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An analytic function and iterated integrals

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Let $f(z)$, $g(z)$ be weight 2 cusp forms for the group $\Gamma_0(N)$, i.e., $f(z)\,\mathrm{d}z$, $g(z)\,\mathrm{d}z$ are holomorphic differentials on the compact Riemann surface $X_0(N)$ obtained from the upper half plane modulo this group. In this paper we introduce an entire function of s denoted $M(s, f, g)$ which is related to the pair f, g in somewhat the same way that the Mellin transform of f

$$M(s, f) = \int_0^\infty y^{s-1} f\,\mathrm{d}y \tag{1.1}$$

is related to f. The value $M(1, f, g)$ at $s = 1$ is an iterated integral $M(1, f, g) = \int_{y=\infty}^0 F(iy)g(iy)\,\mathrm{d}y$ where $F(iy) = \int_\infty^y f(i\eta)\,\mathrm{d}\eta$. We have shown in [2] that iterated integrals of harmonic 1-forms on any Riemann surface X calculate a geometric property of X embedded in its Jacobian $J(X)$: namely we consider the algebraic 1-cycle in $J(X)$ given by $X - \iota(X)$, where ι is the involution of J given by the group theoretic inverse. This cycle is homologous to zero and so has an image in the intermediate Jacobian of $J(X)$: this image is determined ([2]) exactly by iterated integrals on X. The special case $X = X_0(64)$ = degree 4 Fermat curve $x^4 + y^4 = 1$ is discussed in [3]; iterated integrals on the general Fermat curve are discussed in [8].

$M(s, f, g)$ satisfies a functional equation for $s \to 2 - s$, just as $M(s, f)$ satisfies

$$N^{(s-1)/2} M(s, f) = -N^{(1-s)/2} M(2 - s, f \mid W_N)$$

$$\left(\text{where } (f \mid W_N)(z)\,\mathrm{d}z = f\left(-\frac{1}{Nz}\right) d\left(-\frac{1}{Nz}\right)\right). \tag{1.2}$$

We now define $M(s, f, g)$ by (1.3) below and give our results (formulas 1.5, 1.6, 1.7); the proofs are in the next section.

$M(s, f, g)$ is defined by the iterated integral

$$M(s, f, g) = \int_{y=\infty}^{0} \left[\int_{\eta=\infty}^{y} \eta^{s-1} f(i\eta)\, \mathrm{d}\eta \right] y^{1-s} g(\mathrm{i}y)\, \mathrm{d}y \tag{1.3}$$

which we will abbreviate as

$$\int_{\infty}^{0} (y^{s-1} f\, \mathrm{d}y,\ y^{1-s} g\, \mathrm{d}y). \tag{1.4}$$

$M(s, f, g)$ is an entire function of s since f, g vanish rapidly at the cusps 0, ∞, and satisfies a functional equation

$$\begin{aligned} &M(s, f, g) - \tfrac{1}{2}M(s, f)M(2 - s, g) \\ &\quad = -[M(2 - s, f|W_N, g|W_N) - \tfrac{1}{2}M(2 - s, f|W_N)M(s, g|W_N)] \end{aligned} \tag{1.5}$$

equivalently

$$M(s, f, g) + M(2 - s, f|W_N, g|W_N) = -N^{s-1}M(s, f)M(s, g|W_N). \tag{1.5'}$$

In particular if $f|W_N = f$ and $g|W_N = -g$ then (1.2) gives that $M(s, f)M(s, g)$ vanishes at $s = 1$ as do all its even order derivatives, while the odd order derivatives are given by those of $M(s, f, g)$ (Gross–Zagier [1] studies the first derivative); however, the even order derivatives of $M(s, f, g)$ at $s = 1$ need not be zero.

We derive the following integral formula for $M(s, f, g)$ (no assumptions on f, g with regard to W_N are required)

$$M(s, f, g) = -\frac{1}{2\pi i} \int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{M(s', f)M(s', g|W_N)N^{s'-1}\mathrm{d}s'}{s' - s} \tag{1.6}$$

where the path of integration is a vertical line $s' = \sigma' + it$ in which $\sigma' = \operatorname{Re} s' > \operatorname{Re} s$.

Suppose that $-g|W_N$ is the twist $f_\psi = f \otimes \psi$ of f by an odd quadratic Dirichlet character ψ, so that

$$\prod_{p\,\text{prime}} [(1 - p^{-s})(1 - \psi(p)p^{-s})]^{-1} = \sum b_n n^{-s}$$

where $\theta_\psi(z) = \Sigma_0^\infty b_n q^n$ is a holomorphic modular form of weight 1 for $\Gamma_0(N)$ with character ψ. Then $M(s, f)\, M(s, f_\psi)$ can be expressed as a Rankin convolution ([6]) and we obtain the following Petersson inner product expression

$$M(s, f, f_\psi) \;=\; \frac{1}{2\pi i} \iint_{\Gamma_0(N)/\mathscr{H}} \overline{f_\varrho(z)}\, \theta_\psi \mathscr{E}_{1,N}(z, s, \psi)\, \mathrm{d}x\, \mathrm{d}y \tag{1.7}$$

where

$$f(z) \;=\; \sum a_n q^n, \overline{f_\varrho(z)} \;=\; \sum a_n \overline{q^n}$$

$$\mathscr{E}_{1,N}(z, s, \psi) \;=\; \frac{2\pi i}{(2\pi)^s} (2Ny)^{s-1} \sum_{m,n \in \mathbb{Z}}{}' \psi(n) \frac{|Nmz + n|^{2-2s}}{Nmz + n}$$
$$\times\, \Gamma\left(s, 2\pi \frac{|Nmz + n|^2}{2Ny}\right) \tag{1.8}$$

(where $\Gamma(s, x) = \int_x^\infty \mathrm{e}^{-t} t^{s-1}\, \mathrm{d}t$: the incomplete Γ-function, so $\Gamma(1, x) = \mathrm{e}^{-x}$).

$\mathscr{E}_{1,N}(z, s, \psi)$ as function of z is of weight 1 and character ψ^{-1} for $\Gamma_0(N)$. (It is similar to the weight one Eisenstein series.) Clearly $\mathscr{E}_{1,N}$ and θ_ψ depend only on N, and not on f.

For $s = 1$ the formulas become

$$M(1, f, f \otimes \psi) \;=\; \frac{1}{2\pi i} \iint_{\Gamma_0(N)/\mathscr{H}} \overline{f_\varrho}\, \theta_\psi \mathscr{E}_{1,N}(z, 1, \psi)\, \mathrm{d}x\, \mathrm{d}y \tag{1.9}$$

$$\mathscr{E}_{1,N}(z, 1, \psi) \;=\; \mathrm{i} \sum_{m,n}{}' \frac{\psi(n)}{mNz + n} \exp\left[-2\pi \left(\frac{|Nmz + n|^2}{2Ny}\right)\right]. \tag{1.10}$$

We end the introduction with the following general point of view suggested by (1.6) and (1.5′).

Let $l(s)$ be an analytic function defined in a half plane $\operatorname{Re} s > c$ and decreasing rapidly as $\operatorname{Im} s \to \pm\infty$. For instance $l(s)$ could be an L-function of an algebraic variety multiplied by gamma factors and exponentials, or the product of two such functions, or the right-hand side of (1.5′). Define an entire function $\mu(s)$ by

$$\mu(s) \;=\; \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{l(s')}{s' - s}\, \mathrm{d}s' \tag{1.11}$$

(integral over any vertical line to the right of s and in the half plane of definition of $l(s)$). (1.6) shows $M(s, f, g)$ is such a function $\mu(s)$. Now assume further that $l(s)$ extends to an entire function of s and satisfies a functional equation

$$l(s) = \pm l(k-s) \tag{1.12}$$

(or more generally we have two functions $l_1(s)$, $l_2(s)$ and $l_1(k-s) = \pm l_2(s)$).
Then $\mu(s)$ will satisfy

$$\mu(s) \pm \mu(k-s) = l(s) \tag{1.13}$$

which is (1.5′). This may be seen by integrating $l(s')/(s'-s)$ over a vertical rectangle containing s, $k-s$. (In the more general case $\mu_1(s) \pm \mu_2(k-s) = l_1(s)$). Thus $\mu(s)$ is an "unsymmetric form" of $l(s)$ and the values of μ at special points s, $k-s$ determine those of $l(s)$, but not vice versa. For instance at $s = k/2$ if the sign in (1.12) is minus then $l(k/2) = 0$, but (examples show) $\mu(k/2)$ need not be 0. Our result on the Abel-Jacobi map gives a calculation of it in terms of special values at $s = 1$ of the function $\mu(s)$ rather than of the L-functions. It also suggests the question of a geometrical interpretation of $\mu(k/2)$ in other cases.

2. Let ω_1, ω_2 be holomorphic 1-forms on the compact Riemann surface $X_0(N)$. Their restrictions to the y-axis will be denoted $f\,dy$ and $g\,dy$, where f and g are weight 2 cusp forms.

For a path γ starting at x_0, denote by $\int_\gamma (\omega_i, \omega_j)$ the integrated integral $\int_\gamma W_i \omega_j$ where W_i is the function on γ obtained by integrating ω_i (starting at x_0).

Iterated integrals of differentiable 1-forms α, β on a Riemann surface over paths γ (not necessarily closed, but a product $\gamma_1\gamma_2$ is defined only if $\gamma_1(1) = \gamma_2(0)$) obey the following rules:

(a) $\int_\gamma (\alpha, \beta)$ depends only on the homotopy class of γ (with fixed endpoints) if α, β are holomorphic. If α, β are only differentiable then this statement is not true; however $\int_{\gamma\gamma^{-1}} (\alpha, \beta) = 0$ for differentiable α, β.

(b) $\int_{\gamma_1\gamma_2} (\alpha, \beta) = \int_{\gamma_1} (\alpha, \beta) + \int_{\gamma_2} (\alpha, \beta) + \int_{\gamma_1} \alpha \int_{\gamma_2} \beta$. It follows, on taking $\gamma_1 = \gamma$, $\gamma_2 = \gamma^{-1}$, that $\int_{\gamma^{-1}} (\alpha, \beta) = -\int_\gamma (\alpha, \beta) + \int_\gamma \alpha \int_\gamma \beta$; equivalently

$$\int_\gamma (\alpha, \beta) - \frac{1}{2}\int_\gamma \alpha \int_\gamma \beta = -\left[\int_{\gamma^{-1}} (\alpha, \beta) - \frac{1}{2}\int_{\gamma^{-1}} \alpha \int_{\gamma^{-1}} \beta\right].$$

If f is a weight 2 cusp form, we will only be interested in its values on the positive y-axis, denoted $f(\mathrm{i}y)$. The involution $W_N(z) = -1/(Nz)$ then takes y into $1/(Ny)$ and $f(\mathrm{i}y)\,\mathrm{d}y$ into $W_N^*(f(\mathrm{i}y)\,\mathrm{d}y) = f(i/Ny)\,\mathrm{d}(1/Ny) = (f|W_N)(\mathrm{i}y)\,\mathrm{d}y$. Thus

$$(f|W_N)(\mathrm{i}y) = f\left(\frac{i}{Ny}\right)\left(\frac{-1}{Ny^2}\right).$$

Since f is a cusp form

$$|f(\mathrm{i}y)| \leqslant \text{Const.}\ \mathrm{e}^{-2\pi y} \quad \text{as} \quad y \to \infty$$

and since $f|W_N$ is a cusp form

$$|f(\mathrm{i}y)| \leqslant \text{Const.}\ \frac{\mathrm{e}^{-2\pi/Ny}}{y^2} \quad \text{as} \quad y \to 0.$$

Thus the integrals

$$M(s,f) = \int_0^\infty y^{s-1} f\,\mathrm{d}y \tag{2.1}$$

are entire functions of s. On applying W_N one finds the functional equation

$$N^{(s-1)/2} M(s,f) = -N^{(1-s)/2} M(2-s, f|W_N). \tag{2.2}$$

We define $M(s,f,g)$ as the iterated integral

$$M(s,f,g) = \int_{y=\infty}^0 (y^{s-1} f\,\mathrm{d}y,\ y^{1-s} g\,\mathrm{d}y) = \int_\infty^0 F(s,y)\, y^{1-s} g\,\mathrm{d}y \tag{2.3}$$

$$F(s,y) = \int_{t=\infty}^y t^{s-1} f(\mathrm{i}t)\,\mathrm{d}t. \tag{2.4}$$

$F(s, y)$ is a bounded function of y since f is a cusp form, and the integral (2.3) converges for all s since g is a cusp form.

Noting that the involution W_N reverses the path $\infty \geqslant y \geqslant 0$ and applying the last iterated integral formula listed above under (b), we obtain the functional equation (1.5).

Next, we discuss formula (1.6).

(2.5) PROPOSITION. *For any fixed* $\tilde{s}$, *the Mellin transform* $M(s, F(\tilde{s}, y)) = \int_0^\infty y^{s-1} F(\tilde{s}, y)\,\mathrm{d}y$ *is defined for* Re $s > 0$ *and equals* $-(1/s)\,M(s+\tilde{s}, f)$.

Proof: From (2.4), $F(\tilde{s}, y)$ is bounded for all y and is $\leqslant e^{-y}$ in absolute value as $y \to \infty$. Thus $y^{s-1}F(\tilde{s}, y) \in L^1(dy)$ on $(0, \infty)$ for Re $s > 0$ and so the Mellin transform is defined as absolutely convergent integral. The formula for this Mellin transform arises from changing order of integration:

For Re $s > 0$, $\int_0^t y^{s-1}\,dy = t^s/s$, and so

$$\begin{aligned}\frac{M(s+\tilde{s}, f)}{s} &= \int_{t=0}^{\infty} t^{\tilde{s}-1}\frac{t^s}{s}f(it)\,dt \\ &= \int_{t=0}^{\infty} t^{\tilde{s}-1}f(it)\left[\int_{y=0}^{t} y^{s-1}\,dy\right]dt \\ &= \int_{t=0}^{\infty} t^{\tilde{s}-1}f(it)\left[\int_{y=0}^{\infty} \eta(t, y)\,dy\right]dt\end{aligned}$$

where $\eta(t, y) = y^{s-1}$ for $0 < y \leqslant t$, and 0 for $y > t$. For fixed $t > 0$, $\eta(t, y)$ is L^1 in y on $(0, \infty)$ and therefore $t^{\tilde{s}-1}f(it)\eta(t, y)$ is also L^1 in y for fixed t. On integrating first with respect to y and then with respect to t we obtain an absolutely convergent integral. By Fubini, the same holds for integration in the reverse order:

$$\begin{aligned}\frac{M(s+\tilde{s}, f)}{s} &= \int_{y=0}^{\infty}\left[\int_{t=0}^{\infty} t^{\tilde{s}-1}f(it)\eta(t, y)\,dt\right]dy \\ &= \int_{y=0}^{\infty}\left[\int_{t=y}^{\infty} t^{\tilde{s}-1}f(it)y^{s-1}\,dt\right]dy \\ &= -\int_0^{\infty} y^{s-1}F(\tilde{s}, y)\,dy = -M(s, F(\tilde{s}, y)).\end{aligned}$$

Next we have an estimate for the Mellin transform $M(s, f)$, f a cusp form, and the Mellin inversion formula: let $s = \sigma + it$, then

$$M(s, f) = \mathcal{O}(e^{-[(\pi/2-\varepsilon)|t|]}) \tag{2.6}$$

as $|t| \to \infty$ for every $\varepsilon > 0$, uniformly in any strip $a \leqslant \sigma \leqslant b$,

$$f(iy) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} M(s, f)y^{-s}\,ds. \tag{2.7}$$

(To prove these, we note that $f(iz)$ is holomorphic in any angle $-\pi/2 < -\beta < \operatorname{Arg} z < \beta < \pi/2$.

For fixed β, our previous estimates on $|f(\mathrm{i}y)|$ as $y \to 0$ or $y \to \infty$ imply the same estimates with y replaced by $|z|$ for $f(\mathrm{i}z)$:

$$|f(\mathrm{i}z)| \leqslant \begin{cases} c\,\mathrm{e}^{-D|z|} & \text{as} \quad |z| \to \infty \\ c'(\mathrm{e}^{-(k/|z|)})|z|^{-2} & \text{as} \quad |z| \to 0 \end{cases}$$

in particular,

$$|f(z)| = \mathcal{O}(|z|^{-a-\varepsilon})$$

as $|z| \to 0$ for all a, and

$$|f(z)| = \mathcal{O}(|z|^{-b+\varepsilon})$$

as $|z| \to \infty$, all b. These last two inequalities are just the hypotheses in ([7], Chapter I, Theorem 31) and 2.6, 2.7, and the holomorphy of $M(s, f)$ in s for all s is the conclusion).

In order to prove (1.6) we will rewrite

$$M(s_0, f, g) = \int_{\infty}^{0} F(s_0, y)\, y^{1-s_0} g(\mathrm{i}y)\, \mathrm{d}y$$

as a convolution:

$$g(\mathrm{i}y) = -(g \,|\, W_N)\left(\frac{i}{Ny}\right) N^{-1} y^{-2}$$

$$y^{1-s_0} g(\mathrm{i}y) = -N^{-1} y^{-1-s_0} (g \,|\, W_N)\left(\frac{i}{Ny}\right)$$

$$\begin{aligned} M(s_0, f, g) &= -\int_0^{\infty} F(s_0, y)\, y^{1-s_0} g(\mathrm{i}y)\, \mathrm{d}y \\ &= \int_0^{\infty} F(s_0, y)\, N^{-1} y^{-1-s_0} (g \,|\, W_N)\left(\frac{i}{Ny}\right) \mathrm{d}y \\ &= N^{s_0-1} \int_0^{\infty} F(s_0, y)(Ny)^{-s_0} (g \,|\, W_N)\left(\frac{i}{Ny}\right) \frac{\mathrm{d}y}{y} \\ &= (N^{s_0-1})(\phi_1 * \phi_2)\left(\frac{1}{N}\right) \end{aligned}$$

where $\phi_1 * \phi_2$ is the convolution

$$(\phi_1 * \phi_2)(x) = \int_0^\infty \phi_1(y)\phi_2(xy^{-1})\frac{\mathrm{d}y}{y}$$

and

$$\phi_1(y) = F(s_0, y),\ \phi_2(y) = y^{s_0}(g|W_N)(\mathrm{i}y).$$

Now $y^k\phi_1(y) \in L^1(\mathrm{d}y)$ for $k > -1$ and $y^k\phi_2(y) \in L^1(\mathrm{d}y)$ for all k. It follows ([7], Chapter II, Theorem 44) that $x^k\phi_1 * \phi_2(x) \in L_1(\mathrm{d}x)$ on $(0, \infty)$ for $k > -1$ and the Mellin transform $M(s, \phi_1 * \phi_2)$ is defined for $\operatorname{Re} s = k + 1 > 0$ and equals

$$M(s, \phi_1)M(s, \phi_2) = -\frac{M(s + s_0, f)}{s}M(s + s_0, g|W_N)$$

for $\operatorname{Re} s > 0$.

By Mellin inversion ([7], Chapter I, Theorem 28)

$$\begin{aligned}\phi_1 * \phi_2(x) &= \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\substack{\sigma-iT\\ \sigma=\operatorname{Re}s>0}}^{\sigma+iT} M(s, \phi_1 * \phi_2)x^{-s}\,\mathrm{d}s\\ &= -\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{M(s + s_0, f)}{s}M(s + s_0, g|W_N)x^{-s}\,\mathrm{d}s.\end{aligned}$$

Replacing $s + s_0$ by s,

$$= -\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{M(s, f)M(s, g|W_N)}{s - s_0}x^{-(s-s_0)}\,\mathrm{d}s$$

for $\operatorname{Re} s > \operatorname{Re} s_0$.

Putting $x = 1/N$ and multiplying by N^{s_0-1}

$$M(s_0, f, g) = -\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{M(s, f)M(s, g|W_N)}{s - s_0}N^{s-1}\,\mathrm{d}s$$

for $\sigma = \operatorname{Re} s > \operatorname{Re} s_0$.

However 2.6 shows the integral over the vertical line $s = \sigma + it$, $-\infty < t < \infty$ has an integrand which is $\mathcal{O}(\mathrm{e}^{-A|t|})$ and so the limit in T is $\int_{-\infty}^{\infty}$ as in (1.6).

3. We introduce now some further assumptions on the cusp forms which allow the product $M(s, f)\, M(s, g \mid W_N)$ to be represented as a Rankin convolution. As usual define $L(s, f)$ by $M(s, f) = (2\pi)^{-s}\Gamma(s)\, L(s, f)$.

Assume first that for Re s large enough

$$L(s, f) = \prod_{p \text{ prime}} [(1 - \alpha_p p^{-s})(1 - \alpha'_p p^{-s})]^{-1}. \tag{3.1}$$

where $\alpha_p \alpha'_p = p$ for $(p, N) = 1$.

Next let ψ be the Dirichlet character associated with an imaginary quadratic field K of discriminant $D < 0$ and $D \equiv 0 \pmod 4$, where $D \mid N$. ψ is thus a primitive quadratic character mod $|D|$ and odd: $\psi(-1) = -1$. Assume that $-g \mid W_N = f \otimes \psi$ where $L(s, f \otimes \psi)$ is obtained from $L(s, f)$ by replacing α_p, α'_p by $\psi(p)\alpha_p$, $\psi(p)\alpha'_p$ for all p. Finally assume $\psi(p)\, \alpha_p \alpha'_p = 0$ if p divides N.

The assumption on ψ implies that

$$\prod_p [(1 - p^{-s})(1 - \psi(p) p^{-s})]^{-1} = \sum_{n \geqslant 1} b_n n^{-s}$$

where

$$\sum_{n=0}^{\infty} b_n q^n = \theta_\psi(z)$$

is a weight 1 modular form for $\Gamma_0(|D|)$ with character ψ; θ_ψ is the sum of θ series associated to the ideal classes in K. If we make ψ into a character mod N, θ_ψ can be regarded as a form for $\Gamma_0(N)$ with character ψ mod N. See [1] or [9], Chapter 9.

To obtain examples of such f, $f \otimes \psi$ (having opposite signs under the action of W_N) we follow (Oesterlé [4], 2.1, 2.2, or Shimura [5], Theorem 3.64 and following material). Let N_0 be a positive integer, f_0 a weight 2 normalized newform for $\Gamma_0(N_0)$: $L(f_0, s)$ has then an Euler product 3.1 with $\alpha_p \alpha'_p = p$ for $p \nmid N_0$ and $\alpha_p \alpha'_p = 0$ if $p \mid N_0$. With any ψ_1 as above where D is prime to N_0, $f_0 \otimes \psi_1$ has level $N = N_0 D^2$. Let $f_0 \mid W_{N_0} = -\varepsilon(f_0) f_0$ where $\varepsilon(f_0) = \varepsilon = \pm 1$. Then $\varepsilon(f_0 \otimes \psi_1) = \psi_1(-N)\,\varepsilon(f_0)$, and $f_0 \otimes \psi_1$ is again a newform, of level N. Now let ψ_2 be the Dirichlet character associated with the corresponding real quadratic field of discriminant $-D > 0$, where $D/4$ is square free and $\equiv 2 \pmod 4$, so that $-D/4$ satisfies the same condition. Then $f_0 \otimes \psi_2$ also satisfies the conditions of $f_0 \otimes \psi_1$ but the sign $\varepsilon(f_0 \otimes \psi_2) = \psi_2(-N_0)\,\varepsilon(f_0)$. Since $\psi_1(-1) = -\psi_2(-1)$, $f_0 \otimes \psi_1$ and $f_0 \otimes \psi_2$ will have opposite sign exactly when $\psi_1\psi_2(N_0) = 1$. Finally, we take $f = f_0 \otimes \psi_1$, $\psi = \psi_1\psi_2$ the odd character associated to $\mathbb{Q}(\sqrt{-1})$, $f \otimes \psi = f_0 \otimes \psi_2$.

As is well known, the product $L(s, f)\,L(s, f \otimes \psi)$ can be expressed as a Rankin convolution of f and θ_ψ:

$$L(s, f)\,L(s, f \otimes \psi) = \prod_p [(1 - \alpha_p p^{-s})(1 - \alpha_p \psi(p) p^{-s})(1 - \alpha'_p p^{-s}) \times (1 - \alpha'_p \psi(p) p^{-s})]^{-1}.$$

If $L(s, f) = \Sigma_1^\infty a(n) n^{-s}$, $\theta_\psi = \Sigma_0^\infty b(n) q^n$ and we use Lemma 1 of [6] then

$$L(s, f)\,L(s, f \otimes \psi) = \left[\sum_1^\infty a(n) b(n) n^{-s}\right] \prod_p (1 - \alpha_p \alpha'_p \psi(p) p^{-2s})^{-1}. \tag{3.2}$$

If $p \mid N$, $\alpha_p \alpha'_p \psi(p)$ was assumed 0, and if $p \nmid N$, $\alpha_p \alpha'_p \psi(p) = \psi(p) p$. Thus the last factor on the right is just

$$L_N(2s - 1, \psi) = \sum_{(n,N)=1} \psi(n) n^{1-2s},$$

and by ([6], §2)

$$2(4\pi)^{-s}\Gamma(s)\,L(s, f)\,L(s, f \otimes \psi) = \int_{\Gamma_0(N)\backslash\mathscr{H}} \overline{f_\varrho}\,\theta_\psi\,E_{1,N}(z, s - 1, \psi)\,y^{s-1}\,\mathrm{d}x\,\mathrm{d}y \tag{3.3}$$

where, regarding ψ as a character mod N, $\psi(-1) = -1$,

$$E_{1,N}(z, s, \psi) = \sum_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \psi(n)(mNz + n)^{-1}|mNz + n|^{-2s}. \tag{3.4}$$

We want to show now that the series in (3.4) and the integral in (3.3) can be interchanged. The reader who wishes to take this for granted may skip to the lines just preceding (3.6). We make some rough estimates for $E_{1,N}(z, s, \psi)$. The (m, n) term on the right side of (3.4) is bounded in absolute value by $|mNz + n|^{-2(\sigma+1/2)}$ where $\sigma = \operatorname{Re} s$ and so the series is absolutely convergent whenever the weight 0 Eisenstein series for $SL(2, \mathbb{Z})$, $E(Nz, \sigma + \frac{1}{2})$ converges, where

$$y^s E(z, s) = y^s \sum_{(m,n)\neq(0,0)} |mz + n|^{-2s}.$$

Clearly the series for $E(z, s)$ converges if $\operatorname{Re} s > 1$, so the series for $E_{1,N}(z, s, \psi)$ converges absolutely for $\operatorname{Re} s > \frac{1}{2}$.

Next, suppose z belongs to the usual fundamental domain for $SL_2(\mathbb{Z})$: $|x| \leqslant \frac{1}{2}$, $|z| \geqslant 1$. Then

$$\begin{aligned} |mz + n|^2 &= m^2|z|^2 + 2mnx + n^2 \\ &\geqslant m^2|z|^2 - |mn| + n^2 \geqslant m^2|z|^2 - \left(\frac{m^2 + n^2}{2}\right) + n^2 \\ &\geqslant \frac{m^2 + n^2}{2}. \end{aligned}$$

$$y^\sigma \sum{}' |mz + n|^{-2\sigma} \leqslant (2y)^\sigma \sum{}' (m^2 + n^2)^{-\sigma}. \tag{3.5}$$

Since $y^s E(z, s)$ is an $SL(2, \mathbb{Z})$ invariant function, (3.5) holds whenever $y \geqslant 1$, in particular for $z \to \mathrm{i}\infty$. If however $z \to$ a cusp p/q on the x-axis let $\gamma = \begin{bmatrix} * & * \\ q & -p \end{bmatrix}$ in $SL(2, \mathbb{Z})$. Then

$$\begin{aligned} y^\sigma E(z, \sigma) &= \frac{y^\sigma}{|qz - p|^{2\sigma}} E(\gamma z, \sigma) \\ &\leqslant (\tfrac{1}{2}q^2 y)^{-\sigma} \sum{}' (m^2 + n^2)^{-\sigma} \end{aligned}$$

since $|qz - p|^2 \geqslant q^2y^2$ and $\gamma z \to \infty$.

In the integral of (3.3), let $\operatorname{Re} s > \frac{3}{2}$ so that we have the bound $\mathcal{O}(y^{\sigma-1})$ for $y^{s-1}E_{1,N}(z, s - 1, \psi)$ as $z \to \mathrm{i}\infty$, and bounds $\mathcal{O}(y^{-\sigma})$ as z approaches other cusps. At each cusp let q denote a local coordinate $q = \exp(2\pi \mathrm{i}\gamma(z))$, γ linear fractional. Since f is a cusp form and so $\mathcal{O}(|q|)$ as $q \to 0$, θ_ψ is a holomorphic modular form and so $\mathcal{O}(1)$, and $\mathrm{d}x\,\mathrm{d}y$ may be replaced by

$$\frac{\mathrm{d}q\,\mathrm{d}\bar{q}}{q\bar{q}}$$

the integrand is $\mathcal{O}[|q|^{-1}(\log|q|)^{\mathrm{const}}]\,\mathrm{d}q\,\mathrm{d}\bar{q}$ so the integral is absolutely convergent, uniformly in s for constant $\sigma > \frac{3}{2}$. Further since these estimates are valid if we replace $E_{1,N}$ by the series of absolute values of its terms and even by the series (3.5), the integral in (3.3) may be interchanged with the summation of the series (3.4), and the terms of this series are majorized by using the series (3.5). We can thus estimate

$$\int_{\Gamma_0(N)\backslash\mathscr{H}} |f_{\bar{\varrho}}\theta_\psi E_{1,N}(z, s - 1, \psi) y^{s-1}| \mathrm{d}x\,\mathrm{d}y \leqslant C(\sigma)$$

where $C(\sigma)$ is a constant depending only on σ.

Next, the integral in (1.6) is

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{M(s, f)\, M(s, f \otimes \psi)\, N^{s-1}}{s - s_0}\,\mathrm{d}s$$

$$= \int_{\sigma-i\infty}^{\sigma+i\infty} [2(4\pi)^{-s}\Gamma(s) L(s, f) L(s, f \otimes \psi)] \frac{\Gamma(s)}{(2\pi)^s} \frac{(2N)^{s-1}}{s - s_0}\,\mathrm{d}s$$

$$= \int_{\sigma-i\infty}^{\sigma+i\infty} \left[\int_{\Gamma_0(N)/\mathscr{H}} (\ )\,\mathrm{d}x\,\mathrm{d}y\right] \frac{\Gamma(s)}{(2\pi)^s} \frac{(2N)^{s-1}}{s - s_0}\,\mathrm{d}s.$$

If we replace the inner integrand by its absolute value the inner integral is bounded by $C(\sigma)$, and the estimate

$$\Gamma(s) = \mathcal{O}(\mathrm{e}^{-(\pi/2)|t|}|t|^{\sigma-(1/2)}) \quad \text{for} \quad |t| \geqslant 1,$$

σ in a closed interval ($\sigma > \sigma_0$), shows that the iterated integral converges absolutely. By Fubini we may then interchange the order of integration in the variables z, s:

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{M(s, f)\, M(s, f \otimes \psi)\, N^{s-1}}{s - s_0}\,\mathrm{d}s = \int_{\Gamma_0(N)/\mathscr{H}} \overline{f_\varrho(z)}\,\theta_\psi(z)$$

$$\times \left[\int_{s=\sigma-i\infty}^{\sigma+i\infty} E_{1,N}(z, s - 1, \psi)\, y^{s-1} \frac{\Gamma(s)}{(2\pi)^s} \frac{(2N)^{s-1}}{s - s_0}\,\mathrm{d}s\right] \mathrm{d}x\,\mathrm{d}y.$$

Furthermore, by the above discussion, in the inner integral we may interchange integration over s with the summation defining $E_{1,N}$: if we denote this inner integral $\mathscr{E}_{1,N}(z, s_0, \psi)$ then

$$\mathscr{E}_{1,N}(z, s_0, \psi) = \sum_{m,n}{}' \frac{\psi(n)}{mNz + n} \int_{\sigma-i\infty}^{\sigma+i\infty} |mNz + n|^{-2(s-1)} \frac{\Gamma(s)}{2\pi} \frac{(\pi^{-1}Ny)^{s-1}}{s - s_0}\,\mathrm{d}s \tag{3.6}$$

However we have for $u > 0$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)}{s - s_0} u^{-s}\,\mathrm{d}s = u^{-s_0}\int_u^{\infty} \mathrm{e}^{-y} y^{s_0-1}\,\mathrm{d}y = u^{-s_0}\Gamma(s_0, u). \tag{3.7}$$

($\Gamma(s_0, u)$ is the "incomplete Γ-function", where $u > 0$.)
Taking $u = (\pi^{-1}Ny)^{-1}|mNz + n|^2$ gives

$$\mathscr{E}_{1,N}(z, s, \psi) = \mathrm{i}\sum_{m,n}{}' \frac{\psi(n)}{mNz + n}\left(\pi\frac{|mNz + n|^2}{Ny}\right)^{1-s} \Gamma\left(s, \pi\frac{|mNz + n|^2}{Ny}\right) \tag{3.8}$$

which is the same as (1.8). Clearly $\mathscr{E}_{1,N}(z, s, y)$ has the form

$$\mathrm{i}\sum_{m,n}{}' \frac{\psi(n)}{mNz + n} f\left(\frac{y}{|mNz + n|^2}, s\right)$$

where

$$f(y, s) = \left(\frac{Ny}{\pi}\right)^{s-1} \Gamma\left(s, \frac{\pi}{Ny}\right);$$

if $f(y, s)$ is instead taken as y^{s-1} we get $E_{1,N}(z, s - 1, \psi)$.

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