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The rationality of some moduli spaces of plane curves

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Introduction

The main purpose of this paper is to show that the moduli space for plane curves of given degree D is rational, provided that either or $D \equiv 1$ (*mod.* 9) and $D \geqslant 19$, or $D \equiv 1$ (4). This we approach as a problem in invariant theory, for the moduli space is the quotient variety $\mathbf{P}_D/SL_3$, where $\mathbf{P}_D$ is the $(\binom{D+2}{2} - 1)$-dimensional projective space of homogeneous polynomials of degree D in three variables and SL_3 acts by contragredient substitution on the coefficients. In particular, however, we are unable to solve the problem when $D = 4$, in which case the moduli space is (birationally equivant to) that for curves of genus three. (Recall that for $g \leqslant 6$, $\mathcal{M}_g$ can be described birationally as an orbit space $\mathbf{P}/G$, where G is some reductive group acting linearly on a projective space $\mathbf{P}$; from this description we have shown that $\mathcal{M}_4$ and $\mathcal{M}_6$ are rational, while for $\mathcal{M}_1$ and $\mathcal{M}_2$ this was established by the invariant theorists of the last century). We also show that if X is the space of pencils of binary forms whose degree K is even and at least ten, then the quotient X/PGL_2 is rational and, using a result of Mukai [8], that the moduli space for polarized $K3$ surfaces of degree 18 is rational.

This general question of whether $\mathbf{P}/G$ need be rational has been answered negatively by Saltman [11]; in his counter-examples G is finite, and it is still unknown whether quotients of projective spaces (or other homogeneous varieties) by connected, or even classical, groups are rational. However, Bogomolov has shown [2] that for many groups, including the simply connected classical ones, these quotients are stably rational (in other words, their products by suitable projective spaces are rational). On the other hand, Beauville, Colliot-Thélène, Sansuc and Swinnerton- Dyer [1] have constructed irrational threefolds X such that $X \times \mathbf{P}^3$ is rational.

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Preliminaries

NOTATION. (i) Whatever the group G, $\mathbf{1}$ will denote its trivial one-dimensional representation.

(ii) If the reductive group G acts linearly on the quasi-projective variety X, a point $x \in X$ is *very stable* if it is stable and its stabilizer is trivial. The locus of very stable points will be denoted by X^{vs}. (If we need to emphasize the G-linearized polarizing sheaf $\mathscr{L}$, we shall write $X^{vs}(\mathscr{L})$.) We shall say that X is a *good G-space* (or simply *good*) if $X^{vs} \neq \emptyset$. If V is a representation of G and Z the kernel of the action of G on $\mathbf{P}(V)$, then we shall say that V is *very good* if V is a good G-space, $\mathbf{P}(V)$ is a good (G/Z)-space and V is not an eigenspace for any one-parameter subgroup of G (so that Z is a finite subgroup of the centre of G).

By Mumford's theory [8] and Luna's étale slice theorem [8, Appendix 1], G acts freely on X^{vs}; a geometric quotient X^{vs}/G exists and the map π: $X^{vs} \to X^{vs}/G$ is a principal G-bundle. See [8] for the details.

(iii) We let the symbol $\sim$ denote birational equivalence.

THEOREM 1 [8, §7.1]. *Suppose that f: $X \to Y$ is a G-equivariant morphism of quasi-projective k-varieties and that $\mathscr{L} \in \operatorname{Pic} Y$, $\mathscr{M} \in \operatorname{Pic} X$ are G-linear sheaves where $\mathscr{L}$ is ample and $\mathscr{M}$ is ample relative to f. Suppose that $Y^{vs}(\mathscr{L}) \neq 0$. Then for all $n \gg 0$,*

$$f^{-1}(Y^{vs}(\mathscr{L})) \subseteq X^{vs}(\mathscr{M} \otimes \mathscr{L}^{\otimes n}).$$

Set $Y^{vs}(\mathscr{L}) = Y_0$. Then there is a Cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, "\alpha"] \arrow[d] & X_0/G \arrow[d] \\ Y_0 \arrow[r, "\beta"] & Y_0/G, \end{tikzcd}$$

and there exists $\mathscr{L}_1 \in \operatorname{Pic}(Y_0/G)$, $\mathscr{M}_1 \in \operatorname{Pic}(X_0/G)$ such that

$$\mathscr{L} \mid Y_0 \cong \beta^* \mathscr{L}_1, \quad \mathscr{M} \mid X_0 \cong \alpha^* \mathscr{M}_1.$$

The birational triviality of certain principal bundles

If G acts on X, then the quotient map π: $X^{vs} \to X^{vs}/G$ is a G-bundle in the étale topology. We shall be concerned in this section with the question of whether π is a bundle in the Zariski topology; it is well-known that this is equivalent to π having a section defined generically, and to π being generically a bundle in the Zariski topology.

PROPOSITION 2. *If G is a connected reductive group, then the following statements are equivalent.*

(*i*) *There exists a good representation W such that the morphism $W^{vs} \to W^{vs}/G$ is a Zariski G-bundle.*

(*ii*) *Every principal G-bundle $X \to Y$ carrying an ample G-linearized invertible sheaf is a Zariski bundle.*

(*iii*) *For any very good representation V of G, the morphism $\mathbf{P}(V)^{vs} \to \mathbf{P}(V)^{vs}/G$ is a Zariski (G/Z)-bundle, where Z is the kernel of the action of G on $\mathbf{P}(V)$ (Z is a subgroup of the centre of G).*

Proof. (i) $\Rightarrow$ (ii): By Theorem 1, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ X \times \mathbf{P}(W \oplus 1) \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[r, "\sigma"] & (X \times \mathbf{P}(W \oplus 1))/G \arrow[u, "\alpha"'] \arrow[d] \\ \mathbf{P}(W \oplus 1) \arrow[r, "\varrho"] & \mathbf{P}(W \oplus 1)/G \end{tikzcd}$$

whose top and bottom squares are Cartesian. Moreover, using Theorem 1 to descend a suitable line bundle cutting out $\mathcal{O}(1)$ on fibres of α, it follows that α is a ruling. (We have blurred the distinction between regular and rational maps on one hand and that between S and S^{vs} for a G-variety S on the other, in an abuse of notation that will recur throughout this paper.) By hypothesis, ϱ is a ruling, and so by pull-back σ is also a ruling. So there is a generic section γ: $Y \to X \times \mathbf{P}(W \oplus 1)$; then $p_1 \circ \gamma$ is a generic section of π.

(ii) $\Rightarrow$ (iii): By hypothesis, the map α: $V^{vs} \to V^{vs}/G$ is a Zariski bundle. Recall a result of Hall [7] and Rosenlicht [10]: if a torus $T \cong (\mathbf{C}^*)^r$ acts with finite kernel on a variety X, then $X \sim (X/T) \times \mathbf{P}^r$. Let $T = \mathbf{C}^*$; then there is an obvious action of T on V, commuting with that of G, and we have $\mathbf{P}(V)^{vs} \sim V^{vs}/T$, canonically. Consider the commutative diagram

$$\begin{tikzcd} V^{vs} \arrow[r, "\alpha"] \arrow[d] & V^{vs}/G \arrow[d] \\ \mathbf{P}(V)^{vs} \arrow[r, "\gamma"] & \mathbf{P}(V)^{vs}/G, \end{tikzcd}$$

where the vertical maps are quotients by T; by hypothesis, α is a ruling, and β is also a ruling by the result just quoted. Hence γ has a generic section, and so is a Zariski bundle.

(iii) $\Rightarrow$ (i): Take any good representation W (which certainly exists, since for any semi-simple group there are, up to isomorphism and the addition of trivial factors, only finitely many faithful representations that are not good). It follows from the hypothesis that $\mathbf{P}(W \oplus \mathbf{1}) \to \mathbf{P}(W \oplus \mathbf{1})/G$ is a Zariski G-bundle, and so $W \to W/G$ is a Zariski bundle. Q.E.D.

Here are some examples of pairs (G, W) satisfying the hypotheses of Proposition 2, (i):

$G = SL_n$ or Sp_n and W is the space of $n \times n$ matrices on which G acts by left multiplication. Using the description of the ring of invariants given by Weyl [13], it is easy to see that W is ruled over W/G.

It is worth pointing out that in contrast to the case of $G = SL_n$ just mentioned, if V is a representation of PGL_n containing a very stable vector, the quotient map $V \to V/PGL_n$ is never a bundle in the Zariski topology. For by Proposition 2, it is enough to produce one such representation V. The conjugation action of PGL_n of M_n gives a homomorphism $PGL_n \to \mathrm{Aut}(M_n) = GL_{n^2}$, and $GL_{n^2} \hookrightarrow V = \mathbf{C}^{n^4}$; the left multiplication of GL_{n^2} by PGL_n extends to a linear action on V. Now let P be the stabilizer of a point in the action of PGL_n on $\mathbf{P}^{n-1}$. Consider the map α: $GL_{n^2}/P \to GL_{n^2}/PGL_n$; Haboush has shown [6] that this is generically a universal Severi-Brauer scheme. Since non-trivial Severi-Brauer schemes over function fields exist for all dimensions, it follows that α cannot be uniruled, or else α would be generically trivial, by Châtelet's Theorem. Hence the quotient map $V \to V/PGL_n$ cannot be ruled, as we said above.

Plane curves

Our aim in this section is to prove the following result:

THEOREM 3. *The orbit space of the family of degree D plane curves, modulo the action of SL_3, is rational, provided that $D \equiv 1(9)$ and $D \geqslant 19$.*

Note. The hypotheses on D arise because the technique requires that D be prime to 3, and that D be sufficiently large; moreover, there are computational difficulties that have only been overcome when $D \equiv 1(9)$.

We begin by setting up some notation. We denote by $V(D)$ the space of ternary forms of degree D (so that $V(D) = \mathrm{Symm}^D(\mathbf{C}^3)^\vee$, where $\mathbf{C}^3$ is the standard representation of SL_3, and set $\mathbf{P}_D = \mathbf{P}(V(D))$, the family of plane curves of degree D. Throughout, G will denote the group SL_3 and $\bar{G}$ will denote PGL_3.

The idea behind proving that $\mathbf{P}_D/SL_3$ is rational is the following. Suppose that $D \equiv 1(3)$. Then we shall construct a G-equivariant rational map ϕ: $\mathbf{P}_D \to \mathbf{P}_4$ given by a linear system of quartics, and establish various things:

(i) ϕ is dominant.
(ii) The generic fibre F of ϕ is geometrically reduced and irreducible, so that F is a component of the complete intersection F' of fourteen quartics.
(iii) There is a linear space L contained in the triple focus of each quartic containing F' such that $\dim L \geqslant 13$. Thus if we project F' away from L to a complementary linear space M in $\mathbf{P}_D$ via a map π the fibres of π are linear.
(iv) The restriction of π to F is dominant. (We shall express this by saying that the linear space L is *non-degenerate*.)

From (i)–(iv), it follows that F is rational over the function field $\mathbf{C}(\mathbf{P}_4)$.

Pass now to the quotients by G. Set $X = \mathbf{P}_D/\bar{G}$, $Y = \mathbf{P}_4/\bar{G}$, and let ψ: $X \to Y$ denote the induced map. Since $\bar{G}$ acts generically freely on $\mathbf{P}_4$, the generic fibre F^* of ψ is geometrically isomorphic to F. If $K = \mathbf{C}(Y)$, the invariant subfield of $\mathbf{C}(\mathbf{P}_4)$, then F^* lies naturally in $(\mathbf{P}_D) \times K$, and we show that F^* is rational over K by finding a non-degenerate linear space L^* as in (iii) above that is defined over K. To do this, we use the birational triviality of the principal $\bar{G}$-bundle $\mathbf{P}_4 \to Y$. It now follows that F^* is rational over K; say $F^* \sim Y \times \mathbf{P}^N$. To complete the proof, we use the fact that although the rationality of Y is unknown (of course, $Y \sim \mathcal{M}_3$), the product $Y \times \mathbf{P}^8$ is rational; since $N > 8$, it follows that X is rational over $\mathbf{C}$.

The construction of ϕ. The existence of ϕ is equivalent to the appearance of $V(4)$ as a component of the representation $\mathrm{Symm}^4\,(V(D))$ of G, a fact which can presumably be checked by a computation of characters. However, to prove the statements (i)–(iv) above, we must know ϕ explicitly; its mere existence is insufficient. To achieve this, we use the symbolical method, which we now outline; for a full and lucid description, see [4].

First, some more notation. For a multi-index $\boldsymbol{i} = (i_1, i_2, i_3)$ of non-negative integers, we write $i = i_1 + i_2 + i_3$; if $\#\boldsymbol{i} = D$, then $\binom{D}{i_1,\, i_2,\, i_3}$ denotes the trinomial coefficient $D!/(i_1!\,i_2!\,i_3!)$. Let (x_1, x_2, x_3) be homogeneous co-ordinates on $\mathbf{P}^2$, and let $\boldsymbol{x}^{\boldsymbol{i}}$ denote the monomial $x_1^{i_1}x_2^{i_2}x_3^{i_3}$. Then for $f \in V(D)$, we can write $f = \Sigma_{\#\boldsymbol{i}=D} \binom{D}{i_1\; i_2\; i_3} A_{\boldsymbol{i}} \cdot \boldsymbol{x}^{\boldsymbol{i}}$, since char $\mathbf{C} = 0$.

For us, the symbolical method provides a way in which to write down *covariants*, i.e. G-equivariant morphisms from $V(D)$ to other G-spaces. It proceeds by writing f "symbolically" in several different ways as a D'th power of a linear form $a_x = a_1x_1 + a_2x_2 + a_3x_3$, so that $f = a_x^D = b_x^D = c_x^D = \ldots,$

then taking a suitable formal expression in the symbols x, a, b, c, etc. (and maybe various contragredient variables as well, although we shall not need this generality) and finally substituting in the identities $A_i = a_1^{i_1} a_2^{i_2} a_3^{i_3} = b_1^{i_1} b_2^{i_2} b_3^{i_3} = \ldots$ to rewrite the expression as a function only of the coefficients A_i and the variables x_1, x_2, x_3. The action of G on the sets $\boldsymbol{a}$, $\boldsymbol{b}$, etc. of formal coefficients is contragredient to that on the variables x_1, x_2, x_3, so that the linear forms a_x, $b_x \ldots$ and the determinants

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

which will be denoted by (abc), etc., are G-invariant.

Now we can write down the map ϕ. Say $D = 9M + 1$. Then we define ϕ: $V(D) \to V(4)$ by

$$\phi(f) = (abc)^{3M} (abd)^{3M} (acd)^{3M} (bcd)^{3M} a_x b_x d_x c_x.$$

Since all the terms (abc), a_x, etc. are G-invariant, we see that ϕ is G-equivariant. By abuse of notation, we also let ϕ denote the induced rational map $\mathbf{P}_D \to \mathbf{P}_4$. Write $\phi(f) = \Sigma_{\#\alpha=4} G_\alpha \boldsymbol{x}^\alpha$; then ϕ is defined by the linear system of quartics generated by the G_α.

We need some more notation:

$$I = \{\boldsymbol{i} = (i_1, i_2, i_3) \mid i_\beta \in \mathbf{Z},\quad i_\beta \geqslant 0 \quad \text{and} \quad \#\boldsymbol{i} = D\},$$

$$J = \{\boldsymbol{i} \in I \mid i_1 \geqslant 6M + 3\},$$

$$H = I - J,$$

$$\boldsymbol{j} = (M + 1, 4M, 4M),$$

$$A = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \mid \alpha_\beta \in \mathbf{Z},\quad \alpha_\beta \geqslant 0 \quad \text{and} \quad \#\boldsymbol{\alpha} = 4\}.$$

For any $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l} \in I$ and for $\beta = 1, 2, 3$, set $p_\beta = i_\beta + j_\beta + k_\beta + l_\beta$. For any subset $S \subset I$, we let $\mathbf{P}(S)$ denote the linear subspace of $\mathbf{P}_D$ defined by the equations ($A_i = 0 \forall \boldsymbol{i} \notin S$). If $S = \{\boldsymbol{i}\}$, we write P_i for $\mathbf{P}(\{\boldsymbol{i}\})$.

T denotes the maximal torus of SL_3 consisting of diagonal matrices. Define weights w_1, w_2 of monomials $\boldsymbol{x}^k$ by $w_1(\boldsymbol{x}^k) = k_1 - k_2$ and $w_2(\boldsymbol{x}^k) = k_1 - k_3$.

Finally, let B denote the base scheme of the map ϕ. The proof of (i)–(iv) hinges upon an analysis of B.

LEMMA 4. *There is a point P at which B is smooth of codimension 15.*

Proof of Lemma 4. Since SL_3 acts covariantly on the variables $\boldsymbol{x}$ and contravariantly on the coefficients $\{A_i\}$, it follows that $w_1(A_i \cdot A_j \cdot A_k \cdot A_l) = p_2 - p_1$ and $w_2(A_i \cdot A_j \cdot A_k \cdot A_l) = p_3 - p_1$. Also, for $\boldsymbol{\alpha} \in A$, $w_1(x^\alpha) = \alpha_1 - \alpha_2$, $w_2(\mathbf{x}^\alpha) = \alpha_1 - \alpha_3$. Hence if the term $A_iA_jA_kA_l \cdot \boldsymbol{x}^\alpha$ appears in the covariant $\phi\,(f)$ with non-zero coefficient, we must have $p_2 - p_1 = \alpha_2 - \alpha_1$ and $p_3 - p_1 = \alpha_3 - \alpha_1$. Since $\Sigma_\beta\,\alpha_\beta = 4$ and $\Sigma_\beta\, p_\beta = 4(9M + 1)$, it follows that $p_\beta = \alpha_\beta + 12M$ for $\beta = 1, 2, 3$. In particular $p_1 \leqslant 4 + 12M$, and so every one of the quartic hypersurfaces $\{G_\alpha = 0\}$ is triple along the linear space $\mathbf{P}(J)$. In particular, $\mathbf{P}(J) \subset B$; we shall analyze B by projecting away from $\mathbf{P}(J)$. So consider the diagram

$$\begin{tikzcd} B \arrow[d, hook] \arrow[dr, "\sigma"] & \\ \mathbf{P}(I) \arrow[r, "\pi"'] & \mathbf{P}(H) \end{tikzcd}$$

where σ, π are the projections with centre $\mathbf{P}(J)$. Notice that every fibre $\sigma^{-1}(P)$, $P \in \mathbf{P}(H)$, is (at least set-theoretically) a linear space.

Consider the vector $\boldsymbol{j} = (M + 1, 4M, 4M)$. Note that for every $\boldsymbol{\alpha} \in A$, the vector $\boldsymbol{i} = \boldsymbol{\alpha} + 12M \cdot (1, 1, 1) - 3\boldsymbol{j}$ lies in J (provided that $M \geqslant 2$) and the term $A_i \cdot A_j^3 \cdot \boldsymbol{x}^\alpha$ is T-invariant. We shall need the following result:

PROPOSITION 5. *Provided that $M \geqslant 2$, then for every $\boldsymbol{\alpha} \in A$ and for $\boldsymbol{i} = \boldsymbol{\alpha} + 12M\,(1, 1, 1) - 3\boldsymbol{j}$, the term $A_i \cdot A_j^3 \cdot \boldsymbol{x}^\alpha$ occurs in the covariant $\phi\,(f)$ with non-zero coefficient.*

The proof of Proposition 5 is postponed.

Assuming Proposition 5, we see that the fibre $\sigma^{-1}(P_j)$ is a reduced linear space of codimension 15 in the linear space $\pi^{-1}(P_j)$. Then there is a unique irreducible component B_1 of B such that σ induces a dominant map $B_1 \to \mathbf{P}(H)$, and moreover B_1 is of codimension 15 in $\mathbf{P}(I)$. Also, since $\sigma^{-1}(P_j)$ is reduced, it follows that B is reduced at the generic point of B_1. Then take P to be a geometric generic point of B_1. This completes the proof of Lemma 4.

Proof of Theorem 3. For any r, put $\mathbf{P}(V(r)) = \mathbf{P}_r$. Let ψ: $\mathbf{P}_D \to \mathbf{P}_4$ be the rational map induced by ϕ. Let γ: $\tilde{\mathbf{P}}_D \to \mathbf{P}_D$ be the blow-up along B and let $\tilde{\psi}$: $\tilde{\mathbf{P}}_D \to \mathbf{P}_4$ be the induced morphism; $\tilde{\psi}$ is PGL_3-equivariant. We have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathbf{P}}_D \arrow[rr] \arrow[d, "\gamma"'] \arrow[drr, "\tilde{\psi}"] & & \mathbf{P}_D \times \mathbf{P}_4 \arrow[d, "\mathrm{pr}_2"] \arrow[dll, "\mathrm{pr}_1"'] \\
\mathbf{P}_D \arrow[rr, "\psi"'] & & \mathbf{P}_4
\end{tikzcd}$$

where pr_1, pr_2 are the projections.

Since B is locally a complete intersection at P, $\tilde{\psi}$ induces an isomorphism $\gamma^{-1}(P) \to \mathbf{P}_4$. Since also B is smooth at P, it follows that $\tilde{\mathbf{P}}_D$ is smooth along $\gamma^{-1}(P)$. Let ν: $\mathbf{P}_D^* \to \tilde{\mathbf{P}}_D$ be the normalization. Put $\nu^{-1}(\gamma^{-1}(P)) = \Sigma$; then ν induces an isomorphism $\Sigma \to \gamma^{-1}(P)$, and so an isomorphism $\Sigma \to \mathbf{P}_4$. Hence the induced morphism ψ^*: $\mathbf{P}_D^* \to \mathbf{P}_4$ has a section, and so is equal to its Stein factorization. Then by Bertini's theorem, the geometric generic fibre of ψ^* is normal and connected, and so irreducible; hence the geometric generic fibre of $\tilde{\psi}$ is reduced and irreducible.

Let ξ denote the generic point of $\mathbf{P}_4$. Then the generic fibre $F_\xi = \tilde{\psi}^{-1}(\xi)$ maps isomorphically to its image F'_ξ in $\mathbf{P}_D \otimes k(\xi)$ via γ. By construction, F'_ξ is an irreducible component of the intersection E_ξ of 14 quartics in $\mathbf{P}_D \otimes k(\xi)$ all of which are triple along $\mathbf{P}(J) \otimes k(\xi)$. Let τ denote the projection away from $\mathbf{P}(J) \otimes k(\xi)$. Since F'_ξ contains $B_1 \otimes k(\xi)$, σ: $B_1 \to \mathbf{P}(H)$ is dominant, and B_1 is the only component of B on which σ is dominant, it follows that τ induces a dominant map $F'_\xi \to \mathbf{P}(J)$ and E'_ξ is the only component of E_ξ on which τ is dominant. Since every fibre of $\tau|_{F_\xi}$ is a linear space, at least set-theoretically, the same is true of $\tau|_{F'_\xi}$. Pulling back to $\mathbf{P}_D \times \mathbf{P}_4$, we see that projection away from $\mathbf{P}(J) \otimes k(\xi)$ induces a dominant map $F_\xi \to \mathbf{P}(H) \otimes k(\xi)$ whose fibres are linear.

We now pass to the quotients. The locus $X \subset \mathbf{P}_4$ of very stable points is open and non-empty. (The sheaf $\mathcal{O}(3)$ is PGL_3-linearized.) Put $Y = \tilde{\psi}^{-1}(X)$, $Z = \mathbf{P}_D \times X$, $\mathcal{M} = \mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(2) \in \mathrm{Pic}\, Z$. PGL_3 acts on Y and Z, and $\mathcal{M}$ is linearized. Set $X_1 = X/PGL_3$, $Y_1 = Y/PGL_3$, $Z_1 = Z/PGL_3$; by Theorem 1 these quotients exist and there is a commutative diagram

$$\begin{tikzcd}
Z \arrow[rrr, "\alpha"] \arrow[ddd] & & & Z_1 \arrow[ddd, "\delta_1"] \\
& Y \arrow[r] \arrow[ul, hook'] \arrow[ddl, "\tilde{\psi}"] & Y_1 \arrow[ur, hook] \arrow[ddr, "\psi_1"] & \\
& & & \\
X \arrow[rrr, "\beta"'] & & & X_1
\end{tikzcd}$$

where every square is Cartesian. Moreover, $\mathcal{M}$ descends to a sheaf $\mathcal{M}_1 \in \mathrm{Pic}\, Z_1$ that induces $\mathcal{O}(1)$ on each fibre of δ_1. Hence $Z_1 \to X_1$ is a $\mathbf{P}^N$-bundle in the Zariski topology. Let η be the generic point of X_1. Put

$m = \dim \mathbf{P}(J) = \#(J) - 1$. Consider the algebraic $k(\eta)$-scheme $W = \{m\text{-planes } L \text{ in } (Z_1)_\eta = \mathbf{P}_D \otimes k(\eta)$ such that projection away from L onto a complementary $(N - m - 1)$-plane L' induces a dominant map $(Y_1)_\eta \to L'$ with linear fibres, and L lies in the triple locus of every member of some 14-dimensional linear system of quartic hypersurfaces containing $(Y_1)_\eta\}$. W is a locally closed subscheme of the Grassmannian of m-planes in $\mathbf{P}^N$. It follows from our discussion of $F_\xi = (Y_1)_\eta \otimes k(\xi)$ that W has a $k(\xi)$-point. By Proposition 2, $k(\xi)$ is a rational extension of $k(\eta)$, and so W has a $k(\eta)$-point, say Π. Then projection away from Π shows that $(Y_1)_\eta$ is rational over $k(\eta)$. Since $X_1 \times \mathbf{P}^8$ is rational and $\dim Y_1 - \dim X_1 \geqslant 8$, it follows that Y_1 is rational. Q.E.D.

It remains to prove Proposition 5.

Write $D = 9M + 1$. Recall that $\boldsymbol{j} = (M + 1, 4M, 4M)$, and that for each vector $\boldsymbol{\alpha}$ with $\#\boldsymbol{\alpha} = 4$, we define $\boldsymbol{i} = \boldsymbol{\alpha} + 12M(1, 1, 1) - 3\boldsymbol{j}$. Proposition 5 will follow from a stronger result.

PROPOSITION 6. *The coefficient $C_{\boldsymbol{\alpha}}$* of $A_{\boldsymbol{i}} \cdot A_{\boldsymbol{j}}^3 \cdot x^{\boldsymbol{\alpha}}$ *in the covariant $\phi(f)$ is given by $C_{\boldsymbol{\alpha}} = (-1)^M \binom{3M}{M, M, M} \cdot \binom{6M}{2M, 2M, 2M} \cdot \sigma_{\boldsymbol{\alpha}}(M)$, where each $\sigma_{\boldsymbol{\alpha}}$ is a rational function that does not vanish for any integral value of $M \geqslant 2$.*

Proof. We define $r_1, \ldots, r_6$ to be the coefficients of certain monomials in the symbolical expression

$$\Delta = (abc)^{3M}(abd)^{3M}(acd)^{3M}(bcd)^{3M}.$$

r_1 is the coefficient of $a^{(M,0,0)} \cdot b^{(M,4M.4M)} \cdot c^{(M,4M,4M)} \cdot d^{(M+1,4M-1,4M)}$,
r_2 is the coefficient of $a^{(9M-1,1,0)} \cdot b^{(M,4M,4M)} \cdot c^{(M,4M,4M)} \cdot d^{(M+1,4M-1,4M)}$,
r_3 is the coefficient of $a^{(9M-2,2,0)} \cdot b^{(M,4M,4M)} \cdot c^{(M+1,4M-1,4M)} \cdot d^{(M+1,4M-1,4M)}$,
r_4 is the coefficient of $a^{(9M-3,3,0)} \cdot b^{(M,4M-1.4M)} \cdot c^{(M+1,4M-1,4M)} \cdot d^{(M+1,4M-1,4M)}$,
r_5 is the coefficient of $a^{(9M-2,1,1)} \cdot b^{(M,4M,4M)} \cdot c^{(M+1,4M-1,4M)} \cdot d^{(M+1,4M,4M-1)}$,
r_6 is the coefficient of $a^{(9M-3,2,1)} \cdot b^{(M+1,4M-1,4M)} \cdot c^{(M+1,4M-1,4M)} \cdot d^{(M+1,4M,4M-1)}$.

Then each $C_{\boldsymbol{\alpha}}$ is an integral linear combination of $r_1, \ldots, r_6$; we have the relations

$$\boldsymbol{\alpha} = (4, 0, 0)\colon C_{\boldsymbol{\alpha}} = 4r_1. \tag{1}$$

$$\boldsymbol{\alpha} = (3, 1, 0)\colon C_{\boldsymbol{\alpha}} = 4(r_1 + 3r_2). \tag{2}$$

$$\boldsymbol{\alpha} = (2, 2, 0)\colon C_{\boldsymbol{\alpha}} = 12(r_1 + r_3). \tag{3}$$

$$\boldsymbol{\alpha} = (1, 3, 0)\colon C_{\boldsymbol{\alpha}} = 4(3r_3 + r_4). \tag{4}$$

$$\boldsymbol{\alpha} = (0, 4, 0)\colon C_{\boldsymbol{\alpha}} = 4r_4. \tag{5}$$

$$\boldsymbol{\alpha} = (2, 1, 1)\colon C_{\boldsymbol{\alpha}} = 24(r_2 + r_5). \tag{6}$$

$$\boldsymbol{\alpha} = (1, 2, 1)\colon C_{\boldsymbol{\alpha}} = 12(r_3 + 2r_5 + r_6). \tag{7}$$

$$\boldsymbol{\alpha} = (0, 3, 1)\colon C_{\boldsymbol{\alpha}} = 4(r_4 + 3r_6). \tag{8}$$

$$\boldsymbol{\alpha} = (0, 2, 2)\colon C_{\boldsymbol{\alpha}} = 24r_6. \tag{9}$$

The other $C_{\boldsymbol{\alpha}}$ are given by symmetry.

The key to computing the quantities $r_1, \ldots, r_6$ is an identity involving Laguerre polynomials which was pointed out to me by Noam Elkies. Recall that the Laguerre polynomial $L_n^\alpha(x)$ of degree n and index α is

$$L_n^\alpha(x) = \sum_i (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}.$$

The identity in question is

$$L_n^\alpha(x) \cdot L_n^\alpha(y) = \frac{(n+\alpha)!}{n!} \cdot \sum_{k=0}^{n} \frac{L_{n-k}^{\alpha+2k}(x+y)}{(k+\alpha)!} \cdot \frac{x^k y^k}{k!} \tag{*}$$

([5], p. 1089). Differentiating this with respect to y and using the fact that $(L_n^\alpha)' = -L_{n-1}^{\alpha+1}$, we get

$$L_n^\alpha(x) \cdot L_{n-1}^{\alpha+1}(y) = \frac{(n+\alpha)!}{n!} \sum_k \left[\frac{x^k y^k}{k!(k+\alpha)!} L_{n-k-1}^{\alpha+2k+1}(x+y) - \frac{x^k y^{k-1}}{(k-1)!\,(k+\alpha)!} \cdot L_{n-k}^{\alpha+2k}(x+y) \right]. \tag{**}$$

Differentiating once more with respect to y gives

$$L_n^\alpha(x) \cdot L_{n-2}^{\alpha+2}(y) = \frac{(n+\alpha)!}{n!} \sum_k \left[\frac{x^k y^{k-2}}{(k-2)!\,(k+\alpha)!} \cdot L_{n-k}^{\alpha+2k}(x+y) - 2\frac{x^k y^{k-1}}{(k-1)!\,(k+\alpha)!} \cdot L_{n-k-1}^{\alpha+2k+1}(x+y) + \frac{x^k y^k}{k!(k+\alpha)!} L_{n-k-2}^{\alpha+2k+2}(x+y) \right]. \tag{***}$$

We deduce various combinatorial identities from those above by setting $x + y = 0$ and comparing coefficients of powers of x, remembering that $L_n^\alpha(0) = \binom{n+\alpha}{n}$. Then from (*) we get

$$\sum_{i,j} (-1)^i \frac{\binom{n+\alpha}{n-i}\binom{n+\alpha}{n-j}}{i!\,j!} x^{i+j} = \frac{(n+\alpha)!}{n!} \sum_k \frac{(-1)^k \binom{n+\alpha+k}{n-k}}{k!(k+\alpha)!} x^{2k};$$

multiplying through by $r!$ and comparing the coefficients of x^r gives

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i}\binom{n+\alpha}{n-i}\binom{n+\alpha}{n-r+i} = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \dfrac{(n+\alpha)!}{n!} \dfrac{(-1)^k \binom{n+\alpha+k}{n-k} \cdot (2k)!}{k!(k+\alpha)!} & \text{if } r = 2k. \end{cases} \tag{A}$$

From (**), we derive

$$\sum (-1)^i \binom{2k}{i}\binom{n+\alpha}{n-i}\binom{n+\alpha}{n-1-2k+i} = \frac{(-1)^k (n+\alpha)!\,(2k)!\binom{n+\alpha+k}{n-k-1}}{n!\,k!\,(k+\alpha)!} \tag{B}$$

by comparing coefficients of x^{2k}, and

$$\sum (-1)^i \binom{2k-1}{i}\binom{n+\alpha}{n-1}\binom{n+\alpha}{n+i-2k}$$

$$= (-1)^k \frac{(2k-1)!\,(n+\alpha)}{n!}! \cdot \frac{\binom{n+\alpha+k}{n-k}}{(k-1)!\,(k+\alpha)!} \text{ using } x^{2k-1}. \tag{B$'$}$$

From (***), we derive

$$\sum_i (-1)^i \binom{2k}{i}\binom{n+\alpha}{i+\alpha}\binom{n+\alpha}{n-2k+i-2}$$

$$= \frac{(2k)!\,(n+\alpha)!}{n!} \left[\frac{\binom{n+\alpha+k}{n-k-2}}{k!(k+\alpha)!} + \frac{\binom{n+\alpha+k+1}{n-k-1}}{(k-1)!\,(k+\alpha+1)!} \right] \tag{C}$$

by considering the coefficients of x^{2k}; comparing the coefficients of x^{2k-1} yields

$$\sum_i (-1)^i \binom{2k-1}{i}\binom{n+\alpha}{i+\alpha}\binom{n+\alpha}{n-2k+i-1}$$

$$= \frac{-2(2k-1)!\,(n+\alpha)!}{n!} \cdot \frac{\binom{n+k+\alpha}{n-k-1}}{(k-1)!\,(k+\alpha)!}. \tag{C$'$}$$

We shall use these identities, for various values of n, k, α, to compute $r_1, \ldots, r_6$.

Fix some more notation: for any monomial N in the symbols a_1, a_2, a_3, $\ldots$, d_3, let $C(N)$ denote the coefficient of N in Δ. Here need not be a number, if the degree of N is small. Also, let (bc) denote

$$\det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix}.$$

So for example, $C(a_1^{9M}) = (bc)^{3M}(bd)^{3M}(cd)^{3M}(bcd)^{3M}$. Hence $C(a_1^{9M}b_1^M c_1^M d_1^M) = (-1)^M\binom{3M}{M,M,M}(bc)^{4M}(bd)^{4M}(cd)^{4M}$. Expansion of this last expression shows that

$$r_1 = (-1)^M \cdot \binom{3M}{M,M,M} \cdot \sum_i (-1)^i \binom{4M}{i}^3;$$

using (A) with $r = n = 4M$, $\alpha = 0$, we see that

$$r_1 = (-1)^M \cdot \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M}.$$

(Identity (A) in this case is well-known and due to Dixon.)

Rather than expand determinants to evaluate the other r_i, we shall use the fact that Δ is annihilated by the Lie algebra $\mathbf{sl}_3$ (although some expansion, similar to that above, will be required). Put

$$x = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{sl}_3.$$

Recalling that $a_1, \ldots d_3$ are contragredient variables, we have $x(a_1) = a_2$, $x(a_2) = 0$, $x(a_3) = 0$. Since the only monomials Z such that $a_1^{9M-1}a_2b_1^Mb_2^{4M}b_3^{4M}c_1^Mc_2^{4M}c_3^{4M}d_1^Md_2^{4M}d_3^{4M}$ appears in $x(Z)$ are $a_1^{9M}\cdot b_1^M \ldots\ldots d_3^{4M}$, $a_1^{9M-1}a_2b_1^{M+1}b_2^{4M-1}b_3^{4M}c_1^M \ldots\ldots d_3^{4M}$ and the two monomials obtained from the latter by permuting the $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, it follows that $9M\cdot r_1 + 3(M+1)\cdot r_2 = 0$, and so

$$r_2 = -\frac{3M}{(M+1)}\cdot r_1.$$

We next compute r_3. For this, we consider the action of ${}^t x$. The only monomials Z such that $a_1^{9M-1}a_2 b_1^{M+1}b_2^{4M-1}b_3^{4M}c_1^{M+1}c_2^{4M-1}c_3^{4M}d_1^M d_2^{4M}d_3^{4M} = W_1$, say, appears in ${}^t x(Z)$ are

$$a_1^{9M-2}a_2^2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^M d_2^{4M}d_3^{4M} = W_2,$$

$$a_1^{9M-1}a_2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M}c_2^{4M}c_3^{4M} \cdot d_1^M d_2^{4M}d_3^{4M} = W_3,$$

$$a_1^{9M-1}a_2 \cdot b_1^{M}b_2^{4M}b_3^{4M} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^M d_2^{4M}d_3^{4M} = W_4$$

and

$$a_1^{9M-1}a_2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^{M-1} d_2^{4M+1}d_3^{4M} = W_5.$$

We have $C(W_2) = r_3$, $C(W_3) = C(W_4) = r_2$, and

$$2 . C(W_2) + 4M \cdot C(W_3) + 4M \cdot C(W_4) + (4M+1) \cdot C(W_5) = 0.$$

The only monomials differentiating under t_x to

$a_1^{9M} \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M}c_2^{4M}c_3^{4M} \cdot d_1^M d_2^{4M}d_3^{4M}$ are W_5,

$$a_1^{9M} \cdot b_1^{M}b_2^{4M}b_3^{4M} \cdot c_1^{M}c_2^{4M}c_3 \cdot {}^{4M} d_1^M d_2^{4M}d_3^{4M} = W_0,$$

$$a_1^{9M} \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M-1}c_2^{4M+1}c_3^{4M} \cdot d_1^M d_2^{4M}d_3^{4M} = W_6 \text{ and}$$

$$a_1^{9M} \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M}c_2^{4M}c_3^{4M} \cdot d_1^{M-1} d_2^{4M+1}d_3^{4M} = W_7.$$

Note that $C(W_0) = r_1$ and $C(W_6) = C(W_7)$. Expansion of the determinants show that

$$C(W_6) = (-1)^{M-1}\binom{3M}{M+1, M-1, M} \cdot \sum_i (-1)^i \binom{4M}{i}\binom{4M-1}{i-1}\binom{4M+1}{i}.$$

Now

$$\sum_i (-1)^i \binom{4M}{i}\binom{4M-1}{i-1}\binom{4M+1}{i}$$

$$= \sum_i (-1)^i \binom{4M-1}{i-1}\binom{4M}{i}^2 + \sum (-1)^i \binom{4M-1}{i-1}\binom{4M}{i-1}\binom{4M}{i}$$

$$= \sum (-1)^i \binom{4M}{i}^3 - \sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}^2 + \sum (-1)^i \binom{4M-1}{i-1}\binom{4M}{i-1}\binom{4M}{i},$$

and

$$\sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}^2 = \frac{(4M-1)!}{(2M-1)!(2M)!}\cdot\binom{6M}{2M} = \tfrac{1}{2}\binom{6M}{2M,2M,2M},$$

applying identity (B′) with $k = 2M$, $n = 2k$, $\alpha = 0$. Also $\Sigma(-1)^i\binom{4M-1}{i-1}\binom{4M}{i-1}\binom{4M}{i} = -\Sigma(-1)^i\binom{4M-1}{i}\binom{4M}{i}\binom{4M}{i+1} = 0$, by identity (A) with $n = 4M$, $\alpha = 0$, $r = 4M-1$. Hence $C(W_6) = (-1)^{M-1}\binom{3M}{M+1,M-1,M}\cdot\frac{1}{2}\binom{6M}{2M,2M,2M}$. Also, the action of ${}^t x$ shows that

$$C(W_5) + 4M\cdot C(W_0) + (4M+1)\cdot C(W_6) + (4M+1)\cdot C(W_7) = 0.$$

So

$$\begin{aligned} C(W_5) &= 4M\cdot(-1)^{M-1}\binom{3M}{M,M,M}\cdot\binom{6M}{2M,2M,2M} \\ &\quad - (4M+1)(-1)^{M-1}\binom{3M}{M-1,M,M+1}\binom{6M}{2M,2M,2M} \\ &= (-1)^M\binom{3M}{M,M,M}\binom{6M}{2M,2M,2M}\left[\frac{(4M+1)}{M+1} - 4M\right], \end{aligned}$$

and so since $2r_3 + 8M\cdot r_2 + (4M+1)\cdot C(W_5) = 0$, we get

$$\begin{aligned} r_3 &= (-1)^M\binom{3M}{M,M,M}\cdot\binom{6M}{2M,2M,2M}\cdot \\ &\quad\times\left[\frac{12M^2}{M+1} + (4M+1)\cdot\frac{(4M(M+1) - M(4M+1)}{M+1}\right]. \end{aligned}$$

So

$$\sigma_1(M) = 1,\quad \sigma_2(M) = \frac{-3M}{M+1},\quad \sigma_3(M) = \frac{24M^2+3M}{M+1}.$$

We next consider r_4. Let W_8 denote the given monomial whose coefficient is r_4; then ${}^t x(W_8)$ contains

$$a_1^{9M-2}a_2^2\cdot b_1^{M+1}b_2^{4M-1}b_3^{4M}\cdot c_1^{M+1}c_2^{4M-1}c_3^{4M}\cdot d_1^{M+1}d_2^{4M-1}d_3^{4M}.$$

The other monomials whose images under t_x contain this are W_2 and the two other monomials obtained from W_2 by permuting the vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

Then $3 \cdot r_4 = 3 \cdot 4M \cdot C(W_2)$, and $r_4 = -4M . r_3$; hence

$$r_4 = (-1)^M \cdot \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \frac{(-4M).(24M^2 + 3M)}{(M+1)}.$$

Now consider r_5. Let

$$y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in sl_3,$$

and let W_9 denote the given monomial whose coefficient is r_5. Then the monomials Z for which $a_1^{9M-1}a_2 \cdot b_1^M b_2^{4M} b_3^{4M} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} . d_1^{M+1} d_2^{4M} d_3^{4M-1}$ appears in $y(Z)$ are W_9, W_4,

$$a_1^{9M-1}a_2 \cdot b_1^{M-1} b_2^{4M} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M+1} d_2^{4M} d_3^{4M-1} = W_{10}$$

and

$$a_1^{9M-1}a_2 \cdot b_1^{M} b_2^{4M} b_3^{4M} \cdot c_1^{M} c_2^{4M-1} c_3^{4M+1} \cdot d_1^{M+1} d_2^{4M} d_3^{4M-1} = W_{11}$$

We must find $C(W_{10})$ and $C(W_{11})$.
The monomials Z for which

$$a_1^{9M} \cdot b_1^{M-1} b_2^{4M} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M+1} d_2^{4M} d_3^{4M-1}$$

appears in ${}^t x(Z)$ are W_{10},

$$a_1^{9M} \cdot b_1^{M-2} b_2^{4M+1} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M+1} d_2^{4M} d_3^{4M-1} = W_{12},$$

$$a_1^{9M} \cdot b_1^{M-1} b_2^{4M} b_3^{4M+1} \cdot c_1^{M} c_2^{4M} c_3^{4M} \cdot d_1^{M} d_2^{4M+1} d_3^{4M-1} = W_{13}.$$

and

$$a_1^{9M} \cdot b_1^{M-1} b_2^{4M} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M} d_2^{4M+1} d_3^{4M-1} = W_{14}.$$

Expansion of the determinants shows that

$$C(W_{12}) = (-1)^{M+1} \binom{3M}{M-2,M+1,M+1} \cdot \left(-\sum (-1)^i \binom{4M-2}{i} \binom{4M+1}{i+1}^2 \right)$$

$$= (-1)^M \cdot \binom{3M}{M-2,M+1,M+1} \cdot (-1)^{2M-1} \frac{(4M+1)!\,(4M-2)!}{(4M)!\,(2M-1)!\,(2M)!} \cdot \binom{6M}{2M},$$

using identity (B) with

$$\alpha = 1, \quad n = 2k + 2, \quad k = 2M - 1,$$

$$= -(-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \left[\frac{M(M-1)(4M+1)}{2(M+1)^2(4M-1)}\right].$$

Similarly,

$$C(W_{13}) = (-1)^M \binom{3M}{M-1,M,M+1} \cdot \left[-\sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}\binom{4M-1}{i+1}\right]$$

$$= (-1)^M \binom{3M}{M-1,M,M+1} \cdot \sum (-1)^i \binom{4M-1}{i}^2 \binom{4M}{i}$$

$$= (-1)^M \binom{3M}{M-1,M,M+1} \cdot \sum (-1)^i \binom{4M-1}{i}^2 \binom{4M-1}{i-1}$$

$$= (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \frac{M}{6(M+1)},$$

Using (B′) with $k = 2M$, $n = 2k - 1$, $\alpha = 0$. Also,

$$C(W_{14}) = (-1)^{M+1} \binom{3M}{M-1,M+1,M} \cdot \left[-\sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}\binom{4M+1}{i}\right]$$

$$= (-1)^M \binom{3M}{M-1,M+1,M} \cdot \left[\sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}^2\right.$$

$$\left. + \sum (-1)^i \binom{4M-1}{i}\binom{4M}{i-1}\binom{4M}{i}\right].$$

The first sum is $\frac{1}{2}\binom{6M}{2M,2M,2M}$ as above, while the second is $-2 \cdot \binom{4M-1}{2M} \cdot \binom{6M}{4M+1}$ by identity (C′) with $n = 2k$, $k = 2M$, $\alpha = 0$. So

$$C(W_{14}) = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \frac{M(2M+1)}{2(M+1)(4M-1)}.$$

Since

$$C(W_{10}) + (4M+1)\cdot C(W_{12}) + 4M\cdot C(W_{13}) + (4M+1)\cdot C(W_{14}) = 0,$$

we get

$$C(W_{10}) = (-1)^M \binom{3M}{M,M,M}\cdot\binom{6M}{2M,2M,2M}\cdot\left[\frac{M(8M^3-78M^2-38M-6)}{6(M+1)^2(4M-1)}\right].$$

The monomials Z for which ${}^t x(Z)$ contains

$a_1^{9M}\cdot b_1^M b_2^{4M} b_3^{4M}\cdot c_1^M c_2^{4M-1} c_3^{4M+1}\cdot d_1^{M+1} d_2^{4M} d_3^{4M-1} = W_{11}$ are

$a_1^{9M}\cdot b_1^M b_2^{4M} b_3^{4M}\cdot c_1^{M-1} c_2^{4M} c_3^{4M+1}\cdot d_1^{M+1} d_2^{4M} d_3^{M-1} = W_{15}$,

$a_1^{9M}\cdot b_1^M b_2^{4M} b_3^{4M}\cdot c_1^M c_2^{4M-1} c_3^{4M+1}\cdot d_1^M d_2^{4M+1} d_3^{4M-1} = W_{16}$ and

$a_1^{9M}\cdot b_1^{M-1} b_2^{4M+1} b_3^{4M}\cdot c_1^M c_2^{4M-1} c_3^{4M+1}\cdot d_1^{M+1} d_2^{4M} d_3^{4M-1} = W_{17}$.

Again expanding we see that

$$C(W_{15}) = (-1)^{M-1}\binom{3M}{M,M-1,M+1}\cdot\left[\sum(-1)^i\binom{4M-1}{i-1}\binom{4M}{i}\binom{4M+1}{i}\right]$$

$$= (-1)^M\binom{3M}{M,M-1,M+1}\cdot\left[\sum(-1)^i\binom{4M-1}{i}\binom{4M}{i}\binom{4M}{i+1}\right.$$

$$\left. + \sum(-1)^i\binom{4M-1}{i}\binom{4M}{i+1}^2\right].$$

The first sum is zero. Denote the second sum by Y; then

$$Y + \sum(-1)^i\binom{4M-1}{i+1}\binom{4M}{i+1}^2 = \sum(-1)^i\binom{4M}{i+1}^3 = -\binom{6M}{2M,2M,2M}.$$

Also

$$Y = \sum(-1)^i\binom{4M-1}{4M-1-i}\binom{4M}{4M-i-1}^2 = \sum(-1)^i\binom{4M-1}{i+1}\binom{4M}{i+1}^2.$$

So $Y = -\frac{1}{2}\binom{6M}{2M,2M,2M}$, and so

$$C(W_{15}) = (-1)^M\binom{3M}{M,M,M}\cdot\binom{6M}{2M,2M,2M}\cdot\left[\frac{-M}{2(M+1)}\right].$$

Similarly,

$$C(W_{16}) = (-1)^M \binom{3M}{M,M,M} \cdot \sum (-1)^i \binom{4M}{i} \binom{4M}{i-1}^2.$$

Applying identity (C) with $n = 2k + 1$, $\alpha = -1$, $k = 2M$, we see that

$$C(W_{16}) = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \left[\frac{2M(8M+1)}{(4M+1)^2} \right].$$

Since

$$C(W_{11}) + 4M \cdot C(W_{15}) + (4M+1) \cdot C(W_{16})$$
$$+ (4M+1) \cdot C(W_{17}) = 0,$$

we have

$$C(W_{11}) = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \left[\frac{-M(24M^2 + 38M + 5)}{2(M+1)(4M+1)} \right].$$

Now we can compute $r_5 = C(W_9)$, using the fact (coming from the y-invariance of Δ) that

$$C(W_9) + 4M \cdot C(W_4) + (4M+1) \cdot C(W_{10}) + (4M+1) \cdot C(W_{11}) = 0.$$

So

$$r_5 = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M}$$
$$\times \left[\frac{M(256M^4 + 1258M^3 + 776M^2 - 79M - 9)}{(M+1)^2(4M-1)} \right].$$

Finally, we consider r_6. Let W_{18} denote the given monomial whose coefficient is r_6. Then the monomials Z such that $y(Z)$ contains

$$a_1^{9M-2}a_2^2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^{M+1}d_2^{4M}d_3^{4M-1} \text{ are } W_{18},$$

$$a_1^{9M-2}a^2 \cdot b_1^{M}b_2^{4M-1}b_3^{4M+1} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^{M+1}d_2^{4M}d_3^{4M-1} = W_{19},$$

$$a_1^{9M-2}a_2^2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M}c_2^{4M-1}c_3^{4M+1} \cdot d_1^{M+1}d_2^{4M}d_3^{4M-1} = W_{20}$$

and

$$a_1^{9M-2}a_2^2 \cdot b_1^{M+1}b_2^{4M-1}b_3^{4M} \cdot c_1^{M+1}c_2^{4M-1}c_3^{4M} \cdot d_1^{M}d_2^{4M}d_3^{4M} = W_{21}.$$

Note that $C(W_{20}) = C(W_{21}) = r_3$. Next, notice that the monomials Z for which

$$a_1^{9M-1}a_2 \cdot b_1^M b_2^{4M-1} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M+1} d_2^{4M} d_3^{4M-1}$$

appears in ${}^t x(Z)$ are W_{19}, W_{10},

$$a_1^{9M-1}a_2 \cdot b_1^M b_2^{4M-1} b_3^{4M+1} \cdot c_1^M c_2^{4M} c_3^{4M} \cdot d_1^{M+1} d_2^{4M-1} d_3^{4M+1} = W_{22}$$

and

$$a_1^{9M-1}a_2 \cdot b_1^M b_2^{4M} b_3^{4M} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^M d_2^{4M+1} d_3^{4M-1} = W_{23}.$$

Note that $C(W_{22}) = C(W_{11})$.

The monomials Z such that ${}^t x(Z)$ contains

$$a_1^{9M} \cdot b_1^M b_2^{4M-1} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^M d_2^{4M+1} d_3^{4M-1} \text{ are } W_{23}, W_{14},$$

$$a_1^{9M} \cdot b_1^M b_2^{4M-1} b_3^{4M+1} \cdot c_1^M c_2^{4M} c_3^{4M} \cdot d_1^M d_2^{4M+1} d_3^{4M-1} = W_{24}$$

and

$$a_1^{9M} \cdot b_1^M b_2^{4M-1} b_3^{4M+1} \cdot c_1^{M+1} c_2^{4M-1} c_3^{4M} \cdot d_1^{M-1} d_2^{4M+2} d_3^{4M-1} = W_{25}.$$

Note that $C(W_{24}) = C(W_{16})$, while expanding the determinant shows that

$$\begin{aligned} C(W_{25}) &= (-1)^{M+1}\binom{3M}{M,M+1,M-1} \cdot \sum (-1)^i \binom{4M}{i}\binom{4M+1}{i-1}\binom{4M-1}{i-1} \\ &= (-1)^M \binom{3M}{M,M+1,M-1} \cdot \Bigg[\sum (-1)^i \binom{4M-1}{i}\binom{4M}{i+1}\binom{4M}{i-1} \\ &\qquad + \sum (-1)^i \binom{4M-1}{i}\binom{4M}{i}\binom{4M}{i+1} \Bigg]. \end{aligned}$$

The second sum vanishes, while the first is

$$\frac{(4M)!\binom{6M}{2M-1}}{(2M-1)!(2M+1)!},$$

by applying identity (B′) with $n = 2k-1$, $k = 2M$, $\alpha = 1$, and so

$$C(W_{25}) = (-1)^M \cdot \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \frac{4M^3}{(M+1)(2M+1)(4M+1)}.$$

Now

$$C(W_{23}) + 4M \cdot C(W_{14}) + 4M \cdot C(W_{24}) + (4M+2) \cdot C(W_{25}) = 0,$$

and so

$$C(W_{23}) = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \left[\frac{-(110M^4 + 92M^3 + 10M^2)}{(M+1)(4M+1)^2}\right].$$

Next, we compute $C(W_{19})$ from the equation

$$2 . C(W_{19}) + 4M \cdot C(W_{10}) + 4M \cdot C(W_{22}) + (4M+1) \cdot C(W_{23}) = 0;$$

we get

$$C(W_{19}) = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M}$$
$$\cdot \left[\frac{M^2(1448M^4 + 3150M^3 + 1470M^2 - 204M - 52)}{6(M+1)^2(4M-1)(4M+1)}\right]$$

Then from the equation

$$C(W_{18}) + (4M+1) \cdot C(W_{19}) + (4M+1) \cdot C(W_{20}) + 4M \cdot C(W_{21}) = 0,$$

we deduce that

$$r_6 + (8M+1) \cdot r_3 + (4M+1) \cdot C(W_{19}) = 0,$$

and so

$$r_6 = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M}$$
$$\times \left[\frac{-(1448M^6 + 7758M^5 + 6078M^4 + 420M^3 - 286M^2 - 18M)}{6(M+1)^2(4M-1)}\right].$$

Bearing in mind the relations between the coefficients C_α and $r_1, \ldots, r_6$, we have proved that indeed every $C_\alpha = (-1)^M \binom{3M}{M,M,M} \cdot \binom{6M}{2M,2M,2M} \cdot \sigma_\alpha(M)$, where each σ_α is a rational function which it remains to determine.

$$\alpha = (4, 0, 0): \sigma_\alpha(M) = 4. \tag{1}$$

$$\alpha = (3, 1, 0): \sigma_\alpha(M) = 4\left(1 - \frac{9M}{M+1}\right), \tag{2}$$

$$\alpha = (2, 2, 0): \sigma_\alpha(M) = \frac{12(24M^2 + 4M + 1)}{M+1} \tag{3}$$

$$\alpha = (1, 3, 0): \sigma_\alpha(M) = 4 \cdot \frac{(24M^2 + 3M)(3 - 4M)}{M+1} \tag{4}$$

$$\alpha = (0, 4, 0): \sigma_\alpha(M) = \frac{-16M(24M^2 + 3M)}{M+1} \tag{5}$$

$$\alpha = (2, 1, 1): \sigma_\alpha(M) = \frac{24(256M^5 + 1258M^4 + 764M^3 - 88M^2 + 12M)}{(M+1)^2(4M-1)} \tag{6}$$

$$\alpha = (1, 2, 1): \sigma_\alpha(M)$$

$$= \frac{(-1448M^6 - 4686M^5 - 9594M^4 + 10236M^3 - 752M^2 - 108M)}{6(M+1)^2(4M-1)} \tag{7}$$

$$\alpha = (1, 2, 1): \sigma_\alpha(M)$$

$$= \frac{-4(4344M^6 + 10062M^5 + 20248M^4 - 1620M^3 - 930M^2 - 54M)}{6(M+1)^2(4M-1)} \tag{8}$$

$$\alpha = (0, 2, 2): \sigma_\alpha(M)$$

$$= \frac{-4(1448M^6 + 7758M^5 + 6078M^4 - 420M^3 - 286M^2 - 18M)}{(M+1)(4M-1)} \tag{9}$$

It is clear that none of these functions vanishes for any integral value of $M \geqslant 2$. This completes the proof of Proposition 6.

REMARK. For hypersurfaces of degree D in $\mathbf{P}^n$, we reduce the question of whether the quotient is rational to a problem of showing that certain coefficients do not vanish, provided that $(D, n + 1) = 1$ and D is sufficiently large. However, the coefficients seem to be much harder to compute; for example, for surfaces of degree $16M + 1$ in $\mathbf{P}^3$, the most accessible coefficient required is that of

$$(b_2b_3b_4 \cdot c_2c_3c_4 \cdot d_2d_3d_4 \cdot e_2e_3e_4)^{5M} \text{ in } (bcd)^{5M}(bce)^{5M}(bde)^{5M}(cde)^{5M}, \text{ where}$$

$$(bcd) = \det\begin{pmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{pmatrix}, \text{ etc.}$$

On the other hand, if D is not prime to $n + 1$, then the argument breaks down, even for binary forms of even degree (this latter problem has been solved by Katsylo and Bogomolov [3], using a different idea) since whatever covariant $V(D) \to W$ we construct, the map $\mathbf{P}(W) \to \mathbf{P}(W)/SL$ is *not* ruled. Using other techniques, however, we can also prove the following result.

THEOREM 7. *The quotient* $\mathbf{P}(V_d)/SL_3$ *is rational if* $d \equiv 1(4)$.

The proof will be based upon the following key result.

PROPOSITION 8. *Suppose that W is an odd-dimensional representation of the reductive group G, and that there is a finite subgroup Z of the centre of G so that $\bar{G} = G/Z$ acts generically freely on $\mathbf{P}(W)$. Suppose that V is a G-subspace of $\wedge^2 W^\vee$ such that $\bar{G}$ acts on $\mathbf{P}(V)$, $\dim V \geqslant \dim W$ and such that given any $w \in W$, there is an element v of V such that $v(w) = 0$ and v has maximal rank. Assume that there is a line bundle $\mathscr{L}$ on $\mathbf{P}(V) \times \mathbf{P}(W)$ that is $\bar{G}$-linearized and that cuts out $\mathcal{O}(1)$ on the fibres of the projection onto $\mathbf{P}(W)$. Then $\mathbf{P}(V)/G \sim \mathbf{P}(W)/G \times \mathbf{P}^r$, where $r = \dim V - \dim W$.*

Proof. Since $\dim W$ is odd, there is a G-equivariant rational map π: $\mathbf{P}(\wedge^2 W^\vee) \to \mathbf{P}(W)$ given by associating to each 2-form its kernel. The locus of indeterminancy of π is the set of forms whose rank is not maximal, and the fibre $\pi^{-1}(w)$ can be identified with $\mathbf{P}(\wedge^2(W/\langle w \rangle)^\vee)$. After blowing up, π is a G-equivariant projective bundle. By hypothesis, π induces a rational dominant mapping σ: $\mathbf{P}(V) \to \mathbf{P}(W)$ that is generically a projective bundle. By restricting the bundle $\mathscr{L}$ to the graph of σ, descending the result to $\mathbf{P}(V)/\bar{G}$ and using Theorem 1 as before, we see that $\mathbf{P}(V)/\bar{G}$ is generically a projective bundle over $\mathbf{P}(W)/\bar{G}$, as required.

COROLLARY 9. *Suppose that G is a symmetric group S_{2n} and that $V \cong \mathbf{C}^{2n-1}$ is the representation of G as the Weyl group $W(A_{2n-1})$. Then the quotient $\mathbf{P}(\wedge^2 V)/G$ is rational.*

Before proceding with the proof of Theorem 7, we shall recall how to give a symbolical description of other representation of SL_3, and how to use this description to given explicit decompositions of various tensor spaces.

The irreducible representations of SL_3 will be denoted by $V(p, q)$, corresponding to the diagram ${}^{p}\bullet\!\!-\!\!\bullet^{q}$, where p, q are non-negative integers. Here, $V(p, 0) = V_p$, the space of ternary forms of degree p, and $V(0, p) = V_p^\vee$. We can describe $V(p, q)$ as a subspace of $V(p, 0) \otimes V(0, q)$ in symbolical terms, as follows:

$$V(p, q) = \{a_x^p \otimes u_A^q \,|\, a_A \cdot a_x^{p-1} \otimes u_A^{q-1} = 0\},$$

where x_1, x_2, x_3 are cogredient variables, u_1, u_2, u_3 are contragredient variables and $u_A = A_1 u_1 + A_2 u_2 + A_3 u_3$. In non-symbolical terms, we can write an element f of $V(p, q)$ as

$$f = \sum \binom{p}{i_1, i_2, i_3} \cdot \binom{q}{j_1, j_2, j_3} \cdot \alpha_{i,j} \boldsymbol{x}^i \otimes \boldsymbol{u}^j,$$

where $\alpha_{i,j} = a^i \cdot A^j$, and the coefficients $\alpha_{i,j}$ satisfy the linear relations implied by the condition that $a_A \cdot a_x^{p-1} \otimes u_A^{q-1} = 0$. We can interpret $V(p, q)$ as the space of sections of a line bundle $\mathcal{O}_F(p, q)$ over the flag variety $F = \{(x, l) \,|\, x \in \mathbf{P}^2,\ \ell \in (\mathbf{P}^2)^\vee \text{ and } x \in l\}$, as predicted by the Borel-Weil theorem, as follows:

F is a divisor of bidegree $(1, 1)$ on $\mathbf{P}^2 \times (\mathbf{P}^2)^\vee$. Let pr_1, pr_2 be the projections of $\mathbf{P}^2 \times (\mathbf{P}^2)^\vee$ onto its factors, and put $\mathcal{O}(p, q) = \mathrm{pr}_1^* \mathcal{O}(p) \otimes \mathrm{pr}_2^* \mathcal{O}(q)$. Then from the exact sequence $0 \to \mathcal{O}(-1, -1) \to \mathcal{O} \to \mathcal{O}_F \to 0$, we get $0 \to \mathcal{O}(p-1, q-1) \to \mathcal{O}(p, q) \to \mathcal{O}_F(p, q) \to 0$; then the symbolical description above gives an explicit splitting of the map $V(p, 0) \otimes V(0, q) = H^0(\mathcal{O}(p, q)) \to H^0(\mathcal{O}_F(p, q)) = V(p, q)$. From this description, we see that $\dim V(p, q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$.

Proof of Theorem 7. Set $W = V(1, 2n)$, so that $\dim W = (2n+1) \times (2n+3) = N$, say. There is an embedding of $V = V(4n+1, 0)$, the space of ternary $(4n+1)$-ics, into $\wedge^2 W^\vee$, given symbolically by the formula $a_x^{4n+1}(b_x \otimes u_B^{2n},\ c_x \otimes u_C^{2n}) = (abc) a_B^{2n} a_C^{2n}$. To prove the theorem, it will be enough to check that the hypotheses of Proposition 8 hold for this embedding.

There is a G-equivariant embedding $F \hookrightarrow \mathbf{P}^{N-1} = \mathbf{P}(V(1, 2n)^\vee)$, so that $\mathcal{O}_F(1) = \mathcal{O}_F(1, 2n)$. Letting pr_1, p_2 denote the projections of F onto $\mathbf{P}^2$

and $(\mathbf{P}^2)^\vee$, we see that under pr_2 F is embedded as a scroll over $(\mathbf{P}^2)^\vee$; i.e. the fibres of pr_2 are embedded as lines. Let Gr denote the Grassmannian of lines in $\mathbf{P}^{N-1}$, and let $X \subset Gr$ be the subvariety corresponding to the fibres of pr_2. Clearly $X \cong \mathbf{P}^2$, and we wish to determine its degree via the Plücker embedding $X \hookrightarrow Gr \hookrightarrow \mathbf{P}^{p-1}$, where $p = \binom{N}{2}$.

Put $\mathscr{L} = \mathrm{pr}_1^*\mathcal{O}(1)$, $\mathscr{M} = \mathrm{pr}_2^*\mathcal{O}(1)$, so that $\mathcal{O}_F(1) = \mathcal{O}(\mathscr{L} + 2n\mathscr{M})$. Note that $\mathscr{L}^3 = \mathscr{M}^3 = 0$, and $\mathscr{L}^2 \cdot \mathscr{M} = \mathscr{L} \cdot \mathscr{M}^2 = 1$.

Put $A = \{l \in Gr \mid l \text{ lies in a given hyperplane}\}$,
$B = \{l \in Gr \mid l \text{ meets a given linear space of codimension } 3\}$,
$C = \{l \in Gr \mid l \text{ meets a given linear space of codimension } 2\}$,

Then C is a hyperplane section of Gr in its Plücker embedding, and $C^2 = A + B$. Note that $\deg X = X \cdot C^2 = X \cdot A + X \cdot B$.

Lemma 10. $X \cdot B = 12n^2 + 6n$.

Proof. $X \cdot B = \deg F$, and

$$\deg F = (\mathscr{L} + 2n\mathscr{M})^3 = 3\mathscr{L}^2 \cdot 2n\mathscr{M} + 3L \cdot 4n^2\mathscr{M}^2 = 12n^2 + 6n.$$

Lemma 11. $X \cdot A = 4n^2 + 2n + 1$.

Proof. Let $S \in |F \cap H|$ be a smooth hyperplane section of F. Then $X \cdot A$ is the number of fibres of pr_2 that lie in H, which is the number of exceptional curves contracted by the birational morphism $\mathrm{pr}_2 \colon S \to \mathbf{P}^2$. This number in turn is $9 - K_S^2$. By the adjunction formula, $\mathcal{O}(K_S) = \mathcal{O}_S(-\mathscr{L} + (2n-2)\mathscr{M})$, and so $K_S^2 = (-\mathscr{L} + (2n-2)\mathscr{M})^2 \cdot (\mathscr{L} + 2n\mathscr{M}) = -4n^2 - 2n + 8$. The lemma follows.

It follows from these two lemmas that

$$\deg X = 12n^2 + 6n + 4n^2 + 2n + 1 = (4n+1)^2.$$

Since the embedding $X \hookrightarrow Gr \hookrightarrow \mathbf{P}^{p-1}$ is G-equivariant, X must be embedded by a complete linear system, and so is embedded by $H^0(\mathcal{O}_{\mathbf{P}^2}(4n+1))$. In other words, the linear span of X in $\mathbf{P}^{p-1}$ must be $\mathbf{P}(V_{4n+1})$. We shall use this description and the connection with the projective geometry of F to prove that the hypotheses of Proposition 8 do apply to the embedding $V \hookrightarrow \wedge^2 W^\vee$. For this, we shall need two lemmas.

Lemma 12. *Say $N = 2k + 1$, where as above $N = (2n+1)(2n+3)$. Suppose that $p_1, \ldots, p_k$ are generic points on X, corresponding respectively to lines $L_1, \ldots, L_k$ in $\mathbf{P}^{N-1}$ and to 2-forms $x_1, \ldots, x_k$. Then the linear span*

$\langle L_1, \ldots, L_k \rangle$ is a hyperplane $\mathbf{P}^{N-2}$ in $\mathbf{P}^{N-1}$, or equivalently for generic scalars $\lambda_1, \ldots, \lambda_k$, the 2-form $\lambda_1 x_1 + \cdots + \lambda_k x_k$ *is of maximal rank.*

Proof. Suppose in fact that there is a positive integer $r \leqslant k - 1$ such that given generic $p_1, \ldots, p_{r+1}$ as above, the linear span $\langle L_1, \ldots, L_r \rangle$ is $(2r-1)$-dimensional, but $\langle L_1, \ldots, L_{r+1} \rangle$ is at most $2r$-dimensional. If $\dim \langle L_1, \ldots, L_{r+1} \rangle \leqslant 2r - 1$, then given $L_1, \ldots, L_r$, every line L_{r+1} lies in $\langle L_1, \ldots, L_r \rangle$, which contradicts the irreducibility of the G-action on $\mathbf{P}^{N-1}$. So we can suppose that $\dim \langle L_1, \ldots, L_{r+1} \rangle = 2r$; i.e., that L_{r+1} meets $\langle L_1, \ldots, L_{r+1} \rangle$ in a point.

Now suppose that $L_1, \ldots, L_{r+1}, M$ are lines corresponding to generic points of X. Put $\Pi = \langle L_1, \ldots, L_r \rangle$, $\Pi' = \langle L_1, \ldots, L_{r-1}, L_{r+1} \rangle$; both Π and Π' are $(2r-1)$-dimensional. We know that M meets both Π and Π', and so either M is contained in $\langle L_1, \ldots, L_{r+1} \rangle$ or M meets $\Pi \cap \Pi'$. Hence either every line corresponding to a point of X lies in a fixed $\mathbf{P}^{2r}$, or every line meets a fixed $\mathbf{P}^{2r-2}$; both of these, however, contradict the irreducibility of the G-action, and the Lemma follows.

LEMMA 13. *Given a generic $w \in W$, there are points $p_1, \ldots, p_k \in X$ corresponding respectively to 2-forms $x_1, \ldots, x_k$ such that for generic scalars $\lambda_1, \ldots, \lambda_k$, the 2-form $\lambda_1 x_1 + \cdots + \lambda_k x_k$ has maximal rank and $x_i(w) = 0$ for all i.*

Proof. By the previous Lemma, there is a rational map $\varrho\colon X^k = X \times \cdots \times X \to \mathbf{P}(W)$ given by $\varrho(p_1, \ldots, p_k) = \ker(x_1) \cap \ldots \cap \ker(x_k)$, where x_i corresponds to p_i. We need to show that ϱ is dominant.

Suppose that ϱ is not dominant. Then given generic $p_1, \ldots, p_k$ in X, there is a one-parameter deformation $\{(p_{1,t}, \ldots, p_{k,t})\}_{t \in \Gamma}$ of $(p_1, \ldots, p_k)$ such that for all $t \in \Gamma$, $\varrho(p_{1,t}, \ldots, p_{k,t}) = \varrho(p_1, \ldots, p_k)$. I.e. if L_i, $L_{i,t}$ are the lines corresponding to p_i, $p_{i,t}$, then the linear spans $\langle L_1, \ldots, L_k \rangle$, $\langle L_{1,t}, \ldots, L_{k,t} \rangle$ are all equal. Denote this common linear space by H, a hyperplane. Then $F \cap H$ contains a one-dimensional family of lines, parametrized by a cover of Γ. Denote the total space of this family by Δ. Then $\mathcal{O}_F(\Delta) \cong \mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}^2}(d)$, some d. Also, $\mathcal{O}_F(1) \cong \mathrm{pr}_1^* \mathcal{O}(1) \otimes \mathrm{pr}_2^* \mathcal{O}(2n)$, and so $d \leqslant 2n$. So we have k generic points $p_1, \ldots, p_k$ on $\mathbf{P}^2$ and a curve of degree $d \leqslant 2n$ through them. However, $k = 2n^2 + 4n + 1$, while $h^0(\mathcal{O}_{\mathbf{P}^2}(2n)) = 2n^2 + 3n + 1$; so this is impossible, and the Lemma is proved.

Completion of proof of Theorem 7. Let $Y \subseteq \mathbf{P}(\wedge^2 W^\vee)$ be the subvariety swept out by the linear spans $\langle L_1, \ldots, L_k \rangle$, as $L_1, \ldots, L_k$ run over the lines corresponding to points $p_1, \ldots, p_k$ of X. Since $\mathbf{P}(V)$ is the linear span of X, we see that $Y \subseteq \mathbf{P}(V)$, while by the previous two Lemmas the generic point of Y corresponds to a 2-form of maximal rank, and the rational map

$\mathbf{P}(\wedge^2 W^\vee) \to \mathbf{P}(W)$ is dominant when restricted to Y, and so a fortiori when restricted to $\mathbf{P}(V)$.

Next, we need a sheaf $\mathscr{L}$. For this, let α: $\mathbf{P}(V) \times \mathbf{P}(W) \to \mathbf{P}(V)$, β: $\mathbf{P}(V) \times \mathbf{P}(W) \to \mathbf{P}(W)$ be the projections. Then $\alpha^*\mathcal{O}(1) \otimes \beta^*\mathcal{O}(2)$ is PGL_3-linearized and cuts out $\mathcal{O}(1)$ on the fibres of β. It is known that PGL_3 acts generically freely on $\mathbf{P}(W)$, and so by Proposition 8 we have $\mathbf{P}(V)/G \sim \mathbf{P}(W)/G \times \mathbf{P}^r$, where $r = \dim V - \dim W = 2n(2n + 1)$. To complete the proof, we must consider two cases separately.

(i) $4n + 1 \equiv 0(3)$. Then $V(4n + 1, 0)$ and $V(1, 2n)$ are representations of PGL_3. Let $\boldsymbol{g}$ denote the adjoint representation of PGL_3; then PGL_3 acts generically freely on $\mathbf{P}(\boldsymbol{g} \oplus \boldsymbol{g})$ and the quotient $\mathbf{P}(\boldsymbol{g} \oplus \boldsymbol{g})/PGL_3$ is rational, and so $(\mathbf{P}(W)/PGL_3) \times \mathbf{P}(\boldsymbol{g} \oplus \boldsymbol{g})$ is rational. Since $\dim \mathbf{P}(\boldsymbol{g} \oplus \boldsymbol{g}) = 15$ and $2n(2n + 1) \geqslant 20$, it follows that $\mathbf{P}(V)/G$ is rational.

(ii) $4n + 1 \not\equiv 0(3)$. Let F again denote the flag variety with projections pr_1, pr_2: $F \to \mathbf{P}^2$. Put $\mathscr{L} = \mathrm{pr}_1^*\mathcal{O}(1)$. Consider $\bar{G} = PGL_3$ acting on $F \times \mathbf{P}(W)$; let α: $F \times \mathbf{P}(W) \to F$ and β: $F \times \mathbf{P}(W) \to \mathbf{P}(W)$ denote the projections. Then $\alpha^*\mathscr{L} \otimes \beta^*\mathcal{O}(i)$, where $i = 1$ or 2 according to whether $4n + 1 \equiv 1$ or $2(3)$, is $\bar{G}$-linearized and descends to a line bundle that cuts out $\mathcal{O}(1)$ on the fibres of $(F \times \mathbf{P}(W))/\bar{G} \to (\mathbf{P}^2 \times \mathbf{P}(W))/\bar{G}$, the map induced from pr_2: $F \to \mathbf{P}^2$. Hence $(F \times \mathbf{P}(W))/\bar{G} \sim (\mathbf{P}^2 \times \mathbf{P}(W))/\bar{G} \times \mathbf{P}^1$, and similarly $(\mathbf{P}^2 \times \mathbf{P}(W))/\bar{G} \sim \mathbf{P}(W)/\bar{G} \times \mathbf{P}^2$, so that $(F \times \mathbf{P}(W))/\bar{G} \sim \mathbf{P}(W)/\bar{G} \times \mathbf{P}^3$. On the other hand, $(F \times \mathbf{P}(W))/\bar{G} \sim \mathbf{P}(W)/B$, where B is a Borel subgroup of $\bar{G}$, and this is rational by a theorem of Vinberg [12]. Hence $\mathbf{P}(W)/\bar{G} \times \mathbf{P}^3$ is rational, and so $\mathbf{P}(V)/G$ is rational Q.E.D.

Pencils of binary forms of even degree

It is frequently natural to study quotient spaces X/G, where X in some homogeneous variety more general than projective space. In this section we consider the case where X is the space of pencils of binary forms whose degree is given, and $G = PGL_2$.

We fix the following notation: $V(D) = H^0(\mathcal{O}_{\mathbf{P}^1}(D))$, $\mathbf{P}_{\mathrm{D}} = \mathbf{P}(V(D))$ and Gr = Grassmannian of lines in $\mathbf{P}_{\mathrm{D}}$. We assume that $D = 2K$ is even.

THEOREM 14. *If $D \geqslant 10$, then Gr/G is rational.*

Proof. We have the Plücker embedding $Gr \hookrightarrow \mathbf{P}(\wedge^2 V(D)) = \mathbf{P}^N$, say. We shall write down a linear covariant ϕ: $\wedge^2 V(D) \to V(6)$, again using the symbolic method. (The existence of ϕ follows from examining the weights of the representation $\wedge^2 V(D)$, but to prove various non-degeneracy statements we shall need to know ϕ explicitly.) Let $f \in \wedge^2(V(D))$; symbolically, we can write $f = a_x^D \otimes b_x^D$, an alternating tensor (so that interchanging a_x and b_x transforms f into $-f$). We write

$$a_x^D = \sum_{i=0}^{D} \binom{D}{i} A_i x_1^{D-i} x_2^i, \quad b_x^D = \sum_{i=0}^{D} \binom{D}{i} B_i x_1^{D-i} x_2^i.$$

Then the Plücker coordinates $\{\lambda_{ij}\}$ are given by $\lambda_{ij} = A_i B_j - A_j B_i$. Hence $f = \Sigma_{i,j} \binom{D}{i}\binom{D}{j}\lambda_{ij}(x_1^{D-i}x_2^i \wedge x_1^{D-j}x_2^j)$. Define the covariant ϕ by $\phi(f) = (ab)^{D-3} a_x^3 \cdot b_x^3$, where

$$(ab) = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Write $\phi(f) = \Sigma_{\alpha=0}^{6} g_\alpha(\{\lambda_{ij}\}) x_1^{6-\alpha} x_2^\alpha$, where each g_α is a linear function of $\{\lambda_{ij}\}$. Let $T \subset SL_2$ be the maximal torus consisting of diagonal matrices, and let w be its weight defined by $w(x_1, x_2) = (1, -1)$. Then $w(A_i) = -(D - 2i) = 2i - D$, and so $w(\lambda_{ij}) = 2(i + j - D)$, $w(g_\alpha) = 2\alpha - 6$. ϕ induces a rational map ψ: $Gr \to \mathbf{P}_6$; let B be the base scheme of ψ.

LEMMA 15. *There is a point $Q \in B$ at which B is smooth and has codimension 7 in Gr.*

Proof. Set $I = \{(i, j) \mid i, j \in \mathbf{Z}$ and $0 \leqslant i < j \leqslant D\}$. For any subset $S \subseteq I$, define $\mathbf{P}(S) \subseteq \mathbf{P}^N$ by the equations $\lambda_{ij} = 0\ \forall (i, j) \in I - S$. Put $H = \{(0, 1), \ldots, (0, D)\}$, $J = I - H$, and project B away from $\mathbf{P}(J)$ to $\mathbf{P}(H)$ via the rational map τ. Consider the fibre $\tau^{-1}(P_{0,K+2}) = B \cap \mathbf{P}(J \cup \{(0, K + 2)\})$ as a subscheme of $\mathbf{P}(J \cup \{(0, K + 2)\})$. This has homogeneous co-ordinates $\lambda_{0,K+2}$; $\lambda_{12}, \ldots, \lambda_{1D}; \ldots; \lambda_{D-1,D}$; the other λ_{ij} vanish. Amongst the Plücker equations defining Gr, we have $\lambda_{0,K+2} \cdot \lambda_{i,j} - \lambda_{0,i} \cdot \lambda_{K+2,j} + \lambda_{0,j} \cdot \lambda_{K+2,i} = 0$ if i, j are both distinct from $0, K + 2$. So $(\tau^{-1}(P_{0,K+2}) - \mathbf{P}(J))^-$ (the bar denoting closure) is contained in the locus defined by the equations $\lambda_{ij} = 0$ whenever i, j are both distinct from 0, $K + 2$, so the remaining coordinates are

$$\lambda_{0,K+2}, \lambda_{1,K+2}, \ldots, \lambda_{K+1,K+2}, \lambda_{K+2,K+3}, \ldots, \lambda_{K+2,D}.$$

Now I claim that for each $\alpha = 0, \ldots, 6$, the variable $\lambda_{K-5+\alpha,K+2}$ occurs in g_α with non-zero coefficient; this will be verified later. Then from the equations $g_0 = \cdots = g_6 = 0$, we deduce $\lambda_{K-5,K+2} = \cdots = \lambda_{K+1,K+2} = 0$. Hence $(\tau^{-1}(\mathbf{P}_{0,K+2}) - \mathbf{P}(J))^- \cong \mathbf{P}^{D-8}$. Take any point $Q \in \tau^{-1}(P_{0,K+2}) - \mathbf{P}(J)$, and the Lemma is proved.

Proof of Theorem 14. Let π: $\tilde{Gr} \to Gr$ be the blow-up along B and $\tilde{\psi}$: $\tilde{Gr} \to \mathbf{P}_6$ the induced morphism. As in the proof of Theorem 3, $\tilde{\psi}$ is surjective and its generic fibre is geometrically reduced and irreducible. Set $X = \mathbf{P}_6^{vs}$, $Y = \tilde{\psi}^{-1}(X)$, $Z = X \times Gr$, $W = X \times \mathbf{P}^N$, $X_1 = X/G$, $Y_1 = Y/G$, $Z_1 = Z/G$, $W_1 = W/G$. We have a commutative diagram with Cartesian squares:

$$\begin{tikzcd}
W \arrow[rrrrr] \arrow[dddr] & & & & & W_1 \arrow[dddl] \\
& Z \arrow[ul, hook] \arrow[rrr] \arrow[dd] & & & Z_1 \arrow[ur] \arrow[dd] & \\
& & Y \arrow[ul] \arrow[r] \arrow[dl] & Y_1 \arrow[ur] \arrow[dr] & & \\
& X \arrow[rrr] & & & X_1 &
\end{tikzcd}$$

Let η, ξ be the generic points of X, X_1 respectively. Then by descending a suitable line bundle as before, we have $(W_1)_\eta \cong \mathbf{P}^N \otimes k(\eta)$, while $(Z_1)_\eta$ is a form of the Grassmannian $G(2, D + 1) \otimes k(\eta)$ and $(Y_1)_\eta \cap (Z_1)_\eta$ is defined as the intersection of $(Z_1)_\eta$ with a linear space of codimension 6. Denote $G(2, D + 1)$ by Γ.

LEMMA 16. $(Z_1)_\eta \cong \Gamma \otimes k(\eta)$.

Proof. Let $E \to Gr$ be the universal rank 2 vector subbundle of the trivial rank $(D + 1)$ vector bundle F over Gr and let $\tilde{E}$, $\tilde{F}$ be their pull-backs to Z. Then E and F are PGL_2-linearized, and so $\tilde{E}$ and $\tilde{F}$ are also. Hence $\tilde{E}$ descends to a subbundle $\tilde{E}_1$ of the trivial rank $(D + 1)$ vector bundle $\tilde{F}_1$ over Z_1, and so there is a $k(\eta)$-morphism π: $(Z_1)_\eta \to \Gamma \otimes k(\eta)$ such that $\pi^*U \cong \tilde{E}_1$, where U is the universal bundle over $\Gamma \otimes k(\eta)$. After the base change $k(n) \to k(\xi)$, $\pi \otimes k(\xi)$ is an isomorphism, and so π is an isomorphism.

Q.E.D.

Completion of the proof of Theorem 14. We now know that $(Y_1)_\eta$ is the intersection of $\Gamma \otimes k(\eta)$ with a linear space of codimension 6. We shall show that $(Y_1)_\eta$ is rational over $k(\eta)$.

Regard $\Gamma \otimes k(\eta)$ as the set of lines in $\mathbf{P}^D \otimes k(\eta)$. Let $H \subset \mathbf{P}^D \otimes k(\eta)$ be any hyperplane defined over $k(\eta)$. Then there is a rational map γ: $\Gamma \otimes k(\eta) \to H$ defined by $\gamma(l) = l \cap H$ whose fibres are linear spaces. The base locus of γ is a copy of the Grassmannian Δ of lines in H; it has codimension 2 in $\Gamma \otimes k(\eta)$. We wish to choose H so that γ induces a dominant map $(Y_1)_\eta \to H$. This is certainly possible unless the subvariety Q of $\mathbf{P}^D \otimes k(\eta)$ swept out by the lines corresponding to points of $(Y_1)_\eta$ is not the whole of $\mathbf{P}^D \otimes k(\eta)$. We proceed to show that this is impossible.

We have a diagram

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Gr \otimes k(\eta) & & \mathbf{P}^D \otimes k(\eta) \end{tikzcd}$$

where E is the incidence relation and p, q are the projections onto the factors. The subvariety Q is just $p^{-1}((Y_1)_\eta)$, and so has codimension six in E; since q is a $\mathbf{P}^1$-bundle, it follows that the restriction of q to Q is surjective, which is what we need.

So we can find a $k(\eta)$-hyperplane H so that the map $(Y_1)_\eta \to H$ is dominant; the fibres are linear spaces, and so $(Y_1)_\eta$ is rational over $k(\eta)$. But $k(\eta)$ is rational over $\mathbf{C}$, by the classically derived invariant theory for binary sextics, and so to prove the theorem we have only to check the non-vanishing of certain coefficients.

We have $f = a_x^D \otimes b_x^D$, $\phi(f) = (ab)^{D-3} a_x^3 \cdot b_x^3$. The Plücker coordinates λ_{ij} are given by $\lambda_{ij} = a_1^{D-i} a_2^i b_1^{D-j} b_2^j - a_1^{D-j} a_2^j b_1^{D-i} b_2^i$, and so expansion of the expression for $\phi(f)$ shows that the various coefficients are as follows:

(i) $x_1^6 \cdot \lambda_{t,D-t-3}$: coefficient is $2 \cdot (-1)^t \binom{D-3}{t}$;

(ii) $x_1^5 x_2 \cdot \lambda_{t,D-t-2}$: coefficient is $6 \cdot (-1)^t [\binom{D-3}{t} - \binom{D-3}{t-1}]$;

(iii) $x_1^4 x_2^2 \cdot \lambda_{t,D-t-1}$: coefficient is

$$\frac{6 \cdot (-1)^t \cdot (D-3)!}{t!\,(D-t-1)!}[5t^2 - t(5D-5) + (D-1)(D-2)].$$

So if $t = K - 3$, the coefficient is

$$\frac{6 \cdot (-1)^{K-3}(2K-3)!}{(K-3)!\,(K+2)!}(-K^2 - K + 32),$$

which is never zero for integral K;

(iv) $x_1^3 x_2^3 \cdot \lambda_{t,D-t}$: coefficient is

$$\frac{2 \cdot (-1)^t (D-3)!}{t!(D-t)} \cdot (D-2t) \cdot (D^2 - D(10t+3) + 10t^2 + 2).$$

So if $t = K - 2$, the coefficient is

$$\frac{2 \cdot (-1)^{K-2}(2K-3)!}{(K-2)!\,(K+2)!} \cdot 24.6(-K^2 - K + 7),$$

which is never zero for integral K. By symmetry, the coefficients of $x_1^2x_2^4 \cdot \lambda_{K-1,K-2}$, $x_1x_2^5 \cdot \lambda_{K,K-3}$ and $x_2^6 \cdot \lambda_{K+1,K-4}$ are also non-zero. This completes the proof of Theorem 14.

A special example

In this final section, our aim is to solve a problem in the invariant theory for another group of rank two, namely the exceptional group G_2.

Mukai has shown [8] that the moduli space $\mathscr{K}_{18}$ of polarized $K3$ surfaces of degree 18 (equivalently, of genus 10) is birational to the orbit space $Gr(3, \mathfrak{g})/G$, where G is the exceptional Lie group G_2 and $\mathfrak{g} = \mathrm{Lie}(G)$. Our aim here is to show that this quotient is a rational variety. Notice that at the moment, the only other space $\mathscr{K}_{2d}$ known to be rational is $\mathscr{K}_{10}$, while for $d \neq 5, 9$ the problem is open, and in fact for $d \geqslant 10$, the question of unirationality is unresolved.

We shall show that $\mathscr{K}_{18}$ is rational in various stages as follows:

(i) We construct a covariant ψ: $Gr(3, \mathfrak{g}) \to \mathbf{P}(C_0)$, where C_0 is the irreducible 7-dimensional representation of G, which is dominant; then if $x \in \mathbf{P}(C_0)$ is generic, we have $Gr(3, \mathfrak{g})/G \sim \psi^{-1}(x)/\mathrm{Stab}(x)$, since the orbit of x is dense in $\mathbf{P}(C_0)$.
(ii) Put $S = \mathrm{Stab}(x)$. Then the connected component S^0 is isomorphic to SL_3, and $S = S^0 \rtimes \langle\theta\rangle$, $\theta^2 = 1$. As S^0-modules, we have $\mathfrak{g} = \mathfrak{s} \oplus V \oplus V^\vee$, where $\mathfrak{s} = \mathrm{Lie}\, S^0$ and $V \cong \mathbf{C}^3$ is the standard representation; $\mathfrak{s}$ and $V \oplus V^\vee$ are S-modules.
(iii) We construct a dominant S-covariant λ: $Gr(3, \mathfrak{g}) \to \mathbf{P}(V \oplus V^\vee)$, so that if $w \in \mathbf{P}(V \oplus V^\vee)$ is generic, then $\psi^{-1}(x)/S \sim (\psi^{-1}(x) \cap \lambda^{-1}(w))/K$, where $K = \mathrm{Stab}(w) \cap S$.
(iv) There is a K-stable hyperplane H in $\mathfrak{g}$ and a K-equivariant birational equivalence $\psi^{-1}(x) \cap \lambda^{-1}(w) \to Gr(2, H)$.
(v) $Gr(2, H)/K$ is rational.

We begin by recalling some basic facts about the group G. Let C denote the algebra of Cayley numbers; then $G = \mathrm{Aut}\ C$. We shall refer to the basis $\{c_1, \ldots, c_8\}$ of C and the multiplication table for this basis given by Humphreys [14, p. 105]. Define elements $x_2, \ldots, x_8$ of C by $x_2 = c_1 - c_2$,

$x_i = c_i$ for $i \geqslant 3$; then $\{x_2, \ldots, x_8\}$ is the basis of the space C_0 of elements of C whose trace vanishes. The group G preserves C_0.

Let $\pi: C \to C_0$ be the G-equivariant projection and $\varepsilon: C_0 \to C$ the inclusion. Then the composite $C_0 \otimes C_0 \xrightarrow{\varepsilon \otimes \varepsilon} C \otimes C \xrightarrow{m} C \xrightarrow{\pi} C_0$, where m is the multiplication in C, gives a G-equivariant algebra structure on C_0, which we denote by n. A glance at the multiplication table for C shows that n is skew-symmetric, so that there is a G-linear map $p: \wedge^2 C_0 \to C_0$ given by $p(u \wedge v) = n(u, v)$. There is a non-degenerate symmetric bilinear form on C given by the norm; it restricts to give another such form q on C_0. Let H denote the group $SO(C_0, q)$, and $\boldsymbol{h} = \operatorname{Lie} H$; then $\boldsymbol{h} \cong \wedge^2 C_0$ as H-modules. Specifically, define an action of $\wedge^2 C_0$ on C_0 by $(a \wedge b)(z) = q(a, z) \cdot b - q(b, z) \cdot a$, which is just the adjoint representation of $\boldsymbol{h}$. Via the isomorphism $\boldsymbol{h} \cong \wedge^2 C_0$, we can identify the subalgebra $\boldsymbol{g}$ of $\boldsymbol{h}$ with $\ker p$.

Referring to the basis $\{x_2, \ldots, x_8\}$ of C_0 given above, $\{x_3 \wedge x_7, x_3 \wedge x_8, x_4 \wedge x_6, x_4 \wedge x_8, x_5 \wedge x_6, x_5 \wedge x_7, x_3 \wedge x_6 - x_5 \wedge x_8, x_3 \wedge x_6 - x_4 \wedge x_7, x_2 \wedge x_3 - x_7 \wedge x_8, x_2 \wedge x_4 - x_8 \wedge x_6, x_2 \wedge x_5 - x_6 \wedge x_7, x_2 \wedge x_6 - x_5 \wedge x_4, x_2 \wedge x_7 - x_3 \wedge x_5, x_2 \wedge x_8 - x_4 \wedge x_3\}$ is a basis of $\boldsymbol{g}$, and the bilinear form q is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1_3 \\ 0 & -1_3 & 0 \end{pmatrix}.$$

Recall that $\boldsymbol{g}$ contains a copy s of the Lie algebra $\boldsymbol{sl}_3$ so that as $\boldsymbol{sl}_3$-modules, $\boldsymbol{g} \cong \boldsymbol{sl}_3 \oplus V \oplus V^\vee$, where $V \cong \mathbf{C}^3$ is the standard representation of $\boldsymbol{sl}_3$ and $V^\vee$ is its dual. Explicitly, if

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is the standard basis of V and $\{f_1, f_2, f_3\}$ is the dual basis of $V^\vee$, then we can make the following identifications:

$$x_2 \wedge x_3 - x_7 \wedge x_8 = e_1, \quad x_2 \wedge x_4 - x_8 \wedge x_6 = e_2$$

$$x_2 \wedge x_5 - x_6 \wedge x_7 = e_3, \quad x_2 \wedge x_6 - x_5 \wedge x_4 = f_1,$$

$$x_2 \wedge x_7 - x_3 \wedge x_5 = f_2, \quad x_2 \wedge x_8 - x_4 \wedge x_3 = f_3,$$

$$x_3 \wedge x_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_3 \wedge x_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$x_4 \wedge x_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_4 \wedge x_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$x_5 \wedge x_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad x_5 \wedge x_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$x_3 \wedge x_6 - x_4 \wedge x_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$x_3 \wedge x_6 - x_5 \wedge x_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

LEMMA 17. *Consider x_2 as a point in $\mathbf{P}(C_0)$. Let S denote stab(x_2). Then $S \cong SL_3 \rtimes \langle\theta\rangle$, where θ is the involution given by $\theta(g) = {}^t g^{-1}$, and the orbit $O(x_2)$ is dense in $\mathbf{P}(C_0)$.*

Proof. By considering the action of $\wedge^2 C_0$ on C_0 and the basis of g described above, it is clear that Lie$(S^0) = s$, and so $S^0 \cong SL_3$. By counting dimensions, it follows that $O(x_2)$ is dense in $\mathbf{P}(C_0)$. Then there can be at most one G-invariant hypersurface in $\mathbf{P}(C_0)$, and indeed there is one, namely $Q = \{x \in \mathbf{P}(C_0) \mid q(x, x) = 0\}$. Hence $\pi_1(O(x_2)) = \mathbf{Z}/2\mathbf{Z}$, and so S has at most two components. It is clear however, that the element θ of G defined by $\theta(x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_2, x_6, x_7, x_8, x_3, x_4, x_5)$ lies in S but not S^0, and that $\theta g \theta = {}^t g^{-1}$ for all $g \in S^0$. This proves Lemma 17.

Define $\alpha\colon \wedge^3(\wedge^2 C_0) \to \wedge^6 C_0$ by $\alpha((a \wedge b) \wedge (c \wedge d) \wedge (c \wedge d) \wedge (e \wedge f)) = a \wedge b \wedge c \wedge d \wedge e \wedge f$. Note that α induces a rational map $\beta\colon \mathbf{P}(\wedge^3(\wedge^2 C_0)) \to \mathbf{P}(\wedge^6 C_0) \cong \mathbf{P}(C_0)$, where the last isomorphism is G-equivariant.

PROPOSITION 18. *There is a dominant G-equivariant rational map* $\psi\colon Gr(3, \mathbf{g}) \to \mathbf{P}(C_0)$, *obtained as the composite* $Gr(3, \mathbf{g}) \xrightarrow{\iota} \mathbf{P}(\wedge^3 \mathbf{g}) \to \mathbf{P}(\wedge^3(\wedge^2 C_0)) \xrightarrow{\beta} \mathbf{P}(C_0)$, where ι is the Plücker embedding.

Proof. The only thing to check is that ψ is dominant. To see this, note that $\alpha((x_3 \wedge x_6 - x_4 \wedge x_7) \wedge (x_4 \wedge x_8) \wedge (x_5 \wedge x_7)) = x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8$, which corresponds to x_2. Since $O(x_2)$ is dense, the proposition follows.

Let $\gamma\colon \tilde{G}r \to Gr(3, \mathbf{g})$ be the blow-up along the base scheme B of ψ and $\tilde{\psi}\colon \tilde{G}r \to \mathbf{P}(C_0)$ the induced morphism. Put $\tilde{F} = \tilde{\psi}^{-1}(x_2)$ and $F = \gamma(\tilde{F})$; F consists of components of the intersection of $Gr(3, \mathbf{g})$ with a linear subspace of codimension 6.

COROLLARY 19. *$Gr(3, \mathbf{g})/G$ is birationally equivalent to F/S.*

Next, we can define a linear map $\delta\colon \wedge^3(s \oplus V \oplus V^\vee) \to s \otimes \wedge^2 V \oplus s \otimes \wedge^2 V^\vee$ of S-modules by $\delta(x \wedge y \wedge z) = pr_1(x) \otimes pr_2(y) \wedge pr_2(z) + pr_1(x) \otimes pr_3(y) \wedge pr_3(z) - pr_1(y) \otimes pr_2(x) \wedge pr_2(z) - pr_1(y) \otimes pr_3(x) \wedge pr_3(z) + pr_1(z) \otimes pr_2(x) \wedge pr_2(y) + pr_1(z) \otimes pr_3(x) \wedge pr_3(y)$, where pr_i is the projection of $s \oplus V \oplus V^\vee$ onto its i'th factor. Identifying $\wedge^2 V$ with $V^\vee$ and $\wedge^2 V^\vee$ with V, and making the natural contractions $s \otimes V^\vee \to V^\vee$ and $s \otimes V \to V$, yields a linear S-map $\eta\colon \wedge^3 \mathbf{g} = \wedge^3(s \oplus V \oplus V^\vee) \to V \oplus V^\vee$, and so an S-equivariant rational map $\lambda\colon Gr(3, \mathbf{g}) \to \mathbf{P}(V \oplus V^\vee)$. Let $\mu\colon Gr^* \to Gr(3, \mathbf{g})$ be the blow-up along the base scheme D of λ, $\lambda^*\colon Gr^* \to \mathbf{P}(V \oplus V^\vee)$ the induced morphism, $X^* = \lambda^{*-1}(e_1 + f_1)$ and $X = \mu(X^*)$. Put $F_0 = F \cap X$, and $K = \mathrm{Stab}(\mathbf{C}(e_1 + f_1)) \cap S$. Put

$$K_1 = \left\{ \left(\begin{array}{c|cc} \pm 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & & \end{array} \right) \in SL_3 \right\};$$

then $K = K_1 \rtimes \langle \theta \rangle$. In particular, the orbit $O(e_1 + f_1)$ is dense in $\mathbf{P}(V \oplus V^\vee)$. For the moment, we make no claim that the rational map $F \to \mathbf{P}(V \oplus V^\vee)$ is dominant.

In g, the line $\mathbf{C}.\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is K-stable, where in conformity with the notation above,

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_4 \wedge x_7 - 2x_3 \wedge x_6 + x_5 \wedge x_8,$$

and the subspace H generated by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the first twelve elements of the given basis of g is a complement. There is a K-equivariant rational map σ: $Gr(3, g) \to Gr(2, H)$ given by $L \to L \cap H$, which exhibits $Gr(3, g)$ (birationally) as a scroll over $Gr(2, H)$; i.e., if π: $\hat{Gr} \to Gr(3, g)$ is the blow-up along the base scheme E of σ, $\hat{\sigma}$: $\hat{Gr} \to Gr(2, H)$ is the induced morphism, $W \in Gr(2, H)$, $\hat{Y} = \hat{\sigma}^{-1}(W)$ and $Y = \varrho(\hat{Y})$, then Y is a copy of $\mathbf{P}^{11}$, embedded linearly in $\mathbf{P}(\wedge^3 g)$.

The next Lemma is the key non-degeneracy statement that we shall need.

LEMMA 20. *There is an element $W \in Gr(2, H)$ for which there is a unique 3-plane $L \in Gr(3, g)$ satisfying the following conditions: (i) $L \cap H = W$; (ii) $L \in F \cup B$; (iii) $L \in X \cup D$. (Recall that condition (ii) means that if $\{x, y, z\}$ is a bais of L, then $\alpha(x \wedge y \wedge z) \in \mathbf{C} \cdot x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8$, while condition (iii) means that $\delta(x \wedge y \wedge z) \in \mathbf{C}(e_1 + f_1)$.) Moreover, $L \notin B$, $L \not\subseteq H$* and *$L \notin D$.*

Proof. Take W to be the 2-plane spanned by the elements

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + e_1 + e_2 + f_1 + f_3 \text{ and}$$

$$y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + e_1 + f_1.$$

Suppose that $z \in \boldsymbol{g}$, $z \notin W$ and $x \wedge y \wedge z$ satisfies (i)–(iii) above. We can write $z = p + q + r$, where $p = (p_{ij}) \in s$, $q = q_1e_1 + q_2e_2 + q_3e_3 \in V$ and $r = r_1f_1 + r_2f_2 + r_3f_3 \in V^\vee$. Then $\delta(x \wedge y \wedge z) = (p_{12} - (r_2 + r_3))e_1 + p_{22}e_2 + (p_{32} + r_1 - r_3)e_3 + (p_{31} - q_2 - q_3)f_1 + (p_{32} + q_2 - q_1)f_2 + (p_{33} + 2q_3)f_3$, and so having $x \wedge y \wedge z \in X \cup D$ is equivalent to the following equations being satisfied: $p_{12} - (r_2 + r_3) = p_{31} - (q_2 + q_3)$, $p_{22} = 0$, $p_{32} + r_1 - r_3 = 0$, $p_{32} + q_2 - q_1 = 0$, $p_{33} + 2q_3 = 0$. Also, having $x \wedge y \wedge z \in F \cup B$ is the same as having $\alpha(x \wedge y \wedge z) \in \mathbf{C} \cdot (x_3 \wedge x_4 \wedge \ldots \wedge x_8)$, and calculation shows that

$$\begin{aligned}
\alpha(x \wedge y \wedge z) = {} & (-q_3 - 2p_{32} - p_{21} + p_{11} - 2p_{12} - r_3 - 3r_1) \\
& \cdot x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 + (3p_{11} - 2p_{13} + q_2 - r_1 + p_{23}) \\
& \cdot x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_8 + (-3q_1 + p_{23} - 2r_1 - p_{13} \\
& + p_{12} + r_3 - q_3))\, x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_7 \wedge x_8 + (-3q_1 - 2p_{21} - p_{11} \\
& - r_3 - r_1 - p_{23})\, x_2 \wedge x_3 \wedge x_4 \wedge x_6 \wedge x_7 \wedge x_8 + (-q_2 - 2p_{31} + p_{13} \\
& + r_2 - 3q_1)\, x_2 \wedge x_3 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 + (-3q_2 - p_{31} - 2q_1 + p_{23} \\
& - 3r_1 - p_{21} - r_2)\, x_2 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 + (-2q_2 - p_{31} - 2p_{11} \\
& - q_3 - 2r_3 - p_{12} - r_2)\, x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8.
\end{aligned}$$

Finally, replacing z by $z + ax + by$ for suitable $a, b \in \mathbf{C}$, we may assume that $p_{12} + p_{23} + p_{31} + q_1 + q_2 + r_1 + r_3 = 0$ and $p_{13} + p_{32} + p_{21} + q_1 + r_1 = 0$. It is now a matter of checking the non-vanishing of various determinants to see that (up to scalars) there is a unique non-zero vector z satisfying the conditions (i)–(iii), and that then $x \wedge y \wedge z \notin B$, $x \wedge y \wedge z \notin D$ and $x \wedge y \wedge z \not\subseteq H$.

Put $F \cup B = F'$, $X \cup D = X'$. Since the base locus E of σ is a copy of $Gr(3, H)$, identified with the intersection $Gr(3, \mathfrak{g}) \cap \mathbf{P}(\wedge^3 H)$, it follows from Lemma 20 that $X' \cap F' \cap \varrho(\hat{\sigma}^{-1}(W))$ consists of a single element, say L, and that $L \notin E$.

COROLLARY 21. *F/S is birationally equivalent to F_0/K.*

Proof. Since $L \notin D$, the map λ is defined at L, and $\lambda(L) = (e_1 + f_1) \in \mathbf{P}(V \oplus V^\vee)$. Also, $F_0 = \mu(\lambda^{*-1}(e_1 + f_1))$, and since the orbit $O(e_1 + f_1)$ is dense, the Corollary follows.

COROLLARY 22. *F_0/K is birationally equivalent to $Gr(2, H)/K$.*

Proof. Let $\hat{F}_0$ denote the strict transformation of F_0 in $\hat{G}r$. By construction, F_0 meets $\varrho(\hat{\sigma}^{-1}(W))$ in only the point L, at which the rational map σ is defined. Also, since the equations defining F_0 are linear and $\varrho(\hat{\sigma}^{-1}(W))$ is linear, this intersection is transverse. Since F_0/K is irreducible, every component of F_0 is 22-dimensional, and there is a unique component that maps dominantly under σ to $Gr(2, H)$. But also since F_0/K is irreducible, the components of F_0 are permuted transitively by K, and the Corollary follows.

Notice that we can decompose H into a direct sum of K-spaces as follows: $H = \mathfrak{sl}_2 \oplus W \oplus W \oplus Y$, where $\mathfrak{sl}_2 = \mathrm{Lie}(K^0)$, $W = U \oplus U^\vee$ where U is the standard representation of SL_2 and $Y = \mathbf{C} \cdot \{e_1, f_1\}$. Since $H \cong H^\vee$ as K-spaces, we can identify $Gr(2, H)$ with $Gr(11, H)$. There is a K-equivariant rational map $\tau\colon Gr(11, H) \to Gr(4, W' \oplus Y)$, where we have fixed a copy W' of W in H, given by $\tau(M) = M \cap (W' \oplus Y)$. Put $H' = W' \oplus Y$.

LEMMA 23. *$Gr(2, H)/K$ is birationally equivalent to $Gr(4, H')/K \times \mathbf{P}^{14}$.*

Proof. Let $v\colon Gr' \to Gr(11, H)$ be the blow-up of the base scheme of τ, and $\tau'\colon Gr' \to Gr(4, H')$ the induced morphism. Let $\mathscr{L}$ be the line bundle defining the Plücker embedding of $Gr(11, H)$; then τ' is a $\mathbf{P}^{14}$ bundle, and $v^*\mathscr{L}$ is a K-linearized line bundle cutting out $\mathcal{O}(1)$ on each fibre of τ'. Since K acts generically freely on $Gr(4, H)$, Mumford's descent theorem (Theorem 1 above) proves the Lemma.

LEMMA 24. *$Gr(4, H')/K \times \mathbf{P}^5$ is rational.*

Proof. Put $\mathbf{P} = \mathbf{P}(H')$. Then a descent argument as in the proof of Lemma 23 shows that $(Gr(4, H') \times \mathbf{P})/K$ is birationally a $\mathbf{P}^5$-bundle over

$Gr(4, H')/K$; we want to show that $(Gr(4, H') \times \mathbf{P})/K$ is generically a $Gr(4, H')$-bundle over $\mathbf{P}/K$. So let $X \subset \mathbf{P}$ be the open subvariety of the set of stable points on which G acts with trivial stabilizers (and so freely, by the étale Slice Theorem). Put $Gr(4, H') \times X = Y$. Then we can identify Y with the Grassmannian of rank 4 sub-bundles of $H' \otimes {}_{\mathbf{C}}\mathcal{O}_X = \mathcal{F}$, say; let $\mathcal{E}$ be the universal sub-bundle of $H' \otimes {}_{\mathbf{C}}\mathcal{O}_Y$. Then we have a Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y/K = Y_1 \\ \big\downarrow & & \big\downarrow p \\ X & \longrightarrow & X/K = X_1, \end{array}$$

and since $\mathcal{E}$, $\mathcal{F}$ are K-linearized, they descend to vector bundles $\mathcal{E}_1$ and $\mathcal{F}_1$ over Y_1 and X_1 respectively. It follows that Y_1 is a Grassmannian over X_1, as required. Finally, we want to show that $\mathbf{P}/K$ is rational; this, however, follows immediately from Castelnuovo's criterion (or by elementary calculation).

THEOREM 25. *$\mathcal{K}_{18}$ is rational.*

Proof. This follows from Corollaries 19, 21, and 22, and Lemmas 23 and 24.

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