COMPOSITIO MATHEMATICA

FABIO BARDELLI
LUISELLA VERDI

On osculating cones and the Riemann-Kempf singularity theorem for hyperelliptic curves, trigonal curves, and smooth plane quintics

Compositio Mathematica, tome 65, nº 2 (1988), p. 177-199
<http://www.numdam.org/item?id=CM_1988__65_2_177_0>

Compositio Mathematica 65: 177–199 (1988)

On osculating cones and the Riemann–Kempf singularity theorem for hyperelliptic curves, trigonal curves, and smooth plane quintics

FABIO BARDELLI[1] & LUISELLA VERDI[2]
[1]*Dipartimento di Matematica, Università, Strada Nuova 65, 27100 Pavia, Italy;*
[2]*Istituto Matematico "U. Dini", Viale Morgagni 67-A, 50134 Firenze, Italy*

Received 31 December 1986; accepted in revised form 2 October 1987

Introduction

Let C be a smooth projective curve of genus $g \geqslant 3$, $\varphi_K\colon C \to \mathbb{P}^{g-1}$ its canonical map, $\tilde{C} = \varphi_K(C)$, $I(\tilde{C}) = \oplus_d I_d(\tilde{C})$ the homogeneous graded ideal of $\tilde{C}$. It is well known that: φ_K is an embedding and $I(\tilde{C})$, by the Enriques–Petri theorem, is spanned by $I_2(\tilde{C})$ except in the following cases (see [S.D.] or [A.C.G.H.]):

i) C is hyperelliptic: it is the only case in which φ_K is not an embedding, but it is composed with the unique g_2^1 on C.
ii) C is trigonal: in this case (and for $g \geqslant 5$) $I(\tilde{C})$ is spanned by $I_2(\tilde{C})$ and $I_3(\tilde{C})$. The variety defined by $I_2(\tilde{C})$ is the smooth rational 2-dimensional scroll R spanned by the trisecants of $\tilde{C}$. Each trisecant intersects $\tilde{C}$ in a divisor of the unique g_3^1 on $\tilde{C}$;
iii) C is a plane quintic: in this case $I(\tilde{C})$ is spanned by $I_2(\tilde{C})$ and $I_3(\tilde{C})$. The variety defined by $I_2(\tilde{C})$ is the Veronese surface S in $\mathbb{P}^5$, which is spanned by the conics passing through any five coplanar points of $\tilde{C}$. The 5-tuples of coplanar points of $\tilde{C}$ constitute the unique g_5^2 on $\tilde{C}$.

The exceptional cases i), ii), iii) are due to the presence on C of a unique g_2^1, g_3^1, g_5^2 respectively.

Let C^k be the cartesian product of C k-times, $C^{(k)}$ be the symmetric product of C k-times, $J(C)$ be the jacobian of C, $\mu_k\colon C^k \to J(C)$ be the Abel–Jacobi map, $W_k = \mu_k(C^k) \subseteq J(C)$. By the Riemann–Kempf singularity theorem (denoted by R.K.t. in the sequel, see [K] or [A.C.G.H.]), the g_2^1 is represented by a (unique) singular point T_2 of W_2, the g_3^1 is represented by a (unique) singular point T_3 of W_3, the g_5^2 is represented by a (unique) triple point T_5 of W_5. If we denote by $TC_{T_i}W_i$ the tangent cone to W_i at T_i, we have that (by R.K.t.): $\mathbb{P}TC_{T_2}W_2 = \tilde{C}$; $\mathbb{P}TC_{T_3}W_3 = R$; $\mathbb{P}TC_{T_5}W_5 = \mathrm{Ch}(S)$ where

$\mathrm{Ch}(S)$ is the chordal variety of the Veronese surface S; in the cases i), ii), iii), respectively. We note that $\mathrm{SingCh}(S) = S$ (see 4.0 iii)).

In this paper we study the osculating cones of order r $\widetilde{OC}(r)_{T_i}(W_i)$ at T_i: they are schemes whose underlying points sets, denoted simply by $OC(r)_{T_i}(W_i)$, are constituted by the points of the lines of $T_{T_i}(W_i)$ whose intersection multiplicity with W_i at T_i is greater than or equal to r. We introduce them in Section 1, where some useful properties are reviewed. Sections 2, 3 and 4 are devoted to the proof of the following

PROPOSITION 1. *For C hyperelliptic, $g \geqslant 3$ one has*
i) $\mathbb{P}\widetilde{OC}(3)_{T_2}(W_2) = \mathbb{P}\widetilde{OC}(4)_{T_2}(W_2) = \tilde{C}$;
ii) $\mathbb{P}\widetilde{OC}(5)_{T_2}(W_2) = \{\varphi_K(B_1), \dots, \varphi_K(B_{2g+2})\}_{\mathrm{red}}$ *where B_i $i = 1, \dots, 2g+2$ are the ramification points of the double cover π: $C \to \mathbb{P}^1$ associated to the g^1_2.*

PROPOSITION 2. *For C trigonal, $g \geqslant 4$, and with two distinct g^1_3's if $g = 4$, one has*
i) $\mathbb{P}\widetilde{OC}(3)_{T_3}(W_3) = R$;
ii) $\mathbb{P}\widetilde{OC}(4)_{T_3}(W_3) = \tilde{C}$.

PROPOSITION 3. *For C a plane quintic one has*
i) $\mathbb{P}\widetilde{OC}(4)_{T_5}(W_5) = \mathbb{P}\widetilde{OC}(5)_{T_5}(W_5) = \mathrm{Ch}(S)$;
ii) $\mathbb{P}\widetilde{OC}(6)_{T_5}(W_5) \cap S = \tilde{C}$ counted twice.

In each of the previous cases C can be reconstructed from some of the osculating cones $\widetilde{OC}(r)_{T_i}(W_i)$ and their singular loci, in particular for $g = 3$ in the hyperelliptic case, $g = 4$ in the trigonal case, and in the plane quintic case the results above imply the Torelli theorem for these families of curves. The results of Section 2 and 3 of this paper have been announced in [B.V.].

1. Osculating cones and some useful properties

Let $U = \{z \in \mathbb{C}^n : |z| < \varepsilon\}$, $W \subseteq U$ an analytic variety defined by an ideal $I(W)$ of holomorphic functions on U and with $0 \in W$. Let $\gamma\colon \Delta \to U$ be an analytic arc of curve with $\gamma(0) = 0$. $\forall f \in I(W)$ $f \circ \gamma(0) = 0$. The intersection multiplicity of W and γ at 0 is defined as:

$$(W \cdot \gamma)_0 = \min_{f \in I(W)} \{\mathrm{ord}_0\, (f \circ \gamma)\} \tag{1.1}$$

(see Sh., p. 73).

For any $f \in I(W)$ we write the power series expansion of f at 0 as

$$f = \sum_{i=1}^{\infty} f_i \text{ with } f_i \text{ homogeneous polynomial } \deg f_i = i. \tag{1.2}$$

DEFINITION 1.3. The osculating cone of order r to W at 0, denoted by $\widetilde{OC}(r)_0(W)$ henceforth, is the scheme defined by the ideal spanned by the following set of forms:

$$\{f_k\colon \forall f \in I(W) \quad \forall k < r\}.$$

Then we have:

The set of points underlying the scheme $\widetilde{OC}(r)_0(W)$ will be denoted simply by $OC(r)_0(W)$ and is equal to: $\{v \in \mathbb{C}^n\colon \text{the line } l = \{\lambda \cdot v\}_{\lambda \in \mathbb{C}} \text{ is such that } (W \cdot l)_0 \geqslant r\}$. (1.4)

$$OC(2)_0(W) = T_0 W = \text{tangent space to } W \text{ at } 0. \tag{1.5}$$

Let $TC_0 W$ be the tangent cone (as scheme) to W at 0 and $ITC_0 W$ its defining ideal (which is spanned by the initial forms f^{in} $\forall f \in I(W)$). We have:

If $ITC_0 W$ is spanned by forms of degree k, then $\forall Q \in ITC_0(W)$ with $\deg Q = k$, there exists $f \in I(W)$ such that $Q = f^{\text{in}}$. (1.6)

Let $k = \min_{f \in I(W)} \{\deg f^{\text{in}}\}$, then $\widetilde{OC}(k+1)_0(W) \supseteq TC_0 W$; and if $ITC_0 W$ is spanned by forms of degree k one has

$$\widetilde{OC}(k+1)_0(W) = TC_0 W. \tag{1.7}$$

2. The hyperelliptic case

Let C be a hyperelliptic curve of genus $g \geqslant 3$, $\pi\colon C \to \mathbb{P}^1$ the double cover associated to the unique g_2^1 on C. It is known that $|K_C| = \Sigma_{g-1}\, g_2^1$. Let P be a ramification point of π, so $2P \in g_2^1$ and $(2g-2)P \in |K_C|$. Let $\sigma \in H^0(C, \mathcal{O}_C(2P))$ with $\operatorname{div}(\sigma) = 2P$ and $\omega = \sigma^{g-1} \in H^0(C, \mathcal{O}_C(K_C))$, so that $\operatorname{div}(\omega) = (2g-2)P$. Let σ, τ be a basis for $H^0(C, \mathcal{O}_C(2P))$ and $f = \tau/\sigma \in \mathcal{M}(C)$ be the rational function giving the map π. It is easy to see that $\varrho = \sigma^{g-2} \in H^0(C, \mathcal{O}_C(K_C \backslash -2P))$ and that $\{\varrho, \varrho f, \ldots, \varrho f^{g-2}\}$ is a basis for

$H^0(C, \mathcal{O}_C(K_C \setminus -2P))$. In the same way one checks that $\{\omega, \omega f, \ldots, \omega f^{g-1}\}$ is a basis for $H^0(C, \mathcal{O}_C(K_C))$. We will set $\omega_i = \omega f^i$ $i = 0, \ldots, g-1$. Let $i\colon C \to C$ be the hyperelliptic involution of C. The set of holomorphic differentials

$$V = \{\alpha \in H^0(C, \mathcal{O}(K_C)) \mid \mathrm{div}\,(\alpha) = (2g-2)P\}$$

is a 1-dimensional vector space containing ω and invariant for i^*, so from $i^2 = id_{|C}$ it follows $i^*(\omega) = -\omega$ (see [G.H.]). It is also clear that $i^*(f^k) = f^k$ $\forall k \in \mathbb{N}$ and in particular $i^*(\omega_j) = -\omega_j = 0, \ldots, g-1$. The Abel–Jacobi map $\mu_2\colon C^2 \to J(C)$ is given by

$$C^2 \ni (P_1, P_2) \mapsto \left(\sum_{i=1}^{2} \int_P^{P_i} \omega_0, \ldots, \sum_{i=1}^{2} \int_P^{P_i} \omega_{g-1}\right) \in J(C)$$

where P, the base point, is the point chosen above. Let $\Gamma = \{(P_1, P_2) \in C^2\colon P_2 = i(P_1)\}$. Then it is clear that $\mu_2(\Gamma) = 0 \in J(C)$ and so T_2, the singular point of $W_2 = \mu_2(C^2)$, is 0.

Proof of Proposition 1. Over an open set $U_k \in C$ with local coordinate t_k, we will call $\Omega_i^k(t_k)\,dt_k$ the local expression of ω_i. Let us assume that $U_2 = i(U_1)$ and that $i\colon U_1 \to U_2$ is given by $t_2 = -t_1$, so that

$$\Omega_i^2(t_2) = \Omega_i^1(t_1) \quad \text{for } (t_1, t_2) \in (U_1 \times U_2) \cap \Gamma. \tag{2.1}$$

Let $(P_1, P_2) \in U_1 \times U_2$, then $\forall \vartheta \in I(W_2 \cap A)$, where $A \subseteq J(C)$ is a sufficiently small open neighborhood of $0 \in J(C)$, and after eventually shrinking $U_1 \times U_2$ in such a way that $\mu_2(U_1 \times U_2) \subset A$, we have $g = \vartheta \circ \mu_2 \equiv 0$ over $U_1 \times U_2$ and so $\partial^{h+k} g/\partial t_1^h \partial t_2^k = 0$ over $U_1 \times U_2$. We let ϑ_i, ϑ_{ij}, ϑ_{ijk} and so on denote the partials of ϑ with respect to the variables carrying the lower indices.

REMARK 2.2. $\forall \vartheta \in I(W_2 \cap A)$ we have $\vartheta_i(0) = 0$ $i = 0, \ldots, g-1$: in fact by R.K.t. (see [K] or [A.C.G.H.]) $\mathbb{P}TC_0 W_2$ is the rational normal curve $\tilde{C}$ in $\mathbb{P}^{g-1}$: in particular it is not degenerate (not contained in any hyperplane).

We want to evaluate $\partial^{h+k} g/\partial t_1^h \partial t_2^k$ at one point $(t_1, t_2) \in (U_1 \times U_2) \cap \Gamma$, therefore the ϑ's and their partials will be evaluated at 0; and from the relation (2.1), after setting $t_1 = t$ and leaving out the upper indices of the

$\Omega_i^k(t)$'s, one gets:

$$\left.\frac{\partial^2 g}{\partial t_1 \partial t_2}\right|_{t_2=-t_1} = \sum_{i,j=0}^{g-1} \vartheta_{ij}(0)\Omega_i(t)\Omega_j(t) = 0. \tag{2.3}$$

$$\left.\frac{\partial^3 g}{\partial t_1^2 \partial t_2}\right|_{t_2=-t_1} = \sum_{i,j,k=0}^{g-1} \vartheta_{ijk}(0)\Omega_i(t)\Omega_j(t)\Omega_k(t)$$

$$+ \sum_{i,j=0}^{g-1} \vartheta_{ij}(0)\Omega_i'(t)\Omega_j(t) = 0. \tag{2.4}$$

$$\left.\frac{\partial^4 g}{\partial t_1^2 \partial t_2^2}\right|_{t_2=-t_1} = \sum_{ijkl=0}^{g-1} \vartheta_{ijkl}(0)\Omega_i(t)\Omega_j(t)\Omega_k(t)\Omega_l(t)$$

$$+ 2\sum_{ijk=0}^{g-1} \vartheta_{ijk}(0)\Omega_i'(t)\Omega_j(t)\Omega_k(t)$$

$$+ \sum_{ij=0}^{g-1} \vartheta_{ij}(0)\Omega_i'(t)\Omega_j'(t) = 0. \tag{2.5}$$

By differentiating (2.3) and after interchanging indices one can see that the second summand of (2.4) is 0 $\forall\ t$ and so that

$$\sum_{ijk=0}^{g-1} \vartheta_{ijk}(0)\Omega_i(t)\Omega_j(t)\Omega_k(t) = 0 \tag{2.6}$$

By differentiating (2.6) one gets easily that the second summand of (2.5) is 0 $\forall\ t$ and so

$$\sum_{ijkl=0}^{g-1} \vartheta_{ijkl}(0)\Omega_i(t)\Omega_j(t)\Omega_k(t)\Omega_l(t)$$

$$+ \sum_{ij=0}^{g-1} \vartheta_{ij}(0)\Omega_i'(t)\Omega_j'(t) = 0. \tag{2.7}$$

(2.3) and (2.6) say that

$$\tilde{C} \subset \mathbb{P}\widetilde{OC}(4)_0(W_2) \subset \mathbb{P}\widetilde{OC}(3)_0(W_2). \tag{2.8}$$

But $\mathbb{P}\widetilde{OC}(3)_0(W_2) = \mathbb{P}TC_0W_2$ because by R.K.t. $I(TC_0W_2)$ is spanned by quadrics and so (1.7) applies. On the other hand $\mathbb{P}TC_0W_2$ is the rational normal curve $\tilde{C}$ and so, in view of (2.8), one gets i) of Prop. 1. Now we want to show that the set of ramification points of π is exactly the set of common zeroes of all the second summands Σ_ϑ of (2.7), for ϑ varying in $I(W_2 \cap A)$. Let U be the open set of C endowed with the local coordinate t introduced above and $\Omega(t)dt = \omega_{|U}$. Since $\omega_i = \omega f^i$ by substituting in (2.3), we get $\Omega^2(t)\ \Sigma_{ij=0}^{g-1}\ \vartheta_{ij}(0)f^{i+j}(t) = 0$ and, from $\Omega^2(t) \neq 0$, we find

$$\sum_{ij=0}^{g-1} \vartheta_{ij}(0)f^{i+j}(t) = 0. \tag{2.9}$$

By differentiating we get

$$\sum_{ij=0}^{g-1} \vartheta_{ij}(0)(i+j)f^{i+j-1}(t) = 0. \tag{2.10}$$

Σ_ϑ on U is equal to

$$\sum_{ij=0}^{g-1} \vartheta_{ij}(0)\,[(\Omega'(t))^2 f^{i+j}(t) + (i+j)\Omega'(t)\Omega(t)f'(t)f^{i+j-1}(t)$$
$$+\ ij\Omega(t)^2 f'(t)^2 f^{i+j-2}(t)]. \tag{2.11}$$

In view of (2.9) and (2.10), (2.11) is simply

$$\Omega(t)^2 f'(t)^2 \sum_{ij=0}^{g-1} \vartheta_{ij}(0)\ ij f^{i+j-2}(t). \tag{2.12}$$

By R.K.t. the ideal of TC_0W_2 is spanned by the minors of the matrix

	ϱ	ϱf	$\varrho f^2, \ldots, \varrho f^{g-2}$
σ	ω	ωf	$\omega f^2, \ldots, \omega f^{g-2}$
$\tau = \sigma \cdot f$	ωf	ωf^2	$\omega f^3, \ldots, \omega f^{g-1}$

(in fact $\omega = \sigma\varrho$) that is, after setting $z_i = \omega_i$, by the minors of the matrix

$$\begin{pmatrix} z_0, z_1, \ldots, z_{g-2} \\ z_1, z_2, \ldots, z_{g-1} \end{pmatrix}.$$

It follows that $z_0z_2 - z_1^2 \in I(TC_0W_2)$ and so, by (1.6), there exists a $\tilde{\vartheta} \in I(W_2 \cap A)$ such that $z_0z_2 - z_1^2 = \tilde{\vartheta}^{\text{in}}$. Therefore $\tilde{\vartheta}_{0,2}(0) = 1$, $\tilde{\vartheta}_{1,1}(0) = -2$, $\tilde{\vartheta}_{ij}(0) = 0$ for all the other indices i, j. (2.12), for $\vartheta = \tilde{\vartheta}$, is equal to

$$-2\Omega(t)^2(f'(t))^2. \tag{2.13}$$

One can see that (2.13) is zero exactly at the ramification points $B_1, \ldots, B_{2g+2}$ of π: on $C\backslash P$ this is obvious because $\Omega(t)$ never vanishes on $C\backslash P$; at P we have $\text{ord}_P\,\Omega(t) = 2g - 2$, $\text{ord}_P f(t) = -2$, $\text{ord}_P f'(t) = -3$ and therefore

$$\text{ord}_P(\Omega(t)f'(t))^2 = 4g - 10 > 0 \quad \text{for } g \geqslant 3 \tag{2.14}$$

so (2.13) vanishes at P. Thus it suffices to show that $\forall\vartheta \in I(W_2 \cap A)$, (2.12) is 0 at $B_1, \ldots, B_{2g+2}$. Let B_i be one of the ramification points $B_i \neq P$: f is holomorphic in a neighborhood Y of B_i so $\Sigma_{ij=0}^{g-1}\,\vartheta_{ij}(0)\;ijf^{i+j-2}(t)$ is holomorphic on Y, and therefore (2.12) is zero at B_i because it contains the factor $f'(t)^2$ which vanishes to second order at any ramification point that is regular for f. We now compute

$$\text{ord}_P\left\{(\Omega(t)f'(t))^2\sum_{ij=0}^{g-1}\vartheta_{ij}(0)\;ijf^{i+j-2}(t)\right\}.$$

If U is a neighborhood of P and the local coordinate t is such that $f|_U = 1/t^2$, the relation (2.9) becomes

$$\sum_{ij=0}^{g-1}\vartheta_{ij}(0)t^{-2(i+j)} = 0 \tag{2.15}$$

and from this we deduce

$$\vartheta_{g-1,g-2}(0) = \vartheta_{g-1,g-1}(0) = 0. \tag{2.16}$$

From (2.16) the lowest degree for t in $\Sigma_{ij=0}^{g-1}\,\vartheta_{ij}(0)\,ijf(t)^{i+j-2}$ is $-4g + 12$, and so by (2.14)

$$\text{ord}_P\left\{(\Omega(t)f'(t))^2\sum_{ij=0}^{g-1}\vartheta_{ij}(0)\,ijf^{i+j-2}(t)\right\} \geqslant 2. \tag{2.17}$$

Therefore (2.12) is zero at $P\ \forall \vartheta \in I(W_2 \cap A)$. It follows that the first summands of (2.7), for ϑ varying in $I(W_2 \cap A)$, vanish simultaneously exactly at the ramification points of π, and so we get that $\mathbb{P}OC(5)_{T_2}(W_2) = \{\varphi_K(B_1), \ldots, \varphi_K(B_{2g+2})\}$ as points sets. Moreover by the previous arguments, (2.13) and (2.17)

$$\min_{\vartheta \in I(W_2 \cap A)} \{\operatorname{ord}_{B_i} \Sigma\vartheta\} = 2 \quad \text{for } i = 1, \ldots, 2g + 2$$

$$\left(\text{at } P \text{ take for instance } \vartheta_2 = \begin{vmatrix} z_{g-3} & z_{g-2} \\ z_{g-2} & z_{g-1} \end{vmatrix}\right),$$

so, since φ_K has degree 2, we get

$$\min_{\vartheta \in I(W_2 \cap A)} \{(\tilde{C} \cdot \{\vartheta_4 = 0\})_{\phi_K(B_i)}\} = 1$$

for $i = 1, \ldots, 2g + 2$, whence ii) of Prop. 1 is easily deduced.

REMARK 2.18. Proposition 1 gives the Torelli theorem for the family of hyperelliptic curves of genus 3.

3. The trigonal case

Let C be a trigonal curve of genus $g \geqslant 4$ and with two distinct g_3^1's if $g = 4$. Let $\omega_0, \ldots, \omega_{g-1}$ be a basis for $H^0(C, \mathcal{O}_C(K_C))$, P_0 be a base point on C. The Abel–Jacobi map μ_3: $C^3 \to J(C)$ is defined by

$$C^3 \ni (P_1, P_2, P_3) \to \left(\sum_{i=1}^{3} \int_{P_0}^{P_i} \omega_0, \ldots, \sum_{i=1}^{3} \int_{P_0}^{P_i} \omega_{g-1}\right) \in J(C).$$

Let $\Gamma = \{(P_1, P_2, P_3) \in C^3\colon P_1 + P_2 + P_3 \in g_3^1$ (a fixed g_3^1)$\}$. $\mu_3(\Gamma)$ is a singular point T_3 of $W_3 = \mu_3(C^3)$ and, after modifying μ_3 by a suitable translation, we will assume that $T_3 = 0 \in J(C)$.

Proof of Proposition 2. Over an open set $U_k \subseteq C$ with local coordinate t_k we will denote by $\Omega_i^k(t_k)\, dt_k$ the local expression of ω_i. Let $(P_1, P_2, P_3) \in U_1 \times U_2 \times U_3$ and let γ: $\Delta \to U_1 \times U_2 \times U_3$ be an analytic arc of curve given by $t_i = h_i \cdot s + \tilde{t}_i$ $i = 1, 2, 3$ where $h_i \in \mathbb{C}$, $\tilde{t}_i \in U_i$, $s \in \Delta$, and $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \Gamma$.

It is clear that $\forall \vartheta \in I(W_3 \cap A)$, where $A \subseteq J(C)$ is a sufficiently small open neighborhood of $0 \in J(C)$, we have $g = \vartheta \circ \mu_3 \circ \gamma \equiv 0$ over Δ and so also $d^{(n)}g/ds^n \equiv 0$ on Δ. We let ϑ_i, ϑ_{ij}, ϑ_{ijk} be the partials of ϑ as in the proof of Prop. 1 and we let

$$\psi_i(s) = \sum_{j=1}^{3} \Omega_i^j(t_j(s))h_j. \tag{3.1}$$

REMARK 3.2. $\forall \vartheta \in I(W_3 \cap A)\ \vartheta_i(0) = 0 \quad i = 0, \ldots, g-1$.
In fact $\mathbb{P}TC_0 W_3$ by R.K.t. is the smooth rational ruled surface R spanned by the trichords of $\tilde{C}$ in $\mathbb{P}^{g-1}$: in particular R is not degenerate.

We evaluate the following derivatives at $s = 0$ for an arc γ such that $\gamma(0) = (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \Gamma$ (it is therefore understood that the ϑ's and their partials will be evaluated at $0 = \mu_3 \circ \gamma(0)$ and each $\Omega_i^l(t_l)$ will be evaluated at $\tilde{t}_l$):

$$\left.\frac{d^2 g}{ds^2}\right|_{s=0} = \sum_{ij=0}^{g-1} \vartheta_{ij}(0)\psi_i(0)\psi_j(0) = 0 \tag{3.3}$$

and if

$$s_1 = \sum_{i,j,k=0}^{g-1} \vartheta_{ijk}(0)\psi_i(0)\psi_j(0)\psi_k(0);$$

$$s_2 = 3\sum_{i,j=0}^{g-1} \vartheta_{ij}(0)\psi_i'(0)\psi_j(0),$$

$$\left.\frac{\mathrm{d}^3 g}{\mathrm{d}s^3}\right|_{s=0} = s_1 + s_2 = 0. \tag{3.4}$$

We note that (3.3) and $s_i\ i = 1, 2$ are homogeneous polynomials in (h_1, h_2, h_3) of degree 2, 3 respectively. We evaluate (3.3) and (3.4) for $h_1 = 1$ and $h_2 = h_3 = 0$, thus getting:

$$\sum_{ij=0}^{g-1} \vartheta_{ij}(0)\Omega_i^1(\tilde{t}_1)\Omega_j^1(\tilde{t}_1) = 0 \tag{3.5}$$

$$\sum_{ijk=0}^{g-1} \vartheta_{ijk}(0)\Omega_i^1(\tilde{t}_1)\Omega_j^1(\tilde{t}_1)\Omega_k^1(\tilde{t}_1) + 3\sum_{ij=0}^{g-1} \vartheta_{ij}(0)\frac{d\Omega_i^1(\tilde{t}_1)}{dt_1}\Omega_j^1(\tilde{t}_1) = 0. \tag{3.6}$$

(3.5) and (3.6) hold $\forall \tilde{t}_1 \in U_1$: by differentiating (3.5) one gets easily from (3.6)

$$\sum_{ijk=0}^{g-1} \vartheta_{ijk}(0)\Omega_i^1(\tilde{t}_1)\Omega_j^1(\tilde{t}_1)\Omega_k^1(\tilde{t}_1) = 0 \tag{3.7}$$

(3.5) and (3.7) tell us that:

$$\tilde{C} \subset \mathbb{P}\widetilde{OC}(4)_0(W_3) \subset \mathbb{P}\widetilde{OC}(3)_0(W_3). \tag{3.8}$$

Since ITC_0W_3 is spanned by quadrics (by R.K.t.), we get from (1.7) that $\mathbb{P}\widetilde{OC}(3)_0(W_3) = \mathbb{P}TC_0W_3 = R$. So i) of Prop. 2 is proved. We want to prove that $\mathbb{P}\widetilde{OC}(4)_0(W_3) = \tilde{C}$. For this it will suffice to show that:

> For any trichord r of C, $r \subseteq R$, there exists a $\vartheta \in I(W_3 \cap A)$ such that the cubic polynomial $c = \vartheta_3 = \Sigma_{ijk=0}^{g-1} \vartheta_{ijk}(0)\, z_i z_j z_k$ is not identically zero on r. (3.9)

In fact if $r \cap \tilde{C}$ is a set of 3 distinct points c will be zero only at these points; if $r \cap \tilde{C}$ has multiple points of intersection (and this happens for finitely many trichords r), since $\{c = 0\}$ cannot cut along R a divisor of the form $\tilde{C} + \Sigma_{i=1}^{m} r_i$ *plus* a finite set of points (here r_i are some trichords of $\tilde{C} \subseteq R$), $c|_r$ will vanish exactly at the points of $r \cap \tilde{C}$. Moreover $c|_r$ will vanish at each of these points with its corresponding multiplicity (this is easy to show by looking at what happens at a nearby trichord r'). In any case (3.9) implies by the above argument that locally over R $\tilde{C}$ is cut by a cubic hypersurface $\{c = 0\}$ transversal to R and with $c = \vartheta_3$ for a certain $\vartheta \in I(W_3 \cap A)$: we get easily from this $\tilde{C} = \mathbb{P}\widetilde{OC}(4)_0(W_3)$.

We now fix $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \Gamma$ and the corresponding trichord r in $\mathbb{P}^{g-1}$ in such a way that the set $r \cap \tilde{C}$ contains at least two distinct points, and note the following facts:

> $z_i = \psi_i(0)$ is the i-th component of a vector z in $T_0J(C) \simeq \mathbb{C}^g$, whose representing point Z in $\mathbb{P}^{g-1}$ traces the line r as (h_1, h_2, h_3) vary in $T_{(\tilde{t}_1,\tilde{t}_2,\tilde{t}_3)}C^3$, (this is the differential of μ_3 at $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$). (3.10)

Let P_i be the point of C which is given in U_i by the value $t_i = \tilde{t}_i$, $\underline{\omega}(P_i)$ be the vector $(\Omega_0^i(\tilde{t}_i), \ldots, \Omega_{g-1}^i(\tilde{t}_i))$, $\underline{\omega}'(P_i)$ be the vector $((d\Omega_0^i/dt_i)(\tilde{t}_i), \ldots,$

$(d\Omega^{\prime}_{g-1}/dt_i)(\tilde{t}_i))$, Θ the matrix $(3\vartheta_{ij}(0))_{ij=0,\dots,g-1}$.

$$s_2 = (h_1^2, h_2^2, h_3^2)^t({}^t\underline{\omega}'(P_1), {}^t\underline{\omega}'(P_2), {}^t\underline{\omega}'(P_3)) \cdot \Theta$$

$$\cdot ({}^t\underline{\omega}(P_1), {}^t\underline{\omega}(P_2), {}^t\underline{\omega}(P_3)) \cdot {}^t(h_1, h_2, h_3) \qquad (3.11)$$

$\Theta \cdot z$ gives a linear form which is the equation of the tangent hyperplane to the quadric of equation $\Sigma_{i,j=0}^{g-1} \vartheta_{ij}(0)z_i z_j = 0$ at the point Z (and this holds also if the quadric is singular at Z, in this case $\Theta \cdot z \equiv 0$). By varying ϑ in $I(W_3 \cap A)$, one gets a family Λ of hyperplanes $\{H_\lambda\}_{\lambda \in \Lambda}$ and one sees easily that $\bigcap_{\lambda \in \Lambda} H_\lambda = \{$the tangent plane to R at $Z\}$. (3.12)

It is well known that in the family of tangent planes to R at Z, for Z varying in r, any two tangent planes at distinct points of r are distinct. (3.13)

An easy way to see (3.13) is to compute the tangent spaces to R from the parametrization $(\lambda, t) \to f_h(t) + \lambda f_k(t)$ for the scroll R, where f_h and f_k are parametrizations of degree h, k rational normal curves which span disjoint linear spaces $\mathbb{P}^h$ and $\mathbb{P}^k$ in $\mathbb{P}^{g-1}$ with $h + k + 1 = g - 1$.

In view of (3.4) and (3.10) to prove the statement (3.9), it will be enough to show that there exists a $\tilde{\vartheta} \in I(W_3 \cap A)$ such that s_2 is not identically 0 (on r). By assumption among P_1, P_2, P_3 at least two of the P_i's, let's say P_1 and P_2, are distinct and we may assume that P_2 is not a ramification point of the map $C \to \mathbb{P}^1$ given by the g_3^1. The coefficient of $h_1 h_2^2$ in s_2 is given by

$$a_{21} = \underline{\omega}'(P_2) \cdot \Theta \cdot {}^t\underline{\omega}(P_1).$$

The product $\underline{\omega}(P_2) \cdot \Theta \cdot {}^t\underline{\omega}(P_1)$ is zero $\forall \vartheta \in I(W_3 \cap A)$, because by taking all the linear forms $\Theta \cdot {}^t\underline{\omega}(P_1)$ one gets the ideal of the tangent plane π_1 to R at $\varphi_K(P_1)$ by (3.12), and $\underline{\omega}(P_2) \in r \subset \pi_1$. The tangent line l_2 to $\tilde{C}$ at $\varphi_K(P_2)$ is given by parametric equations

$$\lambda\underline{\omega}(P_2) + \mu\underline{\omega}'(P_2).$$

Since $l_2 \neq r$, l_2 cannot be contained in π_1 (otherwise $\pi_1 =$ the tangent plane to R at $\varphi_K(P_2)$ which is absurd by (3.13)), so we can choose $\tilde{\vartheta} \in I(W_3 \cap A)$ such that $l_2 \not\subset \mathrm{Ker}\,\{\tilde{\Theta} \cdot {}^t\underline{\omega}(P_1)\}$. Then since $l_2 = \{\lambda\underline{\omega}(P_2) + \mu\underline{\omega}'(P_2)\}$ we get

$$\tilde{a}_{21} = \underline{\omega}'(P_2) \cdot \Theta \cdot {}^t\omega(P_1) \neq 0$$

and (3.9) is proved in this case.

We are left to deal with the situation $P_1 = P_2 = P_3 = \bar{P}$; it is clear that we can take $U_1 = U_2 = U_3 = U$ and so $\forall i, j = 1, 2, 3$ $\Omega_k^i(t_i) = \Omega_k^j(t_j) = \Omega_k(t)$, since t_i and t_j are the same local coordinate on U. Here $t = 0$ corresponds to $\bar{P}$. By writing

$$\Omega_k(t) = \sum_{l=0}^{\infty} a_l^k t^l$$

and after setting

$$w_1 = \sum_{i=1}^{3} t_i, \quad w_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} t_i t_j, \quad w_3 = t_1 t_2 t_3,$$

and

$$\sum_{i=1}^{3} t_i^l = P_l(w_1, w_2, w_3),$$

one gets that the Abel–Jacobi map (with base point $\bar{P}$) $\mu_{(3)}: U^{(3)} \to J(c)$, where $U^{(3)}$ is the symmetric product of U three times and w_1, w_2, w_3 are local coordinates on $U^{(3)}$, is given by:

$$(w_1, w_2, w_3) \mapsto \left(\ldots, \sum_{l=0}^{\infty} \frac{a_l^k}{l+1} P_{l+1}(w_1, w_2, w_3), \ldots \right).$$

We consider an analytic arc of curve $\tilde{\gamma}: \Delta \to U^{(3)}$ given by $w_i = h_i \cdot s$, with $\gamma(0) = (0, 0, 0) = (P_1, P_2, P_3) \in \Gamma$. $\forall \vartheta \in I(W_3 \cap A)$ $\tilde{g} = \vartheta \circ \mu_{(3)} \circ \tilde{\gamma} = 0$ on Δ. We let

$$\tilde{\psi}_i(s) = \sum_{l=0}^{\infty} \frac{a_l^i}{l+1} \left(\sum_{r=1}^{3} \frac{\partial P_{l+1}}{\partial w_r} h_r \right),$$

$$s_1 = \sum_{ijk=0}^{g-1} \vartheta_{ijk}(0) \tilde{\psi}_i(0) \tilde{\psi}_j(0) \tilde{\psi}_k(0),$$

$$s_2 = 3 \sum_{ij=0}^{g-1} \vartheta_{ij}(0) \tilde{\psi}_i'(0) \tilde{\psi}_j(0);$$

and so we get, (as before in (3.4)):

$$\left. \frac{d^3 g}{ds^3} \right|_{s=0} = s_1 + s_2 = 0. \tag{3.14}$$

REMARK 3.15. By using the identities

$$P_1(w_1, w_2, w_3) = w_1, \; P_2(w_1, w_2, w_3) = w_1^2 - 2w_2 \quad \text{and}$$

$$P_n = w_1 P_{n-1} - w_2 P_{n-2} + w_3 P_{n-3}$$

one can compute easily all the isobaric polynomials $P_n(w_1, w_2, w_3)$ and their partials at (0, 0, 0). Here is a list of the ones we will use:

$$\frac{\partial P_1}{\partial w_1} = 1, \quad \frac{\partial P_2}{\partial w_2} = -2, \quad \frac{\partial P_3}{\partial w_3} = 3, \quad \frac{\partial^2 P_2}{\partial w_1^2} = 2, \quad \frac{\partial^2 P_3}{\partial w_1 \partial w_2} = -3,$$

$$\frac{\partial^2 P_4}{\partial w_2^2} = 4, \quad \frac{\partial^2 P_4}{\partial w_1 \partial w_3} = 4, \quad \frac{\partial^2 P_5}{\partial w_2 \partial w_3} = -5, \quad \frac{\partial^2 P_6}{\partial w_3^2} = 6$$

all the other 1st and 2nd order partials are zero at (0, 0, 0).

By an easy computation and in view of (3.15), one gets

$$\tilde{\psi}_i(0) = \sum_{r=0}^{2} (-1)^r \frac{\Omega_i^{(r)}(0)}{r!} h_{r+1};$$

$$\tilde{\psi}_i'(0) = \sum_{\substack{q=1 \\ r+s=q+1}}^{5} (-1)^{q+1} \frac{\Omega_i^{(q)}(0)}{q!} h_r h_s,$$

where

$$\Omega_i^{(r)}(t) = \frac{d^{(r)}\Omega_i(t)}{dt^r}.$$

s_2 in matrix notation can be written as:

$$(h_1, h_2, h_3)\, {}^t\!\left({}^t\underline{\omega}(0), -{}^t\underline{\omega}'(0), \frac{{}^t\underline{\omega}''(0)}{2}\right) \cdot \Theta$$

$$\cdot \left({}^t\underline{\omega}'(0), -\frac{{}^t\underline{\omega}''(0)}{2!}, \frac{{}^t\omega^{(3)}(0)}{3!}, \frac{{}^t\underline{\omega}^{(3)}(0)}{3!}, -\frac{{}^t\underline{\omega}^{(4)}(0)}{4!}, \frac{{}^t\underline{\omega}^{(5)}(0)}{5!}\right)$$

$$\cdot\, {}^t(h_1^2, 2h_1h_2, h_2^2, 2h_1h_3, 2h_2h_3, h_3^2)$$

where $\underline{\omega}^{(r)}(t)$ is the vector $(\Omega_0^{(r)}(t), \ldots, \Omega_{g-1}^{(r)}(t))$ and $\Theta = \{3\vartheta_{ij}(0)\}$.

The coefficient of h_2^3 in the cubic form s_2 is

$$a_{2,3} = -\underline{\omega}'(0) \cdot \Theta \cdot {}^t\underline{\omega}^{(3)}(0)/3!$$

The map $\varphi_K: C \to \mathbb{P}^{g-1}$ is given on U by $\underline{\omega}(t) = (\Omega_0(t), \ldots, \Omega_{g-1}(t))$. Since $\tilde{C} \subset R$, one has that $\underline{\omega}(t) \cdot \Theta \cdot {}^t\underline{\omega}(t) \equiv 0$ in t $\forall \vartheta \in I(W_3 \cap A)$, and by computing the fourth derivative of this identity, setting $t = 0$, and using the symmetry of Θ one gets:

$$\omega^{(4)}(0) \cdot \Theta \cdot {}^t\underline{\omega}(0) + 4\underline{\omega}'(0) \cdot \Theta \cdot \underline{\omega}^{(3)}(0) + 3\underline{\omega}''(0) \cdot \Theta \cdot {}^t\underline{\omega}''(0) = 0. \tag{3.16}$$

If χ is the tangent plane to R at $\bar{P}$ one has $(\tilde{C} \cdot \chi)_{\bar{P}} = \min_\tau(\tilde{C} \cdot H_\tau)_{\bar{P}}$, where $\{H_\tau\}$ is the family of hyperplanes through χ (each H_τ will be defined by a linear form $\mathrm{L}H_\tau$). $H_\tau \cap R = D_\tau$ is a divisor that on a suitable neighborhood of $\bar{P} \subset R$ has the form $kr + \sigma$ where $k \geqslant 1$ and σ is a local section of R passing through 0: that is a curve section of the ruling of the scroll R contained in this neighborhood of $\bar{P}$. Therefore $(\tilde{C} \cdot \chi)_{\bar{P}} = 4$. After writing

$$\underline{\omega}(t) = \sum_{r=0}^{\infty} \underline{\omega}^{(r)}(0) \frac{t^r}{r!},$$

we have that

$$\mathrm{L}H_\tau(\underline{\omega}(t)) = \sum_{r=0}^{\infty} \mathrm{L}H_\tau(\underline{\omega}^{(r)}(0)) \frac{t^r}{r!}$$

and therefore $\mathrm{L}H_\tau(\underline{\omega}^{(r)}(0)) = 0$ for $r = 0, 1, 2, 3$, but there exists $\bar{\tau}$ such that $\mathrm{L}H_{\bar{\tau}}(\underline{\omega}^{(4)}(0)) \neq 0$. Since there exists a $\tilde{\vartheta} \in I(W_3 \cap A)$ such that $\mathrm{L}H_{\bar{\tau}} = \tilde{\Theta} \cdot {}^t\underline{\omega}(0)$, we see that for this $\tilde{\vartheta}$

$$\underline{\omega}^{(4)}(0) \cdot \tilde{\Theta} \cdot {}^t\underline{\omega}(0) \neq 0. \tag{3.17}$$

If $\{H_\beta\}$ is the family of hyperplanes through r, one sees easily that $\mathrm{L}H_\beta(\underline{\omega}^{(k)}(0)) = 0$ for $k = 0, 1, 2$ by the same argument applied above, and so in particular $\underline{\omega}''(0) \in r$ from which, recalling that $r \subset R = \cap$ {quadrics of equation $z \cdot \Theta \cdot {}^tz = 0$ $\forall \vartheta \in I(W_3 \cap A)$}, we deduce

$$\underline{\omega}''(0) \cdot \Theta \cdot {}^t\underline{\omega}''(0) = 0 \quad \forall \vartheta \in I(W_3 \cap A). \tag{3.18}$$

By (3.17) and (3.18) we see that (3.16) gives $\tilde{a}_{23} = -\underline{\omega}'(0) \cdot \Theta \cdot {}'\underline{\omega}^{(3)}(0)/3! = (1/4!)\, \underline{\omega}^{(4)}(0) \cdot \Theta \cdot {}'\underline{\omega}(0) \neq 0$. Therefore s_2 is not 0 for $\vartheta = \tilde{\vartheta}$, s_1 is not 0 too and (3.9) is proved, as well as ii) of Prop. 2.

REMARK 3.19. From Prop. 2, ii) the Torelli theorem for the family of curves of genus 4 admitting two distinct g_3^1's follows immediately. This is a classical result (see [A.C.G.H.] and [K.2] for this result in char. $p \neq 2$).

4. The plane quintic case

Let C be a smooth plane quintic. C has a unique g_5^2 and we let D be a divisor $D \in g_5^2$, $D = Q_1 + \cdots + Q_5$, $Q_i \neq Q_j$ for $i \neq j$. We choose a basis $\{\sigma_0, \sigma_1, \sigma_2\}$ for $H^0(C, \mathcal{O}_C(D))$ and homogeneous coordinates x_0, x_1, x_2 in $\mathbb{P}^2$ in such a way that the embedding $\sigma\colon C \to \mathbb{P}^2$ is given by $\forall P \in C\ x_i = \sigma_i(P)$ $i = 0, 1, 2$.

Since by the adjunction formula we have $\mathcal{O}_C(2D) = \mathcal{O}_C(K_C)$ we may let

$$\{\omega_0 = \sigma_0^2, \quad \omega_1 = \sigma_0\sigma_1, \quad \omega_2 = \sigma_0\sigma_2, \quad \omega_3 = \sigma_1^2, \quad \omega_4 = \sigma_1\sigma_2, \quad \omega_5 = \sigma_2^2\}$$

be a basis for $H^0(C, \mathcal{O}_C(K_C))$.

It is clear that $\varphi_K = v \circ \sigma$ where v is the Veronese embedding $v\colon \mathbb{P}^2 \to \mathbb{P}^5$ given by

$$z_0 = x_0^2, \quad z_1 = x_0x_1, \quad z_2 = x_0x_2, \quad z_3 = x_1^2, \quad z_4 = x_1x_2, \quad z_5 = x_2^2$$

with $(z_0, \ldots, z_5)$ homogeneous coordinates in $\mathbb{P}^5$. $v(\mathbb{P}^2) = S$ is the Veronese quartic surface in $\mathbb{P}^5$.

We let Ch (S) be the chordal variety of S and

$$M = \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_3 & z_4 \\ z_2 & z_4 & z_5 \end{pmatrix};$$

It is well known (see [Se. R.] pp. 128–130) that:

4.0. i) $I(S)$, the homogeneous ideal of S, is spanned by the six linearly independent 2×2 minors of M;

ii) $\mathrm{Ch}(S)$, the chordal variety of S, is defined by the equation $\det M = 0$;

iii) $\mathrm{SingCh}(S) = S$.

Let $P_0 \in C$ be a base point. The Abel–Jacobi map $\mu_5 \colon C^5 \to J(C)$ is given by

$$C^5 \ni (P^1, \ldots, P^5) \mapsto \left(\sum_{i=1}^{5} \int_{P_0}^{P_i} \omega_0, \ldots, \sum_{i=1}^{5} \int_{P_0}^{P_i} \omega_5 \right) \in J(C).$$

Let $\Gamma = \{(P_1, \ldots, P_5) \in C^5 \colon P_1 + \ldots + P_5 \in g_5^2\}$.

By R.K.t. $\mu_5(\Gamma)$ is a triple point T_5 of the divisor $W_5 = \operatorname{Im} \mu_5$ and, after modifying μ_5 by a suitable translation, we will assume that $T_5 = 0 \in J(C)$.

Proof of Proposition 3. Over an open set $U_k \subset C$ with local coordinate t_k we will denote by $\Omega_i^k(t_k)dt_k = \omega_{i|U_k}$. We let $\gamma \colon \Delta \to \Pi_{i=1}^5 U_i \subset C^5$ be an analytic arc of curve in C^5 given by $t_i = h_i \cdot s + \tilde{t}_i$, where $s \in \Delta$, $h_i \in \mathbb{C}$ $\forall i = 1, \ldots, 5$, and $(\tilde{t}_1, \ldots, \tilde{t}_5) \in \Gamma \cap \Pi_{i=1}^5 U_i$. If $\vartheta = 0$ is a local equation of W_5 in a neighborhood A of $0 \in J(C)$, we have $g = \vartheta \circ \mu_5 \circ \gamma = 0$ over Δ and so $d^{(n)}g/ds^n = 0$ on Δ. We let ϑ_i, ϑ_{ij}, ϑ_{ijk} and so on be the partials of ϑ with respect to the variables carrying the lower indices and we also let

$$\psi_i(s) = \sum_{j=1}^{5} \Omega_i^j(t_j(s))h_j.$$

Since 0 is a triple point of W_5 $\vartheta_i(0) = \vartheta_{ij}(0) = 0$ $i, j = 0, \ldots, g-1$ and therefore, as in the derivation of (3.4), one gets the following derivatives:

$$\left.\frac{d^3g}{ds^3}\right|_{s=0} = \sum_{ijk=0}^{5} \vartheta_{ijk}(0)\psi_i(0)\psi_j(0)\psi_k(0) = 0. \tag{4.1}$$

$$\left.\frac{d^4g}{ds^4}\right|_{s=0} = \sum_{ijkl=0}^{5} \vartheta_{ijkl}(0)\psi_i(0)\psi_j(0)\psi_k(0)\psi_l(0) + 6\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\psi_i'(0)\psi_j(0)\psi_k(0) = 0. \tag{4.2}$$

$$\begin{aligned}\left.\frac{d^5g}{ds^5}\right|_{s=0} = {} & \sum_{ijklm=0}^{5} \vartheta_{ijklm}(0)\psi_i(0)\psi_j(0)\psi_k(0)\psi_l(0)\psi_m(0) \\ & + 10\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0)\psi_i'(0)\psi_j(0)\psi_k(0)\psi_l(0) \\ & + 10\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\psi_i''(0)\psi_j(0)\psi_k(0) \\ & + 15\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\psi_i'(0)\psi_j'(0)\psi_k(0) = 0.\end{aligned} \tag{4.3}$$

(4.1), (4.2) and (4.3) are polynomials in $(h_1, \ldots, h_5)$ of degree 3, 4, 5 respectively. The coefficient of $h_1^2 h_2$ in (4.1) is

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1(\tilde{t}_1)\Omega_j^1(\tilde{t}_1)\Omega_k^2(\tilde{t}_2) = 0. \tag{4.4}$$

(4.4) holds for any $(\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2$ (because one can always find $\tilde{t}_3, \tilde{t}_4, \tilde{t}_5$ such that $(\tilde{t}_1, \ldots, \tilde{t}_5) \in \Gamma$), therefore by the linear independence of the Ω_k^2's we find

$$\sum_{ij=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1 = 0 \quad \forall \tilde{t}_1 \in U_1, \quad \forall k = 0, \ldots, 5. \tag{4.5}$$

The coefficient of h_1^3 in (4.1) is

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1\Omega_k^1 = 0 \quad \forall \tilde{t}_1 \in U_1. \tag{4.6}$$

The first and second derivatives of (4.6) are

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1 \frac{d\Omega_k^1}{dt_1} = 0 \quad \forall \tilde{t}_1 \in U_1, \quad \text{and} \tag{4.7}$$

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1 \frac{d^2\Omega_k^1}{dt_1^2} + 2\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1 \frac{d\Omega_j^1}{dt_1}\frac{d\Omega_k^1}{dt_1} = 0. \tag{4.8}$$

We multiply (4.5) by $(d^2\Omega_k^1)/(dt_1^2)$ and sum over k: so we get

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1 \frac{d^2\Omega_k^1}{dt_1^2} = 0 \tag{4.9}$$

and, in view of (4.8) also:

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1 \frac{d\Omega_j^1}{dt_1}\frac{d\Omega_k^1}{dt_1} = 0. \tag{4.10}$$

The coefficient of h_1^4 in (4.2) is

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0)\Omega_i^1\Omega_j^1\Omega_k^1\Omega_l^1 + 6\sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i^1\Omega_j^1 \frac{d\Omega_k^1}{dt_1} = 0. \tag{4.11}$$

From (4.7) and (4.11) we have

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0)\Omega_i^1\Omega_j^1\Omega_k^1\Omega_l^1 = 0 \tag{4.12}$$

and differentiating, one finds that

$$\sum_{jkl} \vartheta_{ijkl}(0) \sum_{i=0}^{5} \frac{d\Omega_i^1}{dt_1} \Omega_j^1\Omega_k^1\Omega_l^1 = 0. \tag{4.13}$$

The coefficient of h_1^5 in (4.3) in view of (4.9), (4.10) and (4.13) is

$$\sum_{ijklm=0}^{5} \vartheta_{ijklm}(0)\Omega_i^1\Omega_j^1\Omega_k^1\Omega_l^1\Omega_m^1 = 0. \tag{4.14}$$

We now give a geometric interpretation of the relations found above.

By R.K.t. and the choices we made at the beginning $\mathbb{P}TC_0W_5$ has equation

$$\det M = 0. \tag{4.15}$$

On the other hand the power series expansion of ϑ at 0 gives for $\mathbb{P}TC_0W_5$ the equation

$$\sum_{ijk=0}^{5} \vartheta_{ijk}(0)z_iz_jz_k = 0 \tag{4.16}$$

and so (4.15) and (4.16) coincide (up to a scalar).

REMARK 4.17. The equality of (4.15) and (4.16) gives, up to a constant multiplier, that

$$\vartheta_{035}(0) = 1, \quad \vartheta_{124}(0) = 2, \quad \vartheta_{223}(0) = \vartheta_{044}(0) = \vartheta_{115}(0) = -2$$

and all other $\vartheta_{ijk}(0) = 0$.

Then, by the coincidence of (4.15) and (4.16) and by 4.0.iii), one gets that $\mathrm{Sing}\mathbb{P}TC_0W_5 = S$, S being defined either by

$$rkM = 1 \quad \text{or by} \tag{4.18}$$

$$\sum_{ij=0}^{5} \vartheta_{ijk}(0)z_iz_j = 0 \quad \forall k = 0, \ldots, 5. \tag{4.19}$$

Then (4.5) shows that $\tilde{C} \subset S$ and (4.6), (4.12) and (4.14) allow to see that

$$\tilde{C} \subseteq \mathbb{P}OC(6)_0(W_5). \tag{4.20}$$

Since $\mathbb{P}\widetilde{OC}(4)_0(W_5) = \mathbb{P}TC_0W_5$ and (4.15), the equation of $\mathbb{P}TC_0W_5$, defines Ch(S), $i)$ of Prop. 3 follows by applying Prop. 1.6 iii) page 232 in [A.C.G.H.], because the g_5^2 is semicanonical. Furthermore $\tilde{C} \subset S \subset$ Ch $(S) = P\widetilde{OC}(4)_0(W_5) = P\widetilde{OC}(5)_0(W_5)$ so to prove ii) of Prop. 3 it is enough to prove that

$$S \cap Q = \text{twice } \tilde{C} \tag{4.21}$$

where Q is the quintic hypersurface in $\mathbb{P}^5$ defined by the equation

$$q = \sum_{ijklm=0}^{5} \vartheta_{ijklm}(0)z_iz_jz_kz_lz_m = 0$$

We already know that $\tilde{C} \subset S \cap Q$ so to prove (4.21) it will suffice to show that for any line $l \subset \mathbb{P}^2$ with $l \cap \sigma(C) = R_1 + \cdots + R_5$ $R_i \neq R_j$ for $i \neq j$, Q intersects the conic $v(l)$ at each point $v(R_i)$ $i = 1, \ldots, 5$ exactly with multiplicity 2 (and therefore $Q \cap v(l) = 2(v(R_1) + \cdots + v(R_5)))$, since then $v^*(Q) \subset \mathbb{P}^2$ has degree 10 and has a double point at every point of C, hence equals twice C. For this we assume that $R_1 + \cdots + R_5$ is the divisor $\sigma(D) = \Sigma_{i=1}^5 \sigma(Q_i)$, that l is the line $\{x_2 = 0\}$, in particular that $R_1 = (1, 0, 0)$, $R_2 = (0, 1, 0)$, $R_3 = (1, 1, 0)$; and that t, the tangent line to $\sigma(C)$ at R_1, has equation $x_1 - x_2 = 0$. All of this can always be arranged by suitable change of coordinates of $\mathbb{P}^2$. We also let the 5-tuple $(\tilde{t}_1, \ldots, \tilde{t}_5)$ be local coordinates on U^5 for the 5-tuple $(Q_1, \ldots, Q_5)$. Since (4.19) vanishes on S in particular it is clear that

$$\sum_{jk=0}^{5} \vartheta_{ijk}(0)\psi_j(0)\psi_k(0) = 0 \tag{4.22}$$

for

$$\underline{\psi}(0) = (\psi_0(0), \ldots, \psi_5(0)) \in v(l) \subset S \quad \forall i = 0, \ldots, 5$$

and also that

$$\sum_{jkl=0}^{5} \vartheta_{ijkl}(0)\psi_j(0)\psi_k(0)\psi_l(0) = 0 \tag{4.23}$$

for

$$\underline{\psi}(0) \in v(l) \quad \text{and} \quad i = 0, \ldots, 5.$$

In fact

$$\sum_{jkl=0}^{5} \vartheta_{ijkl}(0)\, z_j z_k z_l = 0 \quad i = 0, \ldots, 5$$

define Sing$K$, where K is the quartic hypersurface

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0) z_i z_j z_k z_l = 0,$$

and so by Th.1.6 iii) p. 232 in [A.C.G.H.], since $2g_5^2 = |K_C|$, Ch (S) is a component of K and thus $S = \mathrm{SingCh}(S) \subset \mathrm{Sing}K$. We multiply (4.22) by $\psi_i''(0)$, (4.23) by $\psi_i'(0)$ and we sum over $i = 0, \ldots, 5$: thus we get that the second and the third summand of (4.3) are 0 for $\underline{\psi}(0) \in v(l)$ and so if

$$s_1 = \sum_{ijklm=0}^{5} \vartheta_{ijklm}(0)\psi_i(0)\psi_j(0)\psi_k(0)\psi_l(0)\psi_m(0),$$

$$s_2 = 15 \sum_{ijk=0}^{5} \vartheta_{ijk}(0)\psi_i'(0)\psi_j'(0)\psi_k(0)$$

(4.3) becomes

$$s_1 + s_2 = 0 \quad \text{for } \underline{\psi}(0) \in v(l). \tag{4.24}$$

In order to make (4.24) explicit we write down the differential of μ_5 at $D \in C^5 (D \in g_5^2)$ $d\mu_5|_D$: $\mathbb{P}TC^5_{(Q_1,\ldots,Q_5)} \to \mathbb{P}^5 = \mathbb{P}T_0 J(C)$: this is given in our coordinate system by

$$\begin{pmatrix} z_0 \\ z_1 \\ \\ \\ \\ z_5 \end{pmatrix} = \begin{pmatrix} \psi_0(0) \\ \psi_1(0) \\ \\ \\ \\ \psi_5(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & \omega_0(Q_4) & \omega_0(Q_5) \\ 0 & 0 & 1 & \omega_1(Q_4) & \omega_1(Q_5) \\ 0 & 0 & 0 & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & \omega_5(Q_4) & \omega_5(Q_5) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \\ \\ \\ h_5 \end{pmatrix}$$

The image of $d\mu_5|_D$ (viewed in the projective space $\mathbb{P}^5$) is the plane M containing the conic $v(l)$. M has equations $z_2 = z_4 = z_5 = 0$ and $v(l)$ has equation $z_1^2 - z_0z_3 = 0$. If we restrict $d\mu_5|_D$ to the subspace $V = \{h_4 = h_5 = 0\} \subset TC^5_{(Q_1,\ldots,Q_5)}$ we get a specific linear projective isomorphism α: $\mathbb{P}(V) \to M$ given by $z_0 = h_1 + h_3$, $z_1 = h_3$, $z_3 = h_2 + h_3$; whose inverse α^{-1} is $h_1 = z_0 - z_1$, $h_2 = z_3 - z_1$, $h_3 = z_1$, and $v(l)$ will be transformed by α^{-1} into a conic $\nabla \subset \mathbb{P}(V)$ of equation $h_1h_2 + h_1h_3 + h_2h_3 = 0$. $\alpha^{-1}(v(R_1))$ is the point $(1, 0, 0) \in \nabla \subset \mathbb{P}(V)$. We write the following parametrization λ: $\mathbb{C} \to \nabla$ of the conic ∇: $h_1 = 1$, $h_2 = -u/(1+u)$, $h_3 = u$; $\lambda(0) = (1, 0, 0)$. We observe that s_1 and s_2 are both homogeneous polynomials of degree 5 in the variables h_i's and that $s_1|_{\mathbb{P}(V)} = \alpha^*(q|_M)$ and therefore by (4.24) $-s_2|_\nabla = \alpha^*(q|_{v(l)})$. Summing up we have

$$\begin{array}{ccccc}
 & & \mathbb{P}TC^5_D & \longrightarrow & \mathbb{P}^5 = \mathbb{P}T_0J(C) \\
 & & \uparrow\cup & & \uparrow\cup \\
 & & \mathbb{P}(V) & \xrightarrow[\alpha]{\sim} & M \\
 & & \uparrow\cup & & \uparrow\cup \\
\mathbb{C} & \xrightarrow{\lambda} & \nabla & \xrightarrow[\alpha|_\nabla]{\sim} & v(l)
\end{array}$$

We now read (4.24) over ∇ (or over $\mathbb{C}$ by λ): for this we consider $\lambda^*(s_2|_\nabla)$ and its first and second derivatives at $u = 0$. We state some facts we will need in the computation of these derivatives.

We have the following table:

$$\begin{array}{lll}
h_1(0) = 1 & h_1'(0) = 0 & h_1''(0) = 0 \\
h_2(0) = 0 & h_2'(0) = -1 & h_2''(0) = 2 \\
h_3(0) = 0 & h_3'(0) = 1 & h_3''(0) = 0
\end{array} \tag{4.25}$$

from which we see that every monomial in the $h_i(u)$'s and their derivatives containing factors of the form $h_i(u)h_i'(u)$ or $h_i(u)h_i''(u)$ vanishes at $u = 0$.

Since σ_2 vanishes at the five distinct points $Q_1, \ldots, Q_5$ it vanishes simply at e ch one of them so $\sigma_2'(Q_1) \neq 0$. (4.26)

We recall that $\omega_0 = \sigma_0^2$, $\omega_1 = \sigma_0\sigma_1$ and so on. By computing derivatives and using $\sigma_0(Q_1) = 1$, $\sigma_1(Q_1) = \sigma_2(Q_1) = 0$ we get $\Omega_0'(\tilde{t}_1) = 2\sigma_0'(Q_1)$, $\Omega_1'(\tilde{t}_1) = \sigma_1'(Q_1)$, $\Omega_2'(\tilde{t}_1) = \sigma_2'(Q_1) \neq 0$, $\Omega_3'(\tilde{t}_1) = \Omega_4'(\tilde{t}_1) = \Omega'_5(\tilde{t}_1) = 0$. (4.27)

Now

$$\lambda^*(s_2|_\nabla) = 15 \sum_{ijklmn=0}^{5} \vartheta_{ijk}(0)\Omega_i'(\tilde{t}_l)\Omega_j'(\tilde{t}_m)\Omega_k'(t_n) \cdot h_l^2(u)h_m^2(u)h_n(u), \tag{4.28}$$

so by (4.25) $d/(du)\ \lambda^*(s_2|_\nabla)|_{u=0}$ reduces to

$$15 \sum_{ijklmn=0}^{5} \vartheta_{ijk}(0)\Omega_i'(\tilde{t}_l)\Omega_j'(\tilde{t}_m)\Omega_k(\tilde{t}_n)h_l^2(0)h_m^2(0)h_n'(0). \tag{4.29}$$

To get non-zero summands in (4.29) one has to take $l = m = 1$ and $n = 2$, 3 thus getting

$$15 \sum_{ijk=0}^{5} \vartheta_{ijk}(0)\Omega_i'(\tilde{t}_1)\Omega_j'(\tilde{t}_1) \cdot (\Omega_k(\tilde{t}_3) - \Omega_k(\tilde{t}_2)). \tag{4.30}$$

After looking at the matrix of $d\mu_5|_D$ computed above one gets

$$\Omega_k(\tilde{t}_3) - \Omega_k(\tilde{t}_2) = \begin{cases} 1 & k = 0, 1 \\ 0 & 2 \leqslant k \leqslant 5 \end{cases},$$

so (4.30) actually is

$$15 \sum_{ij=0}^{5} (\vartheta_{ij0}(0) + \vartheta_{ij1}(0))\ \Omega_i'(\tilde{t}_1)\Omega_j'(\tilde{t}_1). \tag{4.31}$$

By (4.26) the products $\Omega_i'(\tilde{t}_1)\Omega_j'(\tilde{t}_1)$ may be not 0 for $0 \leqslant i, j \leqslant 2$. But by (4.17) all the $\vartheta_{ij0}(0)$ and $\vartheta_{ij1}(0)$ with $0 \leqslant i, j \leqslant 2$ are zero, so (4.31) and therefore (4.29) is zero. Thus $d/(du)\lambda^*(s_2|_\nabla)|_{u=0} = 0$. $d^2/(du^2)\lambda^*(s_2|_\nabla)_{u=0}$ reduces by (4.25) to

$$15\left(\sum_{ijklmn=0}^{5} \vartheta_{ijk}(0)\Omega_i'(\tilde{t}_l)\Omega_j'(\tilde{t}_m)\Omega_k(\tilde{t}_n) \cdot 4h_l'(0)^2h_m(0)^2h_n(0)\right.$$
$$\left. + \sum_{ijklmn=0}^{5} \vartheta_{ijk}(0)\Omega_i'(\tilde{t}_l)\Omega_j'(\tilde{t}_m)\Omega_k(\tilde{t}_n)h_l(0)^2h_m(0)^2h_n''(0)\right). \tag{4.32}$$

The first summand may be not zero only for $l = 2, 3$ and $m = n = 1$; $\Omega_k(\tilde{t}_1) \neq 0$ only for $k = 0$ and $\Omega_0(\tilde{t}_1) = 1$, so it becomes

$$60\left(\sum_{ij=0}^{5} \vartheta_{ij0}(0)\Omega_i'(\tilde{t}_2)\Omega_j'(\tilde{t}_1) + \sum_{ij=0}^{5} \vartheta_{ij0}(0)\Omega_i'(\tilde{t}_3)\Omega_j'(\tilde{t}_1)\right). \tag{4.33}$$

(4.33) is zero because $\Omega_j'(\tilde{t}) \neq 0$ only for $0 \leqslant j \leqslant 2$ (by (4.26)) and $\vartheta_{ij0}(0) = 0$ for $0 \leqslant j \leqslant 2$ by (4.17).

The second summand of (4.32) imposes $l = m = 1$ and $n = 2$; since $\Omega_k(\tilde{t}_2) \neq 0$ only for $k = 3$ (as it can be seen in the matrix of $d\mu_5|_D$), we get

$$30 \sum_{ij=0}^{5} \vartheta_{ij3}(0)\Omega_i'(\tilde{t}_1)\Omega_j'(\tilde{t}_1) = 30\vartheta_{223}(0)\Omega_2'(\tilde{t}_1)^2 \neq 0$$

by (4.27) and (4.17). So $d/(du^2)\lambda^*(s_2|_\nabla)|_{u=0} \neq 0$ and the proof is complete.

REMARK 4.34. From Prop. 3 ii) the Torelli theorem for smooth plane quintics follows immediately.

Acknowledgements

Research partially supported by M.P.I. funds.

We would like to thank Roy Smith and Robert Varley for some interesting conversations on the genus 3 case of Proposition 1: this seems to have been known classically (although we do not have any precise reference for this).

References

[A.C.G.H.] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of Algebraic Curves, Grundlehren der Math. Wiss.* 267, Springer Verlag (1984).

[B.V.] F. Bardelli and L. Verdi, Un'osservazione sul teorema di singolarità di Riemann-Kempf per curve iperellittiche e trigonali, *Boll. Un. Mat. It.* (6) 5-A (1986) 201–204.

[G.H.] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience Publ. (Pure and applied Math.), J. Wiley & Sons (1978).

[K] G. Kempf, On the geometry of a theorem of Riemann, *Ann. of Math.* 98 (1973) 178–185.

[K.2.] G. Kempf, The equations defining a curve of genus four, *A.M.S. Proceedings* 97,2 June 1986, Providence R.I.

[S.D.] B. Saint-Donat, On Petri's analysis of the linear system of quadrics through a canonical curve, *Math. Ann.* 206 (1973) 157–175.

[Se.R.] J.G. Semple and L. Roth, *Introduction to Algebraic Geometry*, Oxford Clarendon Press (1949).

[Sh.] I.R. Shafarevic, Basic Algebraic Geometry, *Grundlehren der Math. Wiss.* 213, Springer Verlag, Berlin, Heidelberg, New York (1977).