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# Non-abelian extensions have nonsimple spectrum

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**Abstract.** Let $T$ be a $G$ extension of a measure preserving transformation $T_0$, where $G$ is a compact non-abelian group. The maximal spectral multiplicity of $T$ is greater than or equal to the supremum of the dimensions of the irreducible representations of $G$; i.e., non-abelian extensions have nonsimple spectrum. As a partial converse we show that generically (in an appropriate topology) abelian extensions have simple spectrum. Generalizations, corollaries and applications are discussed.

## 1. Introduction

One important method for producing examples in ergodic theory is the compact group extension construction. Given a measure preserving transformation $T_0$ of a Lesbesgue probability space $(X_0, \mu_0)$, a compact metrizable group $G$, and a measurable function $\varphi: X_0 \to G$, this construction yields a new transformation $T$ with the property that it commmutes with a free measure preserving action of $G$. Conversely, any transformation $T$ which commutes with such an action is the $G$ extension of some transformation $T_0$.

Along with the many specific examples which have been constructed by this method (cf. e.g., [BFK], [Fu], [J], [MwN]), the general properties of compact group extensions have also been studied (cf. e.g., [P], [Ru1], [Ru2], [R4], [N]). In this paper we will consider the general question of what spectral multiplicity properties a compact group extension can have.

For a measure preserving transformation $T$ of $(X, \mu)$, the spectral multiplicity of $T$ is definned to be the spectral multiplicity of the corresponding induced unitary operator $U_T$ on $L^2(X, \mu)$. In the context of operator theory, it follows from the spectral theory that a unitary operator $U$ has nonsimple spectrum if and only if it has a non-abelian commutant. If $T$ commutes with a free measure-preserving $G$ action, then $U_T$ commutes with the unitary

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representation of $G$ induced on $L^2(X, \mu)$ by this $G$ action. When $G$ is non-abelian we get that $U_T$ has nonsimple spectrum, which implies (by definition) the same for $T$. This shows that non-abelian group extensions have nonsimple spectrum.

Actually, we are able to obtain better estimates of the multiplicity than just this. In Proposition 1 we show that lower bounds on the multiplicity function are given by the dimensions of the irreducible representations of $G$. In particular (Theorem 1) the maximal spectral multiplicity $M_T$ satisfies the estimate $M_T \geqslant D_G$, where $D_G$ denotes the supremum of the dimensions of the irreducible representations of $G$.

For groups $G$ with a finite upper bound on the dimensions of the irreducible representations, this inequality gives a finite lower bound on the multiplicity. In particular, such an estimate always holds for finite non-abelian group extensions. This fact is interesting because it is one of the very few general conditions known to imply finite lower bounds on the spectral multiplicity of a transformation $T$. In fact, for many years the question of the existence of any transformations with nonsimple spectrum of finite multiplicity remained open. Although there are now several constructions for such transformations (cf. [O], [R1], [R2], [R3], [K], [G] (multiplicity 2 only) and [MwN]), with the exception of the construction of Katok, [K],* lower bounds on the multiplicity have usually been obtained by ad hoc methods. In contrast, there are several well known general conditions which imply a finite upper bound on the multiplicity (cf. Lemma 1 below).

Complementing the results for non-abelian extensions, we also study the spectral multiplicity of $G$ extensions where the group $G$ is abelian. Even if $T_0$ has simple spectrum it is possible for an abelian extension $T$ of $T_0$ to have nonsimple spectrum (or even infinite spectral multiplicity). However, typically this does not happen. In the natural topology on the space of all abelian $G$ extensions $T$, the set of those $T$ with simple spectrum is generic (i.e., has a dense $G_\delta$ subset).

## 2. Definitions and other preliminaries

Let $T$ be a measure preserving transformation of a Lebesgue probability space $(X, \mu)$ and let $U_T$ be corresponding the induced unitary operator on $L^2(X, \mu)$, (defined $U_T f(x) = f(T^{-1}x)$). By the Spectral Theorem (cf. [K]), $U_T$ is determined up to equivalence by a spectral measure class $\sigma$ and a

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* A. Katok [K] has shown that in general if $T$ if the Cartesian $k$'th power of a transformation $T_0$ then $M_T \geqslant K!$.

$\{+\infty, 1, 2, \ldots\}$ valued multiplicity function $m$ on the circle. The set of *essential spectral multiplicities* of $T$ (cf. [K]), is the set of all $\sigma$ essential values of $m$. We denote this set by $\mathscr{M}_T$. The *maximal spectral multiplicity*, $M_T$, is defined $M_T = \sup \mathscr{M}_T$ (cf. [K]). The transformation $T$ has *simple spectrum* if $M_T = 1$, or equivalently, if $\mathscr{M}_T = \{1\}$. Otherwise, $T$ has *nonsimple spectrum*. When $T$ is not ergodic it automatically has nonsimple spectrum for a trivial reason, so we will usually only be concerned with the ergodic case.

Throughout this paper $G$ will denote a compact metrizable group. In particular, $G$ may be finite. Suppose that $\mathscr{L} = \{L_g\}_{g \in G}$ is a measurable measure preserving left action of $G$ on $(X, \mu)$. The action $\mathscr{L}$ is called ($\mu$-almost) *free* if

$$\mu\{x: L_g x = x \text{ for some } g \neq id\} = 0. \tag{1}$$

The transformation $T$ is said to *commute* with $\mathscr{L}$ if

$$L_g T = T L_g \tag{2}$$

for all $g \in G$ and for $\mu$ almost all $x$. If (1) and (2) hold, $G$ is said to be in the commutant of $T$.

Given a transformation $T$ and an action $\mathscr{L}$ of $G$ satisfying (1) and (2), let $(X_0, \mu_0)$ be the Lebesgue space generated by the partition of $(X, \mu)$ into $G$ orbits. Let $T_0$ be the factor transformation induced on $(X_0, \mu_0)$ by $T$. Up to sets of measure 0, we can write $(X, \mu) = (X \times G, \mu_0 \times \gamma)$, where $\gamma$ is Haar measure on $G$. It follows from (1) and (2) that there exists a measurable function $\varphi: X_0 \to G$ so that

$$T(x, y) = (T_0 x, \varphi(x) y) \tag{3}$$

and

$$L_g(x, y) = (x, y g^{-1}). \tag{4}$$

In general, a transformation $T$ obtained from another transformation $T_0$ by (3) is called a *group extension* (or *G extension*) of $T_0$. The function $\varphi$ which appears in (3) is called the *cocycle* of the extension (cf. [K] for an explanation of this terminology). Clearly, every $G$ extension $T$ commutes with the $G$ action $\{L_g\}$. This shows $G$ is in the commutant of $T$ if and only if $T$ is a $G$ extension of some transformation $T_0$.

Now let us consider the following generalization of (3). Let $T_0$ be a transformation of $(X_0, \mu_0)$. Let $K$ be a closed subgroup of $G$. We define

$(X, \mu) = (X_0 \times G/K, \mu_0 \times \gamma_K)$, where $\gamma_K$ denotes Haar measure projected to $K$. Let $\varphi$: $X_0 \to G$ be measurable and define a measure preserving transformation $T$ on $(X, \mu)$ by

$$T(x, yK) = (T_0 x, \varphi(x) yK). \tag{5}$$

Because $T$ acts isometrically on the compact homogeneous "fibers" of the extension (5), it is called an *isometric extension* of $T_0$.

We note: if $T_0$ is ergodic then there always exist cocycles $\varphi$ so that the $G$ extension (3) and isometric extension (5) are ergodic.

## 3. Lower bounds on multiplicity

We begin by recalling some basic facts from the representation theory of compact groups (cf. [M1]). The left action of $G$ on $G/K$ by translation gives rise to a *quasi-regular representation* of $G$ on $L^2(G/K, \gamma_K)$, namely: $W'_g f(yK) = f(gyK)$. When $K$ is the trivial subgroup that is the *regular representation* of G, which we denote by $W_g$. It follows from the Peter–Weyl Theorem (cf. [M1]), that the regular representation $W_g$ decomposes into an orthogonal direct sum of finite dimensional irreducible unitary representations:

$$W_g = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{d_j} W_g^{j,k}, \tag{6}$$

where $W_g^{j,k} = W_g|_{H_{j,k}}$, and

$$L^2(G, \gamma) = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{d_j} H_{j,k}. \tag{7}$$

In addition, $t \leqslant +\infty$ (with $t < +\infty$ if and only if $G$ is finite), $d_j = \dim H_{j,k} \leqslant d_{j+1}$, and $W_g^{j,k}$ is equivalent to $W_g^{j',k'}$ if and only if $j = j'$. Furthermore, every irreducible unitary representation of $G$ appears in this decomposition, its multiplicity being equal to its dimension.

For $K$ nontrivial, by the theory of induced representations (cf. [M1]), the quasi-regular representation $W'_g$ is the representation obtained by inducing the one dimensional identity representation $Id_K$ of $K$ to $G$. Applying Peter–Weyl Theorem again, we obtain a decomposition:

$$W'_g = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{m_j} W_g^{j,k}. \tag{8}$$

From the Frobenius Reciprocity Theorem (cf. [M1]), we have that $0 \leqslant m_j \leqslant d_j = \dim W_g^{j,k} < +\infty$. Thus $W_g'$ is a subrepresentation of $W_g$.

The following proposition is a generalization of a lemma which is well known for compact abelian extension transformations. It is the main technical ingredient for all of the results of this paper.

PROPOSITION 1. *Suppose a transformation $T$ of $(X, \mu)$ is an isometric extension of a transformation $T_0$ of $(X_0, \mu_0)$ by $G/K$, where $K$ is a closed subgroup of a compact metrizable group $G$. Let $t$ and $m_j$ be as in (8). Then there exists a $U_T$ invariant orthogonal decomposition*

$$L^2(X, \mu) = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{m_j} \mathscr{H}_{j,k} \tag{9}$$

*where $U_T|_{\mathscr{H}_{j,k}}$ and $U_T|_{\mathscr{H}_{j,k}}$ are equivalent under conjugation by a unitary isomorphism.*

*Proof.* For each $j$, $k$, let $S_{j,k}: H_{j,k} \to H_{j,1}$ be the unitary intertwining operator for (8) satisfying

$$S_{j,k} \circ W_g^{j,k} = W_g^{1,k} \circ S_{j,k}.$$

For each $j$, we choose an arbitrary orthonormal basis $\{e_{j,1}^1, \ldots, e_{j,1}^{d_j}\}$ for $H_{j,1}$, and define

$$e_{j,k}^r = S_{j,k} e_{j,1}^r. \tag{10}$$

Then

$$\mathscr{B} = \{e_{j,k}^r : j = 1, \ldots, t,\ k = 1, \ldots, m_j,\ r = 1, \ldots, d_j\}$$

forms an orthonormal basis for $L^2(G/K, \gamma_K)$.

We define $\mathscr{H}_{j,k}$ to be the set of all $f \in L^2(X, \mu)$ of the form

$$f(x, y) = \sum_{r=1}^{m_j} b_r(x) e_{j,k}^r\ (y) \tag{11}$$

where $b_r(x) \in L^2(X_0, \mu_0)$ for $r = 1, \ldots, m_j$, and $y \in G/K$. Note that $f \in \mathscr{H}_{j,k}$ if and only if for $\mu_0$ a.e. $x \in X_0$, $f(x, .) \in H_{j,k}$. It follows that the subspaces $\mathscr{H}_{j,k}$ are orthogonal and span $L^2(X, \mu)$.

For $f \in \mathscr{H}_{j,k}$ we have

$$(U_T f)(x, y) = \sum_{r=1}^{m_j} (U_T b_r e^r_{j,k})(x, y)$$

$$= \sum_{r=1}^{m_j} b_r(T_0 x) W_{\varphi(x)} e^r_{j,k}(y),$$

and since $H_{j,k}$ is $W_{\varphi(x)}$ invariant,

$$W_{\varphi(x)} e^r_{j,k} = \sum_{s=1}^{m_j} w_{s,r}(\varphi(x)) e^s_{j,k},$$

where the functions $w_{s,r}(\varphi(x))$ are matrix elements for $W^{j,k}_{\varphi(x)}$. Thus,

$$(U_T f)(x, y) = \sum_{s=1}^{m_j} c_s(x) e^s_{j,k}(y),$$

with

$$c_s(x) = \sum_{r=1}^{m_j} b_r(T_0 x) w_{s,r}(\varphi(x)). \tag{12}$$

The fact that $c_s \in L^2(X_0, \mu_0)$ follows from the fact that the matrix elements $w_{r,s}$ are continuous, and therefore bounded on $G$.

To finish the proof, we define a unitary operator $R$: $\mathscr{H}_{j,k_1} \to \mathscr{H}_{j,k_2}$ by

$$Rf(x, y) = \sum_{r=1}^{m_j} b_r(x) S_{j,k_2} S^{-1}_{j,k_1} e^r_{j,k_1}(y),$$

$$= \sum_{r=1}^{m_j} b_r(x) e^r_{j,k_2}(y). \tag{13}$$

Thus,

$$R \circ U_T f(x, y) = \sum_{r=1}^{m_j} b_r(T_0 x) S_{j,k_2} S^{-1}_{j,k_1} W_{\varphi(x)} e^r_{j,k_1}(y)$$

$$= \sum_{r=1}^{m_j} b_r(T_0 x) W_{\varphi(x)} S_{j,k_2} S^{-1}_{j,k_1} e^r_{j,k_1}(y)$$

$$= U_T \circ Rf(x, y).$$

■

Let us denote by $\mathscr{D}_{G/K}$ the set of multiplicities $m_j$ of the irreducible representations occurring in the quasi-regular representation associated with $G/K$. Let $D_{G/K} = \sup \mathscr{D}_{G/K}$. For the regular representation of $G$ we use the notations $\mathscr{D}_G$ and $D_G$.

THEOREM 1. *Let G be a compact group. Suppose that T is a G extension, or equivalently, that T commutes with a free measure preserving G action. If G is non-abelian then* $M_T \geqslant D_G > 1$. *In particular, T has nonsimple spectrum.*

*Proof.* Since $K = \{id\}$, (9) holds with $m_j = d_j$ for all $j$. Since $G$ is non-abelian, not every irreducible unitary representation is one dimensional, so $d_j \geqslant 1$ for some $j$. Letting $\mathscr{K}_j = \oplus_{k=1}^{d_j} \mathscr{H}_{j,k}$, we note that the spectral multiplicities of $U_T|_{\mathscr{K}_j}$ is a multiple of $d_j$. ■

In the general case, Proposition 1 and the proof of Theorem 1 really gives a bit more information about the multiplicities.

COROLLARY 1. *For an isometric extension T, the essential values of the spectral multiplicity function of* $U_T|_{\mathscr{K}_j}$ *are multiples of* $m_j$.

We conclude this section by describing sufficient conditions for an isometric extension (which is not in general a group extension), to have nonsimple spectrum.

PROPOSITION 2. *Let G be a compact metrizable group and K be a closed subgroup. Suppose there exist closed subgroups* $K_1$ *and* $K_2$, *with* $K \leqslant K_1 \leqslant K_2 \leqslant G$, *where* $K_1$ *is normal in* $K_2$, *and* $K_2/K_1$ *is non-abelian. Then there is at least one irreducible representation of G which appears with multiplicity greater than 1 in the quasi-regular representation associated with* $G/K$.

*Proof.* The quasi-regular representation has a subrepresentation equivalent to the regular representation of $K_2/K_1$. ■

We call an isometric extension by $G/K$ which satisfies the hypotheses of Proposition 2 *properly non-abelian*. Note that since we do not require $K$ to be normal in $K_1$, or $K_2$ to be normal in $G$, this condition is rather weak. We do not know whether it is also a necessary condition for the existence of multiplicity in the quasi-regular representation.

PROPOSITION 3. *If T is a properly non-abelian isometric extension of some transformation* $T_0$, *then* $M_T \geqslant D_{K_1/K_2} > 1$.

*Note*. Any properly non-abelian isometric extension $T$ of $T_0$ has a factor $T_2$ which is itself a $K_2/K_1$ extension of some other isometric extension $T_1$ of $T_2$.

## 4. Upper bounds and the abelian case

In this section we show how additional assumptions on $T_0$ and $G$ can sometimes be used to eliminate the (real) possibility that $M_T = +\infty$. The following lemma lists some well known conditions which imply finite upper bounds for $M_T$.

LEMMA 1. *If $T$ admits a good r-cycle approximation* (*cf.* [R1]), *or has rank* $\leqslant r$ [C], *or may be realized as an interval exchange transformation involving* $\leqslant r + 1$ *intervals* [O], *then* $M_T \leqslant r$.

For convenience, in any of the three cases above (or in any other situation where we have an *a priori* upper bound $r$ on $M_T$), we will denote $r$ by $R_T$. Note that $R_T = 1$ implies that $T$ is ergodic.

A combination of Lemma 1 with the results of Section 3 yields the following.

COROLLARY 2. *Suppose $T$ is a properly non-abelian isometric extension. Then* $R_T \geqslant D_{K_1/K_2} > 1$. *In particular, if $T$ commutes with a free action of a compact metrizable non-abelian group $G$, then* $R_T \geqslant D_G > 1$, (*i.e., $T$ is not rank 1*).

Recently J. King [Ki] has shown that if $T$ is rank 1 then the comutant $C(T)$ of $T$ is abelian. As we noted in the introduction, a non-abelian comutant is already enough to imply nonsimple spectrum. Thus, applying Lemma 1, we obtain another proof of King's result.* Notice that the compactness assumption in Corollary 2 was not needed. An interpretation of Corollary 2 in terms of $C(T)$ is that the rank $R_T$ imposes restrictions on what compact subgroups $C(T)$ can have. As the next result shows, even finite rank (i.e., $R_T < +\infty$) has implications for $C(T)$.

For many infinite compact groups $G$ there are irreducible unitary representations of $G$ of arbitrarily high dimension, i.e., $D_G = +\infty$. We will call such a group *large*. An example of a large compact group is the group $SU(2)$ of complex $2 \times 2$ unitary matrices with determinant 1.

COROLLARY 3. *Suppose $T$ commutes with the action of a large compact group. Then* $R_T = +\infty$. *In particular, $T$ does not have finite rank.*

* We wish to thank the referee for pointing out the connection between our results and King's.

C. Moore [Mo] has shown that a locally compact group $G$ satisfies a finite upper bound on the dimensions of its irreducible representations if and only if $G$ contains an open subgroup of finite index. It follows that a compact metrizable group is large unless it satisfies this condition.

Next, we discuss a simple example of a situation where these results can be applied to construct an interesting class of transformations. Let $T_0$ be an irrational rotation on the circle, viewed as an interval exchange transformation on [0, 1] involving 2 intervals. Let $\alpha$ denote the point of a discontinuity (i.e., $T_0$ has rotation number $2\pi(1 - \alpha)$). Let $G$ be a finite non-abelian group with card $(G) = s$ and $D_G = r > 1$. Let $\varphi$: $[0, 1] \to G$ be a piece-wise continuous function with $t - 2$ discontinuities, all of which occur at rational points in [0, 1]. Also suppose that $\mathrm{Im}(\varphi)$ generates $G$. Let $T$ be the $G$ extension of $T_0$ with cocycle $\varphi$. This situation has been studied by Veech [V], who has shown that if $\alpha$ is poorly enough approximated by rationals (i.e., if it has bounded partial sums in its continued fraction expansion), then $T$ is ergodic. Now clearly $T$ is an interval exchange transformation with $R_T < st$. Thus we have:

Corollary 4. *The transformation T constructed above is an ergodic interval exchange transformation with* $1 < r \leqslant M_T < st < +\infty$.

Corollary 4 provides a recipe for constructing many examples of ergodic interval exchange transformations with nonsimple spectrum of finite multiplicity. Specific examples showing that there exist interval exchange transformations arbitrary finite $M_T$ appear in [R1].

The next theorem concerns the theory of the typical properties of abelian group extensions.

Definition 1. Let $\Phi$ denote the set of measurable functions $\varphi$: $X_0 \to G$, and let $d_G$ be the translation invariant metric on $G$ of diameter 1. The *$L^1$-topology* on $\Phi$ is the topology given by the metric

$$d(\varphi_1, \varphi_2) = \int_{X_0} d_G(\varphi_1(x), \varphi_2(x))\, d\mu_0.$$

A property of $\varphi \in \Phi$ is called *generic* if it holds for all $\varphi$ belonging to a dense $G_\delta$ subset $\Phi_0$ of $\Phi$. A *partition* $\xi$ of $(X_0, \mu_0)$ is a finite collection of disjoint subsets of $(X_0, \mu_0)$ with equal measure and total measure 1. A transformation $T$ preserves $\xi$ if $TE \in \xi$ whenever $E \in \xi$. A sequence $\xi_n$ of partitions is said to be *generating*, denoted $\xi_n \to \varepsilon$, if for any set $E$, there exists a union $E_n$ of elements of $\xi_n$ with $\mu_0(E \,\Delta\, E_n) \to 0$ as $n \to \infty$.

DEFINITION 2 (cf. [K]). A transformation $T_0$ admits a *good cyclic approximation* $(T_{0,n}, \xi_{0,n})$, where $T_{0,n}$ is a transformation preserving the partition $\xi_{0,n}$, if (i) $\xi_{0,n} \to \varepsilon$, and (ii)

$$\lim_{n\to\infty} q_n \sum_{k=0}^{q_n-1} \mu_0(T_0^k E \,\Delta\, T_{0,n}^k E) = 0,$$

where $E \in \xi_{0,n}$, and $q_n = \text{card}\ \xi_{0,n}$.

THEOREM 2. *If G is abelian and $T_0$ admits a good cyclic approximation, then for a generic set of $\varphi \in \Phi$ the corresponding G extension T has simple spectrum, and in particular T is ergodic.*

*Note.* Since the condition that $T_0$ admits a good cyclic approximation is generic in the weak topology on the set of all measure preserving transformations (cf. [K]), an alternative statement would be that the generic compact abelian group extension has simple spectrum. Furthermore, the theorem can be strengthened to say that the generic compact abelian group extension is, in fact, weakly mixing but not mixing.

*Proof.* Since $G$ is abelian, all of its irreducible representations are 1 dimensional characters, each of which occurs with multiplicity 1 in the regular representation. We denote these characters by $x_j(y)$, $y \in G$, $1 \leqslant j \leqslant t \leqslant +\infty$. The decomposition (9) becomes

$$L^2(X, \mu) = \bigoplus_{j=1}^{t} \mathscr{H}_j, \tag{14}$$

where $\mathscr{H}_j = \{x_j f\colon f \in L^2(X_0, \mu_0)\}$.

Now $T$ is approximated by $T_{0,n}$ which cyclically permutes the partition $\xi_{0,n}$ of $(X_0, \mu_0)$. Let $\Phi_n$ denote the set of functions in $\Phi$ which are constant on the elements of $\xi_{0,n}$. For $\varphi \in \Phi$, $\varphi_n \in \Phi_n$, $k \geqslant 0$, we define

$$\varphi'(k, x) = \varphi(T^{k-1}x) \ldots \varphi(Tx)\varphi(x),$$

and

$$\varphi_n'(k, x) = \varphi_n(T_{0,n}^{k-1}x) \ldots \varphi_n(T_{0,n}x)\varphi_n(x). \tag{15}$$

By the cyclicity of $T_{0,n}$, $\varphi_n'(q_n, x) = \varphi_n'' = \text{constant}$. Given any $g \in G$, we can modify $\varphi_n$, within $\Phi_n$, by changing its value on just one element of $\xi_{0,n}$

to make $\varphi_n'' = g$. From the fact that $\xi_{0,n} \to \varepsilon$, it follows that for any $\varphi \in \Phi$, $g \in G, m > 0$, there exists $n$ sufficiently large and $\varphi_n$, depending on $\varphi, g$, and $m$, with $d(\varphi_n, \varphi) < 1/m$ and $\varphi_n'' = g$.

For any pair $0 \leqslant j \leqslant j' \leqslant t$ there exists $g(j, j') \in G$ so that $x_j(g(j, j')) \neq x_{j'}(g(j, j'))$. Let $\delta = \delta(j, j', q_n)$ be such that $|x_j(y) - 1| < 1/q_n^2$ and $|x_{j'}(y) - 1| < 1/q_n^2$ whenever $d_G(y, id) < \delta$.

We denote $N_\theta(\varphi) = \{\bar{\varphi} \in \Phi: d(\varphi, \bar{\varphi}) < \theta\}$ and define

$$\Phi_0 = \bigcap_{0 \leqslant j < j' \leqslant n} \bigcap_{M=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=M}^{\infty} \bigcup_{m=M}^{\infty} \bigcup_{\varphi \in \Phi} N_{\delta(j,j',q_n)/q_n^3}(\varphi_n(\varphi, g(j, j'), m)).$$

Clearly $\Phi_0$ is dense $G_\delta$ in $\Phi$.

Now $\varphi \in \Phi_0$ if and only if for each pair $j, j'$ there exists a sequence $n(k) \to \infty$ as $k \to \infty$ and $\varphi_{n(k)} \in \Phi_{n(k)}$ such that

$$\lim_{k\to\infty} \frac{q_{n(k)}^2}{\delta(j, j', q_{n(k)})} d(\varphi_{n(k)}, \varphi) = 0, \tag{16}$$

and

$$\varphi_{n(k)}'' = g(j, j'). \tag{17}$$

In particular, (16) is already enough to show that for each $j$, $U_T|_{\mathscr{H}_j}$ has simple spectrum. The argument, which follows, is similar to Theorem 5.1 in [KS]. We pass to a subsequence $n(k)$ satisfying (16) (which for convenience we just denote by $n$). Then we fix an element $C_{0,n}$ of $\xi_{0,n}$ and let $C_{r,n} = T_{0,n}^r C_{0,n}$, $r = 0, \ldots, q_n - 1$. Given $0 < \varepsilon < 1$, it follows that for $n$ sufficiently large, there exist subsets $B_{r,n}$ of $C_{r,n}$ with

$$TB_{r,n} = B_{r+1,n}, \quad r = 0, \ldots, q_n - 2, \tag{18}$$

$$\mu_0(C_{r,n} \backslash B_{r,n}) < \varepsilon/q_n, \tag{19}$$

and

$$d_G(\varphi_n'(r, x), \varphi'(r, x)) < \varepsilon/q_n, \tag{20}$$

for all $x \in B_{s,n}$, $r, s = 0, \ldots, q_n - 1$ (cf. e.g., [KS] or [R3]).

Let $h = x_j f \in \mathscr{H}_j$ with $\|h\|_2 = 1$. Since $E_{0,n} \to \varepsilon$, for $n$ sufficiently large, there exists $h' = x_j f' \in \mathscr{H}_j$, with $f'$ constant on each element of $\xi_{0,n}$,

$\|h'\|_2 = 1$, and $\|h - h'\| < \varepsilon$. Let $T_{1,n}$ denote the extension of $T_{0,n}$ obtained by substituting $\varphi_n$ for $\varphi$ in (3). Then

$$h' = \sum_{k=0}^{q_n-1} \alpha_k U_{T_{1,n}}^k x_j 1_{C_{0,n}},$$

where $1_{C_{0,n}}$ is the characteristic function of $C_{0,n}$. Let

$$h'' = \sum_{k=0}^{q_n-1} \alpha_k U_T^k x_j 1_{B_{0,n}}.$$

We have by (18), (19), and (20),

$$\begin{aligned}
&\|h' - h''\|_2 \\
&= \left\| \sum_{k=0}^{q_n-1} \alpha_k (x_j(\varphi_n'(k, .))1_{C_{n,k}} - x_j(\varphi'(k, .))1_{B_{n,k}}) \right\|_2 \\
&= \left[ \sum_{k=0}^{q_n-1} |\alpha_k|^2 \int_{X_0} |x_j(\varphi_n'(k, x))1_{C_{n,k}}(x) - x_j(\varphi'(k, x))1_{B_{n,k}}(x)|^2 \, d\mu_0 \right]^{1/2} \\
&\leqslant \left[ \sum_{k=0}^{q_n-1} |\alpha_k|^2 \left[ \int_{B_{n,k}} |x_j(\varphi_n'(k, x)\varphi'^{-1}(k, x))|^2 \, d\mu_0 + 2\mu_0(C_{n,k} \backslash B_{n,k}) \right] \right]^{1/2} \\
&\leqslant \left[ \sum_{k=0}^{q_n-1} 3|\alpha_k|^2 \varepsilon / q_n \right]^{1/2} = (3\varepsilon)^{1/2} \|h'\|_2 = (3\varepsilon)^{1/2}.
\end{aligned}$$

So $\|h - h''\|_2 < \varepsilon + (3\varepsilon)^{1/2}$, and $h''$ belongs to the cyclic subspace generated by $x_j 1_{B_{0,n}}$. To complete the proof that the spectrum is simple, we assume that it is not and apply [KS] Lemma 3.1 to obtain a contradiction.

To show that $U_T|_{\mathscr{H}_j}$ and $U_T|_{\mathscr{H}_{j'}}$ have mutually singular spectral types, let $h_n = x_j f_n \in \mathscr{H}_j$ and $h_n' = x_{j'} f_n'$, with $f_n$ and $f_n'$ constant on the elements of $\xi_{0,n}$. Choose a subsequence $n(k) \to \infty$, satisfying (16) and (17) (again for convenience denoted by $n$). Then it follows from (3) and the cyclicity of $T_{0,n}$ that

$$U_{T_{1,n}}^{q_n} h_n = \lambda' h_n,$$

and

$$U_{T_{1,n}}^{q_n} h_n' = \lambda' h_n',$$

where $\lambda = x_j(g(j, j')) \neq \lambda' = x_{j'}(g(j, j'))$. As in the proof of [KS] Theorems 3.3 and 3.4, we obtain a further refinement $n(m)$ and disjoint sets $\Gamma^j_{n(m)}$ and $\Gamma^{j'}_{n(m)}$ consisting of small intervals around the $q_{n(m)}$'th roots of $\lambda$ and $\lambda'$ respectively. Letting $\varrho$ and $\varrho'$ denote, correspondingly, measures of maximal spectral type for $U_T$ on $\mathscr{H}_j$ and $\mathscr{H}_{j'}$, we have $\lim_{m\to\infty} \varrho(\Gamma^j_{n(m)}) = 1$ and $\lim_{m\to\infty} \varrho'(\Gamma^{j}_{n(m)}) = 1$. This implies $\varrho \perp \varrho'$.

Clearly, for an extension $T$ of $T_0$ to have simple spectrum it is necessary for $T_0$ to have simple spectrum. We will now show that even in this case there may exist some abelian extension $T$ of $T_0$ with nonsimple spectrum. One (well known) example with this property is the skew shift transformation on the 2-torus $[0, 1]^2$ defined by $T(x, y) = (x + \alpha, x + y) \bmod 1$, where $\alpha$ is irrational. Here $T_0$ is just an irrational rotation on the circle. The spectral multiplicity, which is infinite in this example, comes from a non-abelian commutant on the operator theoretic level rather than from non-commuting point transformations in the commutant of $T$.

Another example more closely related to the theme of this paper is the following. Let $T_0$ be an arbitrary measure preserving transformation of $(X_0, \mu_0)$, let $\mathbb{Z}/n$ be a finite cyclic ring, and let $\beta$ be a unit of $\mathbb{Z}/n$ with multiplicative order $m$. For an arbitrary cocycle $\psi_1: X_0 \to \mathbb{Z}/m$ we construct the $\mathbb{Z}/m$ extension $T_1$ of $T_0$ defined by

$$T_1(x, y) = (T_0x, \varphi(x) + y),$$

(in additive notation). Then we construct the following $\mathbb{Z}/n$ extensions $T$ of $T_1$,

$$T(x, y, z) = (T_0x, \psi_1(x) + y, \theta(y)\psi_2(x) + z), \tag{21}$$

where $\psi_2: X_0 \to \mathbb{Z}/n$ is arbitrary and $\theta(y) = \beta^y$. An easy computation (cf. [R1]) shows that for any $\psi_1$ and $\psi_2$, $T$ has nonsimple spectrum. This fact was first noticed in the case $m = 2$, $n = 3$, $\psi_3 = 1$ by Oseledec [O], who used it to construct the first example of an ergodic transformation with nonsimple spectrum of finite multiplicity. The general case was later employed by the author in [R1], [R2] and [R3] to obtain other examples (cf. below).

Now suppose that $T_0$ admits a good cyclic approximation. Then it is not hard to show that for the generic cocycle $\psi_1$ the transformation $T_1$ also admits a good cyclic approximation (cf. [R1]), and it follows that $T_1$ has simple spectrum. Thus $T$ is an abelian extension of $T_1$ that has nonsimple spectrum. It is fairly easy to explain where the multiplicity comes from in this example. The transformation $T_1$ already has a nontrivial commutant (it contains $\mathbb{Z}/m$), and the extension by $\mathbb{Z}/n$ is special; the cocycle $\varphi$: $X_0 \times \mathbb{Z}/m \to \mathbb{Z}/n$ given by $\varphi(x, y) = \theta(y)\psi_2(x)$ is "anti-symmetric" with respect to $\mathbb{Z}/m$. Thus the commutant for $T_1$ does not commute with the extension to $T$.

A closer look at this example is even more revealing. It turns out that $T$ is really an extension of $T_0$ by a certain non-abelian group, namely the semi-direct product group $G = \mathbb{Z}/m \times_\theta \mathbb{Z}/n$. (Recall that this is the group of pairs $(y, z) \in \mathbb{Z}/m \times \mathbb{Z}/n$ with the multiplication $(y, z)(y', z') = (y + y', \theta(y')z + z')$.) The cocycle $\varphi$: $X_0 \to G$ for the extension (21) is given by $\varphi(x) = (\psi_1(x), \psi_2(x))$. This shows that $T$ is the general $G$ extension of $T_0$.

Now let $0_1, \ldots, 0_k$ denote the orbits of $\theta$ acting on $\mathbb{Z}/n$ by iteration. We claim $\mathscr{D}_G = \{d_j\colon d_j = \text{card } 0_j\}$. Indeed, it follows from Mackey, [M2] section 2.2, Theorems A and B, that corresponding to each orbit $0_j$ there are $m/d_j$ inequivalent irreducible representations, of $G$, each with dimension $d_j$. These exhaust the irreducibles.

In [R3] we show how to construct examples of such groups where $\mathscr{D}_G$ is subject only to the following conditions: (i) $1 \in \mathscr{D}_G$, (ii) $\mathscr{D}_G$ is finite, and (iii) $\mathscr{D}_G$ is closed under the operation of taking least common multiples. Furthermore, for such groups $G$ one can prove generalization of Theorem 2 (cf. Theorem 3), which shows that the generic $G$ extension $T$ has $\mathscr{M}_T = \mathscr{D}_G$. This provides (in [R3], but by a slightly different proof), many new examples of the possibilities for $\mathscr{M}_T$. In particular, this class of transformations includes the examples with arbitrary finite multiplicity constructed in [R1].

We conclude by stating more general conditions for the generic isometric extension by $G/K$ to have $\mathscr{M}_T = \mathscr{D}_{G/K}$. For a linear transformation $M$, $\text{ev}(M)$ will denote its set of eigenvalues.

THEOREM 3. *Suppose $G/K$ has the following properties*:

(i) *For any two irreducible representations $W_g^1$ and $W_g^2$ which occur in the corresponding quasi-regular representation, there exists $g_0 \in G$ such that $\text{ev}(W_{g_0}^1) \cap \text{ev}(W_{g_0}^2) = \varphi$, and*

(ii) *For each irreducible representation $W_g^1$ occurring in the corresponding quasi-regular representation, there exists $g_0 \in G$ so that $W_{g_0}^1$ has only simple eigenvalues.*

*Also, suppose that $T_0$ admits a good cyclic approximation.*

*Then for a generic set of $\varphi \in \Phi$, the $G/K$ isometric extension $T$ constructed from $T_0$ using $\varphi$ has $\mathscr{M}_T = \mathscr{D}_{G/K}$.*

The proof of Theorem 3 is a generalization of the proof of Theorem 2 (cf. also the proof of Lemma 3.5 in [R3]). The idea is that the spectrum is simple in each of the subspaces $\mathcal{H}_{j,k}$ of $L^2(X, \mu)$ (in the notation of Proposition 1), using (ii), and that the spectral types for these subspaces when $j \neq j'$ are pairwise disjoint, using (i).

So far we have been unable to find groups, other than semi-direct products of finite cyclic groups, where conditions (i) and (ii) hold. For example they fail for the alternating group $A_5$. Thus the examples in [R3] remain the only examples with finite $\mathcal{M}_T$ which can be computed.

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**References**

[BFK] M. Brin, J. Feldman and A. Katok, Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents, *Annals of Math.* 113 (1981) 159–179.

[C] R. Chacon, Approximation and spectral multiplicity, *Lect. Notes in Math.* 160 (1970) 18–22.

[Fu] H. Furstenberg, Strict ergodicity and transformations of the torus, *Amer. J. Math.* 83 (1961) 573–601.

[G] G. Goodson, On the spectral multiplicity of a class of finite rank transformations, *Proc. Amer. Math. Soc.* 93 (1985) 303–306.

[J] A. del Junco, A transformation with simple spectrum which is not rank one, *Can. J. Math.* 29 (1977) 655–663.

[K] A. Katok, *Constructions in Ergodic Theory*. Progress in Mathematics, Burkhauser, Boston (to appear).

[KS] A. Katok and A. Stepin, Approximations in ergodic theory, *Uspekhi Mat. Nuak.* 22 (1967); *Russian Math. Surveys* 22 (1967) 77–102.

[Ki] J. King, The comutant in the weak closure of the powers for rank-1 transformations, *Ergod. Th. and Dynam. Sys.* 6 (1986) 363–384.

[M1] G. Mackey, *Unitary Group Representations in Physics, Probability and Number Theory*, Benjamin/Cummings, Reading, MA (1978).

[M2] G. Mackey, *Induced Representations of Groups and Quantum Mechanics*, W.A. Benjamin, New York (1968).

[MwN] J. Mathew and M. Nadkarni, A measure preserving transformation whose spectrum has a Lebesgue component of multiplicity two, *Bull. London Math. Soc.* 16 (1984) 402–406.

[Mo] C. Moore, Groups with finite dimensional irreducible representations, *Trans. Amer. Math. Soc.* 166 (1972) 401–410.

[O] V. Oseledec, The spectrum of ergodic automorphisms, *Dokl. Akad. Nuak SSSR* 168 (1966) 402–406.

[P] W. Parry, Ergodic properties of transformations and flows on nilmanifolds, *Amer. J. Math.* 91 (1969) 757–771.

[R1] A. Robinson, Ergodic measure preserving transformations with arbitrary finite spectral multiplicities, *Inv. Math.* 72 (1983) 229–314.

[R2] A. Robinson, Mixing and spectral multiplicity, *Ergod. Th. and Dynam. Sys.* 5 (1985) 617–624.

[R3] A. Robinson, Transformations with highly nonhomogeneous spectrum of finite multiplicity, *Israel J. Math.* 56 (1986) 75–88.

[R4] A. Robinson, Ergodic properties that lift to compact abelian extensions, *Proc. Amer. Math. Soc.* 101 (1987) (to appear).

[Ru1] D. Rudolph, k-Fold mixing lifts to weakly mixing isometric extensions, *Ergod. Th. and Dynam. Sys.* 5 (1985) 445–447.

[Ru2] D. Rudolph, Classifying the isometric extensions of a Bernoulli shift, *J. D'Analyse. Math.* 34 (1978) 36–60.

[V] W. Veech, Finite group extensions of irrational rotations, *Israel J. Math.* 21 (1975) 240–259.