(X)(z_i) \frac{\partial}{\partial z_i} \quad (\forall X \in \mathfrak{h}). \tag{3.3}$$

Choose a basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{a}$ and write $X^k = X_1^{k_1}, \ldots, X_n^{k_n}$. Recall the notation

$$\left(\frac{\partial}{\partial z}\right)^\lambda = \left(\frac{\partial}{\partial z_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{k_n} \quad \text{if} \quad \lambda = -\sum_{i=1}^n k_i \lambda_i \in P_-$$

(see [HO], Proposition 2.3). From (3.3) we obtain:

$$\partial(X^k) = \sum_{\substack{\lambda \in P_- \\ \text{finite sum}}} a_\lambda \left(\frac{\partial}{\partial z}\right)^\lambda \quad \text{with } a_\lambda \text{ a series on } A_- \text{ of the form}$$

$a_\lambda(h) = h^{-\lambda} \cdot \sum_{\kappa \in Q_+} c_\kappa h^\kappa$ $(c_\kappa \in \mathbb{C})$. So we can rewrite D as follows:

$$D = \sum_{\substack{\lambda \in P_- \\ \text{finite sum}}} f_\lambda \left(\frac{\partial}{\partial z}\right)^\lambda$$ with f_λ formal series on A_- of the form

$f_\lambda = h^{\varrho(\ell)-\lambda} \cdot \sum_{\kappa \in Q_+} d_\kappa h^\kappa$ $(d_\kappa \in \mathbb{C})$. Because we assume k to be generic

and $\ell \in \mathbb{Z}_- . B$ (so $\varrho(\ell) \in P_-$) we know that D maps Jacobi polynomials with parameter k into Jacobi polynomials with parameter $k + \ell$ (for Jacobi polynomials: see [HO], Definition 3.13 or [H], section 8). It follows that

$$D(z^\lambda) = \sum_{\substack{\mu \in P_- \\ \mu \geq \lambda + \varrho(\ell)}} e_\mu z^\mu .$$

So we conclude that the formal series f_λ are in fact W-invariant Fourier polynomials and that $D \in \mathbb{A}_n$. □

Recall the notations of [Op]: for k, $\ell \in K$ we define $\mathbb{S}(\ell, k) = \{D \in \mathbb{A}_n | D \circ (L(k) + (\varrho(k), \varrho(k))) = (L(k + \ell) + (\varrho(k + \ell), \varrho(k + \ell))) \circ D\}$ and the map $\eta = \eta(\ell, k)$: $\mathbb{S}(\ell, k) \to \mathbb{C}[\mathfrak{h}^*]$

$$h^\varrho \sum_{\kappa \in Q_+} h^\kappa \Delta(p_\kappa) \to \{\lambda \to p_0(\lambda + \varrho(k))\}.$$

THEOREM 3.6. (*Structure theorem for the spaces* $\mathbb{S}(k, \ell)$)

(a) *For all* $k \in K$, $\gamma(k) = \eta(0, k)$: $\mathbb{S}(0, k) = (\mathscr{S} \otimes \mathfrak{U}(\mathfrak{h}))^{W, L(k)} \to \mathbb{C}[\mathfrak{h}^*]^W$ *is an isomorphism of algebras.*

(b) *If* $\ell \notin \mathbb{Z} . B$ *then* $\mathbb{S}(k, \ell) = \{0\}$. *If* $\ell \in \mathbb{Z} . B$ *then* $\mathbb{S}(\ell, k) = G(\ell, k) . \mathbb{S}(0, k)$. *The generator* $G(\ell, k) \in \mathbb{A}_n$ *has degree* $\Sigma_{\alpha \in R^0_+} \max(|\ell_\alpha|, |\ell_{\alpha/2} + \ell_\alpha|)$. *If* $\ell \in \mathbb{Z}_- . B$ *we can take* $\eta(\ell, k)(G(\ell, k))(\lambda) = \dfrac{c(\lambda, k + \ell)}{c(\lambda, k)}$

(c) *The dependence on* $k \in K$ *is of a polynomial nature: let* $p \in \mathbb{C}[\mathfrak{h}^* \times K]$ *such that* $p(., k) \in \operatorname{Im}(\eta(\ell, k))$, $\forall k \in K$. *Then* $(\eta(\ell, k))^{-1}(p(., k)) \in \mathbb{C}[K] \otimes \mathbb{A}_n$, *and its degree in* k *is equal to* $\deg_k(\{(\lambda, k) \to p(\lambda - \varrho(k), k)\})$.

Proof. (a) is immediate from Lemma 3.4 and Lemma 3.5. From ([Op], Proposition 3.4 and Corollary 3.12) it follows that we only need to prove (b) for $\ell = -\ell_i$ $(i = 1, \ldots, m)$ and in that case we can apply again the Lemmas 3.4 and 3.5. (c) is a consequence of the recurrence relations (3.2). □

REMARK 3.7. For more detailed information about shift operators we refer the reader to [Op], Sections 3 and 4.

4. The generalized Calogero–Moser system

As was explained in ([HO], Section 2) there is a close relation between the differential operator $L(k)$ associated with some root system R and the differential operator

$$S(g) = -\frac{1}{2}\sum_{j=1}^{n} \partial(X_j)^2 + \sum_{\alpha\in R_+} \frac{g_\alpha}{(h^{-\alpha/2} - h^{\alpha/2})^2} \quad (g \in K)$$

on the torus H. This relation is a consequence of the automorphism of $\mathcal{S} \otimes \mathfrak{U}(\mathfrak{h})$ given by $P \to \delta^{1/2} \circ P \circ \delta^{-1/2}$ with $\delta = \delta(k) = \Pi_{\alpha\in R_+} (h^{-\alpha/2} - h^{\alpha/2})^{2k_\alpha}$ (viewed as Nilsson class function on $W\backslash H^{\mathrm{reg}}$):

$$-2S(g) = \delta^{1/2} \circ (L(k) + (\varrho(k), \varrho(k))) \circ \delta^{-1/2}$$

if we take

$$g_\alpha = \tfrac{1}{2}(\alpha, \alpha)k_\alpha(k_\alpha + 2k_{2\alpha} - 1).$$

The asymptotic expansion of $S(g)$ on A_- is given by

$$S(g) = -\frac{1}{2}\sum_{j=1}^{n} \partial(X_j)^2 + \sum_{\alpha\in R_+} g_\alpha \sum_{j=1}^{\infty} jh^{j\alpha}. \tag{4.1}$$

In order to study the quantum integrals of the generalized Calogero–Moser system it is useful to write down the recurrence relations that arise from the equation $[S(g), P] = 0$ for a formal differential operator $P = \Sigma_{\kappa\in Q_+} h^\kappa \partial(p_\kappa)$.

PROPOSITION 4.1. $[S(g), P] = 0$ *if and only if the polynomials* p_κ *satisfy the recurrence relations*

$$(2\lambda + \kappa, \kappa)p_\kappa(\lambda) = -\sum_{\alpha\in R_+} 2g_\alpha \sum_{j=1}^{\infty} j(p_{\kappa-j\alpha}(\lambda + j\alpha) - p_{\kappa-j\alpha}(\lambda)). \tag{4.2}$$

□

From Theorem 3.6 we know that, $\forall g \in K$, the recurrence relations (4.2) lead to an element $P \in (\mathcal{S} \otimes \mathfrak{U}(\mathfrak{h}))^W$ with $[S(g), P] = 0$ if we take $p_0 \in \mathbb{C}[\mathfrak{h}^*]^W$. The p_κ are polynomials in λ and g and we can use (4.2) to make an estimate on the degree of $p_\kappa(\lambda, g)$ in the following sense: define

$$\mathcal{P}_k = \left\{ p \in \mathbb{C}[\mathfrak{h}^* \times K] | p = \sum_{|n| + 2|m| \leqslant k} c_{n,m} X^n g^m,\ c_{n,m} \in \mathbb{C} \right\}. \tag{4.3}$$

Then we have:

PROPOSITION 4.2. *Take $p_0 \in \mathbb{C}[\mathfrak{h}^*]^W$ homogeneous of degree k and $p_\kappa(\lambda, g)$ defined by (4.2). Then $p_\kappa \in \mathcal{P}_k$, $\forall \kappa \in Q_+$.*

Proof. Use induction on κ with respect to the partial ordering $<$ on Q_+. □

DEFINITION 4.3. *For $g \in K$ we define the map*

$$\gamma' = \gamma'(g) \colon (\mathcal{S} \otimes \mathfrak{U}(\mathfrak{h}))^{W, S(g)} \to \mathbb{C}[\mathfrak{h}^*]^W \ by$$

$$\sum_{\kappa \in Q_+} h^\kappa \partial(p_\kappa) \to p_0.$$

PROPOSITION 4.4. *Let $\mathcal{D}$ be the algebra $\mathbb{C}[K] \otimes \mathcal{S} \otimes \mathfrak{U}(\mathfrak{h})$ of differential operators on H^{reg} with coefficients in $\mathbb{C}[K] \otimes \mathcal{S}$. Put*

$$\mathcal{D}_k = \left\{ P \in \mathcal{D} | P = \sum_{2|n| + |m| \leqslant k} f_{n,m} \cdot g^n \partial(X^m), f_{n,m} \in \mathcal{S} \right\}.$$

Then $\mathcal{S} = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots$ is a filtration of $\mathcal{D}$, i.e. $\mathcal{D}_{k_1} \cdot \mathcal{D}_{k_2} \subset \mathcal{D}_{k_1 + k_2}$. □

THEOREM 4.5. *The map $\gamma'(g)$ is an isomorphism of algebras for all $g \in K$. Moreover, if $p_0 \in \mathbb{C}[\mathfrak{h}^*]^W$ is homogeneous of degree k then $D(p_0) = \{g \to (\gamma'(g))^{-1}(p_0)\} \in \mathcal{D}_k$.* □

COROLLARY 4.6. *The quantum mechanical systems on $\mathfrak{a}$ described by the Schrödinger operators S_A, S_T or $S_\mathfrak{a}$ (see Section 1, formulas (1.1), (1.2) and (1.3)) are completely integrable for every root system R and multiplicity function g. The integrals are of an algebraic nature. For every homogeneous $p \in \mathbb{C}[\mathfrak{h}^*]^W$ there exists an integral of the form $\partial(p)$ + lower order terms.*

To obtain results on the complete integrability of the classical systems described by the Hamiltonians H_A, H_T, and $H_{\mathfrak{a}}$ on $\mathfrak{a} \times \mathfrak{a}^*$ (see Section 1, formulas (1.5), (1.6) and (1.7)) we study the associated graded ring $\mathscr{E}$ of $\mathscr{D}$ with respect to the filtration given above (Proposition 4.4). Denote by σ the symbolmap, thus for $D \in \mathscr{D}_k$ we have $\sigma_k(D) = D(\text{mod } \mathscr{D}_{k-1}) \in \mathscr{D}_k/\mathscr{D}_{k-1} = \mathscr{E}_k \subset \mathscr{E}$. It is obvious that $[\mathscr{D}_{k_1}, \mathscr{D}_{k_2}] \subset \mathscr{D}_{k_1+k_2-1}$ so $\mathscr{E}$ is commutative and we can define the so-called Poisson bracket on $\mathscr{E}$. First take two homogeneous elements $f_i \in \mathscr{E}_{k_i}$ and put $\{f_1, f_2\} = \sigma_{k_1+k_2-1}([F_1, F_2])$ where $F_i \in \mathscr{D}_{k_i}$ such that $\sigma_{k_i}(F_i) = f_i$. Extend $\{.,.\}$ bilinearly to $\mathscr{E}$.

PROPOSITION 4.7. $\mathscr{E} \cong \mathbb{C}[K] \otimes \mathscr{S} \otimes \mathbb{C}[\mathfrak{h}^*]$ *and the Poisson bracket can be calculated explicitly*:

$$\{f_1, f_2\} = \sum_{j=1}^{n} \left[\frac{\partial f_1}{\partial X_j} \partial(X_j) f_2 - \partial(X_j) f_1 \frac{\partial f_2}{\partial X_j} \right]$$

(*where we interpret* X_j *as coordinates on* $\mathfrak{a}^*$)

Proof. Easy and left to the reader. □

THEOREM 4.8. *The system described by the Hamiltonians* H_A, H_T *and* $H_{\mathfrak{a}}$ *are completely integrable with algebraic integrals. Let* $\{p_1, \ldots, p_n\}$ *be a set of homogeneous generators for* $\mathbb{C}[\mathfrak{h}^*]^W$. *Then there exists a complete set of integrals of the form* p_i + *terms of lower degree in* $\{X_1, \ldots, X_n\}$ $(i = 1, \ldots, n)$.

Proof. Clear from Theorem 4.5 by taking symbols with respect to the filtration $\mathscr{D}_0 \subset \mathscr{D}_1 \subset \ldots$ of $\mathscr{D}$. □

REMARK 4.9. There is an equivalent, but maybe physically more natural way to describe the passage from the quantum level to the classical level. If we take $\hbar$, Planck's constant, as an extra variable we have to take

$$S(g) = -\frac{\hbar^2}{2} \sum_{j=1}^{n} \partial(X_j)^2 + \sum_{\alpha \in R_+} \frac{g_\alpha}{(h^{-\alpha/2} - h^{\alpha/2})^2}$$

Define a filtration by taking $\deg(\hbar) = -1$, $\deg(\partial(X_j)) = 1$ and define p_i = symbol of $-\hbar(-1)^{1/2}\,\partial(X_i)$. Now take symbols with respect to this filtration (so in some sense we take the limit $\hbar \downarrow 0$).

References

[C] F. Calogero: Solution of the one dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Physics* 12 (1971) 419–436.

[H] G.J. Heckman: Root systems and hypergeometric functions II, *Comp. Math.* 64 (1987) 353–373.

[Ha] Harish-Chandra: Spherical functions on a semisimple Lie group I, *Amer. J. Math.* 80 (1958) 241–310.

[Hel] S. Helgason: *Groups and Geometric Analysis*. Academic Press (1984).

[Hö] L. Hörmander: *An Introduction to Complex Analysis in Several Variables*, van Nostrand (1966).

[HO] G.J. Heckman and E.M. Opdam: Root systems and hypergeometric functions I, *Comp. Math.* 64 (1987) 329–352.

[J] C. Jacobi: Problema trium corporum mutis attractionibus cubus distantiarum inverse proportionalibus recta linea se moventium, *Gesammelte Werke* 4, Berlin (1866).

[K] T.H. Koornwinder: Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent differential operators, I–IV, *Indag. Math.* 36 (1974) 48–66 and 358–381.

[Mac] I.G. Macdonald: unpublished manuscript (1985).

[Mar] C. Marchioro: Solution of a three body scattering problem in one dimension, *J. Math. Physics* 11 (1970) 2193–2196.

[Mo] J. Moser: Three integrable Hamiltonian systems connected with isospectral deformation, *Adv. Math.* 16 197–220 (1975).

[Op] E.M. Opdam: Root systems and hypergeometric functions III, *Comp. Math.* 67 (1988) 21–50.

[OP] M.A. Olshanetsky and A.M. Perelemov: Completely integrable Hamiltonian systems connected with semisimple Lie algebras, *Inv. Math.* 37 (1976) 93–108.

[Se] J. Sekiguchi: Zonal spherical functions on some symmetric spaces, *Publ. R.M.S. Kyoto. Univ.* 12 Suppl. (1977) 455–459.