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The Abel transform and shift operators

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1. Introduction

Let G be a noncompact connected real semisimple Lie group with finite center and Iwasawa decomposition $G = KAN$. Let $\mathfrak{a}$ denote the Lie algebra of A. Let Σ denote the set of restricted roots and Σ^+ a choice of positive roots. The Weyl group of Σ will be denoted by W. Put $\varrho = \frac{1}{2}\Sigma_{\alpha\in\Sigma^+} m_\alpha\alpha$, where m_α is the multiplicity of the root α. Let $\mathscr{D}(G//K)$ and $\mathscr{D}_W(A)$ denote the spaces of K-biinvariant C^∞-functions on G with compact support and W-invariant C^∞-functions on A with compact support respectively. For $f \in \mathscr{D}(G//K)$ we define the Abel transform $\mathscr{A}: f \to F_f$ by

$$(\mathscr{A}f)(a) = F_f(a) = \mathrm{e}^{\varrho(\log a)} \int_N f(an)\,\mathrm{d}n, \quad a \in A,$$

where log denotes the inverse of the mapping exp: $\mathfrak{a} \to A$. This transform plays an important role in the theory of the spherical Fourier transform. It is well-known that the Abel transform is a linear homeomorphism of $\mathscr{D}(G//K)$ onto $\mathscr{D}_W(A)$. For the rank-one ([11, 12, 15]) and the complex ([5, 15]) case an explicit inversion of the Abel transform is known. Besides the complex case the only higher rank case where an explicit inversion is known is the case $G = SU(p, q)$ ([13]). In [1] Aomoto determined an explicit integral representation for the Abel transform for $G = SL(n, \mathbb{C})$ and $SL(n, \mathbb{R})$ where for all $\alpha \in \Sigma$ we have $m_\alpha = 2$ and $m_\alpha = 1$ respectively. In [2, Ch. III] we extended his results to the case $G = SU^*(2n)$ where $m_\alpha = 4$ for all $\alpha \in \Sigma$. Note that these three cases all have associated root systems of type A_{n-1}. For $G = SL(3, \mathbb{R})$ Aomoto [1] uses the explicit integral representation to solve the inversion problem, but this solution is not explicit.

In the first part of this paper (Sections 3, 4 and 5) we show how certain known differential operators associated with root systems (the so-called "lowering/raising" or "shift" operators) can be applied in the theory of the Abel transform. In Section 3 we assume the existence of such operators and show their relation with the Abel transform. The actual existence of the operators is the subject of Section 4, where we also give some historical remarks. In Section 5 we concentrate on the case of a root system of type A_2. The existence of the operator, and an explicit expression in "z-coordinates", was established by Vretare [18]. This leads to an inversion of the Abel transform if the root multiplicity is even, i.e., for $G = \mathrm{SL}(3, \mathbb{C})$, $\mathrm{SU}^*(6)$ and $E_{6(-26)}$ (Theorem 1). In general it is still an open problem if the Abel transform can be inverted by a differential operator if all multiplicities are even (for A_3 see below). We will also present the operator in the usual "t-coordinates". This enables us to apply the operator to the Abel transform in Section 6.

In the second part of this paper (Section 6) we start with the explicit integral representation for the Abel transform for $\mathrm{SL}(3, \mathbb{R})$, $\mathrm{SL}(3, \mathbb{C})$ and $\mathrm{SU}^*(6)$ mentioned above. In this integral representation the multiplicity $m = m_\alpha$ occurs as parameter. Write F_f^m to emphasize the dependence on m. We then use the integral representation to define F_f^m for $m \in \mathbb{C}$, $\mathrm{Re}\, m > 0$. If we also use the explicit expression for the shift operator in the "t-coordinates" from Section 5 then we obtain (Theorem 2) a generalization of Theorem 1. There should be a relation between the results on the transform F_f^m and the results (stated without proof) of Sekiguchi in [16] for the root system of type A_2, where e.g., an integral representation for the spherical functions ϕ_λ^m is given. We also mention the work of Heckman and Opdam [7] where, for arbitrary root systems, the spherical functions ϕ_λ^m are constructed. It would be of interest to have much more results on the transform F_f^m for general m, eventually leading to a theory of the spherical Fourier transform independent of the symmetric spaces (cf. the rank-one case in Koornwinder [11]).

The paper closes with two appendices. In Appendix 1 we give the shift operator for A_3 in the "z-coordinates". This leads to the inversion of the Abel transform for $G = \mathrm{SL}(4, \mathbb{C})$ and $\mathrm{SU}^*(8)$ (Theorem 1′). For the proof that we indeed have a shift operator one needs in particular the radial part of the Laplace–Beltrami operator for the root system A_3 in the "z-coordinates", i.e., we have to translate this operator from the usual "t-" to the "z-coordinates". Since there is not much difference in the calculations for A_3 or A_ℓ, we give this operator in the "z-coordinates" for A_ℓ in Appendix 2.

2. Preliminaries

For all unproved statements we refer to Helgason [8, 9]. As usual let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ denote the sets of all complex numbers, real numbers and integers respectively.

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and put $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+$, $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Let G be a noncompact connected real semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G and $\langle \cdot, \cdot \rangle$ the Killing form of $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition and K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its (real) dual, $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$ put $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. If $\lambda \neq 0$ and $\dim \mathfrak{g}_\lambda \neq 0$ then λ is called a (restricted) root and $m_\lambda = \dim \mathfrak{g}_\lambda$ is called its multiplicity. The set of restricted roots will be denoted by Σ. If $\lambda, \mu \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for $H \in \mathfrak{a}$ and put $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$ and call a root positive if it is positive on $\mathfrak{a}^+$. The corresponding Weyl chamber in $\mathfrak{a}^*$ will be denoted by $\mathfrak{a}^*_+$ and the corresponding basis of Σ will be denoted by Δ. Let Σ^+ be the set of positive roots; for $\lambda \in \Sigma^+$ we will also use the notation $\lambda > 0$. Put $\varrho = \frac{1}{2} \Sigma_{\alpha > 0} m_\alpha \alpha$. Let $\Sigma_0 = \{\alpha \in \Sigma \mid \frac{1}{2}\alpha \notin \Sigma\}$ and put $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$. Let $\mathfrak{n} = \Sigma_{\alpha > 0} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ and let N, $\bar{N}$ denote the corresponding analytic subgroups of G. Let $A = \exp \mathfrak{a}$ and log the inverse of the map exp: $\mathfrak{a} \to A$. If $\ell = \dim \mathfrak{a}$ then ℓ is called the real rank of G and the rank of the symmetric space $X = G/K$. Let $H\colon G \to \mathfrak{a}$ be the Iwasawa projection according to the Iwasawa decomposition $G = KAN$, i.e., if $g \in G$ then $H(g)$ is the unique element in $\mathfrak{a}$ such that $g \in K \exp H(g) N$. For an arbitrary subset E of Σ put $-E = \{-\alpha \mid \alpha \in E\}$. If V is a finite set then $|V|$ will denote its cardinality. We put $w = |W|$ where W is the Weyl group of Σ. If $s \in W$ then $\varepsilon(s)$ will denote the determinant of s. Let $C_1, C_2, \ldots, C_r$ be the conjugacy classes in Σ under the action of the Weyl group W. Put $m = (m_1, m_2, \ldots, m_r)$ where m_i is the multiplicity m_α of a root α in C_i $(i = 1, 2, \ldots, r)$. We shall call m a multiplicity function on Σ.

We normalize the Lebesgue measures $\mathrm{d}a$ and $\mathrm{d}\lambda$ on A and $\mathfrak{a}^*$ such that for the Fourier transform

$$(\mathscr{F}f)(\lambda) = f^*(\lambda) = \int_A f(a)\, \mathrm{e}^{-i\lambda(\log a)}\, \mathrm{d}a, \quad \lambda \in \mathfrak{a}^*$$

we have the inversion formula

$$f(a) = \int_{\mathfrak{a}^*} f^*(\lambda)\, \mathrm{e}^{i\lambda(\log a)}\, \mathrm{d}\lambda, \quad a \in A, f \in \mathscr{S}(A).$$

Here $\mathscr{S}(A)$ denotes the space of rapidly decreasing functions on A. On the compact group K the Haar measure $\mathrm{d}k$ is normalized such that the total measure is 1. The Haar measures of the nilpotent groups N, $\bar{N}$ are normalized such that $\theta(\mathrm{d}n) = \mathrm{d}\bar{n}$ and

$$\int_{\bar{N}} \mathrm{e}^{-2\varrho(H(\bar{n}))}\,\mathrm{d}\bar{n} = 1.$$

The Haar measure $\mathrm{d}g$ on G can be normalized such that

$$\int_G f(g)\,\mathrm{d}g = \int_{\mathrm{KAN}} f(kan)\,\mathrm{e}^{2\varrho(\log a)}\,\mathrm{d}k\mathrm{d}a\mathrm{d}n, \quad f \in \mathscr{D}(G).$$

Here $\mathscr{D}(G)$ denotes the space of C^∞-functions on G with compact support.

The spherical functions on G are the functions

$$\phi_\lambda(g) = \int_K \mathrm{e}^{(\mathrm{i}\lambda - \varrho)(H(gk))}\,\mathrm{d}k, \quad g \in G,$$

where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is arbitrary; moreover $\phi_\lambda = \phi_\mu$ if and only if $\lambda = s\mu$ for some $s \in W$. Let $\mathscr{D}(G//K)$ denote the subspace of $\mathscr{D}(G)$ consisting of functions bi-invariant under K. For $f \in \mathscr{D}(G//K)$ define its spherical Fourier transform $\tilde{f}$ by

$$\tilde{f}(\lambda) = \int_G f(g)\,\phi_\lambda(g^{-1})\,\mathrm{d}g, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Then

$$f(g) = w^{-1}\int_{\mathfrak{a}^*} \tilde{f}(\lambda)\,\phi_\lambda(g)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda, \quad g \in G,$$

where $|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda)$ for $\lambda \in \mathfrak{a}^*$ and $\mathbf{c}(\lambda)$ is Harish–Chandra's $\mathbf{c}$-function

$$\mathbf{c}(\lambda) = \int_{\bar{N}} \mathrm{e}^{-(\mathrm{i}\lambda + \varrho)(H(\bar{n}))}\,\mathrm{d}\bar{n}, \quad \mathrm{Re}\,(\mathrm{i}\lambda) \in \mathfrak{a}_+^*.$$

The function $\mathbf{c}(\lambda)$ can be continued as a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ by

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\langle \mathrm{i}\lambda, \alpha_0\rangle}\,\Gamma(\langle \mathrm{i}\lambda, \alpha_0\rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle \mathrm{i}\lambda, \alpha_0\rangle))\,\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle \mathrm{i}\lambda, \alpha_0\rangle))}, \tag{2.1}$$

where $\alpha_0 = \alpha/\langle\alpha, \alpha\rangle$ and the constant c_0 is given by $\mathbf{c}(-\mathrm{i}\varrho) = 1$.

For $f \in \mathscr{D}(G//K)$ we define the Abel transform $\mathscr{A}: f \to F_f$ by

$$(\mathscr{A}f)(a) = F_f(a) = \mathrm{e}^{\varrho(\log a)}\int_N f(an)\,\mathrm{d}n, \quad a \in A.$$

The function F_f is W-invariant and

$$\tilde{f}(\lambda) = (\mathscr{F} \circ \mathscr{A})(\lambda) = \int_A F_f(a)\, \mathrm{e}^{-\mathrm{i}\lambda(\log a)}\, \mathrm{d}a, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Let $\mathscr{D}_W(A)$ denote the space of W-invariant C^∞-functions on A with compact support. It is well-known that the Abel transform $f \to F_f$ is a linear homeomorphism of $\mathscr{D}(G//K)$ onto $\mathscr{D}_W(A)$; moreover, $*$ denoting convolution on G and on A, $F_{f*g} = F_f * F_g$ (see e.g., [9, Ch IV, Corr. 7.4]).

If $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ then e^ν denotes the function on A given by $\mathrm{e}^\nu(a) = \mathrm{e}^{\nu(\log a)}$, $a \in A$. Now let $\Delta(L_X)$ be the radial part of the Laplace–Beltrami operator for the action of K on G/K. Then one has the following explicit expression for $\Delta(L_X)$ (see e.g., [9, Ch II, prop. 3.9]):

$$\Delta(L_X) = L_A + \sum_{\alpha > 0} m_\alpha (\coth \alpha) A_\alpha. \tag{2.2}$$

Here L_A denotes the ordinary Laplacian on A and A_α is considered as a differential operator on $A^+ = \exp(\mathfrak{a}^+)$.

Let Λ be the set of all linear combinations $\Sigma_{\alpha \in \Delta}\, n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}^+$. For $\mu \in \Lambda$, $\mu \neq 0$ let σ_μ denote the hyperplane in $\mathfrak{a}_{\mathbb{C}}^*$ given by

$$\sigma_\mu = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* | \langle \mu, \mu \rangle = 2\mathrm{i}\langle \mu, \lambda \rangle\}.$$

If $\lambda \notin \sigma_\mu$ ($\mu \in \Lambda \backslash \{0\}$) then there is a unique solution Φ_λ on A^+ of the equation

$$\Delta(L_X)\Phi_\lambda = -(\langle \lambda, \lambda \rangle + \langle \varrho, \varrho \rangle)\Phi_\lambda \tag{2.3}$$

of the form

$$\Phi_\lambda = \mathrm{e}^{\mathrm{i}\lambda - \varrho} \sum_{\mu \in \Lambda} \Gamma_\mu(\lambda)\, \mathrm{e}^{-\mu}, \tag{2.4}$$

where $\Gamma_0 \equiv 1$, Γ_μ is defined recursively (see e.g. [9, Ch IV, §5, (12)]) and the series is absolutely convergent on A^+. Let $\mathfrak{a}'$ denote the set of regular elements in $\mathfrak{a}^*$, i.e., $\mathfrak{a}' = \{\lambda \in \mathfrak{a}^* | \langle \lambda, \alpha \rangle \neq 0\ \forall\, \alpha \in \Sigma\}$. For $\lambda \in \mathfrak{a}'$ the spherical function ϕ_λ is a linear combination

$$\phi_\lambda = \sum_{w \in W} \mathbf{c}(w\lambda)\Phi_{w\lambda} \quad \text{on } A^+. \tag{2.5}$$

Take $H \in \mathfrak{a}^+$ and consider for $\lambda \in \mathfrak{a}^*$

$$\Phi_\lambda(\exp H) = \mathrm{e}^{(\mathrm{i}\lambda - \varrho)(H)} \sum_{\mu \in \Lambda} \Gamma_\mu(\lambda)\, \mathrm{e}^{-\mu(H)}.$$

If $H' \in \mathfrak{a}^+$ then there exists a constant $K_{H'}$ such that

$$|\Gamma_\mu(\lambda)| \leqslant K_{H'}\, e^{\mu(H')} \quad \text{for} \quad \mu \in \Lambda,\ \lambda \in \mathfrak{a}^* \tag{2.6}$$

(see e.g. [9, Ch IV, Lemma 5.6]). Consequently (take $H' = \frac{1}{2}H$, $H \in \mathfrak{a}^+$)

$$|\Phi_\lambda\,(\exp H)| \leqslant C_H \quad \text{for} \quad \lambda \in \mathfrak{a}^*, \tag{2.7}$$

where C_H is a constant. We shall use these estimates in Section 3.

3. The Abel transform and shift operators

For the spherical Fourier transform $f \to \tilde{f}$ ($f \in \mathscr{D}(G//K)$) we have the following inversion formula:

$$f(g) = w^{-1} \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\phi_\lambda(g)|\mathbf{c}(\lambda)|^{-2}\, \mathrm{d}\lambda, \quad g \in G.$$

From (2.5), the identity $|\mathbf{c}(\lambda)|^2 = \mathbf{c}(w\lambda)\mathbf{c}(-w\lambda)\ \forall\ w \in W$ and the W-invariance of $\tilde{f}$ we obtain

$$f(a) = w^{-1} \int_{\mathfrak{a}^*} \sum_{w \in W} \tilde{f}(w\lambda)\Phi_{w\lambda}(a)\mathbf{c}(-w\lambda)^{-1}\, \mathrm{d}\lambda, \quad a \in A^+.$$

According to the Paley–Wiener theorem for the spherical transform, the function $\tilde{f}$ satisfies for each $N \in \mathbb{Z}^+$ the inequality

$$\sup_{\lambda \in \mathfrak{a}^*} |\tilde{f}(\lambda)|(1 + |\lambda|)^N < \infty. \tag{3.1}$$

For the $\mathbf{c}$-function we have an estimate

$$|\mathbf{c}(\lambda)|^{-1} \leqslant B_1 + B_2|\lambda|^p, \quad \lambda \in \mathfrak{a}^*, \tag{3.2}$$

where B_1 and B_2 are positive constants and $p = \frac{1}{2}\dim \mathfrak{n}$ (for these results see e.g., [9, Ch IV, §7]). These two estimates together with (2.6) prove the convergence of

$$\int_{\mathfrak{a}^*} \tilde{f}(\nu)\Phi_\nu(a)\mathbf{c}(-\nu)^{-1}\, \mathrm{d}\nu, \quad a \in A^+.$$

Consequently

$$f(a) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\Phi_\lambda(a)\mathbf{c}(-\lambda)^{-1}\, \mathrm{d}\lambda, \quad a \in A^+.$$

Since $\tilde{f}(\lambda) = (F_f)^*(\lambda)$ we obtain

$$f(a) = \int_{\mathfrak{a}^*} (F_f)^*(\lambda)\Phi_\lambda(a)\mathbf{c}(-\lambda)^{-1}\,\mathrm{d}\lambda, \quad a \in A^+. \tag{3.3}$$

Suppose there exists a differential operator D on A^+, independent of λ, such that

$$\Phi_\lambda(a)\mathbf{c}(-\lambda)^{-1} = D(\mathrm{e}^{\mathrm{i}\lambda(\log a)}), \quad \lambda \in \mathfrak{a}^*,\ a \in A^+. \tag{3.4}$$

Then it would follow from (3.3) that

$$f(a) = DF_f(a), \quad a \in A^+, \tag{3.5}$$

if we can interchange differentiation and integration.

In (3.4) the function $\mathrm{e}^{\mathrm{i}\lambda}$ can be considered as Φ_λ for multiplicity function $m \equiv 0$: $\Delta(L_X)$ reduces to the ordinary Laplacian on A if one takes $m = 0$ in (2.2). Therefore one can view the operator D as "shifting" from multiplicity function m to mulplicity function $\equiv 0$. One could then try to break up the search for such an operator into the search for operators with a "smaller shift from multiplicity function m to $m - n$". Since we thus want to vary the multiplicity function m on the root system Σ we shall write from now on

$$\varrho(m) = \frac{1}{2}\sum_{\alpha > 0} m_\alpha \alpha \quad \text{and} \quad L(m) = \Delta(L_X).$$

Let $m\colon \Sigma \to \mathbb{C}$ be an arbitrary W-invariant function. Write $m_\alpha = m(\alpha)$, $\alpha \in \Sigma$ (we will also represent such a function by a vector $m = (m_1, \ldots, m_r)$ in $\mathbb{C}^r$ as in section 2). Then we can use the explicit expression (2.2) for $\Delta(L_X)$ to define $L(m)$ for such a function m (of course $\varrho(m)$ is defined as above). Following Heckman and Opdam [7, §3] one can then define the function Φ_λ^m as the unique solution on A^+ of equation (2.3) and of the form (2.4) with Γ_0 and Γ_μ as before; the proof (see e.g., [9, Ch IV, §5]) does not depend on the value of the parameter m. If $m_\alpha \geqslant 0\ \forall\ \alpha$ then we also define $\mathbf{c}^m(\lambda)$ using the explicit expression (2.1) for the $\mathbf{c}$-function. If $m \equiv 0$ then we take

$$\phi_\lambda^0(a) = \sum_{w \in W} \mathbf{c}^0(w\lambda)\,\mathrm{e}^{\mathrm{i}w\lambda(\log a)}.$$

Since we normalize ϕ_λ by $\phi_\lambda(e) = 1$ we obtain that $\Sigma_{w \in W}\,\mathbf{c}^0(w\lambda) = 1$. By (2.1) the $\mathbf{c}$-function reduces to a constant for $m \equiv 0$. Hence $\mathbf{c}^0(\lambda) \equiv w^{-1}$. Also note that (2.6) and (3.2) still hold for $m_\alpha \geqslant 0$ (with all constants depending on m). If the numbers m_α are equal to the root multiplicities corresponding to a symmetric space of the noncompact type, then we shall say that "m corresponds to a group-case".

Motivated by (3.3) we define the transform $\mathscr{B}^m$, for a W-invariant function $m\colon \Sigma \to \mathbb{R}^+$, by

$$(\mathscr{B}^m g)(a) = \int_{\mathfrak{a}^*} g^*(\lambda)\Phi_\lambda^m(a)\mathbf{c}^m(-\lambda)^{-1}\,\mathrm{d}\lambda, \quad a \in A^+,\ g \in \mathscr{D}_W(A).$$

If we use the Paley–Wiener theorem for the Euclidean Fourier transform then the convergence of this integral follows as before from (2.6) and (3.2). If m corresponds to a group-case then $f \to F_f^m = \mathscr{A}^m f$ is a linear isomorphism from $\mathscr{D}(G//K)$ onto $\mathscr{D}_W(A)$ and thus, by (3.3), $\mathscr{B}^m = (\mathscr{A}^m)^{-1}$ on A^+. Now let $\mathscr{S}(\mathfrak{a})$ denote the symmetric algebra over $\mathfrak{a}$, $\mathbb{D}(A)$ the algebra of all differential operators on A with constant coefficients and $\partial\colon \mathscr{S}(\mathfrak{a}) \to \mathbb{D}(A)$ the isomorphism which sends $p \in \mathscr{S}(\mathfrak{a})$ to the corresponding operator $\partial(p)$. In order to justify (3.5) and, more generally, (3.7) below, we need the following lemma.

LEMMA 1. *Let $p \in \mathscr{S}(\mathfrak{a})$. Then*

$$[\partial(p)(\mathscr{B}^m g)](a) = \int_{\mathfrak{a}^*} g^*(\lambda)(\partial(p)\Phi_\lambda^m)(a)\mathbf{c}^m(-\lambda)^{-1}\,\mathrm{d}\lambda.$$

Proof

$$(\partial(p)\Phi_\lambda^m)(a) = \sum_{\mu\in\Lambda} \Gamma_\mu^m(\lambda)p(\mathrm{i}\lambda - \varrho(m) - \mu)\,\mathrm{e}^{\mathrm{i}\lambda-\varrho(m)-\mu},$$

so by (2.6) (take $H' = \frac{1}{2}H$, $H \in \mathfrak{a}^+$)

$$|(\partial(p)\Phi_\lambda^m)(a)| \leqslant K_{H/2}^m\,\mathrm{e}^{-\varrho(m)(H)}$$
$$\times \sum_{\mu\in\Lambda} \mathrm{e}^{-\mu(H)/2}C(1 + \langle\lambda, \lambda\rangle + \langle\varrho(m) + \mu, \varrho(m) + \mu\rangle)^k,$$

where $K_{H/2}^m$, C and k are suitable constants. Hence, for a positive constant C_H^m,

$$|(\partial(p)\Phi_\lambda^m)(a)| \leqslant C_H^m(1 + |\lambda|^2)^k.$$

Because of (3.2) and the well-known Paley–Wiener estimate for the function g^*, we can now apply dominated convergence. This proves the lemma. ■

Now let m and n: $\Sigma \to \mathbb{R}^+$ be W-invariant functions on Σ. Suppose there exists a differential operator D on A^+, independent of λ, such that

$$\Phi_\lambda^m(a)\mathbf{c}^m(-\lambda)^{-1} = D(\Phi_\lambda^n(a)\mathbf{c}^n(-\lambda)^{-1}) \quad \text{on } A^+, \quad \lambda \in \mathfrak{a}^*. \tag{3.6}$$

Then the lemma implies that

$$\mathscr{B}^m g = D\mathscr{B}^n g \quad \text{on } A^+, \quad g \in \mathscr{D}_W(A). \tag{3.7}$$

For $n = 0$ we would obtain $\mathscr{B}^m g = Dg$ on A^+. If moreover m corresponds to a group-case then $\mathscr{B}^m = (\mathscr{A}^m)^{-1}$ so

$$(\mathscr{A}^m)^{-1} g = Dg \quad \text{on } A^+, \quad g \in \mathscr{D}_W(A). \tag{3.8}$$

Extend the left-hand side of (3.8) to a W-invariant C^∞-function on A with compact support. Since restriction from G to A induces a bijection of $\mathscr{D}(G//K)$ onto $\mathscr{D}_W(A)$, we can consider $\mathscr{A}^m$ as a transform from $\mathscr{D}_W(A)$ onto itself. Consequently $\mathscr{A}^m \circ D = \text{id}$. Now take $g = \mathscr{A}^m f, f \in \mathscr{D}(G//K)$ in (3.8) and again extend to functions in $\mathscr{D}_W(A)$. Combining the results we obtain

$$\mathscr{A}^m \circ D = D \circ \mathscr{A}^m = \text{id}. \tag{3.9}$$

EXAMPLE. Let G be complex so $m \equiv 2$ and put $\delta = \Pi_{\alpha > 0} (e^\alpha - e^{-\alpha})$. Then

$$\Phi_\lambda(a) = (\delta(a))^{-1} e^{i\lambda(\log a)}, \quad a \in A^+,$$

and

$$\mathbf{c}(\lambda) = \pi(\varrho)/\pi(i\lambda),$$

where $\pi(\lambda) = \Pi_{\alpha > 0} \langle \alpha, \lambda \rangle$ and $\varrho = \Sigma_{\alpha > 0}\, \alpha = \varrho(2)$ (see e.g., [9, Ch IV, §5.2]). Put

$$D = \delta^{-1} \prod_{\alpha > 0} \partial(A_\alpha),$$

with ∂ and A_α as before. Then

$$D(e^{i\lambda(\log a)}) = (\delta(a))^{-1} \left(\prod_{\alpha > 0} \langle \alpha, i\lambda \rangle \right) e^{i\lambda(\log a)} = \pi(-\varrho)\Phi_\lambda(a)\mathbf{c}(-\lambda)^{-1},$$

which is (3.6) for $m = 2, n = 0$, up to the constant $w \cdot \pi(-\varrho(2))$ (recall that $\mathbf{c}^0(\lambda) \equiv w^{-1}$). This proves that for G complex the Abel transform can be inverted by the differential operator D, which is well-known. It is already in a paper by Gangolli [5]. The same proof occurs in a paper on the Abel transform by Rouvière [15]. A completely different proof for $G = \mathrm{SL}(n, \mathbb{C})$ occurs in Aomoto [1] (also see [2, Ch III, end of Section 4]) and Hba [6].

Now suppose that in (3.7) m and n correspond to group-cases. If we take $g = \mathscr{A}^n f, f \in \mathscr{D}(G//K)$ then it would follow that

$$(\mathscr{A}^m)^{-1}(\mathscr{A}^n f) \;=\; Df \quad \text{on } A^+.$$

As for the case $n = 0$ we extend to functions in $\mathscr{D}_W(A)$ and apply $\mathscr{A}^m$. Hence

$$\mathscr{A}^n \;=\; \mathscr{A}^m \circ D. \tag{3.10}$$

The problem is to find a differential operator D on A^+ such that (3.6) holds. We shall first give a precise definition of an operator shifting from m to n. Then we will show how such a shift operator can lead to a suitable D.

Let $\mathscr{O}(A^+)$ denote the space of analytic functions on A^+. Take D in $\mathscr{O}(A^+) \otimes \mathbb{D}(A)$, i.e., $D = \Sigma\, g_i \partial(p_i)$ with $g_i \in \mathscr{O}(A^+)$, $p_i \in \mathscr{S}(\mathfrak{a})$ (finite sum) and consider D as a differential operator on A^+. We shall say that D has convergent expansion on A^+ of type $\varrho(k)$ (k: $\Sigma \to \mathbb{R}$ W-invariant) if each g_i has an absolutely convergent expansion on A^+ of the form

$$\mathrm{e}^{-\varrho(k)} \sum_{\mu \in \Lambda} c_\mu^i\, \mathrm{e}^{-\mu}, \quad c_\mu^i \in \mathbb{C},$$

and $c_0^i \neq 0$ for some i. For such an operator we shall write

$$D \;=\; \mathrm{e}^{-\varrho(k)} \sum_{\mu \in \Lambda} \mathrm{e}^{-\mu} \partial(p_\mu) \quad \text{on } A^+, \quad p_\mu \in \mathscr{S}(\mathfrak{a}). \tag{3.11}$$

Definition. Let m and n: $\Sigma \to \mathbb{R}^+$ be W-invariant functions. We shall say that $D \in \mathscr{O}(A^+) \otimes \mathbb{D}(A)$ shifts from m to n if D satisfies

(i) $D \circ (L(n) + \langle \varrho(n), \varrho(n) \rangle) = (L(m) + \langle \varrho(m), \varrho(m) \rangle) \circ D$;

(ii) D has convergent expansion on A^+ of type $\varrho(m - n)$.

Remark. This definition (and also the use of the term "shift operator") follows closely Opdam [14], where a systematic treatment of shift operators is given (see Section 4 for more information on shift operators).

EXAMPLE. Let G be complex and put $D = \delta^{-1} \Pi_{\alpha>0} \partial(A_\alpha)$. Then D is in $\mathcal{O}(A^+) \otimes \mathbb{D}(A)$ and has convergent expansion on A^+ of type $\varrho(2)$. The radial part $L(2) = \Delta(L_X)$ of the Laplace–Beltrami operator can be written as

$$L(2) = \delta^{-1} \circ (L_A - \langle \varrho(2), \varrho(2) \rangle) \circ \delta$$

(see e.g. [9, Ch II, §3, (57)]). To check (ii) we thus have to show that

$$D \circ L_A = (\delta^{-1} \circ L_A \circ \delta) \circ D,$$

which is trivial. So D shifts from 2 to 0.

Let us return to the general case where m and n: $\Sigma \to \mathbb{R}^+$ are arbitrary W-invariant functions. Suppose we have a differential operator which shifts from m to n. Consider the function $D\Phi_\lambda^n$ on A^+. Recall that for Φ_λ^m we have equation (2.3):

$$L(m)\Phi_\lambda^m = -(\langle \lambda, \lambda \rangle + \langle \varrho(m), \varrho(m) \rangle)\Phi_\lambda^m, \quad \lambda \in \mathfrak{a}^*. \tag{3.12}$$

Because of (i) the function $D\Phi_\lambda^n$ also satisfies (3.12). Moreover, by (ii), $D\Phi_\lambda^n$ has an expansion of the form

$$D\Phi_\lambda^n = \mathrm{e}^{i\lambda-\varrho(m)} \sum_{\mu \in \Lambda} \Gamma_\mu^*(\lambda)\, \mathrm{e}^{-\mu} \quad \text{on } A^+, \quad \lambda \in \mathfrak{a}^*.$$

So $D\Phi_\lambda^n$ and Φ_λ^m are proportional. If we write D as in (3.11) (with $k = m - n$) then it follows that

$$D\Phi_\lambda^n = p_0(i\lambda - \varrho(n))\Phi_\lambda^m, \quad \lambda \in \mathfrak{a}^*.$$

Now define for $\lambda \in \mathfrak{a}^*$

$$q(\lambda) = \mathbf{c}^m(-\lambda)p_0(i\lambda - \varrho(n))/\mathbf{c}^n(-\lambda), \tag{3.13}$$

then

$$D(\Phi_\lambda^n/\mathbf{c}^n(-\lambda)) = q(\lambda)(\Phi_\lambda^m/\mathbf{c}^m(-\lambda)), \quad \lambda \in \mathfrak{a}^*. \tag{3.14}$$

Note that $\mathbf{c}^m(-\lambda)/\mathbf{c}^n(-\lambda)$ has no poles and that the $\mathbf{c}$-function has no zeros for $\lambda \in \mathfrak{a}^*$. If $q(\lambda)$ is independent of λ then this would give a differential operator on A^+, independent of λ, such that (3.6) holds. In Section 4 we shall see that for all rank-one and rank-two (and some rank-three) root systems there exist shift operators which shift from m to $m + k$ (k fixed, m arbitrary) and for which q is indeed independent of λ. If $m \equiv 2$ and Σ is reduced (the complex case) then we have already shown that the operator $D = \delta^{-1} \Pi_{\alpha > 0} \partial(A_\alpha)$ shifts from 2 to 0 and that $q = w \cdot \pi(-\varrho(2))$ (also note that $p_0(\lambda) = \pi(\lambda)$).

4. Existence of certain shift operators

We already noted that the definition of an operator which shifts from m to n closely follows Opdam [14]. In fact he defines a "shift operator" as an operator with a fixed shift k from $m + k$ to m, m arbitrary. Furthermore the operator is defined on $A' = \exp \mathfrak{a}'$ (in fact even on the regular elements in $H = \exp \mathfrak{h}$, $\mathfrak{h}$ the complex torus, i.e., $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$). Then it is shown that the operator is W-invariant. Note that if D_1 and D_2 are operators which shift from n_1 to n_2 and from n_2 to n_3 respectively, then it follows from the definition that $D_1 \circ D_2$ shifts from n_1 to n_3. One then hopes that shift operators can be found with "elementary shifts" in the parameter $m = (m_1, m_2, \ldots, m_r) \in (\mathbb{R}^+)^r$ and that these operators generate all possible shifts. For rank-one and rank-two (and some rank-three) cases shift operators have been found (see below); that these are indeed generators for all possible shifts is proved by Opdam in [14].

If the rank of X is one and the associated root system is of type A_1, then the corresponding generator $(1/\mathrm{sh}\, t)\mathrm{d}/\mathrm{d}t$ is of course well-known (cf. $(\mathrm{d}/\mathrm{d}z)F(a, b; c; z) = (ab/c)F(a + 1, b + 1; c + 1; z)$ where $F(a, b; c; z)$ denotes the hypergeometric function). But the introduction of the concept of shift operator (in [18] they are called "lowering/raising" operators) is due to Koornwinder [10]. In the context of orthogonal polynomials in two variables for the root system BC_2, he obtained the operator with a shift 2 in the parameter m_α corresponding to the longest root α (the divisible root). As an important application he finds a differential operator of order four which commutes with $L(m)$. Just as for the hypergeometric function above, the parameters do not necessarily have to correspond to a group-case.

Until now generators have been found for the following cases (we omit rank-one): a second generator for BC_2 by Sprinkhuizen-Kuyper [17]; A_2 and one generator for BC_2 by Vretare [18]; G_2 and the third generator for BC_2 by Opdam [14]; A_3 by the author (see Appendix 1). Again, as most important application in [14] for the shift operator for G_2, Opdam finds a differential operator of order six which commutes with $L(m)$ for arbitrary m. For all rank-one and rank-two cases the operators, and the corresponding p_0 and q, are listed in [14]: the operators in §2, Table 2.6; p_0 in §3, Table 3.15; q in §4,

Table 4.6. Indeed q does not depend on λ for these operators. This also holds for BC_3 and A_3 (for A_3 see Appendix 1). For all these cases the operators are given in the coordinates $z_1, z_2, \ldots, z_\ell$ which are defined as follows. Put $\Sigma' = \{\alpha \in \Sigma | 2\alpha \notin \Sigma\}$ and let $\Delta = \{\beta_1, \beta_2, \ldots, \beta_\ell\}$ be a basis for Σ'. Let $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ be the fundamental weights of Σ', i.e., $2\langle\lambda_i, \beta_j\rangle/\langle\beta_j, \beta_j\rangle = \delta_{ij}$ where δ_{ij} denotes the Kronecker symbol. Let $W(\lambda_i)$ be the stabilizer in W of λ_i, i.e., $W(\lambda_i) = \{w \in W | w\lambda_i = \lambda_i\}$. Then $z_i = \Sigma_{w \in W/W(\lambda_i)} e^{2w\lambda_i}$ $(i = 1, \ldots, \ell)$. It is shown in [14] that shift operators have polynomial coefficients in the z-coordinates. For the root system A_2 however we shall present the shift operator in the given coordinates on $\mathfrak{a}$ (see Section 5). This also enables us to calculate p_0 (and thus q) for the root system A_2 independently of [14]. In general it is a difficult problem to change from z-coordinates to the coordinates on $\mathfrak{a}$ given by Δ.

Let us return to the Abel transform. Suppose one obtains an explicit expression for the Abel transform F_f^m where m can be taken arbitrary i.e., m does not necessarily has to correspond to a group-case. Then (3.10) should hold for all possible shifts. In Section 6 we will show that for the root system A_2 this is indeed the case. We must emphasize that this is the only higher rank case where we can prove (3.10) without reference to the group. For the other root systems of rank $\geqslant 2$ we have to assume that m and n in (3.10) correspond to group-cases. Then (3.10) holds if n is such that we can form a product of the generators with a total shift from m to n.

We shall now list for each irreducible root system mentioned above, the elementary shifts $k = (k_1, k_2, k_3)$ in the parameter $m = (m_1, m_2, m_3)$. We write $m(\alpha) = m_\alpha$. The realizations of the root systems are as in Bourbaki [3] and we shall follow [14] in the numbering of the root multiplicities. We also give the original notation for the shift operators.

- A_ℓ $(\ell = 2, 3)$: $\Sigma^+ \cong \{e_i - e_j | 1 \leqslant i < j \leqslant \ell + 1\}$;
  $m = m_1 = m(e_1 - e_2)$;
  elementary shift: $k = 2$;
  Vretare's notation in [18, §8] $(\ell = 2)$: E_-^γ
  (relation between γ and m: $m = 2\gamma + 1$);
- BC_ℓ $(\ell = 2, 3)$: $\Sigma^+ \cong \{e_i \pm e_j, e_i, 2e_i | 1 \leqslant i < j \leqslant \ell, i = 1, \ldots, \ell\}$;
  $m = (m_1\ m_2, m_3)$ with $m_1 = m(e_1 - e_2)$, $m_2 = m(2e_1)$, $m_3 = m(e_1)$;
  elementary shifts:
  $k = (2, 0, 0)$ for $\ell = 2$;
  Sprinkhuizen-Kuyper's notation in [17, §4]: $E_-^{\alpha,\beta}$;
  $k = (0, 2, 0)$ for $\ell = 2, 3$;
  Koornwinder's notation in [10, II, §5] $(\ell = 2)$: D_-^γ;
  Vretare's notation in [18, §7] $(\ell = 3)$: D_-^γ;

$k = (0, -2, 4)$ for $\ell = 2$;
Opdam's notation in [14, §2]: $G(0, -1, 2)$;
(relation between α, β, γ in [10, 17, 18] and m_i:
$m_1 = 2\gamma + 1$, $m_2 = 2\beta + 1$, $m_3 = 2\alpha - 2\beta$);

- G_2: $\Sigma^+ \cong \{e_1 - e_2, e_1 - e_3, e_2 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}$;
$m = (m_1, m_2)$ with $m_1 = m(e_1 - e_2)$,
$m_2 = m(2e_1 - e_2 - e_3)$;
elementary shifts:
$k = (2, 0)$ and $k = (0, 2)$;
Opdam's notation in [14, §2]: $G(1, 0)$ and $G(0, 1)$;

Based on the classification of all simple noncompact Lie algebras over $\mathbb{R}$, including their root systems and multiplicities (see e.g., [8, Ch X, Table VI]), it is easy to obtain all possible shifts from one group-case to another in the sense of (3.10). The root system A_2 will be treated in the next section. For the root system BC_2 there are quite a few possibilities. Take e.g., $G = \mathrm{SU}(p, 2)$ with $p > 2$. Then $(m_1, m_2, m_3) = (2, 1, 2(p-2))$. So we can shift to $(2, 1, 2(p-4))$ (i.e., $G = \mathrm{SU}(p-2, 2)$ $(p > 4)$) but also to $(0, 1, 2(p-2))$ (i.e., $G = \mathrm{SU}(p-1, 1) \times \mathrm{SU}(p-1, 1)$ $(p > 2)$ and $\Sigma \cong BC_1 \times BC_1$). In particular we can shift to $(0, 1, 0)$ (i.e., $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ and $\Sigma \cong A_1 \times A_1$) if p is even.

5. The shift operator for root system A_2

In Sections 5 and 6 Σ will be the root system of type A_2. In this section we introduce the shift operator and calculate p_0 and q; since q is independent of λ we can apply (3.9) and (3.10). So in Section 5 we restrict ourselves to the group-case; the results are given in Theorem 1. In Section 6 we generalize to arbitrary m.

Put $E = \mathbb{R}^3$. In E we have the standard basis e_1, e_2, e_3 and inner product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal. Let $\mathfrak{a}$ denote the hyperplane in E orthogonal to the vector $e_1 + e_2 + e_3$. The inner product on E induces an inner product on $\mathfrak{a}$ which we shall also denote by $\langle \cdot, \cdot \rangle$. We identify the dual E^* with E (and $\mathfrak{a}^*$ with $\mathfrak{a}$) by means of this inner product. If $\lambda \in \mathfrak{a}$ and f is a function on $\mathfrak{a}$ then we shall frequently write $f(\lambda)$ for the function on $\mathfrak{a}$ which sends $\mu \in \mathfrak{a}$ to $f(\langle \lambda, \mu \rangle)$, i.e., $f(\lambda) = f \circ \lambda$ if we consider λ as element of $\mathfrak{a}^*$. Often $f(\lambda)$ is also considered as function on $A = \exp \mathfrak{a}$: $f(\lambda)(a) = f(\lambda)(\log a)$, $a \in A$ (as in Section 2). We shall also write ∂_λ for the derivative of f in the direction of λ. So if $g \in C^\infty(A)$ then $(A_\lambda g) \circ \exp = \partial_\lambda (g \circ \exp)$ with A_λ as before.

The root system of type A_2 can be identified with the set $\Sigma = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$ in $\mathfrak{a}$. For Σ we take as basis $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}$. The fundamental weights of Σ will be denoted by λ_1, λ_2, i.e., $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. Let Σ^+ be the set of positive roots with respect to Δ. The Weyl group W of Σ is isomorphic to the symmetric group S_3.

We now introduce the shift operator $D(m)$ which shifts the parameter m to $m - 2$ in the sense of Section 3 (so n will equal $m - 2$ in (3.10)–(3.14)). As before put $\delta = \Pi_{\alpha > 0} (e^\alpha - e^{-\alpha})$. Let $D(m)$ be the differential operator on $\mathfrak{a}^+$ defined by

$$D(m) = \delta^{-1}\left(\prod_{\alpha > 0} \partial_\alpha + \tfrac{1}{2}(m - 2) \sum_{\alpha > 0} \varepsilon(\alpha) \cdot \partial_\alpha \circ \coth \alpha \circ \partial_\alpha\right), \tag{5.1}$$

where $\varepsilon(\alpha) = \Pi_{\beta > 0, \beta \neq \alpha} \langle \alpha, \beta \rangle$. The function δ^{-1} has an expansion on $\mathfrak{a}^+$ of the form $\mathrm{e}^{-\varrho(2)} \Sigma_{\mu \in \Lambda} b_\mu \mathrm{e}^{-\mu}$, $b_0 = 1$. Also

$$\partial_\alpha \circ \coth \alpha \circ \partial_\alpha = \coth \alpha \circ \partial_\alpha^2 - (\langle \alpha, \alpha \rangle / \mathrm{sh}^2 \alpha)\partial_\alpha$$

and $\coth \alpha = 1 + 2 \Sigma_{k \geqslant 1} \mathrm{e}^{-2k\alpha}$ while $1/\mathrm{sh}^2 \alpha = \Sigma_{k \geqslant 1} c_k \mathrm{e}^{-2k\alpha}$ with $c_1 = 4$. Consequently $D(m)$ is in $\mathcal{O}(\mathfrak{a}^+) \otimes \mathbb{D}(A)$ and has convergent expansion on $\mathfrak{a}^+$ of type $\varrho(2)$. As in Section 4 we introduce the coordinates z_1 and z_2 where $z_i = \Sigma_{w \in W/W(\lambda_i)} \mathrm{e}^{2w\lambda_i}$ $(i = 1, 2)$. We also use the notation $\partial_i = \partial/\partial z_i$ $(i = 1, 2)$.

Proposition 1. *In the coordinates z_1, z_2 the operator $D(m)$ is given by*

$$2^{-3}D(m) = \partial_1^3 + \partial_2^3 + z_1\partial_1^2\partial_2 + z_2\partial_1\partial_2^2 + (\tfrac{1}{2}m + 1)\partial_1\partial_2 \tag{5.2}$$

This is precisely the "lowering operator" E_-^γ in Vretare [18, §8]. At this point we do not want to interrupt the main line of argument with the calculations which are necessary for the proof of Proposition 1. Therefore these calculations can be found at the end of this section.

Now it is well-known (see e.g. [10, IV, (5.14)]) that

$$\begin{aligned} 3 \cdot 2^{-3}L(m) = {} & (z_1^2 - 3z_2)\partial_1^2 + (z_1z_2 - 9)\partial_1\partial_2 + (z_2^2 - 3z_1)\partial_2^2 \\ & + (3m/2 + 1)(z_1\partial_1 + z_2\partial_2). \end{aligned} \tag{5.3}$$

The advantage of the z-coordinates is now clear: in the calculations only polynomials in the z_i occur (in fact until now all calculations on shift operators have been made in the z-coordinates).

PROPOSITION 2

$$D(m) \circ (L(m-2) + \langle \varrho(m-2), \varrho(m-2) \rangle)$$
$$= (L(m) + \langle \varrho(m), \varrho(m) \rangle) \circ D(m),$$

so $D(m)$ is a shift operator for the root system A_2 with shift 2.

In the z-coordinates the calculations are straightforward. The use of a computer is convenient but not yet a necessity.

An alternative way to find the operator $D(m)$ and to prove Proposition 2 is to use the explicit expression (2.2) for $L(m)$ and then try to see what form $D(m)$ should have in order to satisfy (i) in the definition of shift operator. This is in fact the way we found expression (5.1) for $D(m)$. The calculations are complicated and lengthy; we shall not present them here.

In order to apply the results (3.9) and (3.10) we have to show that q (see (3.13)) does not depend on λ. For this result we could refer to [14, §4]; however, to make this paper self-contained we shall calculate q using the explicit expression (5.1). We shall write $q(m)$ and $p_0(m; \lambda)$ instead of $q(\lambda)$ and $p_0(\lambda)$. From (5.1) and the expansions on $\mathfrak{a}^+$ of the functions δ^{-1}, coth α and $1/\mathrm{sh}^2\alpha$ it is clear that

$$p_0(m; \lambda) = \prod_{\alpha > 0} \langle \alpha, \lambda \rangle + \tfrac{1}{2}(m-2) \sum_{\alpha > 0} \varepsilon(\alpha) \langle \alpha, \lambda \rangle^2.$$

If we determine the signs $\varepsilon(\alpha)$ then it follows that

$$p_0(m, \lambda) = \prod_{\alpha > 0} \langle \alpha, \lambda \rangle + \tfrac{1}{2}(m-2)(-\langle \alpha_1, \lambda \rangle^2 + \langle \alpha_1 + \alpha_2, \lambda \rangle^2 - \langle \alpha_2, \lambda \rangle^2)$$
$$= \langle \alpha_1, \lambda \rangle \langle \alpha_2, \lambda \rangle (\langle \alpha_1 + \alpha_2, \lambda \rangle + m - 2).$$

Since $\varrho(m-2) = (m-2)\varrho(1)$ and $\langle \varrho(1), \alpha_1 \rangle = \langle \varrho(1), \alpha_2 \rangle = 1$ we obtain that

$$p_0(m; \mathrm{i}\lambda - \varrho(m-2)) = \prod_{\alpha > 0} (\langle \alpha, \mathrm{i}\lambda \rangle - (m-2)). \tag{5.4}$$

Next we have to determine $\mathbf{c}^m(-\lambda)/\mathbf{c}^{m-2}(-\lambda)$. If $m = 2$ (the complex case) then we have shown that $q(2) = w\Pi_{\alpha > 0} \langle -\varrho(2), \alpha \rangle$ so for the root system A_2 we have $q(2) = -2^4 3!$. Suppose that, in general, we have a root system with one root length. Take $\langle \alpha, \alpha \rangle = 2$ and $m = m_\alpha > 2$. Then we have, with normalization $\mathbf{c}(-\mathrm{i}\varrho) = 1$,

$$\mathbf{c}^m(\lambda) = \prod_{\alpha>0} \frac{\Gamma(\frac{1}{2}m + \frac{1}{2}\langle\varrho(m), \alpha\rangle)\Gamma(\frac{1}{2}\langle \mathrm{i}\lambda, \alpha\rangle)}{\Gamma(\frac{1}{2}\langle\varrho(m), \alpha\rangle)\Gamma(\frac{1}{2}m + \frac{1}{2}\langle \mathrm{i}\lambda, \alpha\rangle)}.$$

Here Γ denotes the usual gamma function. Because of $\Gamma(z) = (z-1)\Gamma(z-1)$ and $\varrho(m-2) = \varrho(m) - 2\varrho(1)$ we obtain

$$\mathbf{c}^m(\lambda)/\mathbf{c}^{m-2}(\lambda) = \prod_{\alpha>0} 2/(m - 2 + \langle \mathrm{i}\lambda, \alpha\rangle)$$

$$\times \prod_{\alpha>0} \frac{\Gamma(\frac{1}{2}m + \frac{1}{2}\langle\varrho(m), \alpha\rangle)\Gamma(\frac{1}{2}\langle\varrho(m), \alpha\rangle - \langle\varrho(1), \alpha\rangle)}{\Gamma(\frac{1}{2}m + \frac{1}{2}\langle\varrho(m), \alpha\rangle - (1 + \langle\varrho(1), \alpha\rangle))\Gamma(\frac{1}{2}\langle\varrho(m), \alpha\rangle)}.$$

Recall that $\langle\varrho(1), \alpha\rangle = 1$ if α is simple (see e.g., [3, Ch VI, no 1.10]), so $\langle\varrho(1), \alpha\rangle \in \mathbb{N}$ if $\alpha \in \Sigma^+$. Since for $n \in \mathbb{N}$

$$\Gamma(z)/\Gamma(z-n) = (z-1)(z-2)\cdots(z-n) = (z-n)_n$$

($(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol), we obtain

$$\mathbf{c}^m(-\lambda)/\mathbf{c}^{m-2}(-\lambda) = (-2)^{|\Sigma^+|} \prod_{\alpha>0} (\langle\alpha, \mathrm{i}\lambda\rangle - (m-2))^{-1}$$

$$\times \prod_{\alpha>0} \frac{(\frac{1}{2}(m-2)(1 + \langle\varrho(1), \alpha\rangle))_{\langle\varrho(1),\alpha\rangle+1}}{(\frac{1}{2}(m-2)\langle\varrho(1), \alpha\rangle)_{\langle\varrho(1),\alpha\rangle}}.$$

In particular for the root system A_2:

$$q(m) = \mathbf{c}^m(-\lambda)p_0(m; \mathrm{i}\lambda - \varrho(m-2))/\mathbf{c}^{m-2}(-\lambda)$$

$$= (-2)^3(m-2)_2(m-2)_2(\tfrac{1}{2}(3m-6))_3/$$

$$((\tfrac{1}{2}m-1)_1(\tfrac{1}{2}m-1)_1(m-2)_2).$$

Since $q(2) = w \cdot \pi(-\varrho(2)) = -2^4 3!$ it follows that

$$q(m) = -2^2 3(m-1)(3m-4)(3m-2), \quad m \geqslant 2. \tag{5.5}$$

Recalling (3.6)–(3.14) we have obtained the following result:

THEOREM 1. *Let $X = G/K$ be one of the following symmetric spaces of the noncompact type*: $\mathrm{SL}(3, \mathbb{C})/\mathrm{SU}(3)$, $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$, $E_{6(-26)}/F_4$. *So the associated root system of X is of type A_2 and the multiplicity m equals* 2, 4 *and* 8 *respectively. Let F_f^m denote the corresponding Abel transform ($f \in \mathscr{D}(G//K)$). Put $D(m, 2k) = D(m) \circ D(m-2) \circ \cdots \circ D(m-2k+2)$ and $q(m, 2k) = q(m) \cdot q(m-2) \cdot \cdots \cdot q(m-2k+2)$ ($k \in \mathbb{N}$, $2k \leqslant m$), where $D(m)$ and $q(m)$ are given by* (5.1) *and* (5.5). *Then we have for m, $m - 2k \in \{2, 4, 8\}$ ($k \in \mathbb{N}$)*

$$F^m_{D(m,2k)f} = q(m, 2k) F_f^{m-2k},$$

and also

$$D(m, m) F_f^m = F^m_{D(m,m)f} = q(m, m) f.$$

REMARKS

1. We have also found the shift operator for root system A_3. However, for this case we only have the operator in the z-coordinates; using some of the results of Opdam [14], Theorem 1 then also follows for $X = \mathrm{SL}(4, \mathbb{C})/\mathrm{SU}(4)$ and $\mathrm{SU}^*(8)/\mathrm{Sp}(4)$ (see Appendix 1).
2. As stated before the complex case ($m = 2$) is well-known.
3. Inversion of the Abel transform for $m = 4$ and root system A_2 has very recently also been obtained by Hba [20], using a different method (which was already used in [6] for the case $G = \mathrm{SL}(n, \mathbb{C})$). In fact he first inverts a rank-one Abel transform by the differential operator $((1/\mathrm{sh}\ t)\mathrm{d}/\mathrm{d}t)^2$ (this rank-one transform corresponds to the outer integral in (6.1) in the next section). Then he finds the fourth order differential operator which inverts the remaining "partial" Abel transform (which corresponds to the inner integral in (6.1)). A direct calculation shows that $D(4) \circ D(2)$ is indeed equal to the differential operator P in [20].
4. Theorem 1 is a special case of Theorem 2 in Section 6.

We now return to the *proof of Proposition 1.*

Let $D(m)$ be the differential operator defined by the right-hand side of (5.1). Then we want to show that in the coordinates

$$z_i = \sum_{w \in W/W(\lambda_i)} \mathrm{e}^{2w\lambda_i} \quad (i = 1, 2),$$

we have

$$2^{-3} D(m) = \partial_1^3 + \partial_2^3 + z_1 \partial_1^2 \partial_2 + z_2 \partial_1 \partial_2^2 + (\tfrac{1}{2}m + 1)\partial_1 \partial_2.$$

As before we put $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$. Then

$$\lambda_1 = 3^{-1}(2e_1 - e_2 - e_3) = 3^{-1}(2\alpha_1 + \alpha_2),$$
$$\lambda_2 = 3^{-1}(e_1 + e_2 - 2e_3) = 3^{-1}(\alpha_1 + 2\alpha_2),$$

and thus

$$\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = 2\lambda_2 - \lambda_1, \quad \alpha_1 + \alpha_2 = \lambda_1 + \lambda_2.$$

We have

$$\begin{aligned} \partial_{\alpha_1} &= \partial_{\alpha_1}(z_1)\partial_1 + \partial_{\alpha_1}(z_2)\partial_2, \\ \partial_{\alpha_2} &= \partial_{\alpha_2}(z_1)\partial_1 + \partial_{\alpha_2}(z_2)\partial_2. \end{aligned} \tag{5.6}$$

Now

$$z_1 = e^{2\lambda_1} + e^{2\lambda_2 - 2\lambda_1} + e^{-2\lambda_2}, \quad z_2 = e^{2\lambda_2} + e^{2\lambda_1 - 2\lambda_2} + e^{-2\lambda_1},$$

and since $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ we obtain

$$\begin{aligned} \partial_{\alpha_1}(z_1) &= 2(e^{2\lambda_1} - e^{2\lambda_2 - 2\lambda_1}), & \partial_{\alpha_1}(z_2) &= 2(e^{2\lambda_1 - 2\lambda_2} - e^{-2\lambda_1}), \\ \partial_{\alpha_2}(z_1) &= 2(e^{2\lambda_2 - 2\lambda_1} - e^{-2\lambda_2}), & \partial_{\alpha_2}(z_2) &= 2(e^{2\lambda_2} - e^{2\lambda_1 - 2\lambda_2}), \\ \partial_{\alpha_1 + \alpha_2}(z_1) &= 2(e^{2\lambda_1} - e^{-2\lambda_2}), & \partial_{\alpha_1 + \alpha_2}(z_2) &= 2(e^{2\lambda_2} - e^{-2\lambda_1}). \end{aligned} \tag{5.7}$$

In particular

$$\det(\partial_{\alpha_i}(z_j)) = 4 \sum_{w \in W} \varepsilon(w)\, e^{2w\varrho} = 4\delta,$$

since $\varrho = \lambda_1 + \lambda_2 = e_1 - e_3$. Here, as before, $\varepsilon(w) = \det w$ and $\delta = \Pi_{\alpha > 0}(e^{\alpha} - e^{-\alpha})$. Note that if Σ is an arbitrary root system with basis Δ then we have in general

$$\det(\partial_{\alpha_i}(z_j)) = \left(\prod_{\alpha \in \Delta} \langle \alpha, \alpha \rangle\right) \cdot \delta,$$

since the highest term in the partial ordering on $\mathfrak{a}$ in the left-hand side is equal to $(2^{\ell} \det(\langle \lambda_j, \alpha_i \rangle))\, e^{2\varrho}$. Since $\det(\partial_{\alpha_i}(z_j)) = 4\delta$ we deduce from

(5.6) that

$$\partial_1 = \frac{\partial}{\partial z_1} = (4\delta)^{-1}(\partial_{\alpha_2}(z_2)\partial_{\alpha_1} - \partial_{\alpha_1}(z_2)\partial_{\alpha_2}),$$
$$\partial_2 = \frac{\partial}{\partial z_2} = (4\delta)^{-1}(-\partial_{\alpha_2}(z_1)\partial_{\alpha_1} + \partial_{\alpha_1}(z_1)\partial_{\alpha_2}). \tag{5.8}$$

In Section 3 we showed that for $m = 2$ (the complex case) the operator $\delta^{-1}\,\Pi_{\alpha>0}\,\partial_\alpha$ satisfies (i) for $m = 2$, $n = 0$ in the definition of shift operator. Proposition 2, which is a straightforward calculation, shows that also the right-hand side of (5.2) for $m = 2$ satisfies (i) for $m = 2, n = 0$. This makes the following proposition plausible.

PROPOSITION 3

$$\delta^{-1}\prod_{\alpha>0}\partial_\alpha = 2^3(\partial_1^3 + \partial_2^3 + z_1\partial_1^2\partial_2 + z_2\partial_1\partial_2^2 + 2\partial_1\partial_2).$$

Proof. First a remark on the operator $\delta^{-1}\,\Pi_{\alpha>0}\,\partial_\alpha$ for arbitrary root systems. Note that it sends W-invariant exponential polynomials to W-invariant exponential polynomials. So this operator has polynomial coefficients in the z-coordinates. Furthermore, if w_0 denotes the longest Weyl group element, i.e., the element $w_0 \in W$ such that $w_0\alpha < 0\ \forall\ \alpha \in \Sigma^+$, then $\delta^{-1}\,\Pi_{\alpha>0}\,\partial_\alpha$ is invariant under $-w_0$. For the root system A_ℓ this leads to an invariance in the z-coordinates under the transformations $z_j \leftrightarrow z_{\ell-j+1}$ $(j = 1, \ldots, \ell)$ since $-w_0(\lambda_j) = \lambda_{\ell-j+1}$ in this case (see e.g., (A.4) in Appendix 2 for the λ_j). This reduces the calculations. We now return to the case A_2. It follows from (5.6) that

$$\begin{aligned}\prod_{\alpha>0}\partial_\alpha &= \sum_{i=1}^{2}\left[\prod_{\alpha>0}\partial_\alpha(z_i)\right]\partial_i + \sum_{i=1}^{2}\left[\sum_{\beta>0}\left(\prod_{\alpha>0,\alpha\neq\beta}\partial_\alpha(z_i)\right)\partial_\beta(z_i)\right]\partial_i^2 \\
&+ \left[\sum_{\beta>0}\left\{\left(\prod_{\alpha>0,\alpha\neq\beta}\partial_\alpha(z_1)\right)\partial_\beta(z_2) + \left(\prod_{\alpha>0,\alpha\neq\beta}\partial_\alpha(z_2)\right)\partial_\beta(z_1)\right\}\right]\partial_1\partial_2 \\
&+ \left[\sum_{\beta>0}\left(\prod_{\alpha>0,\alpha\neq\beta}(\partial_\alpha(z_1))\right)\partial_\beta(z_2)\right]\partial_1^2\partial_2 \\
&+ \left[\sum_{\beta>0}\left(\prod_{\alpha>0,\alpha\neq\beta}(\partial_\alpha(z_2))\right)\partial_\beta(z_1)\right]\partial_1\partial_2^2 + \sum_{i=1}^{2}\left[\prod_{\alpha>0}(\partial_\alpha(z_i))\right]\partial_i^3.\end{aligned}$$

We already calculated $\partial_{\alpha_i}(z_j)$ $(1 \leqslant i, j \leqslant 2)$ and $\partial_{\alpha_1+\alpha_2}(z_i)$ $(i = 1, 2)$ in (5.7). Furthermore we have

$$\partial_{\alpha_1}\partial_{\alpha_2}(z_1) = -4\,\mathrm{e}^{2\lambda_2-2\lambda_1}, \quad \partial_{\alpha_1}\partial_{\alpha_2}(z_2) = -4\,\mathrm{e}^{2\lambda_1-2\lambda_2},$$

$$\partial_{\alpha_1}\partial_{\alpha_1+\alpha_2}(z_1) = 4\,\mathrm{e}^{2\lambda_1}, \quad \partial_{\alpha_1}\partial_{\alpha_1+\alpha_2}(z_2) = 4\,\mathrm{e}^{-2\lambda_1},$$

$$\partial_{\alpha_2}\partial_{\alpha_1+\alpha_2}(z_1) = 4\,\mathrm{e}^{-2\lambda_2}, \quad \partial_{\alpha_2}\partial_{\alpha_1+\alpha_2}(z_2) = 4\,\mathrm{e}^{2\lambda_2},$$

and

$$\prod_{\alpha>0} \partial_\alpha(z_i) = 0 \quad (i = 1, 2).$$

Of course this last equality holds for arbitrary root systems since $\Pi_{\alpha>0}\langle\alpha, w\lambda_i\rangle = 0$. In particular the coefficient of ∂_i $(i = 1, 2)$ equals zero. A very simple calculation shows that this is also the case for the coefficient of ∂_i^2 $(i = 1, 2)$. The coefficient of ∂_1^3 equals

$$\prod_{\alpha>0}(\partial_\alpha(z_1)) = 2^3(\mathrm{e}^{2\lambda_1} - \mathrm{e}^{2\lambda_2-2\lambda_1})(\mathrm{e}^{2\lambda_2-2\lambda_1} - \mathrm{e}^{-2\lambda_2})(\mathrm{e}^{2\lambda_1} - \mathrm{e}^{-2\lambda_2}) = 2^3\delta.$$

The coefficient of $\partial_1\partial_2$ becomes

$$\begin{aligned} 2^3(&-\mathrm{e}^{2\lambda_2-2\lambda_1}(\mathrm{e}^{2\lambda_2} - \mathrm{e}^{-2\lambda_1}) + \mathrm{e}^{2\lambda_1}(\mathrm{e}^{2\lambda_2} - \mathrm{e}^{2\lambda_1-2\lambda_2}) \\ &+ \mathrm{e}^{-2\lambda_2}(\mathrm{e}^{2\lambda_1-2\lambda_2} - \mathrm{e}^{-2\lambda_1}) - \mathrm{e}^{2\lambda_1-2\lambda_2}(\mathrm{e}^{2\lambda_1} - \mathrm{e}^{-2\lambda_2}) \\ &+ \mathrm{e}^{-2\lambda_1}(\mathrm{e}^{2\lambda_2-2\lambda_1} - \mathrm{e}^{-2\lambda_2}) + \mathrm{e}^{2\lambda_2}(\mathrm{e}^{2\lambda_1-2\lambda_2} - \mathrm{e}^{-2\lambda_1})) = 2^4\delta, \end{aligned}$$

since the highest term in the partial ordering on $\mathfrak{a}$ is $2(\lambda_1 + \lambda_2)$. It remains to calculate the coefficient of $\partial_1^2\partial_2$:

$$\begin{aligned} 2^3((&\mathrm{e}^{2\lambda_1} - \mathrm{e}^{2\lambda_2-2\lambda_1})(\mathrm{e}^{2\lambda_2-2\lambda_1} - \mathrm{e}^{-2\lambda_2})(\mathrm{e}^{2\lambda_2} - \mathrm{e}^{-2\lambda_1}) \\ &+ (\mathrm{e}^{2\lambda_1} - \mathrm{e}^{2\lambda_2-2\lambda_1})(\mathrm{e}^{2\lambda_1} - \mathrm{e}^{2\lambda_2})(\mathrm{e}^{-2\lambda_2} - \mathrm{e}^{2\lambda_1-2\lambda_2}) \\ &+ (\mathrm{e}^{2\lambda_2-2\lambda_1} - \mathrm{e}^{-2\lambda_2})(\mathrm{e}^{2\lambda_1} - \mathrm{e}^{-2\lambda_2})(\mathrm{e}^{2\lambda_1-2\lambda_2} - \mathrm{e}^{-2\lambda_1})) \\ = 2^3(&\mathrm{e}^{4\lambda_1+2\lambda_2} - \mathrm{e}^{6\lambda_1-2\lambda_2} - \mathrm{e}^{6\lambda_2-4\lambda_1} + \mathrm{e}^{4\lambda_2-6\lambda_1} - \mathrm{e}^{-2\lambda_1-4\lambda_2} + \mathrm{e}^{2\lambda_1-6\lambda_2}). \end{aligned}$$

Since the highest term in the partial ordering on $\mathfrak{a}$ is $4\lambda_1 + 2\lambda_2 = 2(\lambda_1 + \lambda_2) + 2\lambda_1$, this coefficient should be equal to $(2^3 z_1 + \text{const.}) \cdot \delta$.

A simple calculation shows that this is indeed the case with const. $= 0$. This proves the proposition. ■

From the proposition follows that we now only have to translate the operator $\partial_1\partial_2$. In fact we first found the operator $D(m)$ in the coordinates on $\mathfrak{a}$ given by Δ, using a direct method. Afterwards we were able to translate $\partial_1\partial_2$ to these coordinates and also to translate $\Sigma_{\alpha>0}\,\varepsilon(\alpha)\cdot\partial_\alpha\circ\coth\alpha\circ\partial_\alpha$ (cf. (5.1)) to the z-coordinates. Since the latter is the easiest (cf. Proposition 3) we only give the necessary calculations for the translation to the z-coordinates.

PROPOSITION 4

$$\delta^{-1}\sum_{\alpha>0}\varepsilon(\alpha)\cdot\partial_\alpha\circ\coth\alpha\circ\partial_\alpha = 2^3\partial_1\partial_2.$$

Proof. First note that, as in Proposition 3, this operator is invariant under $-w_0$. Using (5.6) we obtain

$$\begin{aligned}\sum_{\alpha>0}\varepsilon(\alpha)\cdot\partial_\alpha\circ\coth\alpha\circ\partial_\alpha &= \sum_{i=1}^{2}\left[\sum_{\alpha>0}\varepsilon(\alpha)\cdot\partial_\alpha\circ\coth\alpha\circ\partial_\alpha(z_i)\right]\partial_i\\ &+\sum_{i=1}^{2}\left[\sum_{\alpha>0}\varepsilon(\alpha)\cdot\partial_\alpha(z_i)\cdot\coth\alpha\cdot\partial_\alpha(z_i)\right]\partial_i^2\\ &+2\left[\sum_{\alpha>0}\varepsilon(\alpha)\cdot\partial_\alpha(z_1)\cdot\coth\alpha\cdot\partial_\alpha(z_2)\right]\partial_1\partial_2.\end{aligned}$$

By (5.7) we have

$$\coth\alpha_1\cdot\partial_{\alpha_1}(z_1) = 2(e^{2\lambda_1-\lambda_2}+e^{\lambda_2-2\lambda_1})(e^{2\lambda_1}-e^{2\lambda_2-2\lambda_1})/(e^{2\lambda_1-\lambda_2}-e^{\lambda_2-2\lambda_1})$$
$$= 2(e^{4\lambda_1-\lambda_2}-e^{3\lambda_2-4\lambda_1})/(e^{2\lambda_1-\lambda_2}-e^{\lambda_2-2\lambda_1}) = 2(e^{2\lambda_1}+e^{2\lambda_2-2\lambda_1}).$$

If we also calculate the other $\coth\alpha\partial_\alpha(z_i)(\alpha>0)$ then we obtain the following list:

$$\coth\alpha_1\cdot\partial_{\alpha_1}(z_1) = 2(e^{2\lambda_1}+e^{2\lambda_2-2\lambda_1});$$

$$\coth\alpha_1\cdot\partial_{\alpha_1}(z_2) = 2(e^{-2\lambda_1}+e^{2\lambda_1-2\lambda_2});$$

$$\coth\alpha_2\cdot\partial_{\alpha_2}(z_1) = 2(e^{-2\lambda_2}+e^{2\lambda_2-2\lambda_1});$$

$$\coth \alpha_2 \cdot \partial_{\alpha_2}(z_2) = 2(e^{2\lambda_2} + e^{2\lambda_1 - 2\lambda_2});$$

$$\coth(\alpha_1 + \alpha_2) \cdot \partial_{\alpha_1+\alpha_2}(z_1) = 2(e^{2\lambda_1} + e^{-2\lambda_2});$$

$$\coth(\alpha_1 + \alpha_2) \cdot \partial_{\alpha_1+\alpha_2}(z_2) = 2(e^{2\lambda_2} + e^{-2\lambda_1}).$$

Using this list (and (5.7) again) the coefficients of the ∂_i, ∂_i^2 $(i = 1, 2)$ and $\partial_1\partial_2$ follow immediately. Let us calculate the only non-zero coefficient:

$$\begin{aligned}\sum_{\alpha>0} \varepsilon(\alpha) \cdot \partial_\alpha(z_1) \cdot \coth \alpha \cdot \partial_\alpha(z_2) &= -4(e^{2\lambda_1} - e^{2\lambda_2 - 2\lambda_1})(e^{-2\lambda_1} + e^{2\lambda_1 - 2\lambda_2}) \\ &\quad + 4(e^{2\lambda_1} - e^{-2\lambda_2})(e^{2\lambda_2} + e^{-2\lambda_1}) \\ &\quad - 4(e^{2\lambda_2 - 2\lambda_1} - e^{-2\lambda_2})(e^{2\lambda_2} + e^{2\lambda_1 - 2\lambda_2}) \\ &= 4(-e^{4\lambda_1 - 2\lambda_2} + e^{2\lambda_2 - 4\lambda_1} + e^{2\lambda_1 + 2\lambda_2} \\ &\quad - e^{-2\lambda_1 - 2\lambda_2} - e^{4\lambda_2 - 2\lambda_1} + e^{2\lambda_1 - 4\lambda_2}) = 4\delta.\end{aligned}$$

So Proposition 4 follows. ■

Now Propositions 3 and 4 imply that

$$\begin{aligned}&\delta^{-1}\left(\prod_{\alpha>0} \partial_\alpha + \tfrac{1}{2}(m-2) \sum_{\alpha>0} \varepsilon(\alpha) \cdot \partial_\alpha \circ \coth \alpha \circ \partial_\alpha\right) \\ &= 2^3(\partial_1^3 + \partial_2^3 + z_1\partial_1^2\partial_2 + z_2\partial_1\partial_2^2 + (\tfrac{1}{2}m + 1)\partial_1\partial_2),\end{aligned}$$

and this proves (5.2) in Proposition 1.

6. The generalized Abel transform for root system A_2

In this section we define the Abel transform F_f^m for $m \in \mathbb{C}$, Re $m > 0$, and then prove that (3.10) holds with $n = m - 2$, $m \in \mathbb{C}$, Re $m > 2$. Notation and set-up is as in Section 5. Let $\mathcal{D}_W(\mathfrak{a})$ denote the space of W-invariant C^∞-functions on $\mathfrak{a}$ with compact support. For $f \in \mathcal{D}_W(\mathfrak{a})$ and $m \in \mathbb{C}$, Re $m > 0$ the Abel transform $\mathcal{A}^m f = F_f^m$ of f is the function on

$$\mathfrak{a}^+ = \{(s_1, s_2, s_3) \in \mathfrak{a} \,|\, s_1 > s_2 > s_3\}$$

defined by:

$$F_f^m(s_1, s_2, s_3) = \frac{\pi^{3m/2}2^{m+4}}{\Gamma(\frac{1}{2}m)^3}$$

$$\times \int_{y_3=-\infty}^{s_3} \mathrm{sh}\,(y_2 - y_3)^{-(m-2)}(\mathrm{ch}\,(y_2 - y_3) - \mathrm{ch}\,(s_2 - s_3))^{(m-2)/2}\,\mathrm{d}y_3$$

$$\iint_{x_1 > y_2 > x_2 > y_3 > x_3} f(x_1, x_2, x_3) \prod_{1 \leqslant i < j \leqslant 3} \mathrm{sh}\,(x_i - x_j)$$

$$\times \left\{ - \prod_{i=1}^{3} (\mathrm{ch}\,(2x_i - y_2 - y_3) - \mathrm{ch}\,(y_2 - y_3)) \right\}^{(m-2)/2} \mathrm{d}x_2 \mathrm{d}x_3. \tag{6.1}$$

In the inner integral x_1 is such that $x_1 + x_2 + x_3 = 0$ and in the outer integral y_2 is such that $y_2 + y_3 = s_2 + s_3$. Note that since $y_3 < s_3$ we have $y_2 - y_3 > s_2 - s_3 > 0$. Also

$$- \prod_{i=1}^{3} (\mathrm{ch}\,(2x_i - y_2 - y_3) - \mathrm{ch}\,(y_2 - y_3))$$

$$= -2^3 \prod_{i=1}^{3} \mathrm{sh}\,(x_i - y_2)\,\mathrm{sh}\,(x_i - y_3) > 0.$$

Extend F_f^m to a function on $\mathfrak{a}' = \{(s_1, s_2, s_3) \in \mathfrak{a} | s_i - s_j \neq 0$ $(1 \leqslant i < j \leqslant 3)\}$ by W-invariance. The notation F_f^m is justified by the next proposition.

PROPOSITION 5. *For $m = 1, 2, 4$ and 8 F_f^m equals the Abel transform on the symmetric spaces of the noncompact type* SL(3, $\mathbb{R}$)/SO(3), SL(3, $\mathbb{C}$)/SU(3), SU*(6)/Sp(3) *and $E_{6(-26)}/F_4$ respectively (up to a constant depending on m).*

Proof. For each of these symmetric spaces we can identify the abelian Lie algebra $\mathfrak{a}$ (defined as usual) with the subspace $\mathfrak{a}$ in E defined as in Section 5. We also identify the associated root system of type A_2 in the dual $\mathfrak{a}^*$ with the subset Σ in $\mathfrak{a}$ and the positive Weyl chamber with the set $\mathfrak{a}^+$ above. Let exp: $\mathfrak{a} \to A$ be as usual. If $g \in \mathscr{D}_W(A)$ then $f = g \circ \exp \in \mathscr{D}_W(\mathfrak{a})$. Now write

$$H_g^m(\exp H) = F_f^m(H), \quad H \in \mathfrak{a}^+,$$

with F_f^m defined by the right-hand side of (6.1). We have to show that H_g^m equals F_f^m for $m = 1, 2, 4$ and 8 where F_f^m now denotes the group-theoretical Abel transform (and not the right-hand side of (6.1)). It is not hard to show that H_g^m is precisely the expression for the Abel transform F_f^m in Aomoto [1] if $m = 1$ [1, (3.18)] or 2 [1, (3.11)] (see [2, Ch III, Section 6]; in [2, Ch III] we also extended Aomoto's results to the case $m = 4$, i.e., $G = \mathrm{SU}^*(6)$). Now let $m = 2, 4$ or 8. In Theorem 1 we proved that there exists a differential operator D on A^+ such that $DF_f^m = F_{Df}^m = \text{const.} \cdot f$ on A. Recall that we extended Df and DF_f^m to functions in $\mathscr{D}_W(A)$. Now $DF_f^m = \text{const.} \cdot f$ for $f \in \mathscr{D}(G//K)$ implies that D is surjective on $\mathscr{D}_W(A)$. Since $Df \in \mathscr{D}_W(A)$ we can also consider H_{Df}^m. In Theorem 2 we shall prove (by direct calculations) that also $H_{Df}^m = \text{const.} \cdot f$ on $\mathfrak{a}'$ ($m = 2, 4$ or 8). So we obtain

$$c_1 f = DF_f^m = F_{Df}^m = c_2 H_{Df}^m \quad \text{on } \mathfrak{a}' \quad (f \in \mathscr{D}(G//K),\ m = 2, 4, 8),$$

for constants c_1 and c_2. By the surjectivity of D we thus have

$$F_g^m = \text{const.} \cdot H_g^m \quad \text{on } \mathfrak{a}' \quad \text{for } g \in \mathscr{D}(G//K) \text{ and } m = 2, 4, 8.$$

∎

REMARK. Changing the order of integration in (6.1) one can write F_f^m as integral over $\mathfrak{a}^+$ with respect to a kernel; this kernel can be expressed as hypergeometric function in several variables. See Aomoto [1] for the case $m = 1$; in [2, Ch III, Section 6] we extended his results to $m \in \mathbb{C}$, $\operatorname{Re} m > 0$ and also simplified the case $m = 1$.

We now state the main result of this section.

THEOREM 2. *Let $f \in \mathscr{D}_W(\mathfrak{a})$ and define F_f^m by* (6.1) *for $m \in \mathbb{C}$,* $\operatorname{Re} m > 0$. *Let $D(m)$ be given by* (5.1). *Then*

$$F_{D(m)f}^m = (-2\pi)^3 F_f^{m-2} \quad \textit{on } \mathfrak{a}',\ \operatorname{Re} m > 2,$$

and

$$F_{D(2)f}^2 = (-2\pi)^3 f \quad \textit{on } \mathfrak{a}'.$$

In particular the transform $f \to F_f^m$ can be inverted by the differential operator $D(m, m) = D(m) \circ D(m-2) \circ \cdots \circ D(2)$ if m is even, i.e.,

$$F_{D(2k,2k)f}^{2k} = (-2\pi)^{3k} f \quad \textit{on } \mathfrak{a}' \quad (k \in \mathbb{N}).$$

REMARK. It follows from Proposition 5 that Theorem 1 is now a special case of Theorem 2. However, there is a difference in the constants: in Theorem 1 the constant $q(m)$ occurs. This is due to different normalizations of measures. When (6.1) was derived in [1] (and also in [2, Ch III]) the Haar measure on the subgroup N was not normalized by $\mathrm{d}n = \theta\,(\mathrm{d}\bar{n})$, $\int_{\bar{N}} \mathrm{e}^{-2\varrho(H(\bar{n}))}\,\mathrm{d}\bar{n} = 1$ (see Section 2). Write $\mathrm{d}_0 n$ for the Haar measure on N induced by the Killing form. Then it is shown in [4, §§3.7–3.9] that $\mathrm{d}n = \gamma\, \mathrm{d}_0 n$ with γ given by [4, (3.60)]. Applied to the case $\Sigma \cong A_2$ we obtain

$$\gamma = \gamma(m) = \pi^{-3m/2} \prod_{\alpha > 0} \frac{\Gamma(\frac{1}{2}m + \frac{1}{2}\langle \varrho(m), \alpha\rangle)}{\Gamma(\frac{1}{2}\langle \varrho(m), \alpha\rangle}.$$

Now compare with (5.4)–(5.5) then $\gamma(m-2)/\gamma(m) = (-2\pi)^3 (q(m))^{-1}$.

Proof. We shall always assume that $x_1 + x_2 + x_3 = 0$ and $y_2 + y_3 = s_2 + s_3$. If f is a function on $\mathfrak{a}$ then it is sometimes convenient to put $g(x_2, x_3) = f(-x_2 - x_3, x_2, x_3)$; note that

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_i}\right) f(x_1, x_2, x_3)|_{x_1 = -x_2 - x_3} = -\frac{\partial}{\partial x_i} g(x_2, x_3) \quad (i = 2, 3).$$

The proof of Theorem 2 is "simply" a direct calculation based on the explicit expression (5.1) of the operator $D(m)$, i.e.,

$$D(m) = 2^{-3}\left[\prod_{1 \leqslant i < j \leqslant 3} \operatorname{sh}(x_i - x_j)\right]^{-1} \left\{ \prod_{1 \leqslant i < j \leqslant 3} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \right.$$
$$+ \tfrac{1}{2}(m-2) \sum_{1 \leqslant i < j \leqslant 3} (-1)^{i+j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)$$
$$\left. \circ \coth(x_i - x_j) \circ \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \right\}.$$

Note that $D(m)$ can also be considered as a W-invariant differential operator on $\mathfrak{a}'$. From now on "$i < j$" will always mean "$1 \leqslant i < j \leqslant 3$", unless otherwise stated.

Since the calculations, needed for the proof, are rather lengthy, it is very useful to introduce several notational conventions. We put

$$D_1 = \prod_{i<j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right),$$

$$D_2 = \sum_{i<j} (-1)^{i+j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \circ \coth(x_i - x_j) \circ \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right),$$

and also

$$\delta_1 = \prod_{i<j} \operatorname{sh}(x_i - x_j).$$

Then

$$D(m) = 2^{-3}\delta_1^{-1}(D_1 + \tfrac{1}{2}(m-2)D_2).$$

Also write

$$p = y_2 + y_3, \quad q = y_2 - y_3, \quad s = s_2 - s_3,$$

and

$$T = \prod_{i=1}^{3} (\operatorname{ch}(2x_i - y_2 - y_3) - \operatorname{ch}(y_2 - y_3)) = \prod_{i=1}^{3} (\operatorname{ch}(2x_i - p) - \operatorname{ch} q).$$

Recall that in (6.1) we have $q > 0$ and $T < 0$.
Furthermore we put $c(m) = \pi^{3m/2}2^{m+4}/(\Gamma(\frac{1}{2}m))^3$ and we write

$$D(y_3) = \{(x_1, x_2, x_3) \in \mathfrak{a}^+ \,|\, x_1 > y_2 > x_2 > y_3 > x_3\}.$$

With these notations we have

$$F^m_{D(m)f}(s_1, s_2, s_3) = 2^{-3}\,\mathrm{c}(m) \int_{y_3=-\infty}^{s_3} \operatorname{sh}^{-(m-2)}q(\operatorname{ch} q - \operatorname{ch} s)^{(m-2)/2}\, dy_3$$

$$\iint_{D(y_3)} ((D_1 + \tfrac{1}{2}(m-2)D_2)f(x_1, x_2, x_3))(-T)^{(m-2)/2}\, dx_2\, dx_3.$$

For different values of the variable y_3 we shall give a sketch (Fig. 1) of the domain of integration $D(y_3)$ in the plane $x_1 + x_2 + x_3 = 0$. We fix a point $s = (s_1, s_2, s_3) \in \mathfrak{a}^+$.

Now use integration by parts. Since f has compact support in $\mathfrak{a}$ and $T = 0$ on $\partial(D(y_3))$ we obtain for Re m sufficiently large (e.g., Re $m > 6$)

$$F^m_{D(m)f}(s_1, s_2, s_3) = -2^{-3}\mathrm{c}(m) \int_{y_3=-\infty}^{s_3} ((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2}\, dy_3$$

$$\iint_{D(y_3)} ((D_1 - \tfrac{1}{2}(m-2)D_2)\,(-T)^{(m-2)/2})f(x_1, x_2, x_3)\, dx_2\, dx_3.$$

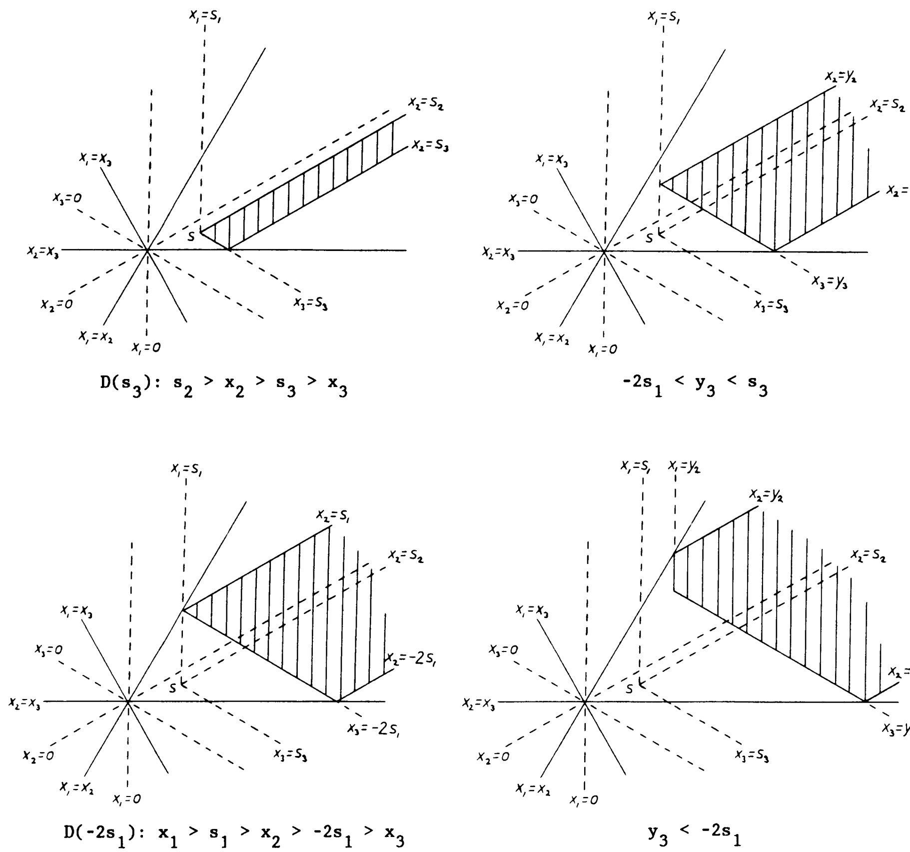

Fig. 1.

We will prove the following result in Proposition 6 below:

$$(D_1 - \tfrac{1}{2}(m-2)D_2)\,(T^{(m-2)/2}) \;=\; -4(m-2)^2(m-3)\ \mathrm{ch}\ q\delta_1 T^{(m-4/2)}$$

$$+\ 2(m-2)^2\ \mathrm{sh}\ q\delta_1 \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)(T^{(m-4)/2}).$$

Then it follows that

$$F^m_{D(m)f}(s_1, s_2, s_3) \;=\; -2^{-1}\mathrm{c}(m)(m-2)^2(m-3)$$

$$\times \int_{y_3=-\infty}^{s_3} ((\mathrm{ch}\ q - \mathrm{ch}\ s)/\mathrm{sh}^2 q)^{(m-2)/2}\ \mathrm{ch}\ q\ \mathrm{d}y_3$$

$$\iint_{D(y_3)} \delta_1(-T)^{(m-4)/2} f(x_1, x_2, x_3)\, dx_2\, dx_3 + 2^{-2} c(m)(m-2)^2$$

$$\times \int_{y_3=-\infty}^{s_3} ((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2} \operatorname{sh} q\, dy_3$$

$$\iint_{D(y_3)} \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right) (\delta_1(-T)^{(m-4)/2} f(x_1, x_2, x_3))\, dx_2\, dx_3.$$

The second term in this sum can be written as (up to the constant $2^{-2}c(m)(m-2)^2$)

$$\int_{y_3=-\infty}^{s_3} ((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2} \operatorname{sh} q\, dy_3$$

$$\left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right) \left\{ \iint_{D(y_3)} \delta_1(-T)^{(m-4)/2} f(x_1, x_2, x_3)\, dx_2\, dx_3 \right\}, \tag{6.2}$$

since $T = 0$ on $\partial(D(y_3))$ and we assumed $\operatorname{Re} m > 6$. Next we can perform an integration by parts with respect to the variable y_3. Then (6.2) becomes

$$-\int_{y_3=-\infty}^{s_3} \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right) \left\{((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2} \operatorname{sh} q \right\} dy_3$$

$$\iint_{D(y_3)} \delta_1(-T)^{(m-4)/2} f(x_1, x_2, x_3)\, dx_2\, dx_3.$$

Put

$$Q(y_2, y_3) = 2(m-3)((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2} \operatorname{ch} q$$

$$+ \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right) \left\{((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-2)/2} \operatorname{sh} q \right\},$$

then we have shown that

$$F^m_{D(m)f}(s_1, s_2, s_3) = -2^{-2} c(m)(m-2)^2 \int_{y_3=-\infty}^{s_3} Q(y_2, y_3)\, dy_3$$

$$\iint_{D(y_3)} \delta_1(-T)^{(m-4)/2} f(x_1, x_2, x_3)\, dx_2\, dx_3.$$

An easy calculation shows that

$$Q(y_2, y_3) = (m-2)((\operatorname{ch} q - \operatorname{ch} s)/\operatorname{sh}^2 q)^{(m-4)/2},$$

and consequently

$$F^m_{D(m)f}(s_1, s_2, s_3) = -2^{-2}c(m)(m-2)^3(c(m-2))^{-1}F^{m-2}_f(s_1, s_2, s_3).$$

Since $c(m)/c(m-2) = 2^2(2\pi)^3(m-2)^{-3}$ we obtain

$$F^m_{D(m)f} = (-2\pi)^3 F^{m-2}_f \text{ on } \mathfrak{a}^+. \tag{6.3}$$

By W-invariance the equality (6.3) holds on $\mathfrak{a}'$. By analytic continuation with respect to m we obtain the theorem for $\operatorname{Re} m > 2$; it is easy to check the case $m = 2$. ■

It remains to prove the following proposition.

PROPOSITION 6

$$(D_1 - kD_2)(T^k) = -16k^2(2k-1) \operatorname{ch} q\delta_1 T^{k-1}$$
$$+ 8k^2 \operatorname{sh} q\delta_1 \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)(T^{k-1}), \quad k \in \mathbb{C}.$$

Proof. The only functions in T depending on the x_i are $\operatorname{ch}(2x_i - p)$ $(i = 1, 2, 3)$. Since these functions and their derivatives occur very frequently in the proof of this proposition, we introduce the notation

$$c_i = \operatorname{ch}(2x_i - p), \quad s_i = \operatorname{sh}(2x_i - p) \quad (i = 1, 2, 3).$$

Two simple calculations show that

$$D_1 T^k = kT^{k-3}\Big[(k-1)(k-2)\prod_{i<j}((\partial_i - \partial_j)T) + (k-1)T$$
$$\times \sum_{i<j;\, m \neq i,j} (-1)^{i+j+1}((\partial_i - \partial_j)T)((\partial_i - \partial_m)(\partial_j - \partial_m)T) + T^2 D_1 T\Big],$$

and

$$D_2 T^k = kT^{k-2}\left[(k-1)\sum_{i<j}(-1)^{i+j}((\partial_i - \partial_j)T)^2 \right.$$
$$\left. \times \coth(x_i - x_j) + TD_2 T\right].$$

So

$$(D_1 - kD_2)(T^k) = k(k-1)T^{k-3}R_1(T) + kT^{k-1}(D_1 - kD_2)(T), \tag{6.4}$$

with

$$R_1(T) = (k-2)\prod_{i<j}((\partial_i - \partial_j)T) + T\sum_{i<j;m\neq i,j}(-1)^{i+j+1}((\partial_i - \partial_j)T)$$
$$\times ((\partial_i - \partial_m)(\partial_j - \partial_m)T) + kT\sum_{i<j}(-1)^{i+j+1}((\partial_i - \partial_j)T)^2$$
$$\times \coth(x_i - x_j). \tag{6.5}$$

For the remainder of the proof we shall introduce one extra notational convention: put

$$sc_{ij} = (c_j - \operatorname{ch} q)s_i - (c_i - \operatorname{ch} q)s_j.$$

Note that $sc_{ij} = -sc_{ji}$. This term sc_{ij} occurs frequently since

$$(\partial_i - \partial_j)(T) = 2(c_m - \operatorname{ch} q)((c_j - \operatorname{ch} q)s_i - (c_i - \operatorname{ch} q)s_j)$$
$$= 2(c_m - \operatorname{ch} q)sc_{ij}, \quad m \neq i, j. \tag{6.6}$$

It will also be convenient to have some alternative expressions for sc_{ij} available. First, since

$$c_i s_j - s_i c_j = \operatorname{sh}(2x_j - 2x_i), \tag{6.7}$$

we have

$$sc_{ij} = \operatorname{sh}(2x_i - 2x_j) - \operatorname{ch} q\,(s_i - s_j). \tag{6.8}$$

From

$$s_i - s_j = 2 \operatorname{sh}(x_i - x_j) \operatorname{ch}(x_i + x_j - p) \tag{6.9}$$

and

$$\operatorname{sh}(2x_i - 2x_j) = 2 \operatorname{sh}(x_i - x_j) \operatorname{ch}(x_i - x_j)$$

then follows that

$$sc_{ij} = 2 \operatorname{sh}(x_i - x_j)(\operatorname{ch}(x_i - x_j) - \operatorname{ch} q \operatorname{ch}(x_i + x_j - p)), \tag{6.10}$$

and thus

$$\coth(x_i - x_j) sc_{ij} = 2 \operatorname{ch}(x_i - x_j)(\operatorname{ch}(x_i - x_j) - \operatorname{ch} q \operatorname{ch}(x_i + x_j - p)). \tag{6.11}$$

With these facts one can easily prove the following (remarkable) lemma:

LEMMA 2

$$D_1 T = 2D_2 T = -32 \operatorname{ch} q \delta_1. \tag{6.12}$$

Proof. With our conventions we have that

$$T = \prod_{i=1}^{3} (c_i - \operatorname{ch} q).$$

First we calculate $D_1 T$. By (6.6) we have

$$(\partial_2 - \partial_3)(T) = 2(c_1 - \operatorname{ch} q) sc_{23},$$

so

$$(\partial_1 - \partial_3)(\partial_2 - \partial_3)(T) = 4(s_1 s_2 (c_3 - \operatorname{ch} q) - s_1 s_3 (c_2 - \operatorname{ch} q)$$

$$- s_2 s_3 (c_1 - \operatorname{ch} q) + (c_1 - \operatorname{ch} q)(c_2 - \operatorname{ch} q) c_3),$$

and thus

$$D_1 T = 8((c_1 s_2 - s_1 c_2)(c_3 - \operatorname{ch} q) + (c_3 s_1 - s_3 c_1)(c_2 - \operatorname{ch} q)$$
$$+ (c_2 s_3 - s_2 c_3)(c_1 - \operatorname{ch} q))$$
$$= -8 \operatorname{ch} q(-\operatorname{sh}(2x_1 - 2x_2) + \operatorname{sh}(2x_1 - 2x_3) - \operatorname{sh}(2x_2 - 2x_3)), \tag{6.13}$$

where we used (6.7) for the second equality. Now note that

$$\delta_1 = \prod_{i<j} \operatorname{sh}(x_i - x_j) = \tfrac{1}{4}(-\operatorname{sh}(2x_1 - 2x_2) + \operatorname{sh}(2x_1 - 2x_3)$$
$$- \operatorname{sh}(2x_2 - 2x_3)), \tag{6.14}$$

so

$$D_1 T = -32 \operatorname{ch} q \delta_1.$$

Next we calculate $D_2 T$. From (6.6) and (6.11) we have

$$\coth(x_i - x_j)((\partial_i - \partial_j)T)$$
$$= 4 \operatorname{ch}(x_i - x_j)(c_m - \operatorname{ch} q)(\operatorname{ch}(x_i - x_j) - \operatorname{ch} q \operatorname{ch}(x_i + x_j - p)),$$
$$m \neq i, j. \tag{6.15}$$

Consequently

$$(\partial_i - \partial_j) \circ \coth(x_i - x_j) \circ (\partial_i - \partial_j)(T)$$
$$= 8(c_m - \operatorname{ch} q)(\operatorname{sh}(2x_i - 2x_j)$$
$$- \operatorname{ch} q \operatorname{sh}(x_i - x_j) \operatorname{ch}(x_i + x_j - p)), \quad m \neq i, j.$$

Using (6.7) and (6.9) we obtain for $m \neq i, j$

$$(\partial_i - \partial_j) \circ \coth(x_i - x_j) \circ (\partial_i - \partial_j)(T)$$
$$= -8(c_m - \operatorname{ch} q)(c_i s_j - s_i c_j) - 4c_m \operatorname{ch} q(s_i - s_j) + 4 \operatorname{ch}^2 q(s_i - s_j).$$

Put in the signs $(-1)^{i+j}$ and add the terms $i < j$. If we compare with (6.13) then it follows that

$$\begin{aligned} D_2 T &= D_1 T - 4 \operatorname{ch} q(-(s_1 - s_2)c_3 + (s_1 - s_3)c_2 - (s_2 - s_3)c_1) \\ &\quad + 4 \operatorname{ch}^2 q(-(s_1 - s_2) + (s_1 - s_3) - (s_2 - s_3)) \\ &= D_1 T - 4 \operatorname{ch} q(c_2 s_1 - s_2 c_1 + c_1 s_3 - s_1 c_3 + c_3 s_2 - s_3 c_2) \\ &= D_1 T + 16 \operatorname{ch} q \delta_1 . \end{aligned} \tag{6.16}$$

The last equality again follows from (6.7) and (6.14). This proves the lemma. ■

It remains to calculate $R_1(T)$ given by (6.5). The notation sc_{ij} and formulas (6.6)–(6.11) will again be very useful. From (6.6) follows that the first term in $R_1(T)$, up to the constant $(k - 2)$, equals

$$\begin{aligned} \prod_{i<j} ((\partial_i - \partial_j)T) &= 8T \prod_{i<j} sc_{ij} \\ &= 8T\{(c_1 - \operatorname{ch} q)^2 sc_{23} s_2 s_3 + (c_3 - \operatorname{ch} q)^2 sc_{12} s_1 s_2 \\ &\quad + (c_2 - \operatorname{ch} q)^2 sc_{31} s_1 s_3\}. \end{aligned} \tag{6.17}$$

Now

$$\begin{aligned} (\partial_i - \partial_m)(\partial_i - \partial_j)T &= 4((c_m - \operatorname{ch} q)(c_j - \operatorname{ch} q)c_i - (c_m - \operatorname{ch} q)s_i s_j \\ &\quad + (c_i - \operatorname{ch} q)s_j s_m - (c_j - \operatorname{ch} q)s_i s_m), \end{aligned}$$

so

$$\begin{aligned} &((\partial_j - \partial_m)T)((\partial_i - \partial_m)(\partial_i - \partial_j)T) \\ &= 8(c_i - \operatorname{ch} q)(c_j - \operatorname{ch} q)(c_m - \operatorname{ch} q)\{(c_m - \operatorname{ch} q)c_i s_j \\ &\quad - (c_j - \operatorname{ch} q)c_i s_m\} + 8(c_i - \operatorname{ch} q)\{(c_i - \operatorname{ch} q)(c_m - \operatorname{ch} q)s_j^2 s_m \\ &\quad - (c_i - \operatorname{ch} q)(c_j - \operatorname{ch} q)s_j s_m^2 + (c_j - \operatorname{ch} q)^2 s_i s_m^2 - (c_m - \operatorname{ch} q)^2 s_i s_j^2\}. \end{aligned}$$

This gives us, after some reordering, the second term in $R_1(T)$:

$$T \sum_{i<j;m\neq i,j} (-1)^{i+j+1}((\partial_i - \partial_j)T)((\partial_i - \partial_m)(\partial_j - \partial_m)T)$$

$$= 8T^2\{(c_3 - \operatorname{ch} q)(c_1 s_2 - c_2 s_1) + (c_2 - \operatorname{ch} q)(c_3 s_1 - c_1 s_3)$$

$$+ (c_1 - \operatorname{ch} q)(c_2 s_3 - c_3 s_2)\} + 16T\{(c_1 - \operatorname{ch} q)^2 sc_{23} s_2 s_3$$

$$+ (c_2 - \operatorname{ch} q)^2 sc_{31} s_1 s_3 + (c_3 - \operatorname{ch} q)^2 sc_{12} s_1 s_2\}.$$

If we compare with (6.13) then we see that the first term in parentheses $\{-\}$ equals

$$2^{-3} D_1 T = -4 \operatorname{ch} q\delta_1,$$

while the second term in parentheses $\{-\}$ is the same as in (6.17). If we use (6.6) for the last term in $R_1(T)$, then we obtain that

$$R_1(T) = 8kT \sum_{i<j;m\neq i,j} (-1)^{i+j}(c_m - \operatorname{ch} q)^2 sc_{ji} s_i s_j - 32T^2 \operatorname{ch} q\delta_1$$

$$+ 4kT \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(c_m - \operatorname{ch} q)^2 sc_{ij}^2 \coth (x_i - x_j).$$

Put

$$R_2(T) = (R_1(T) + 32T^2 \operatorname{ch} q\delta_1)/8kT$$

and use (6.11) to rewrite $R_2(T)$ as

$$R_2(T) = \sum_{i<j;m\neq i,j} (-1)^{i+j}(c_m - \operatorname{ch} q)^2 sc_{ji}$$

$$\times \{s_i s_j + \operatorname{ch} (x_i - x_j)(\operatorname{ch} (x_i - x_j) - \operatorname{ch} q \operatorname{ch} (x_i + x_j - p))\}.$$

Since

$$s_i s_j = \operatorname{sh} (2x_i - p) \operatorname{sh} (2x_j - p)$$

$$= \operatorname{ch}^2(x_i + x_j - p) - \operatorname{ch}^2(x_i - x_j),$$

we conclude that

$$R_2(T) = \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(c_m - \mathrm{ch}\, q)^2 2\, \mathrm{sh}\,(x_i - x_j)(\mathrm{ch}\,(x_i - x_j) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i + x_j - p))\, \mathrm{ch}\,(x_i + x_j - p)(\mathrm{ch}\,(x_i + x_j - p) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i - x_j)).$$

Here we also used expression (6.10) for $sc_{ji} = -sc_{ij}$. Now we can replace $2\,\mathrm{sh}\,(x_i - x_j)\,\mathrm{ch}\,(x_i + x_j - p)$ by $s_i - s_j$ again (cf. (6.9)). This leads to

$$R_2(T) = \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(s_i - s_j)(c_m - \mathrm{ch}\, q)^2(\mathrm{ch}\,(x_i - x_j) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i + x_j - p))(\mathrm{ch}\,(x_i + x_j - p) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i - x_j)).$$

LEMMA 3

$$R_2(T) = -4\, \mathrm{ch}\, q\delta_1 T + \mathrm{sh}\, q\delta_1 \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)(T).$$

Proof. Put

$$U_{ij} = (\mathrm{ch}\,(x_i - x_j) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i + x_j - p))(\mathrm{ch}\,(x_i + x_j - p) - \mathrm{ch}\, q\, \mathrm{ch}\,(x_i - x_j))$$

then

$$U_{ij} = (1 + \mathrm{ch}^2 q)\, \mathrm{ch}\,(x_i - x_j)\, \mathrm{ch}\,(x_i + x_j - p) - \mathrm{ch}\, q\, (\mathrm{ch}^2(x_i - x_j) + \mathrm{ch}^2(x_i + x_j - p)).$$

Use $\mathrm{ch}^2 w + \mathrm{ch}^2 z = \mathrm{ch}\,(w + z)\, \mathrm{ch}\,(w - z) + 1$ and $2\, \mathrm{ch}\, w\, \mathrm{ch}\, z = \mathrm{ch}\,(w + z) + \mathrm{ch}\,(w - z)$ to rewrite this as

$$U_{ij} = \tfrac{1}{2}(1 + \mathrm{ch}^2 q)(c_i + c_j) - \mathrm{ch}\, q(c_i c_j + 1).$$

Since $\frac{1}{2}(1 + \mathrm{ch}^2 q) = \mathrm{ch}^2 q - \frac{1}{2}\mathrm{sh}^2 q$, we obtain for $m \neq i, j$

$$
\begin{aligned}
(c_m - \mathrm{ch}\, q)U_{ij} &= \mathrm{ch}^2 q(c_i c_m + c_j c_m) - \mathrm{ch}\, q\, c_i c_j c_m - \mathrm{ch}\, q\, c_m \\
&\quad - \tfrac{1}{2}\mathrm{sh}^2 q(c_i c_m + c_j c_m) - \mathrm{ch}^3 q(c_i + c_j) + \mathrm{ch}^2 q\, c_i c_j \\
&\quad + \mathrm{ch}^2 q + \tfrac{1}{2}\mathrm{ch}\, q\, \mathrm{sh}^2 q(c_i + c_j) \\
&= -\mathrm{ch}\, q\, c_i c_j c_m + \mathrm{ch}^2 q(c_i c_j + c_i c_m + c_j c_m) - \mathrm{ch}^3 q(c_i + c_j + c_m) \\
&\quad + \mathrm{ch}^4 q + \mathrm{ch}^2 q - \mathrm{ch}^4 q + \mathrm{ch}^3 q\, c_m - \mathrm{ch}\, q\, c_m \\
&\quad + \tfrac{1}{2}\mathrm{ch}\, q\, \mathrm{sh}^2 q(c_i + c_j + c_m) - \tfrac{1}{2}\mathrm{sh}^2 q\, (c_i c_j + c_i c_m + c_j c_m) \\
&\quad + \tfrac{1}{2}\mathrm{sh}^2 q\, c_i c_j - \tfrac{1}{2}\mathrm{ch}\, q\, \mathrm{sh}^2 q\, c_m \\
&= -\mathrm{ch}\, q\{c_1 c_2 c_3 - \mathrm{ch}\, q(c_1 c_2 + c_1 c_3 + c_2 c_3) + \mathrm{ch}^2 q(c_1 + c_2 + c_3) - \mathrm{ch}^3 q\} \\
&\quad + \tfrac{1}{2}\mathrm{ch}\, q\, \mathrm{sh}^2 q(c_1 + c_2 + c_3) - \tfrac{1}{2}\mathrm{sh}^2 q(c_1 c_2 + c_1 c_3 + c_2 c_3) \\
&\quad - \mathrm{ch}^2 q\, \mathrm{sh}^2 q + \tfrac{1}{2}\mathrm{sh}^2 q\, c_i c_j + \tfrac{1}{2}\mathrm{ch}\, q\, \mathrm{sh}^2 q\, c_m \\
&= -\,\mathrm{ch}\, qT + \tfrac{1}{2}\mathrm{sh}^2 q\{\mathrm{ch}\, q(c_1 + c_2 + c_3) - 2\mathrm{ch}^2 q \\
&\quad - (c_1 c_2 + c_1 c_3 + c_2 c_3)\} + \tfrac{1}{2}\mathrm{sh}^2 q(c_i c_j + \mathrm{ch}\, q\, c_m).
\end{aligned}
$$

Next observe that (cf. (6.16))

$$
\begin{aligned}
&\sum_{i<j;\, m \neq i,j} (-1)^{i+j+1}\, (s_i - s_j)(c_m - \mathrm{ch}\, q) \\
&= \sum_{i<j;\, m \neq i,j} (-1)^{i+j+1}\, (s_i - s_j)c_m = 4\delta_1,
\end{aligned}
\tag{6.18}
$$

hence

$$
\begin{aligned}
R_2(T) \;&=\; -4\,\mathrm{ch}\, q\delta_1 T + 2\,\mathrm{sh}^2 q\delta_1 \\
&\quad \times \{\mathrm{ch}\, q\, (c_1 + c_2 + c_3) - 2\,\mathrm{ch}^2 q - (c_1 c_2 + c_1 c_3 + c_2 c_3)\} \\
&\quad + \tfrac{1}{2}\mathrm{sh}^2 q \sum_{i<j;\, m \neq i,j} (-1)^{i+j+1}(s_i - s_j)(c_m - \mathrm{ch}\, q)(c_i c_j + \mathrm{ch}\, q\, c_m).
\end{aligned}
$$

Now

$$\sum_{i<j;m\neq i,j} (-1)^{i+j+1}(s_i - s_j)(c_m - \operatorname{ch} q)(c_i c_j + \operatorname{ch} q\, c_m)$$

$$= \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(s_i - s_j)((c_m^2 - c_i c_j)\operatorname{ch} q - \operatorname{ch}^2 q\, c_m + c_1 c_2 c_3)$$

$$= -\operatorname{ch}^2 q \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(s_i - s_j)c_m$$

$$+ \operatorname{ch} q \sum_{i<j;m\neq i,j} (-1)^{i+j+1}(s_i - s_j)(c_m^2 - c_i c_j)$$

$$= -4\operatorname{ch}^2 q\delta_1 + \operatorname{ch} q\{c_1(c_1 s_2 - c_2 s_1 + c_3 s_1 - c_1 s_3)$$

$$+ c_2(c_1 s_2 - c_2 s_1 + c_2 s_3 - c_3 s_2) + c_3(c_3 s_1 - c_1 s_3 + c_2 s_3 - c_3 s_2)\}$$

$$= -4\operatorname{ch}^2 q\delta_1 + 4\operatorname{ch} q(c_1 + c_2 + c_3)\,\delta_1 .$$

For the third equality we used (6.18) and for the last equality (6.7), (6.14) and the fact that (e.g. again by (6.7))

$$c_1 \operatorname{sh}(2x_2 - 2x_3) - c_2 \operatorname{sh}(2x_1 - 2x_3) + c_3 \operatorname{sh}(2x_1 - 2x_2) = 0.$$

Thus we obtain

$$R_2(T) = -4\operatorname{ch} q\delta_1 T + 2\operatorname{sh}^2 q\delta_1$$

$$\times \{2\operatorname{ch} q\,(c_1 + c_2 + c_3) - 3\operatorname{ch}^2 q - (c_1c_2 + c_1c_3 + c_2c_3)\},$$

and indeed

$$\left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)(T) = \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)\left(\prod_{i=1}^{3}(c_i - \operatorname{ch} q)\right)$$

$$= -2\operatorname{sh} q(c_1c_2 + c_1c_3 + c_2c_3)$$

$$+ 4\operatorname{ch} q\operatorname{sh} q\,(c_1 + c_2 + c_3) - 6\operatorname{ch}^2 q\operatorname{sh} q,$$

which proves Lemma 3. ■

We deduce from Lemma 3 and the definition of $R_2(T)$ that

$$\begin{aligned} R_1(T) &= 8kTR_2(T) - 32 \operatorname{ch} q\delta_1 T^2 \\ &= -32(k+1) \operatorname{ch} q\delta_1 T^2 + 8k \operatorname{sh} q\delta_1 T \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} \right)(T). \end{aligned}$$

Now recall (6.4) and (6.12) (= Lemma 2), then we obtain

$$\begin{aligned} (D_1 - kD_2)(T^k) = & -32k(k-1)(k+1) \operatorname{ch} q\delta_1 T^{k-1} \\ & + 8k^2(k-1) \operatorname{sh} q\delta_1 T^{k-2} \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} \right)(T) \\ & + 16k(k-2) \operatorname{ch} q\delta_1 T^{k-1}, \end{aligned}$$

and consequently

$$\begin{aligned} (D_1 - kD_2)(T^k) = & -16k^2(2k-1) \operatorname{ch} q\delta_1 T^{k-1} \\ & + 2^3k^2 \operatorname{sh} q\delta_1 \left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} \right)(T^{k-1}), \end{aligned}$$

which finally proves Proposition 6. ∎

Appendix 1: The shift operator for root system A_3

Put $E = \mathbb{R}^4$. In E we have the standard basis e_1, e_2, e_3, e_4 and inner product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal. Let $\mathfrak{a}$ denote the hyperplane in E orthogonal to the vector $e_1 + e_2 + e_3 + e_4$. The inner product on E induces an inner product on $\mathfrak{a}$ which we shall also denote by $\langle \cdot, \cdot \rangle$. We identify the dual E^* with E (and $\mathfrak{a}^*$ with $\mathfrak{a}$) by means of this inner product.

The root system of type A_3 can be identified with the set $\Sigma = \{e_i - e_j \mid 1 \leqslant i, j \leqslant 4, i \neq j\}$ in $\mathfrak{a}$. For Σ we take as basis $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4\}$. Let Σ^+ be the set of positive roots with respect to Δ. The Weyl group W of Σ is isomorphic to the symmetric group S_4. The fundamental weights of Σ will be denoted by λ_1, λ_2, λ_3 i.e. $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. As in Sections 4 and 5 we put

$$z_i = \sum_{w \in W/W(\lambda_i)} e^{2w\lambda_i} \quad (i = 1, 2, 3).$$

For $m \in \mathbb{C}$ we define $L(m)$ and $\varrho(m)$ as in Section 3. Then $\varrho(m) = \frac{1}{2}m\varrho(2) = \frac{1}{2}m(3e_1 + e_2 - e_3 - 3e_4)$.

We now define $D(m)$, the operator which shifts the parameter m to $m - 2$ in the sense of Section 3. Write ∂_i for $\partial/\partial z_i$ then

$$2^{-6}D(m) = D_3(m) + D_4(m) + D_5(m) + D_6(m),$$

with

$$D_3(m) = \tfrac{1}{2}(m+1)(m+2)\partial_2^3 + \tfrac{1}{4}(m+1)(m+2)^2\partial_1\partial_2\partial_3,$$

$$\begin{aligned}
D_4(m) = {} & \tfrac{3}{2}(m+2)z_2\partial_2^4 + \tfrac{1}{2}(m+1)(m+2)(\partial_1^3\partial_3 + \partial_1\partial_3^3) \\
& + \tfrac{1}{2}(m+2)(m+7)(z_1\partial_1\partial_2^3 + z_3\partial_2^3\partial_3) + \tfrac{3}{4}(m+2)(m+6)(\partial_1^2\partial_2^2 + \partial_2^2\partial_3^2) \\
& + \tfrac{3}{4}(m+2)^2(z_1\partial_1^2\partial_2\partial_3 + z_3\partial_1\partial_2\partial_3^2) + \tfrac{1}{4}(m+2)(5m+14)z_2\partial_1\partial_2^2\partial_3,
\end{aligned}$$

$$\begin{aligned}
D_5(m) = {} & (z_2^2 + (m+3)z_1z_3 - 4)\partial_2^5 + (2m+7)(\partial_1^4\partial_2 + \partial_2\partial_3^4) \\
& + \tfrac{1}{2}(m+2)(z_1\partial_1^4\partial_3 + z_3\partial_1\partial_3^4) + ((\tfrac{3}{2}m+7)z_1z_2 + (m+2)z_3)\partial_1\partial_2^4 \\
& + ((\tfrac{3}{2}m+7)z_2z_3 + (m+2)z_1)\partial_2^4\partial_3 + (\tfrac{7}{2}m+13)(z_1\partial_1^3\partial_2^2 + z_3\partial_2^2\partial_3^3) \\
& + (\tfrac{3}{2}m+3)(z_3\partial_1^3\partial_3^2 + z_1\partial_1^2\partial_3^3) + ((\tfrac{3}{2}m+6)z_1^2 + (2m+8)z_2)\partial_1^2\partial_2^3 \\
& + ((\tfrac{3}{2}m+6)z_3^2 + (2m+8)z_2)\partial_2^3\partial_3^2 + ((\tfrac{1}{2}m+1)z_1^2 + (2m+6)z_2)\partial_1^3\partial_2\partial_3 \\
& + ((2m+6)z_2^2 + (m+6)z_1z_3 + 8)\partial_1\partial_2^3\partial_3 \\
& + ((\tfrac{1}{2}m+1)z_3^2 + (2m+6)z_2)\partial_1\partial_2\partial_3^3 \\
& + ((\tfrac{5}{2}m+7)z_1z_2 + (\tfrac{3}{2}m+9)z_3)\partial_1^2\partial_2^2\partial_3 \\
& + ((2m+5)z_1z_3 + (4m+10))\partial_1^2\partial_2\partial_3^2 \\
& + ((\tfrac{5}{2}m+7)z_2z_3 + (\tfrac{3}{2}m+9)z_1)\partial_1\partial_2^2\partial_3^2,
\end{aligned}$$

and

$$\begin{aligned}
D_6(m) = {} & \partial_1^6 + (z_1z_2z_3 - z_1^2 - z_3^2)\partial_2^6 + \partial_3^6 + 3z_1\partial_1^5\partial_2 + z_2(\partial_1^5\partial_3 + \partial_1\partial_3^5) \\
& + (z_1z_2^2 + z_1^2z_3 - 4z_1)\partial_1\partial_2^5 + (z_2^2z_3 + z_1z_3^2 - 4z_3)\partial_2^5\partial_3 + 3z_3\partial_2\partial_3^5
\end{aligned}$$

$$+ (3z_1^2 + 2z_2)\partial_1^4\partial_2^2 + (z_1z_3 - 1)(\partial_1^4\partial_3^2 + \partial_1^2\partial_3^4) + (3z_3^2 + 2z_2)\partial_2^2\partial_3^4$$

$$+ (2z_1^2z_2 + z_2^2 + z_1z_3 - 4)\partial_1^2\partial_2^4 + (2z_2z_3^2 + z_2^2 + z_1z_3 - 4)\partial_2^4\partial_3^2$$

$$+ (2z_1z_2 + 3z_3)\partial_1^4\partial_2\partial_3 + (z_2^3 + 2z_1z_2z_3 + 3z_3^2 + 3z_1^2 - 4z_2)\partial_1\partial_2^4\partial_3$$

$$+ (2z_2z_3 + 3z_1)\partial_1\partial_2\partial_3^4 + (z_1^3 + 4z_1z_2)\partial_1^3\partial_2^3 + (z_1^2 + z_3^2 - 2z_2)\partial_1^3\partial_3^3$$

$$+ (z_3^3 + 4z_2z_3)\partial_2^3\partial_3^3 + (z_1^2z_2 + 2z_2^2 + 4z_1z_3 + 8)\partial_1^3\partial_2^2\partial_3$$

$$+ (z_1^2z_3 + 2z_2z_3 + 2z_1)\partial_1^3\partial_2\partial_3^2 + (2z_1z_2^2 + z_1^2z_3 + 4z_2z_3 + 8z_1)\partial_1^2\partial_2^3\partial_3$$

$$+ (2z_2^2z_3 + z_1z_3^2 + 4z_1z_2 + 8z_3)\partial_1\partial_2^3\partial_3^2 + (z_2z_3^2 + 2z_2^2 + 4z_1z_3 + 8)\partial_1\partial_2^2\partial_3^3$$

$$+ (z_1z_3^2 + 2z_1z_2 + 2z_3)\partial_1^2\partial_2\partial_3^3 + (3z_1z_2z_3 + 12z_2)\partial_1^2\partial_2^2\partial_3^2.$$

We obtained this operator as follows. Start with the operator $\delta^{-1}\prod_{\alpha>0}\partial_\alpha$ with δ and ∂_α as in Section 5. Recall that $\delta^{-1}\prod_{\alpha>0}\partial_\alpha$ shifts from $m = 2$ to $m = 0$. By use of a computer we translate this operator in the z-coordinates. This is the operator $D(2)$. Now recall (i) in the definition of shift operator in Section 3, i.e.

$$\begin{aligned} &D(m)\circ(L(m-2) + \langle\varrho(m-2), \varrho(m-2)\rangle) \\ &\quad = (L(m) + \langle\varrho(m), \varrho(m)\rangle)\circ D(m). \end{aligned} \tag{A.0}$$

If we use this identity for successive values of m then it is possible (again using a computer) to determine the dependence on m. In these calculations we assume that m is linear in $D_5(m)$, quadratic in $D_4(m)$ and cubic in $D_3(m)$. Another way to obtain the operator is to use the gradation introduced by Opdam [14, §2]. Then use Proposition 2.3 in [14] to determine which terms $z_1^{k_1}z_2^{k_2}z_3^{k_3}\partial_1^{\ell_1}\partial_2^{\ell_2}\partial_3^{\ell_3}$ can occur. It turns out that this gives precisely the same terms as in the operator $D(2)$. The dependence on m then follows as above, where we now also have to determine the constants in $D_6(m)$. Note that in either of these methods we check (A.0) in the process of finding the operator $D(m)$.

Remark. For all our computer-calculations we used the algebraic programming system "Reduce" (version 3.0, April 1983) on the IBM 370/3083 computer. It took approximately 10 sec. cpu-time to check (A.0) for this case (for A_2 this was 0.2 sec.).

In order to use (A.0) one needs the operator $L(m)$ in the z-coordinates.

PROPOSITION 7

$$
\begin{aligned}
L(m) \;=\;& (3z_1^2 - 8z_2)\partial_1^2 + (4z_2^2 - 8z_1z_3 - 16)\partial_2^2 + (3z_3^2 - 8z_2)\partial_3^2 \\
&+ (4z_1z_2 - 24z_3)\partial_1\partial_2 + (2z_1z_3 - 32)\partial_1\partial_3 \\
&+ (4z_2z_3 - 24z_1)\partial_2\partial_3 + (2m + 1)(3z_1\partial_1 + 4z_2\partial_2 + 3z_3\partial_3).
\end{aligned}
$$

Since there is not much difference in the calculations for A_3 or A_ℓ, $\ell \geqslant 3$ we give the operator $L(m)$ in the z-coordinates for A_ℓ in Appendix 2.

Since we have an operator $D(m)$ which satisfies (A.0) we can now apply Proposition 2.5 in [14]. Then it follows that $D(m)$ is a shift operator with shift 2, i.e. $D(m)$ shifts from m to $m - 2$. Furthermore

$$
p_0(i\lambda - \varrho(m - 2)) \;=\; \prod_{\alpha > 0} (\langle \alpha, i\lambda \rangle - (m - 2)).
$$

This follows as in the proof of [14, Theorem 3.6] or by [14, (3.25)]. Note that for $m = 2$ the corresponding p_0 is indeed $\Pi_{\alpha > 0} \langle \alpha, i\lambda \rangle$. Precisely as in Section 5 it follows that q is independent of λ (the expression for $\mathbf{c}^m(-\lambda)/\mathbf{c}^{m-2}(-\lambda)$ in Section 5 was derived under the assumption that the root system had one root length). In fact, with notations as in Section 5, we have

$$
\begin{aligned}
q(m) \;&=\; 2^6 \prod_{\alpha > 0} \frac{(\frac{1}{2}(m - 2)(1 + \langle \varrho(1), \alpha \rangle))_{\langle \varrho(1), \alpha \rangle + 1}}{(\frac{1}{2}(m - 2)\langle \varrho(1), \alpha \rangle)_{\langle \varrho(1), \alpha \rangle}} \\
&=\; 2^8 3(m - 1)^2(3m - 4)(3m - 2)(2m - 3)(2m - 1), \quad m \geqslant 2.
\end{aligned}
$$

Note that $q(2) = 2^8 3 \cdot 4! = w\, \Pi_{\alpha > 0} \langle \alpha, -\varrho(2) \rangle$.

So, as in Section 5, we obtain an inversion of the Abel transform by a differential operator if m is even.

THEOREM 1′. *Let $X = G/K$ be one of the following symmetric spaces of the noncompact type:* SL(4, $\mathbb{C}$)/SU(4), SU*(8)/Sp(4). *So the associated root system of X is of type A_3 and the multiplicity m equals* 2 *and* 4 *respectively. Let F_f^m denote the corresponding Abel transform ($f \in \mathcal{D}(G//K)$). Then*

$$
F^4_{D(4)f} \;=\; q(4)F_f^2 \quad \textit{and} \quad F^2_{D(2)f} \;=\; q(2)f,
$$

with

$$q(4) \;=\; 2^{12}3^3 5^2 7 \quad \textit{and} \quad q(2) \;=\; 2^{11}3^2.$$

Appendix 2: The radial part of the Laplace-Beltrami operator for A_ℓ in the z-coordinates

Put $E = \mathbb{R}^{\ell+1}$. In E we have the standard basis $e_1, e_2, \ldots, e_{\ell+1}$ and inner product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal. Let $\mathfrak{a}$ denote the hyperplane in E orthogonal to the vector $e_1 + e_2 + \cdots + e_{\ell+1}$. The inner product on E induces an inner product on $\mathfrak{a}$ which we shall also denote by $\langle \cdot, \cdot \rangle$. We identify the dual E^* with E (and $\mathfrak{a}^*$ with $\mathfrak{a}$) by means of this inner product.

The root system of type A_ℓ can be identified with the set $\Sigma = \{e_i - e_j \mid 1 \leqslant i, j \leqslant \ell + 1, i \neq j\}$ in $\mathfrak{a}$. For Σ we take as basis $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_\ell = e_\ell - e_{\ell+1}\}$. Let Σ^+ be the set of positive roots with respect to Δ. The Weyl group W of Σ is isomorphic to the symmetric group $S_{\ell+1}$. The fundamental weights of Σ will be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, i.e., $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. As in Sections 4 and 5 we put

$$z_i \;=\; \sum_{w \in W/W(\lambda_i)} e^{2w\lambda_i} \quad (i = 1, 2, \ldots, \ell).$$

For $m \in \mathbb{C}$ we define $L(m)$ by the right-hand side of (2.2) (cf. Section 3) i.e.,

$$L(m) \;=\; L_{\mathfrak{a}} + mR,$$

where $L_{\mathfrak{a}}$ is the ordinary Laplacian on $\mathfrak{a}$ and

$$R \;=\; \sum_{\alpha > 0} \coth \alpha \cdot \partial_\alpha .$$

We use $(t_1, t_2, \ldots, t_{\ell+1})$ as coordinates on E; on $\mathfrak{a}$ we have $t_1 + t_2 + \cdots + t_{\ell+1} = 0$. In this appendix we shall write

$$\partial_i \;=\; \frac{\partial}{\partial t_i}, \quad \partial_{z_j} \;=\; \frac{\partial}{\partial z_j} \quad (i = 1, 2, \ldots, \ell + 1;\ j = 1, 2, \ldots, \ell).$$

Considered as differential operators on $\mathfrak{a}$, $L_{\mathfrak{a}}$ and R are given by

$$L_{\mathfrak{a}} \;=\; \sum_{i=1}^{\ell+1} \partial_i^2, \quad R \;=\; \sum_{1 \leqslant i < j \leqslant \ell+1} \coth(t_i - t_j)(\partial_i - \partial_j). \tag{A.1}$$

PROPOSITION 8

$$L(m) = 4\sum_{j=1}^{\ell}\{(j - j^2/(\ell+1))z_j^2 - \sum_{n=1}^{j} 2nz_{j-n}z_{j+n}\}\partial_{z_j}^2$$

$$+ 8\sum_{1\leqslant j<k\leqslant \ell}\{(j - jk/(\ell+1))z_jz_k - \sum_{n=1}^{j}(k - j + 2n)z_{j-n}z_{k+n}\}\partial_{z_j}\partial_{z_k}$$

$$+ 2((2/(\ell+1)) + m)\sum_{j=1}^{\ell} j(\ell - j + 1)z_j\partial_{z_j},$$

where $z_j \equiv 0$ *if* $j > \ell + 1$, $z_j \equiv 1$ *if* $j = 0$ *or* $j = \ell + 1$.

EXAMPLES. If $\ell = 1$ then $L(m) = 2((z^2 - 4)(\mathrm{d}/\mathrm{d}z)^2 + (m+1)z\,(\mathrm{d}/\mathrm{d}z))$; if $\ell = 2$ then we recover (5.3), which is a result of Koornwinder [10, IV, (5.14)]. For $\ell = 3$ we obtain the expression in Proposition 7 (Appendix 1).

This appendix contains the proof of Proposition 8.
We have

$$\partial_i = \sum_{j=1}^{\ell}\partial_i(z_j)\partial_{z_j} \quad (i = 1, 2, \ldots, \ell+1), \tag{A.2}$$

so

$$\partial_i^2 = \partial_i\circ\sum_{k=1}^{\ell}\partial_i(z_k)\partial_{z_k} = \sum_{k=1}^{\ell}\partial_i^2(z_k)\partial_{z_k} + \sum_{j=1}^{\ell}(\partial_i(z_j))^2\partial_{z_j}^2$$

$$+ 2\sum_{1\leqslant j<k\leqslant \ell}\partial_i(z_j)\partial_i(z_k)\partial_{z_j}\partial_{z_k}.$$

This gives

$$L_{\mathfrak{a}} = \sum_{j=1}^{\ell}\left(\sum_{i=1}^{\ell+1}(\partial_i(z_j))^2\right)\partial_{z_j}^2 + 2\sum_{1\leqslant j<k\leqslant \ell}\left(\sum_{i=1}^{\ell+1}\partial_i(z_j)\partial_i(z_k)\right)\partial_{z_j}\partial_{z_k}$$

$$+ \sum_{j=1}^{\ell}\left(\sum_{i=1}^{\ell+1}\partial_i^2(z_j)\right)\partial_{z_j}. \tag{A.3}$$

The coefficient of ∂_{z_j} can easily be determined. Since

$$\partial_i(z_j) = 2\sum_{w\in W/W(\lambda_j)}\langle w\lambda_j, e_i\rangle\,\mathrm{e}^{2w\lambda_j},$$

we obtain

$$\partial_i^2(z_j) = 4 \sum_{w \in W/W(\lambda_j)} \langle w\lambda_j, e_i \rangle^2 \, \mathrm{e}^{2w\lambda_j}.$$

Thus the coefficient of ∂_{z_j} equals

$$\sum_{i=1}^{\ell+1} \partial_i^2(z_j) = 4 \sum_{w \in W/W(\lambda_j)} \left(\sum_{i=1}^{\ell+1} \langle w\lambda_j, e_i \rangle^2 \right) \mathrm{e}^{2w\lambda_j} = 4|\lambda_j|^2 z_j.$$

Note that this result holds for arbitrary root systems. For A_ℓ we have as fundamental weights:

$$\lambda_j = (1/(\ell + 1))\,((\ell - j + 1)(e_1 + e_2 + \cdots + e_j)$$

$$- j(e_{j+1} + e_{j+2} + \cdots + e_{\ell+1})). \tag{A.4}$$

In particular $|\lambda_j|^2 = j(\ell - j + 1)/(\ell + 1)$. This gives the first-order term in the z-coordinates:

$$\sum_{i=1}^{\ell+1} \partial_i^2(z_j) = (4/(\ell + 1)) \sum_{j=1}^{\ell} j(\ell - j + 1) z_j \partial_{z_j}. \tag{A.5}$$

We now concentrate on the second-order term. Consider the fundamental weight

$$\lambda_1 = (1/(\ell + 1))(\ell e_1 - e_2 - e_3 - \cdots - e_{\ell+1}).$$

The orbit of $2\lambda_1$ under W consists of the $\ell + 1$ vectors

$$\sigma_i = (2/(\ell + 1))(-e_1 - \cdots - e_{i-1} + \ell e_i - e_{i+1} - \cdots - e_{\ell+1})$$

$$(i = 1, 2, \ldots, \ell + 1). \tag{A.6}$$

Note that $\sigma_1 + \sigma_2 + \cdots + \sigma_{\ell+1} = 0$, $\sigma_i - \sigma_j = 2(e_i - e_j)$ and

$$\langle e_j, \sigma_i \rangle = -2/(\ell + 1) \quad \text{if} \quad i \neq j, \quad \langle e_j, \sigma_j \rangle = 2\ell/(\ell + 1). \tag{A.7}$$

Based on (A.4) it is now easy to describe the Weyl group orbit of the vector $2\lambda_i$ $(i = 1, 2, \ldots, \ell)$ in terms of the σ_i $(i = 1, 2, \ldots, \ell + 1)$. One obtains for $j = 1, 2, \ldots, \ell$:

$$2\lambda_j = \sigma_1 + \sigma_2 + \cdots + \sigma_j = -\sigma_{j+1} - \sigma_{j+2} - \cdots - \sigma_{\ell+1},$$

and

$$W \cdot 2\lambda_j = \{\sigma_{i_1} + \sigma_{i_2} + \cdots + \sigma_{i_j} \mid 1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant \ell + 1\}$$

$$= \{-\sigma_{i_1} - \sigma_{i_2} - \cdots - \sigma_{i_{\ell-j+1}} \mid 1 \leqslant i_1 < i_2 < \cdots$$
$$< i_{\ell-j+1} \leqslant \ell + 1\}.$$

Since $z_j = \Sigma_{w \in W/W(\lambda_j)} e^{2w\lambda_j}$ we conclude from the description of the orbit $W \cdot 2\lambda_j$ that z_j is the j-th. elementary symmetric polynomial u_j in the variables $x_i = e^{\sigma_i}$ $(i = 1, 2, \ldots, \ell + 1)$ for $j = 1, 2, \ldots, \ell$. For convenience we also define $z_{\ell+1} = e^{\sigma_1 + \sigma_2 + \cdots + \sigma_{\ell+1}} \equiv 1$. For $j = 1, \ldots, \ell + 1$ we then have $z_j = u_j(x_1, x_2, \ldots, x_\ell)$, where $x_i = e^{\sigma_i}$ $(i = 1, \ldots, \ell + 1)$ and

$$u_j(x_1, x_2, \ldots, x_\ell) = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant \ell+1} x_{i_1} x_{i_2} \ldots x_{i_j}.$$

We also define $u_0 \equiv 1$ and $u_j \equiv 0$ for $j > \ell + 1$. One can now reduce the problem of the calculation of the second-order term in (A.3) to a problem on symmetric polynomials. First we calculate $\partial_i(z_j)$ explicitly. By (A.7) we have

$$\langle e_i, \sigma_{i_1} + \cdots + \sigma_{i_j} \rangle = \begin{cases} -2j/(\ell + 1) & \text{if } \forall k \in \{1, \ldots, j\}: i_k \neq i \\ 2(\ell - j + 1)/(\ell + 1) & \text{if } \exists k \in \{1, \ldots, j\}: i_k = i. \end{cases} \tag{A.8}$$

So

$$\tfrac{1}{2}(\ell + 1)\partial_i(z_j) = (-j) \sum_{\substack{1 \leqslant i_1 < \cdots < i_j \leqslant \ell+1 \\ i_k \neq i}} e^{\sigma_{i_1} + \cdots + \sigma_{i_j}}$$

$$+ (\ell - j + 1) \sum_{\substack{1 \leqslant i_1 < \cdots < i_j \leqslant \ell+1 \\ i_k = i}} e^{\sigma_{i_1} \cdots + \sigma_{i_j}}$$

$$= -jz_j + (\ell + 1)e^{\sigma_i} \sum_{\substack{1 \leqslant i_1 < \cdots < i_{j-1} \leqslant \ell+1 \\ i_k \neq i}} e^{\sigma_{i_1} + \cdots + \sigma_{i_{j-1}}}$$

$$= -jz_j + (\ell + 1)e^{\sigma_i}\left(z_{j-1} - e^{\sigma_i} \sum_{\substack{1 \leqslant i_1 < \cdots < i_{j-2} \leqslant \ell+1 \\ i_k \neq i}} e^{\sigma_{i_1} + \cdots + \sigma_{i_{j-2}}}\right)$$

$$= \cdots$$

$$= -jz_j + (\ell + 1)e^{\sigma_i}(z_{j-1} - e^{\sigma_i}(z_{j-2} - e^{\sigma_i}(\cdots - e^{\sigma_i}(z_1 - e^{\sigma_i})\cdots))),$$

and thus

$$\tfrac{1}{2}(\ell + 1)\partial_\iota(z_\jmath) = (\ell + 1)(e^{\sigma_\iota}z_{\jmath-1} - e^{2\sigma_\iota}z_{\jmath-2} + e^{3\sigma_\iota}z_{\jmath-3}$$

$$+ \cdots + (-1)^{\jmath-2}e^{(\jmath-1)\sigma_\iota} z_1 + (-1)^{j-1}e^{\jmath\sigma_\iota}) + (-jz_\jmath). \tag{A.9}$$

Now write

$$S_{p,q} = \tfrac{1}{4}(\ell + 1)^2 \sum_{i=1}^{\ell+1} \partial_\iota(z_p)\partial_\iota(z_q), \quad 1 \leqslant p, q \leqslant \ell. \tag{A.10}$$

With our notation $x_\iota = e^{\sigma_\iota}$, $z_i = u_\iota$ $(i = 1, 2, \ldots, \ell + 1)$ we have

$$S_{p,q} = \sum_{\iota=1}^{\ell+1} \{(\ell + 1)(x_\iota u_{p-1} - x_\iota^2 u_{p-2}$$

$$+ \cdots + (-1)^{p-2}x_i^{p-1}u_1 + (-1)^{p-1}x_i^p) - pu_p\}$$

$$\times \{(\ell + 1)(x_\iota u_{q-1} - x_\iota^2 u_{q-2} + \cdots + (-1)^{q-2}x_i^{q-1}u_1 + (-1)^{q-1}x_i^q) - qu_q\}$$

$$= (\ell + 1)^2 T_{p,q} - (\ell + 1)qu_q \sum_{i=1}^{\ell+1}\left(\sum_{a=1}^{p} (-1)^{a-1}x_i^a u_{p-a}\right)$$

$$+ (\ell + 1)pq\, u_p u_q - (\ell + 1)\, pu_p \sum_{i=1}^{\ell+1}\left(\sum_{b=1}^{q} (-1)^{b-1}x_i^b u_{q-b}\right),$$

where

$$T_{p,q} = \sum_{\iota=1}^{\ell+1}\sum_{a=1}^{p}\sum_{b=1}^{q} (-1)^{a+b}x_i^{a+b}u_{p-a}u_{q-b}.$$

Let s_k $(k \in \mathbb{Z}^+)$ denote the power sums in the variables x_i $(i = 1, 2, \ldots, \ell + 1)$, i.e., $s_k = \Sigma_{i=1}^{\ell+1} x_i^k$, $s_0 \equiv 1$. Then one has the following well-known Newton-identities for the elementary symmetric polynomials u_ι and the power sums s_k, which can be found in many textbooks on algebra (e.g., [19, §33]):

$$s_k = \sum_{\jmath=1}^{k-1} (-1)^{i-1}u_i s_{k-i} + (-1)^{k-1}ku_k, \quad k \geqslant 1. \tag{A.11}$$

If in (A.11) we have $k > \ell + 1$ then we use our convention $u_j \equiv 0$ for $j > \ell + 1$. In particular we have

$$pu_p = \sum_{a=1}^{p} (-1)^{a-1} s_a u_{p-a} \quad (1 \leqslant p \leqslant \ell + 1), \tag{A.12}$$

and hence

$$S_{p,q} = (\ell + 1)^2 T_{p,q} + (\ell + 1)\left(pqu_p u_q - qu_q \sum_{a=1}^{p} (-1)^{a-1} s_a u_{p-a} \right.$$
$$\left. - pu_p \sum_{b=1}^{q} (-1)^{b-1} s_b u_{q-b} \right) = (\ell + 1)^2 T_{p,q} - (\ell + 1) pq\, u_p u_q, \tag{A.13}$$

with

$$T_{p,q} = \sum_{a=1}^{p} \sum_{b=1}^{q} (-1)^{a+b} s_{a+b} u_{p-a} u_{q-b}.$$

Now apply (A.11) with $k = q + a$ $(1 \leqslant a \leqslant p)$, then we obtain

$$\sum_{b=1}^{q} (-1)^{b-1} s_{a+b} u_{q-b}$$
$$= u_q s_a + \cdots + (-1)^{a-1} u_{q+a-1} s_1 + (-1)^a (q + a) u_{q+a},$$

and thus

$$(-1)^{a+1} \sum_{b=1}^{q} (-1)^{b-1} s_{a+b} u_{q-b} = \sum_{b=1}^{a} (-1)^{b-1} s_b u_{q+a-b} - (q + a) u_{q+a}.$$

So

$$T_{p,q} = \sum_{a=1}^{p} u_{p-a} \sum_{b=1}^{a} (-1)^{b-1} s_b u_{q+a-b} - \sum_{a=1}^{p} (q + a) u_{q+a} u_{p-a}.$$

In the double summation we put $n = a - b$. Then

$$\sum_{a=1}^{p} \sum_{b=1}^{a} (-1)^{b-1} s_b u_{p-a} u_{q+a-b} = \sum_{a=1}^{p} \sum_{n=0}^{a-1} (-1)^{a-n-1} s_{a-n} u_{p-a} u_{q+n}$$
$$= \sum_{n=0}^{p-1} u_{q+n} \left(\sum_{a=n+1}^{p} (-1)^{a-n-1} s_{a-n} u_{p-a} \right) = \sum_{n=0}^{p-1} u_{q+n} (p - n) u_{p-n},$$

where we used (A.12) for the last equality. Consequently

$$\begin{aligned} T_{p,q} &= \sum_{n=0}^{p-1} (p-n)u_{q+n}u_{p-n} - \sum_{n=1}^{p} (q+n)u_{q+n}u_{p-n} \\ &= pu_p u_q + \sum_{n=1}^{p} (p-q-2n)u_{p-n}u_{q+n}. \end{aligned}$$

It follows from (A.13), and the definition of $S_{p,q}$ in (A.10), that

$$\begin{aligned} \sum_{\iota=1}^{\ell+1} \partial_\iota(z_q)\partial_\iota(z_q) &= (2/(\ell+1))^2 S_{p,q} \\ &= 4\left\{(p - pq/(\ell+1))z_p z_q + \sum_{n=1}^{p} (p-q-2n)z_{p-n}z_{q+n}\right\} \\ &\quad (1 \leqslant p, q \leqslant \ell+1). \end{aligned} \tag{A.14}$$

Here $z_j \equiv 0$ if $j > \ell + 1$, $z_j \equiv 1$ if $j = \ell + 1$ and $z_0 \equiv 1$. If we combine (A.14) with (A.5) then we obtain the operator L_α in (A.1) (and (A.3)) in the z-coordinates (cf. end of this appendix).

PROPOSITION 9. *Let R be defined as in* (A.1). *Then*

$$R = \sum_{j=1}^{\ell} 2j(\ell - j + 1)z_j\partial_{z_j}.$$

Proof. We deduce from (A.2) and (A.9) that

$$\begin{aligned} \tfrac{1}{2}(\partial_i - \partial_j) &= \frac{1}{2}\sum_{p=1}^{\ell} (\partial_i - \partial_j)(z_p)\partial_{z_p} \\ &= \sum_{p=1}^{\ell} \{(x_i - x_j)u_{p-1} - (x_i^2 - x_j^2)u_{p-2} + (x_i^3 - x_j^3)u_{p-3} \\ &\quad + \cdots + (-1)^{p-2}(x_i^{p-1} - x_j^{p-1})u_1 + (-1)^{p-1}(x_i^p - x_j^p)u_0\}\partial_{z_p}. \end{aligned}$$

Since

$$\coth(e_i - e_j) = \coth \tfrac{1}{2}(\sigma_i - \sigma_j) = \frac{x_i + x_j}{x_i - x_j},$$

we obtain

$$R = 2\sum_{p=1}^{\ell}\left\{\sum_{1\leqslant i<j\leqslant \ell+1}\left(\frac{x_i + x_j}{x_i - x_j}\right)\{(x_i - x_j)u_{p-1} - (x_i^2 - x_j^2)u_{p-2}\right.$$

$$\left. + \cdots + (-1)^{p-2}(x_i^{p-1} - x_j^{p-1})u_1 + (-1)^{p-1}(x_i^p - x_j^p)u_0\}\right\}\partial_{z_p}.$$

Now

$$\frac{x_i + x_j}{x_i - x_j}(x_i^n - x_j^n) = \left(\sum_{k=0}^{n-1} x_i^k x_j^{n-k-1}\right)(x_i + x_j)$$

$$= x_i^n + x_j^n + 2\sum_{k=1}^{n-1} x_i^k x_j^{n-k}$$

and

$$\sum_{1\leqslant i<j\leqslant \ell+1}(x_i^n + x_j^n) = \ell s_n,$$

where, as before, s_n denotes the n-th. power sum. If we use the Newton-identities (A.12) again, then it follows that the coefficient of ∂_{z_p} $(p = 1, 2, \ldots, \ell)$ in R equals

$$2\ell p u_p + 2U_p,$$

with

$$U_p = 2\sum_{1\leqslant i<j\leqslant \ell+1}\sum_{n=2}^{p}(-1)^{n-1}\left(\sum_{k=1}^{n-1} x_i^k x_j^{n-k}\right)u_{p-n}.$$

Now let n be odd, $n \geqslant 3$. Then

$$\sum_{1\leqslant i<j\leqslant \ell+1}\left(\sum_{k=1}^{n-1} x_i^k x_j^{n-k}\right)$$

$$= \sum_{1\leqslant i,j\leqslant \ell+1}(x_i x_j^{n-1} + x_i^2 x_j^{n-2} + \cdots + x_i^{(n-1)/2}x_j^{(n+1)/2}) - \tfrac{1}{2}(n-1)s_n$$

$$= (s_1 s_{n-1} + s_2 s_{n-2} + \cdots + s_{(n-1)/2}s_{(n+1)/2}) - \tfrac{1}{2}(n-1)s_n.$$

The case n even, $n \geqslant 2$ can be treated similarly and thus we obtain

$$2 \sum_{1 \leqslant i < j \leqslant \ell+1} \left(\sum_{k=1}^{n-1} x_i^k x_j^{n-k} \right)$$

$$= (s_1 s_{n-1} + s_2 s_{n-2} + \cdots + s_{n-2} s_2 + s_{n-1} s_1) - (n - 1)s_n$$

$$(n = 2, 3, \ldots, p).$$

Hence

$$U_p = \sum_{n=2}^{p} (-1)^{n-1} \left(\sum_{k=1}^{n-1} s_k s_{n-k} - (n - 1)s_n \right) u_{p-n}$$

$$= \sum_{n=2}^{p} (-1)^{n-1} \sum_{k=1}^{n-1} s_k s_{n-k} u_{p-n} + \sum_{n=2}^{p} (-1)^n (n - 1) s_n u_{p-n}.$$

In the double summation we first change the order of summation and then put $b = n - k$. This leads to

$$U_p = \sum_{k=1}^{p-1} (-1)^k s_k \left(\sum_{b=1}^{p-k} (-1)^{b-1} s_b u_{p-k-b} \right) + \sum_{n=2}^{p} (-1)^n (n - 1) s_n s_{p-n}.$$

Apply the Newton-identities (A.12), with p replaced by $p - k$, then it follows that

$$U_p = \sum_{n=1}^{p-1} (-1)^n s_n (p - n) u_{p-n} + \sum_{n=2}^{p} (-1)^n (n - 1) s_n u_{p-n}$$

$$= -(p - 1) \left(\sum_{n=1}^{p} (-1)^{n-1} s_n u_{p-n} \right),$$

and thus, again by (A.12),

$$U_p = -p(p - 1)u_p.$$

This shows that the coefficient of ∂_{z_p} ($p = 1, 2, \ldots, \ell$) in R equals

$$2\ell p u_p - 2p(p - 1)u_p = 2p(\ell - p + 1)u_p,$$

which proves Proposition 9. ■

To *summarize* the results of this appendix:
if we combine (A.5) and (A.14) then we obtain

$$L_\mathfrak{a} = 4\sum_{j=1}^{\ell}\left\{(j - j^2/(\ell + 1))z_j^2 - \sum_{n=1}^{j} 2nz_{j-n}z_{j+n}\right\}\partial_{z_j}^2$$

$$+ 8\sum_{1\leqslant j<k\leqslant \ell}\left\{(j - jk/(\ell + 1))z_jz_k - \sum_{n=1}^{j}(k - j + 2n)z_{j-n}z_{k+n}\right)\partial_{z_j}\partial_{z_k}$$

$$+ (4/(\ell + 1))\sum_{j=1}^{\ell} j(\ell - j + 1)z_j\partial_{z_j};$$

furthermore Proposition 9 gives

$$R = \sum_{j=1}^{\ell} 2j(\ell - j + 1)z_j\partial_{z_j},$$

and since

$$L(m) = L_\mathfrak{a} + mR$$

we obtain Proposition 8.

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Notes added in proof

1. In the paper "Système différentiel hypergéométrique de type BC_p" (C. R. Acad. Sci. Paris Sér. I 304 (1987), 363–366), Debiard gives the shift operator D_-^γ for BC_ℓ with shift (0, 2, 0) (in the sense of Section 4), but without proof. In the last chapter of his recent Ph.D. thesis (*Generalized hypergeometric functions associated with root systems*, Univ. of Leiden (1988)) Opdam proves the existence of shift operators for all root systems. From the results in section 3 (and the classification of all simple noncompact Lie algebras over $\mathbb{R}$ and their root systems and multiplicities) then follows that the Abel transform can be inverted by a differential operator if all root multiplicities are even.

2. Using the same method as in Appendix 2 we also obtained the radial part of the Laplace–Beltrami operator in the z-coordinates for BC_ℓ (unpublished).

3. In a forthcoming paper we will show that the generalized Abel transform as defined in (6.1) also satisfies the expected transmutation property with respect to the radial part $L(m) = \Delta(L_X)$ of the Laplace–Beltrami operator, i.e.,

$$F^m_{L(m)f} = (L_{\mathfrak{a}} - \langle \varrho(m), \varrho(m) \rangle) F^m_f \quad \text{on } \mathfrak{a}^+, \quad m \in \mathbb{C}, \ \operatorname{Re} m > 0.$$

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