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Let $\mathscr{G}$ be a connected semi-simple algebraic group defined over $\mathbf{Q}$ such that the symmetric space D of $G := \mathscr{G}(\mathbf{R})$ is Hermitian, and let Γ be an arithmetic subgroup of $\mathscr{G}(\mathbf{Q})$. Then $X^\circ := \Gamma\backslash D$ is a complex analytic space with only quotient singularities which is naturally compactified as a projective variety X by means of the Satake, Baily-Borel compactification [BB]. Let further be given a finite dimensional representation E of G. It determines a metrized local system $\mathbf{E}$ over the regular Γ-orbit space X_{reg} in X° [Z3]. Let $\mathscr{L}^{\cdot}_X(\mathbf{E})$ denote the sheaf complex on X_{reg} which assigns to every open subset U of X the space of smooth differential forms α on $U \cap X_{\text{reg}}$ with coefficients in $\mathbf{E}$ having the property that for every compact subset K of U, $\alpha \mid K \cap X_{\text{reg}}$ and $d\alpha \mid K \cap X_{\text{reg}}$ are *square integrable* (the definition is recalled below). By the generalized De Rham theorem its restriction to X_{reg} is a resolution of the local system $\mathbf{E}$. Zucker proved in [Z3] that this complex is fine and verified in a number of cases (where $X - X^\circ$ is finite) that it represents the intersection complex on X with coefficients in $\mathbf{E}$. He conjectured this to hold in general. Since then this has been verified when G has $\mathbf{Q}$-rank one or two by Borel and Casselman [B2], [BC2], and for some other cases by Zucker [Z4]. Here we prove his general conjecture:

THEOREM. *$\mathscr{L}^{\cdot}_X(\mathbf{E})$ represents the intersection complex on X with values in* $\mathbf{E}$.

The proof hinges on a purity property of the intersection complex on X with values in $\mathbf{E}$ (Thm. 3.8), which yields a stronger vanishing theorem than is actually necessary for proving Zucker's conjecture. We first prove the corresponding property for the intersection complex on a toric resolution of X whose exceptional divisor has normal crossings (4.1). This in turn, is derived from the analogous result for variations of polarized Hodge structure over a product of punctured unit discs, which is due to Cattani–Kaplan–Schmid [CKS] and Kashiwara–Kawai [KK]. Purity on X then follows from an application of the decomposition theorem, in the form as proved by M. Saito [S3, S4]. Our proof also identifies the weight filtration on the local intersection cohomology groups (relative their canonical mixed Hodge structure) with a natural filtration defined on the local L^2-intersection cohomology groups which is defined in representation-theoretic terms and is ubiquitous in previous work on the subject (4.2).

Almost simultaneous with our announcement [L], a proof of Zucker's conjecture was announced by Saper and Stern [SS]. Their proof appears to be completely different from ours.

Some conventions. If M is a smooth manifold of dimension m, then we denote its de Rham sheaf complex of smooth forms by $\mathscr{E}^{\cdot}_M$. In case M is oriented and has a Riemann metric then we have the star operator $*:\mathscr{E}^k_M \to \mathscr{E}^{m-k}_M$. If also is given a *metrized local system* $\mathbf{V}$ on M, i.e., a local system on M of finite dimensionless complex vector spaces equipped with a hermitian metric on the underlying vector bundle, then for $\alpha \in \mathscr{E}^k_M \otimes \mathbf{V}$, denote by $|\alpha|^2 \in \mathscr{E}^m_M$, the m-form obtained from α and $(* \otimes \mathbf{1})(\alpha)$, by taking the exterior product in the first entry and applying the inner product of $\mathbf{V}$ in the second. This form is $\geqslant 0$ relative the volume form on M and (by definition) a section α of $\mathscr{E}^k_M \otimes \mathbf{V}$ over an open U subset of M is *square integrable* if $|\alpha|^2$ is integrable over U. The generic meaning of the subscript (2) (as for instance in $H^k_{(2)}(M, \mathbf{V})$ will be that we are only considering (or computing with) forms which are square integrable and have ditto exterior derivative.

We shall follow the custom of denoting a Lie group by a Roman capital and its Lie algebra by the corresponding gothic lower case letter.

1. A first reduction

The theorem holds for Γ, whenever it holds for a subgroup of Γ finite index, essentially because both intersection cohomology and L^2-cohomology behave in the same (expected) way under passage to finite quotients. Hence we can assume from the outset that Γ is *neat* in the sense of [B1] and that G is simply-connected. Then Γ is torsion free and $X^\circ = X_{\text{reg}}$. If $G = G_1 \times \cdots \times G_k$ is a decomposition into irreducible components over $\mathbf{Q}$, then $(\Gamma \cap G_1) \times \cdots \times (\Gamma \cap G_k)$ is of finite index in Γ. So we may (and will) assume that $\mathbf{G}$ is irreducible over $\mathbf{Q}$.

We recall that X is obtained as the orbit space $\Gamma\backslash D^*$, where D^* is the union of the rational boundary components of D, equipped with the Satake topology. Since Γ is neat, the partition of D^* into boundary components induces a Zariski constructible stratification of X. We make the inductive assumption that it has been shown that $\mathscr{L}^{\cdot}_X(\mathbf{E})$ is an intersection complex on union of the strata of complex codimension smaller than d. Fix a connected stratum S of complex codimension d, and let F be a rational boundary component dominating it. So the G-stabilizer of F is a maximal proper $\mathbf{Q}$-parabolic subgroup of G; we denote it by P. The unipotent radical N of P is also defined over $\mathbf{Q}$. Hence $\Gamma_N := \Gamma \cap N$ is a cocompact discrete subgroup of N, so that $\Gamma_N \backslash N$ is a compact nilmanifold. In what follows we shall do little more than quoting from [Z3] (see also [C]).

For every point of D, there is a unique geodesic emanating from that point which has a limit on F. This is also true if D is replaced by any of its rational boundary components which have F in its closure. So on the union of those boundary components (denoted by Star(F)), these maps make up a «geodesic deformation retraction» π_F: Star(F) $\to F$. Set $Z_F := Z_\Gamma(F)\backslash \mathrm{Star}(F)$. According to Satake and Baily–Borel, Z_F is a locally compact Hausdorff space and admits a natural structure as a normal analytic space, so that each boundary component in Star(F) maps analytically into it. Furthermore, π_F induces a projection $\Pi_F : Z_F \to F$ which is analytic too. The group $(\Gamma \cap P)/Z_\Gamma(F)$ acts freely and properly discontinuously on Z_F as a group of analytic automorphisms. So $Z_S := (\Gamma \cap P)\backslash \mathrm{Star}(F)$ has also the structure of a normal analytic space and likewise π_F induces an analytic retraction $\pi_S : Z_S \to S$. According to Satake there exists a closed neighborhood B_F of F in Star(F), invariant under both N and $\Gamma_P := \Gamma \cap P$, such that every Γ-orbit intersects B_F in a Γ_P-orbit. Then $B_S := \Gamma_P \backslash B_F$ may be regarded either as a closed neighborhood of S in Z_S or as one in Star(S). By shrinking B_F we can ascertain that the following two conditions are also met (cf. [Z3]).

(a) The restriction $\pi_S : B_S \to S$ is proper and locally trivial in the stratified sense, such that the same trivializations make $(B_S \cap X^\circ, \mathbf{E} \mid B_S \cap X^\circ)$ locally trivial in the quasi-isometric sense.

(b) Every geodesic emanating from a point of B_F and having a limit on F, stays inside B_F, and together they give each fiber of $\pi_S \mid B_S$ the structure of a topological cone.

Because of the L^2-Künneth formula [Z3], these properties allow us to concentrate on a single fibre of π_S when proving the inductive step. More precisely, if we fix $\infty \in S$ and let Z denote the fibre of π_S over ∞, and B its intersection with B_S, then it is enough to prove that $H^k(\mathscr{L}^{\cdot}_M(\mathbf{E})(B))$ maps isomorphically onto $H^k(\mathscr{L}^{\cdot}_Z(\mathbf{E})(B - \{\infty\}))$ for $k < d$ and vanishes otherwise. From an analytic point of view these L^2-cohomology groups become more manageable if we exploit a natural fibration of $B^\circ := B \cap X^\circ$ with typical fibre the nilmanifold $\Gamma_N \backslash N$, cf. [Z3] and [C]. To this end, choose $\infty' \in F$ over ∞, and let Ω denote the fibre of the geodesic projection $\pi_F : D \to F$ over ∞'. As a domain with $Z_G(F)$-action, Ω is perhaps best understood when realized as a Siegel domain of the second kind. This is what we shall do in the following section.

2. The Siegel model for Ω

We recall some properties of P. The group N is connected 2-step unipotent, so its center U and $V := N/U$ are both vector space groups. The identity component of the maximal connected $\mathbf{Q}$-split torus A' of P/N is one-dimensional (and hence $\mathbf{Q}$-isomorphic to $(\mathbf{R}^\times)^\circ$). On $\mathfrak{u} := \mathrm{Lie}(U)$ resp. $\mathfrak{v} := \mathrm{Lie}(V)$, A' acts (via the adjoint representation of P) with character χ'^2 resp. χ', where χ' generates the character group of A'. For every $\omega \in \mathrm{Star}(F)$ there is a unique lift A^ω of A' in $P/Z_P(F_\omega)$, where F_ω denotes the boundary component containing ω, such that A^ω-conjugation leaves the $P/Z_P(F_\omega)$-stabilizer of ω invariant. So A' acts naturally on the right on $\mathrm{Star}(F)$. This action induces one on Z_S. Since the A^ω-orbit of ω is the unique geodesic through ω which has a limit point on F, this action is called the *geodesic action* in [BS], see also [Z3]. From now on we fix some lift A of A' in P which is defined over $\mathbf{Q}$ and let $\chi: A \to (\mathbf{R}^\times)^\circ$ denote the composite of $A \to A'$ with χ'. So the adjoint action of A on $\mathfrak{n} := \mathrm{Lie}(N)$ is semi-simple with characters χ and χ^2, and its eigen spaces split $\mathfrak{n}$ as a $\mathbf{Q}$-vector space: $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{u}$. Clearly, the centralizer $Z_P(A)$ of A in P maps isomorphically onto P/N, so that $P = N . Z_P(A)$ is a Levi decomposition (defined over $\mathbf{Q}$). We are now ready to describe Ω as a Siegel domain of the second kind.

The element $\infty' \in F$ determines a complex structure J on $\mathfrak{v}$ which is invariant under $Z_G(F)$. Let $E: \mathfrak{v} \times \mathfrak{v} \to \mathfrak{u}$ denote the anti-symmetric bilinear map induced by the Lie bracket on $\mathfrak{n}$ multiplied by $1/2$. Then E is J-invariant and so $H: v \to u_{\mathbf{C}}$, $H(v_1, v_2) = E(Jv_1, v_2) + \mathrm{i}E(v_1, v_2)$ is a Hermitian map. The adjoint representation of $Z_G(F)$ has a distinguished open orbit C in u which is a nondegenerate open convex cone in u, and H is positive with respect to C, i.e., $H(v, v) := \mathrm{cl}(C) - \{0\}$ unless $v = 0$. The following rule realizes N as a group of affine-linear transformations of $\mathfrak{v} \times \mathfrak{u}_{\mathbf{C}}$:

$$\exp(v_0, u_0): (v, z) \mapsto (v + v_0, z + u_0 + \mathrm{i}H(v, v_0) + \mathrm{i}2 . H(v_0, v_0)).$$

If we let $Z_G(A \times F)$ act diagonally on $\mathfrak{v} \times \mathfrak{u}_{\mathbf{C}}$, then together these groups determine an actionof $Z_G(F)$ on $\mathfrak{v} \times \mathfrak{u}_{\mathbf{C}}$ as a group of complex affine-linear transformations. This group leaves invariant the domain

$$\{(v, z) \in v \times u_{\mathbf{C}}: \mathrm{Im}(z) \in H(v, v) + C\}.$$

The point of this construction is that there is a $Z_G(F)$-equivariant isomorphism of Ω onto this domain. In what follows we shall simply identify the two.

By means of the Siegel model the geometry of Ω as a $Z_\Gamma(F)$-space becomes more transparant. For instance the A-action on Ω is given by

$a.(v, z) = (\chi(a)v, \chi^2(a)z)$. The geodesic action is most conveniently described via the diffeomorphism $N \times C \to \Omega$, $(n, y) \to n.(0, iy)$: on $N \times C$ it is simply $(n, y).a = (n, \chi(a)^2 y)$. This diffeomorphism also shows that N acts freely on Ω and that the orbit space of this action can be identified with C. The image of $Z_G(F \times \mathfrak{u})$ in $GL(\mathfrak{v})$ leaves the hermitian map H invariant, hence is compact, and as is clear from the Siegel description of Ω, the kernel of this representation is just N. So $Z_\Gamma(F \times \mathfrak{u})$ maps to a finite subgroup of $GL(\mathfrak{v})$. As this subgroup preserves the rational structure on $\mathfrak{v}$, the fact that Γ is neat implies that this group is trivial, so that $Z_\Gamma(F \times \mathfrak{u}) = \Gamma_N$. If we denote the image of $Z_\Gamma(F)$ in $GL(\mathfrak{u})$ by $\Gamma(u)$, the projection of Ω onto C defined above induces a projection

$$p: Z_\Gamma(F)\backslash\Omega \to \Gamma(\mathfrak{u})\backslash C.$$

It follows from the preceding that p is a fibre bundle of which each fibre has the structure of a homogeneous space isomorphic to $\Gamma_N\backslash N$. We shall write Z° for its total space.

This restricts to a fibered structure on $B^\circ = B \cap X^\circ$: just notice that $B_F \cap \Omega$ is a covering of B° with covering group $Z_\Gamma(F)$, and that, since B_F was chosen N-invariantly, $\Omega \cap B_F$ will be the pre-image of some $\Gamma(\mathfrak{u})$-invariant subset K of C. So $B^\circ = p^{-1}(\Gamma(\mathfrak{u})\backslash K))$.

Since B° is a union of geodesic half rays, so is K; in other words K is invariant under multiplication by scalars $\geqslant 1$. It is known that every half ray in C meets K [AMRT]. Moreover, we can choose K so that its boundary is smooth and meets every half ray transversally. So there exists a smooth map $f:\Gamma(\mathfrak{u})\backslash C \to (0, \infty)$, such that $\Gamma(\mathfrak{u})\backslash K$ is defined by $f \geqslant 1$ and $f(y.a') = \chi'(a')f(y)$.

3. Reduction to purity

In this section we will show that the result we are after, and which was stated at the end of section 1, can be phrased in topological terms. The main ingredient is Zucker's Künneth theorem for L^2-cohomology (or rather a special case of it), which we shall reformulate in a form which is convenient for our purpose.

A notational convention first. If A' acts linearly on a vector space H, then we denote H_l the subspace of H where A' acts with character $(\chi')^{-l}$ (note the sign!), and refer to it as the *weight l space*. Similarly for a linear A-action on h. Notice, that if $N.A$ happens to act on H, and H is finite dimensional, then the weight filtration $W_l H := \oplus_{l' \leqslant l} H_{l'}$ is independent of the lift A of A'; this is because any two such are conjugate under an element of N, and A acts on $\mathfrak{n}$ with negative weights. Although we will hardly use this remark, it is helpful for recognizing intrinsic properties.

Let $\mathscr{A}^{\cdot}$ denote the subcomplex of $p_*(\mathscr{E}_{Z^\circ} \otimes \mathbf{E})$ consisting of the forms which, when pulled back to Ω are N-invariant. A generalization of the van Est Theorem [Z3, B3] proves that the inclusion is a quasi-isomorphism. The double-sided A-action on Ω, $z \in \Omega \to a.\omega.(a')^{-1} \in \Omega$ preserves N-orbits. If we lift that action to $\Omega \times E$, by letting A act on E via G, then it preserves the N-invariant $\mathbf{E}$-valued forms, and thus induces an A-action on $\mathscr{A}^{\cdot}$. We shall denote this action by Ψ. It is semi-simple, so that $\mathscr{A}^{\cdot} = \oplus \mathscr{A}_l^{\cdot}$.

(3.1) LEMMA. *This decomposition is orthogonal in the sense that $\alpha \in \mathscr{A}_{l,\eta}^{\cdot}$ and $\beta \in \mathscr{A}_{l',\eta}^{\cdot}$ with $l \neq l'$ are orthogonal in every point of $p^{-1}(\eta)$.*

Proof. Let $\omega \in \Omega$ be of the form $(0, iy)$, with $y \in C$, so that $A = A^\omega$. The derivative of $Z_G(F) \to \Omega$, $g \to g.\omega$, identifies the orthogonal complement of the Lie algebra of $Z_G(F)_\omega$ in that of $Z_G(F)$ with the tangent space of Ω at ω, and this identification is a metric one. Now Ad(A) acts self-adjointly on the Lie-algebra of $Z_G(F)$, and virtually by construction the same is true for the A-action on E, when the latter is equipped with the Hermitian inner product defined by ω. From this the lemma readily follows.

The averaging argument also works in an L^2-setting: it produces for every open W in $\Gamma(\mathfrak{u})\backslash C$, a quasi-isomorphism $\mathscr{A}^{\cdot}(W)_{(2)} \to (\mathscr{E}_{Z^\circ}^{\cdot} \otimes \mathbf{E})(p^{-1}W)_{(2)}$. It follows that A' acts semi-simply on $H_{(2)}^k(p^{-1}W, \mathbf{E})$, and determines a finite grading on it. This grading prescribes the behaviour of the L^2-integrand under geodesic translation as the following lemma shows.

(3.2) LEMMA. *For $\alpha \in \mathscr{A}_l^{\cdot}$ and $a' \in A'$, we have $|\alpha.a'|^2 = \chi'(a')^{2d-2l}(|\alpha|^2.a')$.*

Proof. As the left-action of A on $\Omega \times E$ preserves the metric and $|\alpha|^2$ pulls back to an N-invariant $2d$-form, the lemma is equivalent to $|\psi(a).\alpha|^2| = \chi(a)^{2l-2d}(\psi(a).|\alpha|^2)$. The weight of any N-invariant $2d$-form is the same as that of the top exterior power of the dual of $\mathfrak{n}$. This is clearly equal to $2 \dim \mathfrak{u} + \dim \mathfrak{v} = 2d$. Whence the lemma.

In particular geodesic translation does not affect the L^2-cohomology groups. We give C the metric induced by $C \to \Omega$, $y \to (0, iy)$. This metric is $Z_G(F)$-invariant. Zucker shows [Z3] that for every $\lambda > 1$ there exists a smooth function on C with values in $[0, 1]$ which has support in K, is constant 1 on λK, and has bounded derivative. This is equivalent to the condition that f has bounded logarithmic derivative (in fact, this is what he proves).

3.3. LEMMA. *For every compact interval I in A', the map* $(\partial B^{\circ}, \mathbf{E}) \times I \to (B^{\circ}, E)$ *given by the right action of A', is a quasi-isometry onto its image (relative any metric on A').*

Proof. For $\partial_1, \partial_2 \in \mathrm{Lie}(Z_G(F))$, let D_1, D_2 denote the corresponding vector fields on Ω. Then for all $a \in A$, we have $\langle D_1 . a', D_2 . a'\rangle = \langle \mathrm{Ad}(a^{-1})\partial_1, \mathrm{Ad}(a^{-1})\partial_2\rangle$, which remains bounded as long as a stays in the pre-image of I in A. Similarly, if $\langle\, ,\rangle_\omega$ denotes the hermitian metric on E defined by $\omega \in \Omega$, then there exist constants c_1, c_2 such that $c_1\langle\, ,\rangle_\omega \leqslant \langle\, ,\rangle_{\omega,a'} \leqslant c_2\langle\, ,\rangle_\omega$ for all $\omega \in \Omega$ and $a' \in I$. This is because A^ω acts self-adjointly on $(E, \langle\, ,\rangle_\omega)$.

It remains to show that the infinitesimal generator D' of the A'-action on Z° normal to the foliation defined by f (oriented by the condition $\langle D, D'\rangle > 0$), is bounded from below by a positive number. Since the pull back of D' to Ω is $Z_G(F)$-invariant, $\langle D, D'\rangle$ is constant and hence is bounded. Consider the identity $\langle D, D'\rangle = D(\log(f\circ p))/\surd(\langle d\log(f\circ p), d\log(f\circ p)\rangle)$. The numerator is just 1, because $f(y.a') = \chi'(a')f(y)$. The denominator equals the square root of $\langle d\log f, d\log f\rangle \circ p$, which we know to be bounded from above. So $\langle D, D'\rangle$ is bounded from below by a positive number.

(3.4) COROLLARY. *We have canonical isomorphisms* $H^k_{(2)}(\partial B^\circ, \mathbf{E}) \approx IH^k(\partial B, \mathbf{E}) \approx IH^k(Z - \{\infty\}, \mathbf{E})$.

Proof. If I is a relatively compact in A', then it follows from the preceding lemma and the L^2-Künneth theorem that there is natural map from $H^k_{(2)}(\partial B^\circ, \mathbf{E})$ to $H^k_{(2)}(\partial B^\circ . I, \mathbf{E})$ which is an isomorphism. Since $\partial B \times A' \to Z - \{\infty\}$ is a stratified homeomorphism, we have the analogous result for intersection homology. The corollary now follows from our induction hypothesis, which allows us to identify $H^k_{(2)}(\partial B^\circ . I, \mathbf{E})$ with $IH^k(\partial B . I, \mathbf{E})$.

In particular $H^k_{(2)}(\partial B^\circ, \mathbf{E})$ is finite dimensional. Since Z° is complete as a Riemann manifold, so is its submanifold ∂B°. As is well-known [G], the $\mathbf{E}$-valued L^2-cohomology of ∂B° is then (uniquely) harmonically represented: the space of harmonic forms $h^k_{(2)}(\partial B^\circ, \mathbf{E})$ maps isomorphically onto $H^k_{(2)}(\partial B^\circ, \mathbf{E})$. Every such harmonic form pulls back to a form on $\Omega \cap \partial U_F$, so is certainly$N$-invariant. It is clear that the ψ-action of A maps harmonic forms to harmonic forms. With the resulting action of A on $h^k_{(2)}(\partial B^\circ, \mathbf{E})$ the isomorphism of the latter onto $H^k_{(2)}(\partial B^\circ, \mathbf{E})$ is A-equivariant. We orient ∂B° as the boundary of the complex manifold B°. This defines a $*$-operator which maps $h^k_{(2)}(\partial B^\circ, \mathbf{E})$ anti-isomorphically onto $h^{2d-1-k}_{(2)}(\partial B^\circ, \mathbf{E})$; in somewhat more intrinsic terms, the natural bilinear map

$$h^k_{(2)}(\partial B^\circ, \mathbf{E}) \times \overline{h^{2d-1-k}_{(2)}(\partial B^\circ, \mathbf{E})} \to h^{2d-1}_{(2)}(\partial B^\circ),$$

given by the exterior product and the hermitian form on $\mathbf{E}$, is perfect. It is also graded. As $h^{2d-1}_{(2)}(\partial B^\circ)$ has the same weight as the top exterior power of the dual of n, i.e., $2d$ (see the proof of lemma (3.2)), this identifies $h^k_{(2)}(\partial B^\circ, \mathbf{E})_l$ with the anti-dual of $h^{2d-1-k}_{(2)}(\partial B^\circ, \mathbf{E})_{2d-l}$.

(3.5) PROPOSITION (Zucker [Z3]). *For $l > d$, $H^k_{(2)}(B^\circ, \mathbf{E})_l$ vanishes, for $l = d$ it is naturally isomorphic to $H^{k-1}_{(2)}(\partial B^\circ, \mathbf{E})_d \otimes H^1_{(2)}(A'_+)$, whereas for $l < d$, it maps isomorphically onto $H^k_{(2)}(\partial B^\circ, \mathbf{E})_l$. Here A' is equipped with a translation invariant metric, and A'_+ denotes the semi-group A' defined by $\chi' \geqslant 1$.*

For our set-up the line of argument given in [CKS] is perhaps most appropriate. Since the result is well covered in the cited references, we limit ourselves to a description of the basic idea: lemma (3.3) allows the construction of a «relative» Green operator, which produces a quasi-isomorphism of $(\mathscr{E}^\cdot_{Z^\circ} \otimes \mathbf{E})(B^\circ)_{(2)}$ onto the subcomplex of forms which are harmonic when restricted to any fibre of $f \circ p$. This subcomplex is invariant under the ψ-action, and thus is graded by that action. Lemma (3.2) then enables us to interpret lth graded piece as the tensor product of the corresponding piece of $H^\cdot_{(2)}(\partial B^\circ, \mathbf{E})$ with $(\mathscr{E}^\cdot_{A'} \otimes \mathbf{L}(l-d))(A'_+)_{(2)}$, where $\mathbf{L}(m)$ denotes the constant local system $A' \times \mathbf{C}$ on A', metrized by the condition that the section "1" has norm $(\chi')^m$. The proposition then follows from the known values of the groups $H^k_{(2)}(A'_+, \mathbf{L}(m))$: they vanish unless $m < 0$ and $k = 0$ (we get $\mathbf{C}$) or $m = 0$ and $k = 1$ (for which we get an infinite dimensional space).

Via (3.4), $IH^k(Z - \{\infty\}, \mathbf{E})$ acquires a grading. Following an idea occurring in [CKS] and [KK], we will show that this grading admits a topological interpretation. This is based on the following lemma.

(3.6) LEMMA. *If $a \in A$ such that $\chi(a)$ is a positive integer with $\chi(a) - 1$ sufficiently divisible, then $a\Gamma_P a^{-1}$ is contained in Γ_P.*

Proof. Let $\gamma \in \Gamma_P$ and write γ according to the decomposition $P = Z_G(A).\exp(\mathfrak{v}).\exp(\mathfrak{u}): \gamma = \gamma_0\gamma_1\gamma_2$. Then $a\gamma a^{-1} = \gamma_0\gamma_1^{\chi(a)}\gamma_2^{\chi(a)^2} = \gamma\gamma_1^{\chi(a)-1}\gamma_2^{\chi(a)^2-1}$. Since the decomposition is defined over $\mathbf{Q}$, $\log(\gamma_1)$ and $\log(\gamma_2)$ are rational elements of n, and hence there exists a positive integer k such that γ_1^k and γ_2^k belong to $\Gamma \cap N$. So if $\chi(a) - 1$ is an integral multiple of k, then $a\gamma a^{-1} \in \Gamma_P$. Since Γ_P is finitely generated, the lemma follows.

Fix $a \in A$ such that $\chi(a) - 1$ is a sufficiently divisible positive integer as in the lemma. Then the lemma implies that the left action of a on $\mathrm{Star}(F) \times E$ respects Γ_P-orbits and (hence) $Z_\Gamma(F)$-orbits. So this induces an endomorphism Φ_a of $(Z, \mathbf{E})$ (in fact, Φ_a can be interpreted as a «local» Hecke correspondence; see the final remarks of Section 4). If we use the right-left action instead $(\omega, e) \to (a \cdot \omega \cdot (a')^{-1}, a \cdot e)$, then the resulting endomorphism Ψ_a is just Φ_a composed with geodesic translation over $(a')^{-1}$. Evidently Ψ_a preserves each fibre of p. The induced endomorphism Ψ_a^* of $p_*(\mathscr{E}^{\cdot}_{Z^\circ} \otimes \mathbf{E})$ is simply ψ_a^{-1}. So Ψ_a^* acts semi-simply on $H^k_{(2)}(\partial B^\circ, \mathbf{E})$, and $H^k_{(2)}(\partial B^\circ, \mathbf{E})_l$ is its $\chi(a)^l$-eigen space. Since the right-left action preserves each boundary component of $\mathrm{Star}(F)$, Ψ_a preserves each stratum of Z; in particular $\Psi_a^{-1}(\infty) = \{\infty\}$.

(3.7) Lemma. *The endomorphisms Φ_a and Ψ_a of Z are finite (i.e., are proper and have finite fibres.).*

Proof. One easily verifies that $\Psi_a^{-1}(B) = B$. Since B is compact, Ψ_a is proper over B. Combining this with the fact that $\Psi_a^{-1}(\infty) = \{\infty\}$, yields that Ψ_a is finite over some neighborhood of ∞ in Z. As Ψ_a commutes with geodesic translation, we may take this neighborhood to be A'-invariant. But clearly such a neighborhood coincides with Z. The corresponding statement for Φ_a follows from the fact that it is the composition of Ψ_a and geodesic translation over a' (which is a homeomorphism of Z onto Z).

The above lemma implies that Ψ_a induces endomorphisms (all denoted by Ψ_a^*) of $IH^k(\partial B^\circ, \mathbf{E})$, $IH^k(B^\circ, \mathbf{E})$ and $IH^k(Z, \mathbf{E})$. Similarly, Φ_a induces endomorphisms Φ_a^* of $IH^k(Z, \mathbf{E})$ and $IH^k(Z - \{\infty\}, \mathbf{E})$. Since the geodesic translations establish an isotopy between Ψ_a and Φ_a, they induce the same endomorphisms of $IH^k(Z, \mathbf{E})$ and $IH^k(Z - \{\infty\}, \mathbf{E})$. The isomorphism $H^k_{(2)}(\partial B^\circ, \mathbf{E}) \approx IH^k(Z - \{\infty\}, \mathbf{E})$ of (3.4) is natural and hence Ψ_a^*-equivariant. It follows that the endomorphism $\Psi_a^* = \Phi_a^*$ of $IH^k(Z - \{\infty\}, \mathbf{E})$ is semi-simple and that under the isomorphism $H^k_{(2)}(\partial B^\circ, \mathbf{E})_l$ corresponds to the $\chi(a)^l$-eigen space of Φ_a^* in $IH^k(Z - \{\infty\}, \mathbf{E})$. Virtually by definition, $IH^k(Z, \mathbf{E})$ maps isomorphically onto $IH^k(Z - \{\infty\}, \mathbf{E})$ for $k > d$ and vanishes otherwise. So Φ_a^* acts also semi-simply on $IH^k(Z, \mathbf{E})$.

Clearly, if A acts on finite dimensional vector space h with character χ^l, then its contragradient (= transpose inverse) representation on the dual H^* of H has character χ^{-l}, whereas the transpose of the automorphism of H induced by a acts on H^* as multiplication by $\chi(a)^l$. This justifies extending our weight conventions to situations where Φ_a^* acts naturally on a vector space, but A possibly no longer does. For instance, such a vector space is said *to have weights* $\leqslant k$, $k \in \mathbf{R}$, if every eigen value of Φ_a^* is of the form $\chi(a)^l$ with l an integer $\leqslant k$.

(3.8) THEOREM (Purity). *The weights of* $IH^k(Z, \mathbf{E})$ (*with respect to* Φ_a^*) *are* $\leqslant k$.

We show how this theorem (whose proof we postpone to section 4) completes the inductive step in the proof of Zucker's conjecture. In view of the reduction discussed in section 1 and lemma (3.3), we must show that $H_{(2)}^k(B^\circ, \mathbf{E})$ maps isomorphically onto $H_{(2)}^k(\partial B^\circ, \mathbf{E})$ for $k < d$ and vanishes otherwise. Referring to proposition (3.5), we see that this follows if we can show that $H_{(2)}^k(\partial B^\circ, \mathbf{E})_l = 0$ for $(k < d, l \geqslant d)$, and $(k \geqslant d, l \leqslant d)$. Because of the (Poincaré) duality property mentioned above, the second vanishing condition follows from the first one. The discussion which preceded (3.8) implies that this is equivalent to $IH^k(Z, \mathbf{E})$ (for $k < d$) having weights $< d$. But this is clearly a consequence of the theorem.

4. Proof of the purity theorem

We shall need a toric resolution $\tilde{Z}$ of Z as described in [AMRT]. Let us write $\mathfrak{u}_{\mathbf{Z}}$ for the lattice $\log(\Gamma \cap U)$ in $\mathfrak{u}$. Let C_+ denote the convex hull of $\mathrm{cl}(C) \cap \mathfrak{u}_{\mathbf{Q}}$. It contains C and according to Ash and Mumford there exists a $\Gamma(\mathfrak{u})$-invariant collection Σ_+ of simplicial cones in C_+ such that

(i) The relative interiors of the members of Σ_+ form a partition of C_+.
(ii) Every face of a member of Σ_+ belongs to Σ_+.
(iii) The intersection of two members of Σ_+ belongs to Σ_+.
(iv) $\Gamma(\mathfrak{u})$ has only finitely many orbits in Σ_+.
(v) Every member of Σ_+ is spanned by part of a basis of $u_{\mathbf{Z}}$.

Condition (iv) implies that Σ_+ is locally finite on C. Since every $\sigma \in \Sigma_+$ contains a canonical «barycentric» half line, namely the one spanned by the sum of its integral generators, we have a well defined barycentric subdivision of Σ_+. It is clear that this barycentric subdivision also satisfies the properties (i)–(v). Since Γ is neat, so is $\Gamma(\mathfrak{u})$, and hence we can (by taking a sufficiently fine subdivision of Σ_+) also arrange that:

(vi) No $\Gamma(\mathfrak{u})$-orbit meets any member of Σ_+ in more than one point, and if $\sigma, \tau \in \Sigma_+$, then $\Gamma(\mathfrak{u})\tau \cap \sigma$ is a face of σ.

So Σ_+ determines a decomposition of $\Gamma(\mathfrak{u}) \setminus C_+$ whose members are isomorphic images of those of Σ_+, and which is closed under intersection. By property (iv) this decomposition is finite. Likewise, the collection $\Sigma := \{\sigma \cap C : \sigma \in \Sigma_+\}$ determines a decomposition of $\Gamma(\mathrm{u})\backslash C$ with similar properties. We denote that decomposition by Σ', and $\sigma \in \Sigma \mapsto \sigma' \in \Sigma'$ will be the obvious map (but occasionally we also denote a general member of Σ' by σ'). Given $\sigma \in \Sigma$, let $\mathrm{St}(\sigma)$ denote the union of the relative interiors of the members of Σ which have a nonempty intersection with σ. This is an open neighborhood of σ in C, and we have $\mathrm{St}(\sigma) \cap \mathrm{St}(\tau) = \mathrm{St}(\sigma \cap \tau)$. Another barycentric subdivision gives:

(vii) For all $\sigma, \tau \in \Sigma$, the $\Gamma(u)$-orbit of $\mathrm{St}(\tau)$ meets $\mathrm{St}(\sigma)$ in a subset of the form $\mathrm{St}(\sigma_1)$.

Then we have defined an open covering $\{\mathrm{St}(\sigma') : \sigma' \in \Sigma'\}$ of $\Gamma\backslash C$ whose members are the isomorphic images of members of $\{\mathrm{St}(\sigma) : \sigma \in \Sigma\}$ and satisfy $\mathrm{St}(\sigma') \cap \mathrm{St}(\tau') = \mathrm{St}(\sigma' \cap \tau')$. So the collection of $\mathrm{St}(\sigma')$ for which σ' has the maximal dimension n already covers.

Let T denote the algebraic torus $(\Gamma \cap U) \otimes_{\mathbf{Z}} \mathbf{C}^\times \approx \mathrm{u}_{\mathbf{C}}/\mathrm{u}_{\mathbf{Z}}$. We assume the reader to be familiar with the fact that Σ determines a nonsingular torus embedding $T_\Sigma \supset T$. The collection Σ_+ of faces of members of Σ indexes the T-orbits in T_Σ: each orbit is of the form $T(\sigma) := \mathrm{u}_{\mathbf{C}}/(\mathrm{u}_{\mathbf{Z}} + \langle\sigma\rangle_{\mathbf{C}})$ for a unique $\sigma \in \Sigma_+$. Let T_σ be the kernel of the natural homomorphism $T \to T(\sigma)$, and let $\hat{T}_\sigma$ denote its closure in T_Σ. Property (v) implies that the pair $(\hat{T}_\sigma, T_\sigma)$ is isomporphic to $(\mathbf{C}^k, (\mathbf{C}^\times)^k)$, where $k = \dim(\sigma)$. The union $\cup\{T(\tau) : \sigma \supset \tau\}$ is a stratified T-invariant neighborhood of $T(\sigma)$ in T_Σ, which naturally retracts onto $T(\sigma)$. This retraction is (algebraically) trivial over $T(\sigma)$ with fibre $\hat{T}_\sigma$. In particular, $T_\Sigma - T$ is a normal crossing divisor on T_Σ.

Let ord: $T \to \mathrm{u}$ be the map $x + iy + \mathrm{u}_{\mathbf{Z}} \mapsto y$ and put $T = \mathrm{ord}^{-1}(C)$ and $T_\sigma = \mathrm{ord}^{-1}(\sigma)$. Notice that both are semigroups. Let T_Σ denote the T_Σ-interior of the closure of T in T. If $\sigma \in \Sigma$ then it is easily seen that T_Σ contains $T(\sigma)$. It will be of interest to consider a certain open coverings of C and T_Σ which we presently describe.

Let $T(\sigma)$ denote the T_Σ-interior of the closure of $\mathrm{ord}^{-1}(\mathrm{St}(\sigma))$ in T_Σ. Then one verifies that:

(a) $T(\sigma)$ is a neighborhood of $T(\sigma)$ in $\cup\{T(\tau) : \sigma \supset \tau\}$,
(b) the natural retraction of $\cup\{T(\tau) : \sigma \supset \tau\}$ onto $T(\sigma)$ restricted to $T(\sigma)$ is topologically locally trivial such that each fibre can be identified with a contractible neighborhood of the punctual stratum of $\hat{T}_\sigma$ in $\hat{T}_\sigma$, and
(c) $\{T(\sigma) : \sigma \in \Sigma\}$ is an open covering of T_Σ such that $T(\sigma) \cap T(\tau) = T(\sigma \cap \tau)$.

Since Σ is $\Gamma(\mathrm{u})$-invariant, T_Σ comes with an action of that group which preserves the above covering. It follows from property (vi) that no orbit meets $T(\sigma)$ in more than one point. So $\Gamma(\mathfrak{u})$ acts freely on T_Σ, its orbit space is an analytic manifold, and every $T(\sigma)$ maps isomorphically onto its image in $\Gamma(\mathfrak{u})\backslash T_\Sigma$.

Let M denote the Γ_N-orbit space of $\mathfrak{v} \times \mathfrak{u}_{\mathbf{C}}$. Notice that M is in a natural way a T-principal bundle over the complex torus $\mathfrak{v}_{\mathbf{Z}}\backslash\mathfrak{v}_{\mathbf{C}}$ (where $\mathfrak{v}_{\mathbf{Z}}$ is of course the image of $\log(\Gamma_N)$ in $\mathfrak{v}_{\mathbf{C}}$). Without any difficulty a relative version of the preceding can be carried out for M (with the map «ord» being replaced by the composition of the diffeomorphism of M onto $(\Gamma_N\backslash N) \times \mathfrak{u}$ and the projection of the latter onto its second factor). Thus we set $M_\Sigma := M \times^T T_\Sigma$, $M := \Gamma_N\backslash\Omega$ (= the pre-image of C under this projection) and M_Σ be the M_Σ-interior of the closure of M in M_Σ. All this comes with a natural $\Gamma(\mathfrak{v})$- action. The stratum $M(\sigma) := M \times^T T(\sigma)$ of M_Σ is acted on by the quotient $N(\sigma)$ of N by its central subgroup $\exp(\langle\sigma\rangle_{\mathbf{R}})$. If we denote $\Gamma_{N(\sigma)}$ the image of Γ_N in $N(\sigma)$, then each orbit is isomorphic to the compact nilmanifold $\Gamma_{N(\sigma)}\backslash N(\sigma)$, and is a deformation retract of $M(\sigma)$. For every $\sigma \in \Sigma$ we get an open subset $M(\sigma)$ of M_Σ satisfying the following properties:

(a) $M(\sigma)$ is a neighborhood of $M(\sigma)$ in $\cup\{M(\tau)\colon \tau \supset \sigma\}$,
(b) the natural retraction of $\cup\{M(\tau)\colon \tau \supset \sigma\}$ onto $M(\sigma)$ restricted to $M(\sigma)$, $r_\sigma\colon M(\sigma) \to M(\sigma)$, is topologically locally trivial and each fibre can be identified with an open convex neighborhood of the punctual stratum of $\hat{T}_\sigma$ in $\hat{T}_\sigma$,
(c) $\{M(\sigma)\colon \sigma \in \Sigma\}$ is an open covering of M_Σ such that $M(\sigma) \cap M(\tau) = M(\sigma \cap \tau)$.

Also note no $\Gamma(\mathfrak{u})$-orbit meets $M(\sigma)$ in more than one point. So $\Gamma(\mathfrak{u})$ acts freely on M_Σ, its orbit space (denoted $\tilde{Z}$) is an analytic manifold, and every $M(\sigma)$ maps isomorphically onto its image (denoted $\tilde{Z}(\Gamma(\mathfrak{u})\,.\,\sigma)$) in $\tilde{Z}$.

In [AMRT] it is shown that the identity map of Z° extends to a resolution $\pi\colon \tilde{Z} \to Z$ of Z, and that for a suitable choice of Σ, π will be even projective. As this last property is invariant under barycentric subdivision, we can assume this to be the case. A straightforward verification shows that the endomorphism Φ_a of Z° extends to one of $\tilde{Z}$, while preserving every $\tilde{Z}(\sigma')$. Also, Φ_a acts on $T(\sigma)$ and $M(\sigma)$, and commutes with r_σ.

(4.1) THEOREM (Purity on $\tilde{Z}$). *The weights of* $IH^k(\tilde{Z}, \mathbf{E})$ *are* $\leqslant k$.

Before we begin the proof, we state and prove a lemma:

LEMMA. *Let Σ° be an n-dimensional simplical complex whose simplical realization $|\Sigma^\circ|$ is a PL-manifold, and let $A: \sigma \to A_\sigma$ be a contravariant functor from the POset of simplices of Σ° to category of abelian groups. Let $C^\cdot$ denote the alternating Cech complex associated to A and the (closed) covering $\{|\sigma|: \dim \sigma = n\}$ of $|\Sigma^\circ|$, and let $C_p^\cdot$ denote the subcomplex of cochains $f \in C^\cdot$ which vanish on ordered tuples $(\sigma_0, \sigma_1, \sigma_2, \ldots)$ with $\operatorname{codim}(\sigma_0 \cap \sigma_1 \cap \sigma_2 \ldots) \leqslant p$. Then the cochain map $C^\cdot \to C^\cdot/C_p^\cdot$ is injective on cohomology in degrees $\leqslant p$.*

Proof. Clearly $C^\cdot = C_{-1}^\cdot \supset C_0^\cdot \supset \cdots \supset C_{n-1}^\cdot \supset C_n^\cdot = 0$. We first notice that $C_p^\cdot/C_{p+1}^\cdot$ splits naturally as a complex: it is the direct product of subcomplexes of the form $\operatorname{Hom}(D_\sigma, A_\sigma)$ (for certain chain complexes D_σ), where σ runs over the simplices of Σ° of codimension $p + 1$. In fact D_σ can be described as follows: if Δ denotes the abstract simplex whose vertex set is the collection n-simplices in Star(σ), and Δ' denotes the subcomplex of Δ consisting of those faces $([\sigma_0], [\sigma_1], \ldots)(\sigma_i \in \operatorname{Star}(\sigma))$ with $\sigma_0 \cap \sigma_1 \cap \cdots \neq \sigma$, and $\Delta_.$ resp. $\Delta'_.$ denote the chain complexes generated them, then D_σ is simply $\Delta_./\Delta'_.$. Now $\Delta_.$ resp. $\Delta'_.$ has the integral homology of Star(σ) resp. Star(σ)-int($|\sigma|$). So $H^k(C_p^\cdot/C_{p+1}^\cdot)$ is the direct product of the relative cohomology groups $H^k(\operatorname{Star}(\sigma), \operatorname{Star}(\sigma)\text{-int}(|\sigma|); A_\sigma)$ where σ runs over the simplices of Σ° of codimension $p + 1$. Since all these summands are trivial in degree $\neq p + 1$, it follows that the natural cochain map from $C_{p+1}^\cdot$ to $C^\cdot/C_p^\cdot$ is injective on cohomology in degrees $\leqslant p$. The lemma follows by induction.

Proof of (4.1). Consider the Cech spectral sequence of the intersection complex relative the covering $\{\tilde{Z}(\sigma'): \dim \sigma' = n\}$:

$$E_1^{p,q} = \oplus IH^q(\tilde{Z}(\sigma_0') \cap \cdots \cap \tilde{Z}(\sigma_p'), \mathbf{E}) \Rightarrow IH^{p+q}(\tilde{Z}, \mathbf{E}), \tag{1}$$

where the sum extends over the collection of oriented (distinct) $(p + 1)$-tuples $(\sigma_0', \ldots, \sigma_p')$ in Σ' of dimension n with non-empty intersections. We apply the lemma above by taking for Σ° the abstract simplicial complex defined by Σ' and for A the functor $\sigma' \mapsto IH^q(\tilde{Z}(\sigma'), \mathbf{E})$. If we let $C^\cdot$ and $C_p^\cdot$ have the meaning of that lemma, then $C^\cdot = E_1^{\cdot,q}$, $H^p(C^\cdot) = E_2^{p,q}$, and $(C^\cdot/C_p^\cdot)$ is obtained from the left hand side of (1) by restricting the summation to those $(p + 1)$-tuples $(\sigma_0', \ldots, \sigma_p')$ which have the property that $\operatorname{codim}(\sigma_0' \cap \cdots \cap \sigma_p') \leqslant p$. In view of the lemma only such summands contribute to $E_2^{p,q}$. As Φ_a^* acts on the sequence (1), and respects each summand of $E_1^{p,q}$, it therefore suffices to show that $IH^q(\tilde{Z}(\sigma'), \mathbf{E})$ has weights $\leqslant \operatorname{codim}(\sigma') + q$. We choose σ over σ' so that $\tilde{Z}(\sigma')$ may be identified with $M_\Sigma(\sigma)$, and prove the corresponding result for $M_\Sigma(\sigma)$.

Remember that r_σ exhibits the pair $(M_\Sigma(\sigma), M_\Sigma(\sigma) \cap M)$ as a topologically locally trivial fibration over $M(\sigma)$ with fibre a neighborhood of the punctual stratum in $(\hat{T}_\sigma, T_\sigma)$. Consider the Leray sequence for r_σ:

$$E_2^{p,q} = H^p(M(\sigma), R^q r_{\sigma *} \mathscr{I}\mathscr{C}^{\cdot}_\sigma(\mathbf{E})) \Rightarrow IH^{p+q}(M_\Sigma(\sigma), \mathbf{E}), \tag{2}$$

where $\mathscr{I}\mathscr{C}^{\cdot}_\sigma(\mathbf{E})$ denotes «the» intersection complex of $\hat{T}_\sigma$ with values in $\mathbf{E} \mid T_\sigma$. Now $R^q r_{\sigma *} \mathscr{I}\mathscr{C}^{\cdot}(\mathbf{E})$ is a local system over $M(\sigma)$ with stalk $IH^q(\hat{T}_\sigma, \mathbf{E})$. Since $M(\sigma)$ admits a compact nilmanifold $\approx \Gamma_{N(\sigma)} \backslash N(\sigma)$ as a deformation retract, we can replace in the spectral sequence $M(\sigma)$ by that nilmanifold. This identifies $E_2^{p,q}$ with pth (Lie algebra) cohomology group of the standard complex $\Lambda^{\cdot} \mathfrak{n}(\sigma)^* \otimes IH^q(\hat{T}_\sigma, \mathbf{E})$, where $\mathfrak{n}(\sigma)$ acts on $IH^q(\hat{T}_\sigma, \mathbf{E})$ in such a way that it is compatible with the monodromy representation of $\Gamma_{N(\sigma)}$. The obvious Φ_a^*-actions on $\mathfrak{n}(\sigma)$, $\hat{T}_\sigma$ and $\mathbf{E} \mid T_\sigma$ determine one on the standard complex and it is easily seen that the induced action on its cohomology is equivariant with respect to the previous identifications.

At this point we wish to apply the purity theorem of Cattani–Kaplan–Schmid [CKS:1.13] and Kashiwara–Kawai [KK:4.0.1], and conclude that $IH^q(\hat{T}_\sigma, \mathbf{E})$ has weights $\leqslant q$. Their hypotheses require $\mathbf{E} \mid T_\sigma$ to be a polarized variation of complex Hodge structure with unipotent monodromy. (In [CKS] it is assumed that the variation of Hodge structure is real, but this is of no consequence—e.g. pass to $\mathbf{E} \otimes_{\mathbf{R}} \mathbf{C}$.) Following Zucker [Z2], $\mathbf{E}$ is a polarized variation of complex Hodge structure on X°, and the same holds therefore for $\mathbf{E} \mid T_\sigma$. If we identify the fundamental group of T_σ with $\langle \sigma \rangle_{\mathbf{R}} \cap u_{\mathbf{Z}}$, then the monodromy corresponding to $u \in \langle \sigma \rangle_{\mathbf{R}} \cap \mathfrak{u}_{\mathbf{Z}}$ is given by the action of $\exp(u)$ on E, in particular it is unipotent. Let $\sigma \cap \mathfrak{u}_{\mathbf{Z}}$ be spanned (as a semi-group) by the linear independent elements $u_1, \ldots, u_k$ (so $k = \dim(\sigma)$), and denote by $N_1, \ldots, N_k$ the endomorphisms of E they induce as elements of g. Then $N_1, \ldots, N_k$ commute and are nilpotent. Following Deligne (cf. [KK: §3]), $IH^q(\hat{T}_\sigma, \mathbf{E})$ is naturally isomorphic to the qth cohomology group of the subcomplex

$$\oplus \{u_{j_1}^* \wedge \cdots \wedge u_{j_l}^* \otimes N_{j_1} \ldots N_{j_l}(E) \colon 1 \leqslant j_1 < \cdots < j_l \leqslant k\}, \tag{3}$$

of the Kosul complex $\Lambda^{\cdot}(\langle \sigma \rangle_{\mathbf{R}}^*) \otimes E$. Since the *transpose* action of a on $\langle \sigma \rangle_{\mathbf{R}}^*$ is just multiplication with $\chi(a)^2$, the purity theorem quoted above asserts that the eigen values of Φ_a^* on $IH^q(\hat{T}_\sigma, \mathbf{E})$ are as claimed.

If we split $\mathfrak{n}(\sigma) = \mathfrak{v} \oplus \mathfrak{u}(\sigma)$ as an A-representation, then it is immediate from the fact that $\mathfrak{w}^*$ resp. $\mathfrak{u}(\sigma)^*$ has weight 1 resp. 2, that $\Lambda^p \mathfrak{n}; \sigma)^*$ has weights $\leqslant (p - r) + 2r = p + r$, with $r \leqslant \dim \mathfrak{u}(\sigma) = \operatorname{codim}(\sigma)$. Hence the Lie algebra cohomology group $H^p(\mathfrak{n}(\sigma), IH^q; \hat{T}_\sigma))$ has weights

$\leqslant \operatorname{codim}(\sigma) + p$. Feeding this in the spectral sequence (2) gives that $IH^r(M_\Sigma(\sigma), \mathbf{E})$ has weights $\leqslant \operatorname{codim}(\sigma) + r$. The proposition now follows since it had already been reduced to this last result.

We can now prove the purity theorem (3.8). We derive it from purity on $\tilde{Z}$ by means of the decomposition theorem.

Proof of (3.8). As is well known, the representation E admits some number field $K \supset \mathbf{Q}$ as a field of definition. Hence it occurs as a direct summand in the finite dimensional representation $E(K) \otimes_{\mathbf{Q}} \mathbf{C}$, which is defined over $\mathbf{Q}$. So without loss of generality we may assume that $K = \mathbf{Q}$. Then $\mathbf{E}$ acquires the structure of a variation of polarized $\mathbf{Q}$-Hodge structure. Recalling that $\pi: \tilde{Z} \to Z$ is projective, it follows from *M.* Saito's version of the decomposition theorem ([S3: remark (5.4)] and [S4: Thm. 3.21]), that $IH^k(Z, \mathbf{E})$ is a naturally (in particular Φ_a^*-equivariantly) isomorphic to a subquotient of $IH^k(\tilde{Z}, \mathbf{E})$. Hence purity on $\tilde{Z}$ (4.1) implies purity on Z (3.8).

The proofs of the above purity results also enable us to identify the weight filtration on $IH^\cdot(Z, \mathbf{E})$ and $IH^\cdot(\tilde{Z}, \mathbf{E})$ in the sense of mixed Hodge theory.

(4.2) Proposition. *If m is the weight of* $\mathbf{E}$ *as a variation of Hodge structure, then the eigen spaces of* Φ_a^* *split these mixed Hodge structures into pure Hodge structures: the weight l subspace of either cohomology group is a pure Hodge structure of weight* $l + m$. *If E can be defined over a certain subfield of* $\mathbf{C}$, *then the splitting is defined over the same field (Remember however that the splitting depends on the lift A of A′, and therefore might not expected to be canonical!).*

Proof. Since $a \in G(\mathbf{Q})$, its eigen space decomposition in E is defined over the same field as E and so the last clause is clear. Hence as in the proof of (3.8), we may without loss of generality assume that E is defined over $\mathbf{Q}$, so that E can be regarded as a variation of polarized $\mathbf{Q}$-Hodge structure. Let us now return to the proof of (4.1). Fix some $\sigma \in \Sigma$. Any half line in C (e.g., the barycentric halfline of σ) determines a (limiting) mixed Hodge structure on E. According to [CKS], its underlying weight filtration is precisely the filtration defined by the A-action, shifted over m. If we give u the pure Hodge structure of type $(-1, -1)$, then the Koszul complex $\Lambda^\cdot(\langle\sigma\rangle_{\mathbf{R}})^* \otimes E$ is a complex of mixed Hodge structures, and so is its subcomplex (4.1-3). This puts a mixed Hodge structure on $IH^q(\hat{T}_\sigma, \mathbf{E})$. It is clear that the two weight filtrations still only differ by a shift over m. Since Φ_a induces in $IH^q(\hat{T}_\sigma, \mathbf{E})$ an endomorphism of mixed Hodge structure of (by functoriality or by a straightforward computation), each eigen space $IH^q(\hat{T}_\sigma, \mathbf{E})_l$ is a mixed Hodge summand. From the preceding it follows that it is pure of weight $l + m$. We next give $\mathfrak{v}$ the pure Hodge structure of type $((-1, 0), (0, -1))$

defined by its complex structure J (see section 2). Since the bilinear map $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{u}$ induced by the complexified Lie bracket of n is J-invariant, the corresponding map $\mathfrak{v} \otimes \mathfrak{v} \to \mathfrak{u}$ is a morphism of Hodge structures. The A-eigen space decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{u}$ gives n a (split) mixed Hodge structure, and similarly for $\mathfrak{n}(\sigma)$. This induces a mixed Hodge structure on the Lie algebra cohomology group $H^p(\mathfrak{n}(\sigma), IH^q(\hat{T}_\sigma))$, for we have one on the standard complex by means of which it is computed. It is clear that $H^p(\mathfrak{n}(\sigma), IH^q(\hat{T}_\sigma))_l$ is a pure direct summand of weight $l + m$. Since the spectral sequences (4.1–1.2) are spectral sequences of mixed Hodge structures with Φ_a^*-action (with in fact the first sequence degenerating at the E_2-term), the proposition follows for $IH^k(\tilde{Z}, \mathbf{E})$. According to Saito [S1, 3], $IH^k(Z, \mathbf{E})$ is a direct summand of $IH^k(\tilde{Z}, \mathbf{E})$ in the mixed Hodge category and so the proof is complete.

In view of the identification of L^2-cohomology as intersection cohomology, the previous proposition implies that $H^k_{(2)}(\partial B^\circ, \mathbf{E})_l$ carries a natural pure Hodge structure of weight $l + m$. It would be interesting to describe its Hodge filtration in «classical terms», e.g., in terms of automorphic forms.

Final remarks. Recall that the *Hecke algebra* $H(G(\mathbf{Q}), \Gamma)$ defined by Γ in $G(\mathbf{Q})$ is the space complex valued functions on $G(\mathbf{Q})$, which are constant on double cosets modulo Γ and have support in a finite union of such cosets. The product is defined by convolution relative a Haar measure which is assigns 1 to Γ-left cosets. As is well known, each double coset GgG, $g \in G(\mathbf{Q})$, defines a correspondence of X to itself:

$$X = \Gamma\backslash D^* \xrightarrow{\cong} g\Gamma g^{-1}\backslash D^* \longleftarrow (g\Gamma g^{-1} \cap \Gamma)\backslash D^* \longrightarrow \Gamma\backslash D^* = X,$$

(where the first map is induced by translation over g and the other arrows are the obvious ones). If we replace D^* by $D \times E$ (with $G(\mathbf{Q})$ acting diagonally on it), then we see that this correspondence lifts to – what we might still call – a correspondence of the pair (X, E). As the maps in the above diagram are finite, this correspondence induces an endomorphism of every sufficiently natural incarnation of the intersection complex on X with values in $\mathbf{E}$ and the resulting endomorphism $IH^{\cdot}(X, \mathbf{E})$ is independent of that incarnation. This defines a representation of $H(G(\mathbf{Q}), \Gamma)$ in $IH^{\cdot}(X, \mathbf{E})$ (compare [BL]). Similarly $H(G(\mathbf{Q}), \Gamma)$ acts on $H^{\cdot}_{(2)}(X, \mathbf{E})$. Since the Hecke algebra already acts on the complex $\mathscr{L}^{\cdot}_X(\mathbf{E})$, it follows that the identification of $H^{\cdot}_{(2)}(X, \mathbf{E})$ with $IH^{\cdot}(X, \mathbf{E})$ (resulting from our main theorem) is $H(G(\mathbf{Q}), \Gamma)$-equivariant.

The same construction can be carried out locally. For instance, we find a representation of the Hecke algebra $H(Z_{G(\mathbf{Q})}(F), Z_\Gamma(F))$ in $IH^{\cdot}(Z, \mathbf{E})$ (or more intrinsically in the local intersection cohomology group at ∞ with values in $\mathbf{E}$, to which $IH^{\cdot}(Z, \mathbf{E})$ maps isomorphically) and likewise on $H^{\cdot}(\mathscr{L}^{\cdot}_Z(\mathbf{E})(Z))$. Here too, the identification between these cohomology is Hecke-equivariant. It is worth noting that if $a \in A(\mathbf{Q})$ is as in the lemma (3.6), then the local correspondence of (Z, ∞) to itself defined by it, is in fact a morphism and equals Φ_a.

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