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JANINA KOTUS

FOPKE KLOK

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## A sufficient condition for $\Omega$ -stability of vector fields on open manifolds

JANINA KOTUS<sup>1</sup> & FOPKE KLOK<sup>2</sup>

<sup>1</sup>*Institute of Mathematics, Technical University of Warsaw, 00-661 Warsaw, Poland;*

<sup>2</sup>*Philips, Corp ISA/CAD Centre, Postubs 218, 5600 MD Eindhoven, The Netherlands*

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### 1. Introduction

A vector field on an open manifold which satisfied the Kupka–Smale properties is  $\Omega$ -stable if its Auslander recurrent set equals the union of its singularities and closed orbits. For a two dimensional manifold this was already shown in [4], here the generalization for arbitrary dimension will be given. In [4] it was also proved that on the plane  $R^2$  such vector fields are generic. Although this could be generalized to a broader class of surfaces (cf. [4, 6]), genericity in higher dimensions is unobtainable, even for compact manifolds. However, the conditions for  $\Omega$ -stability presented here are weaker than the conditions we know of, and which are all concerned with the structural stability on open manifolds [2, 5].

### 2. Definitions and statement of the result

Let  $M$  be an open manifold,  $\dim M = n < \infty$ . By  $H^r(M)$  we denote the space of complete  $C^r$  vector fields on  $M$  with the  $C^r$  Whitney topology ( $r \geq 1$ ).  $X, Y$  denote elements of  $H^r(M)$ ,  $\Phi_X$  denotes the flow induced by  $X$ . For  $x \in M$ ,  $O_X(x)$  ( $O_X^+(x)$ ,  $O_X^-(x)$ ) is the trajectory of  $x$  (the positive semi-trajectory, the negative semi-trajectory) under  $\Phi_X$ .

Because we will make use of Liapunov functions, we will need the notion of Auslander recurrency, which is defined in terms of the prolongation (limit) set. A general reference for this is [1]. For a vector field  $X$  and  $x \in M$  we define the first positive prolongation set  $D_{1,x}^1(x)$  by:

$$D_{1,x}^1(x) = \{y \in M: \exists x_n \rightarrow x, t_n > 0 \text{ such that } \Phi_X(x_n, t_n) \rightarrow y\}$$

and the first positive prolongation limit set  $J_{1,X}^1(x)$  by:

$$J_{1,X}^1(x) = \{y \in M: \exists x_n \rightarrow x, t_n \rightarrow +\infty \text{ such that } \Phi_X(x_n, t_n) \rightarrow y\}.$$

By induction we define for  $k > 1$

$$D_{1,X}^k(x) = \{z \in M: z \in D_{1,X}^1(y) \text{ for some } y \in D_{1,X}^{k-1}(x)\}$$

$$J_{1,X}^k(x) = \{z \in M: z \in J_{1,X}^1(y) \text{ for some } y \in J_{1,X}^{k-1}(x)\}.$$

Assume that the sets  $D_{\beta,X}^k(x), J_{\beta,X}^k$  are defined for  $k \in \mathbb{N}$  and ordinal numbers  $\beta < \alpha$ . We define the prolongation set  $D_{\alpha,X}^1(x)$  and the prolongation limit set  $J_{\alpha,X}^1(x)$  for the ordinal number  $\alpha$  by:

$$D_{\alpha,X}^1(x) = \{y \in M: \exists x_n \rightarrow x, y_n \rightarrow y, k_n \in \mathbb{N}, \text{ ordinals } \beta_n < \alpha \\ \text{with } y_n \in D_{\beta_n,X}^{k_n}(x_n)\}.$$

$$J_{\alpha,X}^1(x) = \{y \in M: \exists x_n \rightarrow x, y_n \rightarrow y, k_n \in \mathbb{N}, \text{ ordinals } \beta_n < \alpha \\ \text{with } y_n \in J_{\beta_n,X}^{k_n}(x_n)\}.$$

Then we have  $D_{\alpha,X}^k(x) = J_{\alpha,X}^k(x) \cup O_X^+(x)$  for each  $k$ . Let  $\text{Per}(X), \Omega(X), R(X)$  denote respectively the set of periodic points, the set of non-wandering points and the set of Auslander's recurrent points of  $X$ , i.e.

$$\text{Per}(X) = \{x \in M: \Phi_X(x, t) = x \text{ for some } t > 0\}$$

$$\Omega(X) = \{x \in M: \exists x_n \rightarrow x, t_n \rightarrow +\infty \text{ such that } \Phi_X(x_n, t_n) \rightarrow x\}$$

$$R(X) = \{x \in M: \exists \alpha, k \text{ such that } x \in J_{\alpha,X}^k(x)\}.$$

In general  $\text{Per}(X) \subset \Omega(X) \subset R(X)$  and  $\Omega(X) = \{x \in M: x \in J_{1,X}^1(x)\}$ .

As in [4], our definition of  $\Omega$ -stability ( $R$ -stability) reads:

**DEFINITION:** A vector field  $X \in H'(M)$  is  $\Omega$ -stable ( $R$ -stable) if for every neighbourhood  $U$  of  $\Omega(X)$  ( $R(X)$ ) there exists a neighbourhood  $U^*$  of  $X$  such that for each  $Y \in U^*$ :

- (i)  $\Omega(Y) \subset U$  ( $R(Y) \subset U$ )
- (ii) if a connected component of  $U$  contains points of  $\Omega(X)$  ( $R(X)$ ), then it also contains points of  $\Omega(Y)$  ( $R(Y)$ )
- (iii) there exists a homeomorphism  $h_Y: \Omega(X) \rightarrow \Omega(Y)$  ( $R(X) \rightarrow R(Y)$ ) transforming trajectories of  $X$  into trajectories of  $Y$ .

The usual definition of  $\Omega$ -stability for compact manifolds only states the third condition. This definition is closer to the notion of “absolute stability” introduced by Guckenheimer [3].

We denote by  $H_{k-s}^r(M)$  the space of Kupka–Smale vector fields, i.e.  $X \in H_{k-s}^r(M)$  if  $X \in H^r(X)$  and satisfies:

- (i) for each  $x \in \text{Per}(X)$  the trajectory  $O_X(x)$  is hyperbolic
- (ii) for each  $x, y \in \text{Per}(X)$  the unstable manifold  $W^u(x)$  and stable manifold  $W^s(y)$  are in general position.

For definitions see [7].

In the next section, we will prove the following:

**THEOREM:** *If  $X \in H_{k-s}^r(M)$  and  $R(X) = \text{Per}(X)$  then  $X$  is  $R$ -stable.*

**COROLLARY:** *Each  $X$  satisfying the assumptions of the Theorem is  $\Omega$ -stable.*

### 3. Proof of the theorem

We start with recalling two lemmas from [4]. Lemma 1 enables us to control vector fields outside neighbourhoods of the set of Auslander’s recurrent points. Lemma 2 is concerned with the local  $\Omega$ -stability.

**LEMMA 1:** *Let  $X \in H^r(M)$  and  $R(X) = \emptyset$ . Then there exists a  $C^1$  Liapunov function  $L$  on  $M$  with  $X(L) < 0$ .*

*Proof:* From [1], theorem 2.14.10 (p. 239), we know that there exists a strict Liapunov function  $L$ , which is continuous. Because  $C^1$  functions are dense in the space of continuous function, it follows from the proof of this theorem that we can obtain such an  $L$  which is  $C^1$ , with  $X(L) < 0$ .

**LEMMA 2:** *Let  $X \in H_{k-s}^r(M)$  and  $R(X) = \text{Per}(X)$ . Then each compact set  $K \subset M$  meets only finitely many orbits  $O_X(x)$  for  $x \in \text{Per}(X)$ .*

*Proof:* It is clear that there is only a finite number of critical points in  $K$ . Suppose that there exist infinitely many closed orbits  $(O_X(x_n))$ ,  $n \in \mathbb{N}$ , which intersect  $K$ . We may assume that  $x_n \in K$  and  $x_n \rightarrow x \in K$ . Thus  $x \in \Omega(X) \subset R(X)$ . Since  $R(X) = \text{Per}(X)$  and  $X \in H_{k-s}^r(M)$  then  $x$  is a saddle critical point or  $O_X(x)$  is a closed saddle orbit. But in both cases the stable manifold  $W^s(x)$  and unstable  $W^u(x)$  are contained in the set of accumulation points of sequence  $(O_X(x_n))$ ,  $n \in \mathbb{N}$ . It implies that each trajectory  $O_X(z) \subset W^s(x)$  and each trajectory  $O_X(z) \subset W^u(x)$  is contained in  $\Omega(X)$ . This contradiction proves lemma.

The next Remark follows from Lemma 2.

REMARK 1: For each vector field  $X$  as in Lemma 2 we may choose neighbourhoods  $U_n$  of each critical point  $O_X(x_n)$  or each closed orbit  $O_X(x_n)$  with mutually disjoint closures. Then  $R$ -stability for such a vector field reduces to the existence of a neighbourhood  $U^*$  of  $X$  with the property that for each vector field  $Y \in U^*$ ,  $R(Y) \subset \bigcup_{n \in I} U_n$ ,  $I \subset N$ , and  $Y$  possesses a unique critical element  $O_Y(y_n)$  of the same kind (critical point, closed orbit) as  $O_X(x_n)$  in each  $U_n$ .

LEMMA 3: *If  $X \in H_{K^s}^1(M)$  and  $R(X) = \text{Per}(X)$  then  $X$  is  $R$ -stable.*

*Proof:* Let  $(O_X(x_n))$ ,  $n \in I \subset N$ , denote the critical elements of  $X$ , i.e.  $x_n \in \text{Per}(X)$  for each  $n \in I$ , and let  $L$  be a  $C^1$  Liapunov function on  $M - \bigcup_{n \in I} O_X(x_n)$  with  $X(L) < 0$ . We start with choosing a neighbourhood  $U_n$  of each trajectory  $O_X(x_n)$ ,  $n \in I$ , with compact closure as follows:

If  $O_X(x_n)$  is an attractor (i.e.  $O_X(x_n)$  is either a stable closed orbit or a sink), then we choose  $U_n$  with its boundary  $\partial U_n$  transverse to  $X$ . We define  $U_n$  analogously if  $O_X(x_n)$  is a repellor, i.e.  $O_X(x_n)$  is either an unstable closed orbit or a source.

If  $O_X(x_n)$  is a saddle critical point we choose a neighbourhood  $U_n$  on which the vector field is topologically equivalent to its linear part.

If  $O_X(x_n)$  is a saddle closed orbit we choose a neighbourhood  $U_n$  such that the Poincaré map on each section  $\Sigma$  in  $U_n$  is topologically equivalent to its derivative in  $O_X(x) \cap \Sigma$ .

Let  $W_\varepsilon^{s(u)}(x_n) = W^{s(u)}(x_n) \cap U_n$ . Then  $W_\varepsilon^{s(u)}(x_n)$  consists of two components if  $x_n$  is a critical point and  $\dim W_\varepsilon^{s(u)}(x_n) = 1$  or  $O_X(x_n)$  is a closed orbit and  $\dim W_\varepsilon^{s(u)}(x_n) = 2$ . In the other cases the local stable (unstable) manifold of  $x_n$  is connected. We choose small sections  $S_n^s, S_n^u$  with disjoint closures contained in the boundary  $\partial U_n$ ,  $\dim S_n^s = n - 1$ ,  $\dim S_n^u = n - 1$  and transverse to  $W_\varepsilon^s(x_n), W_\varepsilon^u(x_n)$  respectively at their boundary. So each trajectory from  $W_\varepsilon^{s(u)}(x_n)$  meets  $S_n^{s(u)}$  at exactly one point. Moreover, we choose  $S_n^{s(u)}$  such that  $\max \{L(x): x \in S_n^s\} < \min \{L(x): x \in S_n^u\}$ . Let  $G(x_n)$  be a set of trajectories of  $X$  such that arcs of these trajectories and the transverse section  $S_n^{s(u)}$  constitute the boundary of a neighbourhood  $V_n$  of  $O_X(x_n)$  with  $V_n \subset U_n$ .  $G(x_n)$  is a smooth  $(n - 1)$  dimensional manifold; if  $x_n$  is a singularity, every component of  $G(x_n)$  is diffeomorphic to  $S^{s-1} \times S^{u-1} \times I$  ( $S^i$  denotes the  $i$ -dimensional sphere,  $W^s(x_n)$  is assumed  $s$ -dimensional,  $W^u(x_n)$   $u$ -dimensional,  $I = [0, 1]$ ) and if  $x_n$  is a closed orbit it is diffeomorphic to  $S^{s-1} \times S^{u-1} \times S^1 \times I$ . The number of components of  $G(x_n)$  equals one if  $s - 1 > 0$  and  $u - 1 > 0$ , it equals 2 if  $s - 1 = 0$  or  $u - 1 = 0$ , and it equals 4 if both  $s - 1 = 0$  and  $u - 1 = 0$ . Let  $F(x_n)$  be a neighbourhood of  $G(x_n) \cap U_n$  which consists of parts of trajectories of  $X$  and  $F(x_n) \cap \partial U_n \subset S_n^s \cup S_n^u$ . See Figs. 1 and 2.

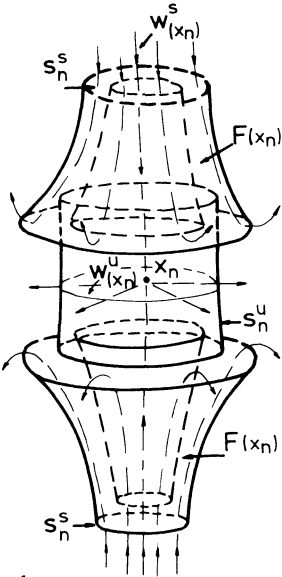


Fig. 1.

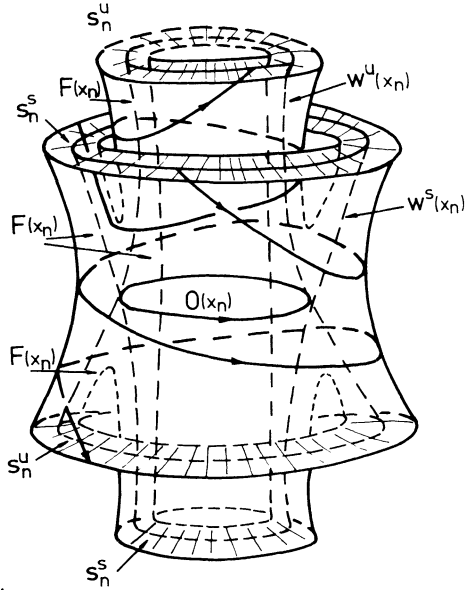


Fig. 2.

At the end, we define neighbourhoods  $W_n$  of each orbit  $O_X(x_n)$  as follows. If  $O_X(x_n)$  is a saddle critical point or a saddle closed orbit then  $W_n \subset V_n$  and  $W_n \cap F(x_n) = \emptyset$ . In the other cases we put  $W_n = U_n$ .

Let  $U_1^*$  be a neighbourhood of  $X$  such that:

- (1) for each  $Y \in U_1^*$  and for each non-saddle orbit  $O_X(x_n)$ ,  $n \in I$ , there exists a unique orbit  $O_Y(y_n)$  in  $U_n$  of the same type as  $O_X(x_n)$  and  $Y$  is transverse to  $\partial U_n$ .
- (2) for each  $Y \in U_1^*$  and for each saddle orbit  $O_X(x_n)$ ,  $n \in I$ , there exists a unique saddle orbit  $O_Y(y_n)$  in  $U_n$ . Moreover,  $Y$  has the set of trajectories  $G(y_n)$  with  $O_Y(y) \cap U_n \subset F(x_n)$  for each  $O_Y(y) \subset G(y_n)$  and  $Y$  is transverse to  $S_n^{s(u)}$ .

Let  $U_2^*$  be a neighbourhood of  $X$  such that if  $Y \in U_2^*$  then  $Y(L)|_{M - \cup_{n \in I} W_n} < 0$ . We show that  $R(Y) = \cup_{n \in I} O_Y(y_n)$  for each  $Y \in U^* = U_1^* \cap U_2^*$ . By construction of  $U_2^*$   $L$  strictly decreases on trajectories of  $Y$ , which are totally contained in  $M - \cup_{n \in I} W_n$ , so each trajectory in  $R(Y)$  intersects at least one  $W_n$ . It follows from the assumption concerning  $L$  that  $L(y) < L(x)$  for any two points  $x, y \in M - \cup_{n \in I} U_n$  with  $y \in O_Y^+(x)$ . From this, using continuity of  $L$ , it is easy to see that for a point  $z \in J_{\alpha, Y}^k(x)$ ,  $x, z \in M - \cup_{n \in I} U_n$ , we have  $L(z) \leq L(x)$ . Suppose that  $R(Y) \neq \text{Per}(Y)$ . Since  $R(Y) \supset \cup_{n \in I} O_Y(y_n)$  for  $Y \in U_1^*$ , there exists a trajectory  $O_Y(z) \subset R(Y)$  crossing the boundary of  $W_{\bar{n}}$  for some saddle orbit  $O_Y(y_{\bar{n}})$ . Assume first that  $O_Y(z)$  is not contained in  $W_{\bar{n}}^s(y_{\bar{n}}) \cup W_{\bar{n}}^u(y_{\bar{n}})$ . This implies that  $O_Y(z)$  intersects  $S_{\bar{n}}^s, S_{\bar{n}}^u$  successively in  $z_1$  and  $z_2$ , so  $L(z_2) < L(z_1)$ . But since  $O_Y(z) \subset R(Y)$ ,  $z_1 \in J_{\alpha, Y}^k(z_2)$  for some

$\alpha$  and  $k$  and thus  $L(z_1) \leq L(z_2)$ , which is a contradiction. The other possibility, where  $O_Y(z) \subset W^s(y_{\bar{n}}) \cup W^u(y_{\bar{n}})$  is proved to be impossible with the same arguments. Let  $O_Y(z) \subset W^s(y_{\bar{n}})$  then for each  $O_Y(y) \subset W^u(y_{\bar{n}})$  we have  $L(z_2) < L(z_1)$  for  $z_1 \in O_Y(z) \cap S_{\bar{n}}^s$ ,  $z_2 \in O_Y(y) \cap S_{\bar{n}}^u$ . Since  $z_1 \in J_{\alpha, Y}^k(z_2)$  then  $L(z_1) \leq L(z_2)$  and we have again a contradiction. The last case, if  $O_Y(z) \subset W^u(y_{\bar{n}})$ , is symmetric to  $O_Y(z) \subset W^s(y_{\bar{n}})$ . Therefore  $R(Y) = \bigcup_{n \in I} O_Y(y_n)$  and the lemma is proved.

Finally, we want to point out that a stronger result in the sense of structural stability is unobtainable, as can be seen from the work [8]. In that paper, an example is given of the vector field satisfying the assumption of the Theorem, but possessing complicated relations among infinitely many saddles at infinity. This complicated structure of the saddles at infinity causes the lack of structural stability proved in [8].

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