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Resolutions of unbounded complexes

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Abstract. Various types of resolutions of unbounded complexes of sheaves are constructed, with properties analogous to injectivity, flatness, flabbiness, etc. They are used to remove some boundedness conditions for the existence of the derived functors of functors such as Hom, tensor product, sections over an open subset, inverse and direct images, and for the validity of various formulae involving these derived functors.

This paper is devoted to the study of various classes of complexes of sheaves and the construction of resolutions of arbitrary complexes of sheaves. It aims at removing some boundedness conditions for the existence of some derived functors and the validity of various formulae.

In particular the following results are established.

THEOREM A. *Let X, Y, Z be topological spaces, $\mathcal{O}_X$, $\mathcal{O}_Y$, $\mathcal{O}_Z$ sheaves of commutative rings on X, Y, Z respectively, and $\mathfrak{D}(X)$, $\mathfrak{D}(Y)$, $\mathfrak{D}(Z)$ the corresponding derived categories of complexes of sheaves. Let also $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces. Then the following hold.*

(i) *For every open subset U of X and every family of supports Φ in X the derived functor $\mathrm{R}\Gamma_\Phi(U; -)$ is well-defined. In particular the hypercohomology $\mathbb{H}^{\cdot}_\Phi(U; \mathscr{A}^{\cdot})$ is well-defined. When computing these functors, the sheaves can be treated as sheaves of abelian groups, and it is sufficient to look at their restrictions to the open subset U.*

(ii) *The functors $\mathrm{RHom}^{\cdot}$, $\mathrm{R}\mathscr{H}om^{\cdot}$ and $\otimes^{\mathrm{L}}$ are well-defined, and for every $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot} \in \mathfrak{D}(X)$,*

$$\mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot} \otimes^{\mathrm{L}} \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot}) \cong \mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{B}^{\cdot}, \mathscr{C}^{\cdot})),$$

$$\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot} \otimes^{\mathrm{L}} \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot}) \cong \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{B}^{\cdot}, \mathscr{C}^{\cdot})),$$

$$\mathscr{A}^{\cdot} \otimes^{\mathrm{L}} (\mathscr{B}^{\cdot} \otimes^{\mathrm{L}} \mathscr{C}^{\cdot}) \cong (\mathscr{A}^{\cdot} \otimes^{\mathrm{L}} \mathscr{B}^{\cdot}) \otimes^{\mathrm{L}} \mathscr{C}^{\cdot}.$$

(iii) *The functors* $\mathrm{R}f_*$ *and* $\mathrm{L}f^*$ *are well-defined, they are adjoint, and for* $\mathscr{A}^\cdot \in \mathfrak{D}(Y)$, $\mathscr{B}^\cdot \in \mathfrak{D}(X)$ *we have*

$$\mathrm{R}f_* \mathrm{R}\mathscr{H}om^\cdot(\mathrm{L}f^*\mathscr{A}^\cdot, \mathscr{B}^\cdot) \cong \mathrm{R}\mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathrm{R}f_*\mathscr{B}^\cdot).$$

Moreover $\mathrm{R}(g \circ f)_* = \mathrm{R}g_* \circ \mathrm{R}f_*$ *and* $\mathrm{L}(g \circ f)^* = \mathrm{L}f^* \circ \mathrm{L}g^*$.

A standard result which the author would have liked to extend to unbounded complexes is the proper base change formula. Its usual proof relies however on the notion of soft sheave for which no suitable equivalent has been found for complexes.

Let now the underlying spaces be locally compact. The functors $\mathrm{R}\Gamma_c$ and $\mathrm{R}f_!$, which are shown to be well-defined, can then be expected to have nice properties, but the results contained in this paper are unfortunately not sufficient to establish this in general. Care can be taken of the possible complications due to the structure sheaf, but not of those arising from the topology of the base space. We are thus led to consider the following condition on a locally compact space T.

CONDITION $(*)$. If $\mathscr{A}^\cdot$ is an acyclic complex of c-soft sheaves on T, then in each degree the kernel of the differential is c-soft.

This condition asserts that resolutions by complexes of c-soft sheaves can be used to compute the hypercohomology with compact support of unbounded complexes on T, provided they exist. It is satisfied at least if T is locally finite-dimensional.

THEOREM B. *Let the ringed spaces* X, Y *and* Z *be locally compact, and assume that the morphisms of ringed spaces* $f\colon X \to Y$, $g\colon Y \to Z$ *and* $g \circ f$ *are such that all their fibers satisfy* $(*)$. *Then the following hold.*
(i) *The functor* $\mathrm{R}f_!$ *has an adjoint* $f^!$, *and for* $\mathscr{A}^\cdot \in \mathfrak{D}(X)$, $\mathscr{B}^\cdot \in \mathfrak{D}(Y)$ *we have*

$$\mathrm{R}\mathscr{H}om^\cdot(\mathrm{R}f_!\mathscr{A}^\cdot, \mathscr{B}^\cdot) \cong \mathrm{R}f_*\mathrm{R}\mathscr{H}om^\cdot(\mathscr{A}^\cdot, f^!\mathscr{B}^\cdot).$$

Moreover $\mathrm{R}(g \circ f)_! = \mathrm{R}_{g!} \circ \mathrm{R}f_!$ *and* $(g \circ f)^! = f^! \circ g^!$.
(ii) *For every* $\mathscr{A}^\cdot \in \mathfrak{D}(X)$ *and* $\mathscr{B}^\cdot, \mathscr{C}^\cdot \in \mathfrak{D}(Y)$ *there are natural isomorphisms*

$$\mathrm{R}f_!(\mathscr{A}^\cdot \otimes^{\mathrm{L}}_{\mathscr{O}_X} \mathrm{L}f^*\mathscr{B}^\cdot) \cong \mathrm{R}f_!\mathscr{A}^\cdot \otimes^{\mathrm{L}}_{\mathscr{O}_Y} \mathscr{B}^\cdot,$$

$$f^!\mathrm{R}\mathscr{H}om^\cdot(\mathscr{B}^\cdot, \mathscr{C}^\cdot) \cong \mathrm{R}\mathscr{H}om^\cdot(\mathrm{L}f^*\mathscr{B}^\cdot, f^!\mathscr{C}^\cdot).$$

(iii) *If*

$$\begin{tikzcd} X' \arrow[r, "q'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

is a cartesian diagram of commutative ringed spaces with q flat, then there are natural isomorphisms of functors

$$\mathrm{L}q^* \circ \mathrm{R}f_! \cong \mathrm{R}f'_! \circ \mathrm{L}q'^*,$$

$$q^! \circ \mathrm{R}f_* \cong \mathrm{R}f'_* \circ q'^!.$$

These results are proved in section 6.

The functor $\mathrm{R}f_!$ is constructed as a right derived functor. The apparent imbalance in these formulae with the occurence of one left and three right derived functors associated to a morphism of ringed spaces can be remedied by considering $\mathrm{R}f_!$ as a left derived functor. This is described in 6.16.

More generally, let $\mathfrak{A}$ be an abelian category, $\mathfrak{C}$ the corresponding category of $\mathbb{Z}$-graded complexes with differentials of degree $+1$ and chain maps as morphisms. Let also $\mathfrak{K}$ be the category which has the same objects as $\mathfrak{C}$ but homotopy classes of chain maps as morphisms. We say that a morphism $f\colon A^\cdot \to B^\cdot$ in $\mathfrak{C}$ or $\mathfrak{K}$ is a *quasi-isomorphism* if it induces an isomorphism $H^\cdot(A^\cdot) \to H^\cdot(B^\cdot)$; we say also in this case that f (or $A^\cdot$) is a *left resolution* of $B^\cdot$, or that f (or $B^\cdot$) is a *right resolution* of $A^\cdot$. Notice that if $A^i = 0$ for $i \neq 0$ and $B^i = 0$ for $i < 0$, then f is a right resolution of $A^\cdot$ if and only if the sequence

$$0 \to A^0 \xrightarrow{f_0} B^0 \to B^1 \to B^2 \to \cdots$$

is exact.

Let $F\colon \mathfrak{K}(\mathfrak{A}) \to \mathfrak{K}(\mathfrak{B})$ be a covariant (resp. contravariant) functor. We assume that F is additive, compatible with the shift of degree and preserves the exact triangles. This is the case for example if F is induced by an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Following Deligne [2], the right derived functor of F is defined at a complex $A^\cdot$ if $A^\cdot$ has a right (resp. left) resolution $X^\cdot$ which is unfolded for F. That is, every right (resp. left) resolution $B^\cdot$ of $X^\cdot$ has itself a right (resp. left) resolution $Y^\cdot$ such that F induces a quasi-isomorphism between $F(X^\cdot)$ and $F(Y^\cdot)$ in $\mathfrak{K}(\mathfrak{B})$. Left derived functors are defined in a similar way. There are some sign conventions which must be observed and which become particularly tricky when several variables are involved. All these questions are thoroughly discussed in [2], which we regard as the standard reference for derived categories. The results described above follow from the existence of various classes of resolutions which have properties analogous to injectivity, flatness, flabbiness, etc, and which are unfolded for the various functors under consideration.

Let $f\colon P^{\cdot} \to A^{\cdot}$, $g\colon Q^{\cdot} \to A^{\cdot}$ be two left resolutions of $A^{\cdot}$. Suppose that $P^{\cdot}$ is bounded above (i.e., there exists $N \in \mathbb{N}$ such that $P^i = 0$ for $i < N$) and that each P^i is projective. Then in $\mathfrak{K}$ there exists a unique morphism $\phi\colon P^{\cdot} \to Q^{\cdot}$ such that $f = g \circ \phi$. Thus bounded above left resolutions by complexes of projective objects of $\mathfrak{A}$ are unique in $\mathfrak{K}$ up to a unique isomorphism, when they exist. They may therefore be used to define derived functors. If the boundedness assumption is dropped, this uniqueness property may fail. The standard example [3] is the complex of free $\mathbb{Z}/4\mathbb{Z}$-modules

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots. \tag{1}$$

It is acyclic, hence can be used as a resolution of the complex 0, which consists also of free modules. However (1) is not homotopic to 0 since tensoring (1) by $\mathbb{Z}/2\mathbb{Z}$ we get the complex

$$\cdots \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \cdots \tag{2}$$

which is not acyclic.

Thus for an unbounded complex $A^{\cdot}$, the individual nature of the objects A^i ($i \in \mathbb{Z}$) is not necessarily reflected by properties of $A^{\cdot}$. It seems therefore better to look at properties of $A^{\cdot}$ itself.

Definition. A complex $A^{\cdot}$ (in $\mathfrak{C}$ or $\mathfrak{K}$) is *K-projective* (resp. *K-injective*) if for every acyclic complex $X^{\cdot}$, the complex of abelian groups $\mathrm{Hom}^{\cdot}(A^{\cdot}, X^{\cdot})$ (resp. $\mathrm{Hom}^{\cdot}(X^{\cdot}, A^{\cdot})$) is acyclic.

In other words, $A^{\cdot}$ is K-projective (resp. K-injective) if the functor $\mathrm{Hom}^{\cdot}(A^{\cdot}, -)$ (resp. $\mathrm{Hom}^{\cdot}(-, A^{\cdot})$) preserves exactness.

A *K-projective resolution* of a complex $A^{\cdot}$ is a quasi-isomorphism $P^{\cdot} \to A^{\cdot}$ with $P^{\cdot}$ K-projective. A *K-injective resolution* of $A^{\cdot}$ is a quasi-isomorphism $A^{\cdot} \to I^{\cdot}$ with $I^{\cdot}$ K-injective. We shall prove in particular:

Theorem C. *Let R be a ring (associative, $1 \in R$), and let $\mathfrak{A}$ be the category of all left R-modules. Then every complex in $\mathfrak{K}$ has a K-projective resolution and a K-injective resolution.*

THEOREM D. *Let $\mathcal{O}$ be a sheaf of rings on a topological space X and let $\mathfrak{A}$ be the category of all sheaves of left $\mathcal{O}$-modules on X. Then every complex in $\mathfrak{K}$ has a K-injective resolution.*

We shall show also that complexes of sheaves always have left resolutions which behave well with respect to tensor product.

Some of the results and methods discussed in this paper apply also to complexes of sheaves on etale sites. The existence of K-injective resolutions in this setting is discussed in 4.6.

0. Notation and recollections

We fix some notation and review some miscellaneous facts needed in the sequel.

0.1. In addition to $\mathfrak{C}$ and $\mathfrak{K}$, we shall also consider the derived category $\mathfrak{D}$ of $\mathfrak{A}$ [5, 1.4]. Depending on the needs of the context, we write also $\mathfrak{C}(\mathfrak{A})$, $\mathfrak{K}(\mathfrak{A})$ and $\mathfrak{D}(\mathfrak{A})$ instead of $\mathfrak{C}$, $\mathfrak{K}$, $\mathfrak{D}$ respectively, or also $\mathfrak{C}(X)$, $\mathfrak{K}(X)$, $\mathfrak{D}(X)$ if $\mathfrak{A}$ is the category $\mathfrak{Mod}(X)$ of all sheaves of $\mathcal{O}$-modules on a ringed space $(X, \mathcal{O})$.

Recall that by definition $\mathfrak{C}$, $\mathfrak{K}$ and $\mathfrak{D}$ have the same objects. In particular a complex $A^{\cdot} \in \mathfrak{K}$ is an object of $\mathfrak{C}$.

0.2. Let $A^{\cdot} \in \mathfrak{C}$, d its differential. Then $A^{\cdot}[1]$ is the complex defined by $A[1]^i = A^{i+1}$ $(i \in \mathbb{Z})$, with differential $-d$.

If $f\colon A^{\cdot} \to B^{\cdot}$ is a chain map, the cone of f is the complex $C_f^{\cdot}$ defined by $C_f^i = A^{i+1} \oplus B^i$ with differential given by $d(a, b) = (-d(a), f(a) + d(b))$ $(a \in A^{i+1}, b \in B^i)$.

0.3. The category of all abelian groups is denoted $\mathfrak{Ab}$.

0.4. Let $A^{\cdot}, B^{\cdot} \in \mathfrak{C}$. The complex $\mathrm{Hom}^{\cdot}(A^{\cdot}, B^{\cdot}) \in \mathfrak{C}(\mathfrak{Ab})$ is defined as follows:

$$\mathrm{Hom}^n(A^{\cdot}, B^{\cdot}) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(A^i, B^{i+n}), \tag{1}$$

and its differential is given by

$$df = d_{B^{\cdot}} \circ f - (-1)^n f \circ d_{A^{\cdot}} \quad (f \in \mathrm{Hom}^n(A^{\cdot}, B^{\cdot})). \tag{2}$$

We have then

$$H^i(\mathrm{Hom}^{\cdot}(A^{\cdot}, B^{\cdot})) = \mathrm{Mor}_{\mathfrak{K}}(A^{\cdot}, B^{\cdot}[i]). \tag{3}$$

0.5. A short exact sequence

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$$

in $\mathfrak{C}$ is *semi-split* if it is split in each degree.

0.6. Let $\mathfrak{E} \subset \mathfrak{C}$ be a class of complexes. A left (resp. right) $\mathfrak{E}$-resolution of a complex $A^{\cdot} \in \mathfrak{C}$ is a quasi-isomorphism $E^{\cdot} \to A^{\cdot}$ (resp. $A^{\cdot} \to E^{\cdot}$) with $E^{\cdot} \in \mathfrak{E}$.

0.7. Morphisms in $\mathfrak{A}$ (resp. $\mathfrak{C}$) will always be called homomorphisms (resp. chain maps).

0.8. Direct and inverse systems are always assumed to be filtered.

0.9. Unless otherwise stated, modules are left modules. This applies also to $\mathcal{O}$-modules, where $\mathcal{O}$ is a sheaf of rings. In paragraphs 5 and 6 all ringed spaces are assumed to be commutative, so that this specification is irrelevant there.

If $\mathfrak{A}$ is the category of all sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$, $\mathscr{A} \in \mathfrak{A}$ and $Z \subset X$ is locally closed, we let $\mathscr{A}_{Z \subset X}$ be the extension by zero of $\mathscr{A}|_Z$ to X. We use a similar notation for complexes of $\mathcal{O}_X$-modules. In case $\mathscr{A} = \mathcal{O}_X$, we write $\mathcal{O}_{Z \subset X}$ instead of $(\mathcal{O}_X)_{Z \subset X}$.

0.10. Let $A^{\cdot} \in \mathfrak{C}$. The following criterion will be useful to check in some cases that $A^{\cdot}$ is acyclic.

Let $\mathfrak{E}$ be a class of objects of $\mathfrak{A}$ such that every object of $\mathfrak{A}$ can be embedded in some object of $\mathfrak{E}$. Assume that $\operatorname{Hom}^{\cdot}(A^{\cdot}, E) \in \mathfrak{C}(\mathfrak{Ab})$ *is acyclic for every $E \in \mathfrak{E}$. Then $A^{\cdot}$ is acyclic.*

0.11. In order to handle certain inverse limits in $\mathfrak{Ab}$, we shall need the following variant of the Mittag-Leffler criterion.

Let I be a well ordered set. We say that an inverse system $(M_i)_{i \in I}$ in $\mathfrak{Ab}$ satisfies $(*)$ if the following hold:

$(*_1)$ If $i \in I$ has no predecessor, then $M_i = \varprojlim_{j<i} M_j$.

$(*_2)$ If $i \in I$ has a predecessor $i-1$, then the homomorphism $M_i \to M_{i-1}$ is surjective.

LEMMA. *Let I be a well ordered set and let the inverse systems $(A_i)_{i \in I}$, $(B_i)_{i \in I}$, $(C_i)_{i \in I}$ and $(D_i)_{i \in I}$ in* $\mathfrak{Ab}$ *satisfy $(*)$. Let*

$$(A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} D_i)_{i \in I} \tag{1}$$

be morphisms of inverse systems, with $g_i \circ f_i = 0$ *and* $h_i \circ g_i = 0$ *for every* $i \in I$, *and let*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{2}$$

be the limit of (1). *If* $i \in I$ *has a predecessor* $i - 1$, *let* A'_i, B'_i, C'_i *and* D'_i *be the respective kernels of the homomorphisms* $A_i \to A_{i-1}$, $B_i \to B_{i-1}$, $C_i \to C_{i-1}$ *and* $D_i \to D_{i-1}$. *Let* $j \in I$ *have the following property. For every* $i > j$ *which has a predecessor* $i - 1$, *the sequence*

$$A'_i \to B'_i \to C'_i \to D'_i \tag{3}$$

is exact. Then the natural homomorphism

$$(\operatorname{Ker} g/\operatorname{Im} f) \to (\operatorname{Ker} g_j/\operatorname{Im} f_j) \tag{4}$$

is an isomorphism.

This follows from Zorn's lemma and diagram chasing.

REMARK. If in the lemma we can take for j the smallest element of I, then the conclusion of the lemma is that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{5}$$

is exact.

1. Elementary properties of K-projective and K-injective complexes

1.1. Recall first

DEFINITION. A complex $A^\cdot$ (in $\mathfrak{C}$ or $\mathfrak{K}$) is *K-projective* (resp. *K-injective*) if for every acyclic complex $S^\cdot \in \mathfrak{C}$, the complex $\operatorname{Hom}^\cdot(A^\cdot, S^\cdot)$ (resp. $\operatorname{Hom}^\cdot(S^\cdot, A^\cdot)$) is acyclic.

Notice that if an acyclic complex $A^\cdot$ is K-projective or K-injective, then it is contractible (consider $\operatorname{Id}_{A^\cdot} \in \operatorname{Hom}^0(A^\cdot, A^\cdot)$). We shall see in 1.4 (resp. 1.5) that the K-projective (resp. K-injective) complexes are precisely those which in the terminology of [6, I.2.5.4] are "free on the left" (resp. "free on the right"). The definition above, which is due to J. Bernstein, is more convenient for the applications in this paper.

1.2. PROPOSITION. *Let* $A^\cdot \in \mathfrak{C}$ *be such that* $A^i = 0$ *for* $i \neq 0$. *Then* $A^\cdot$ *is K-projective (resp. K-injective) if and only if* A^0 *is a projective (resp. injective) object of* $\mathfrak{A}$.

This is clear.

1.3. PROPOSITION. (i) *If two of the vertices of a distinguished triangle of* $\mathfrak{K}$ *are K-projective (resp. K-injective), then so is the third one.*

(ii) $A^\cdot \in \mathfrak{K}$ *is K-projective (resp. K-injective) if and only if* $A^\cdot[1]$ *is so.*

This is clear.

1.4. PROPOSITION. *For every* $A^\cdot \in \mathfrak{C}$ *the following conditions are equivalent.*

(a) $A^\cdot$ *is K-projective.*

(b) *For every* $S^\cdot \in \mathfrak{K}$, *the natural homomorphism*

$$\mathrm{Mor}_{\mathfrak{K}}(A^\cdot, S^\cdot) \to \mathrm{Mor}_{\mathfrak{D}}(A^\cdot, S^\cdot) \tag{1}$$

is an isomorphism.

(c) *For every diagram in* $\mathfrak{K}$

$$\begin{tikzcd} & X^\cdot \arrow[d, "s"] \\ A^\cdot \arrow[r, "f"] & Y^\cdot \end{tikzcd} \tag{2}$$

with s *a quasi-isomorphism, there exists a unique morphism* $g\colon A^\cdot \to X^\cdot$ *such that* $s \circ g = f$ *in* $\mathfrak{K}$.

(d) *For every quasi-isomorphism* $u\colon S^\cdot \to A^\cdot$ *in* $\mathfrak{K}$, *there exists a morphism* $v\colon A^\cdot \to S^\cdot$ *such that* $u \circ v = 1_{A^\cdot}$ *in* $\mathfrak{K}$.

The equivalence of (a) and (b) follows from 0.4(3), the definition of morphisms in $\mathfrak{D}$ and a simple cone argument. We have (b) $\Rightarrow$ (c) because s gives an isomorphism in $\mathfrak{D}$, and (c) $\Rightarrow$ (d) is obtained by taking $Y^\cdot = A^\cdot, f = 1_{A^\cdot}$ in (c). It remains to check (d) $\Rightarrow$ (b).

A morphism from $A^\cdot$ to $S^\cdot$ in $\mathfrak{D}$ is represented by a diagram in $\mathfrak{K}$

$$A^\cdot \xrightarrow{s} B^\cdot \xrightarrow{f} S^\cdot \tag{3}$$

with s a quasi-isomorphism. By (d) there exists a morphism $t\colon A^\cdot \to B^\cdot$ in $\mathfrak{K}$ such that $s \circ t = 1_{A^\cdot}$, and $f \circ t$ represents the same morphism in $\mathfrak{D}$ as (3). Thus (1) is surjective. Let now $g \in \mathrm{Mor}_{\mathfrak{K}}(A^\cdot, S^\cdot)$ map to $0 \in \mathrm{Mor}_{\mathfrak{D}}(A^\cdot, S^\cdot)$. Then there exists a quasi-isomorphism $u\colon C^\cdot \to A^\cdot$ in $\mathfrak{K}$ such that $g \circ u = 0$. By (d), there exists then $v \in \mathrm{Mor}_{\mathfrak{K}}(A^\cdot, C^\cdot)$ such that $u \circ v = 1_{A^\cdot}$. Then $g = g \circ 1_{A^\cdot} = g \circ u \circ v = 0$. Thus (1) is also injective.

REMARK. The conditions (b), (c) and (d) make sense in the more general setting of localization of categories, c.f. [6, I.2.5].

Similarly, we have:

1.5. PROPOSITION. *For every $A^\cdot \in \mathfrak{C}$ the following conditions are equivalent*:
(a) *$A^\cdot$ is K-injective.*
(b) *For every $S^\cdot \in \mathfrak{K}$, the natural homomorphism*

$$\mathrm{Mor}_{\mathfrak{K}}(S^\cdot, A^\cdot) \to \mathrm{Mor}_{\mathfrak{D}}(S^\cdot, A^\cdot) \tag{1}$$

is an isomorphism.
(c) *For every diagram in $\mathfrak{K}$*

$$\begin{array}{ccc} Y^\cdot & \xrightarrow{f} & A^\cdot \\ {\scriptstyle s}\downarrow & & \\ X^\cdot & & \end{array}$$

with s a quasi-isomorphism, there exists a unique morphism $g\colon X^\cdot \to A^\cdot$ such that $g \circ s = f$ in $\mathfrak{K}$.
(d) *For every quasi-isomorphism $u\colon A^\cdot \to S^\cdot$ in $\mathfrak{K}$, there exists a morphism v: $S^\cdot \to A^\cdot$ such that $v \circ u = 1_{A^\cdot}$ in $\mathfrak{K}$.*

2. Special inverse or direct systems

2.1. DEFINITIONS. Let $\mathfrak{J} \subset \mathfrak{C}$ be a class of complexes.
(a) An inverse system $(I_n^\cdot)_{n\in E}$ in $\mathfrak{C}$ is a $\mathfrak{J}$-*special inverse system* if it satisfies the following conditions.
 (i) E is well ordered.
 (ii) If $n \in E$ has no predecessor, then $I_n^\cdot = \varprojlim_{m<n} I_m^\cdot$.
 (iii) If $n \in E$ has a predecessor $n-1$, then the natural chain map $I_n^\cdot \to I_{n-1}^\cdot$ is surjective, its kernel $C_n^\cdot$ belongs to $\mathfrak{J}$, and the short exact sequence

$$0 \to C_n^\cdot \to I_n^\cdot \to I_{n-1}^\cdot \to 0$$

is semi-split.

(b) The class $\mathfrak{J}$ is *closed under special inverse limits* if every $\mathfrak{J}$-special inverse system in $\mathfrak{C}$ has a limit which is contained in $\mathfrak{J}$, and every complex isomorphic in $\mathfrak{C}$ to a complex in $\mathfrak{J}$ is contained in $\mathfrak{J}$.

2.2. EXAMPLES. (a) Suppose that $\mathfrak{J}$ is closed under special inverse limits and that $A^\cdot \in \mathfrak{J} \Leftrightarrow A^\cdot[1] \in \mathfrak{J}$. Then if $u: A^\cdot \to B^\cdot$ is a chain map and $A^\cdot, B^\cdot \in \mathfrak{J}$, the cone $C_u^\cdot$ is also contained in $\mathfrak{J}$. To see this, use the inverse system indexed by $\{0, 1, 2\}$ with $A_0^\cdot = 0$, $A_1^\cdot = A^\cdot[1]$, $A_2^\cdot = C_u^\cdot = \varprojlim A_i^\cdot$.
(b) Using Zorn's lemma, every direct product in $\mathfrak{C}$ can be turned into a special inverse system.
(c) Let $\mathfrak{J}_0$ be a class of objects of $\mathfrak{A}$. Assume that the class $\mathfrak{J}$ of objects of $\mathfrak{C}$ is closed under special inverse limits, and that every single degree complex $A^\cdot \in \mathfrak{C}$ such that $A^i \in \mathfrak{J}_0$ for every $i \in \mathbb{Z}$ is contained in $\mathfrak{J}$. Then every bounded below complex $A^\cdot \in \mathfrak{C}$ such that $A^i \in \mathfrak{J}_0$ for every $i \in \mathbb{Z}$ is contained in $\mathfrak{J}$.

2.3. LEMMA. *The class of all acyclic complexes in* $\mathfrak{C}(\mathfrak{Ab})$ *is closed under special inverse limits.*

This follows from 0.11.

2.4. PROPOSITION. *Let* $\mathfrak{B}$ *be an abelian category and let* $\mathfrak{J} \subset \mathfrak{C}(\mathfrak{B})$ *be closed under special inverse limits. Assume that inverse limits exist in* $\mathfrak{A}$ *and let* F: $\mathfrak{C}(\mathfrak{A}) \to \mathfrak{C}(\mathfrak{B})$ *be a covariant functor which commutes with inverse limits and preserves semi-split short exact sequences. Then* $F^{-1}(\mathfrak{J}) \subset \mathfrak{C}(\mathfrak{A})$ *is closed under special inverse limits.*

This follows immediately from the definitions.

2.5. COROLLARY. *Let* $\mathfrak{T} \subset \mathfrak{C}$ *be a class of complexes. Then the class of all complexes* $A^\cdot \in \mathfrak{C}$ *such that* $\operatorname{Hom}^\cdot(T^\cdot, A^\cdot)$ *is acyclic for every* $T^\cdot \in \mathfrak{T}$ *is closed under special inverse limits. In particular, the class of all K-injective complexes is closed under special inverse limits.*

This follows from 2.3 and the proposition applied to the functors $\operatorname{Hom}^\cdot(T^\cdot, -)$ $(T^\cdot \in \mathfrak{T})$.

2.6. DEFINITIONS. Let $\mathfrak{P} \subset \mathfrak{C}$ be a class of complexes.
(a) A direct system $(P_n^\cdot)_{n\in E}$ in $\mathfrak{C}$ is a $\mathfrak{P}$-*special direct system* if it satisfies the following conditions:
(i) E is well ordered.

(ii) If $n \in E$ has no predecessor, then $P_n^{\cdot} = \varinjlim_{m<n} P_m^{\cdot}$.

(iii) If $n \in E$ has a predecessor $n-1$, then the natural chain map $P_{n-1}^{\cdot} \to P_n^{\cdot}$ is injective, its cokernel $C_n^{\cdot}$ belongs to $\mathfrak{P}$, and the short exact sequence

$$0 \to P_{n-1}^{\cdot} \to P_n^{\cdot} \to C_n^{\cdot} \to 0$$

is semi-split.

(b) $\mathfrak{P}$ is *closed under special direct limits* if every $\mathfrak{P}$-special direct system in $\mathfrak{C}$ has a limit which is contained in $\mathfrak{P}$, and every complex isomorphic in $\mathfrak{C}$ to a complex of $\mathfrak{P}$ is contained in $\mathfrak{P}$.

It is clear that the examples in 2.2 can be dualized. It is clear also that the class of all acyclic complexes in $\mathfrak{C}(\mathfrak{Ab})$ is closed under special direct limits. We have also the following analogues of 2.4, 2.5:

2.7. Proposition. *Let $\mathfrak{B}$ be an abelian category and let $\mathfrak{J} \subset \mathfrak{C}(\mathfrak{B})$ be closed under special inverse limits. Assume that direct limits exist in $\mathfrak{A}$ and let F: $\mathfrak{C}(\mathfrak{A}) \to \mathfrak{C}(\mathfrak{B})$ be a contravariant functor which transforms direct limits into inverse limits and preserves semi-split short exact sequences. Then $F^{-1}(\mathfrak{J}) \subset \mathfrak{C}(\mathfrak{A})$ is closed under special direct limits.*

2.8. Corollary. *Let $\mathfrak{T} \subset \mathfrak{C}$ be a class of complexes. Then the class of all complexes $A^{\cdot} \in \mathfrak{C}$ such that* $\mathrm{Hom}^{\cdot}(A^{\cdot}, T^{\cdot})$ *is acyclic for every $T^{\cdot} \in \mathfrak{T}$ is closed under special direct limits. In particular, the class of all K-projective complexes is closed under special direct limits.*

This follows from 2.3. and 2.7.

2.9. Notation. Assume that inverse (resp. direct) limits exist in $\mathfrak{C}$ and let $\mathfrak{E}$ be a class of complexes in $\mathfrak{C}$. We let $\underleftarrow{\mathfrak{E}}$ (resp. $\underrightarrow{\mathfrak{E}}$) be the smallest class of complexes in $\mathfrak{C}$ which is closed under special inverse (resp. direct) limits and contains $\mathfrak{E}$.

2.10. There is an analogue of 2.4 for contravariant functors, and of 2.7 for covariant functors, in which the class $\mathfrak{J} \subset \mathfrak{C}(\mathfrak{B})$ is assumed to be closed under special direct limits.

3. Existence of K-projective or K-injective resolutions

A. Left resolutions

3.1. Let $\mathfrak{P}$ be a class of complexes in $\mathfrak{C}$. We shall assume in this section that $\mathfrak{P}$ has the following property.

(1) *Every bounded above complex* $A^{\cdot} \in \mathfrak{C}$ *has a left resolution* $P^{\cdot} \to A^{\cdot}$ *with* $P^{\cdot} \in \mathfrak{P}$

Equivalently, for every complex $A^{\cdot} \in \mathfrak{C}$ and every integer $n \in \mathbb{Z}$, there exist $P^{\cdot} \in \mathfrak{P}$ with $H^j(P^{\cdot}) = 0$ for $j > n$ and a chain map $f\colon P^{\cdot} \to A^{\cdot}$ which induces an isomorphism $H^j(P^{\cdot}) \to H^j(A^{\cdot})$ for every $j \leqslant n$.

3.2. EXAMPLES. (a) If $\mathfrak{A}$ has enough projectives, we can take for $\mathfrak{P}$ the class of all bounded above complexes $P^{\cdot} \in \mathfrak{C}$ with P^i projective for every $i \in \mathbb{Z}$. It is well-known that $\mathfrak{P}$ consists in this case of K-projective complexes (this follows also from 1.2, the dual of 2.2(c) and 2.8). By 2.8, the complexes in $\underrightarrow{\mathfrak{P}}$ are then also K-projective.

(b) Let $\mathfrak{A}$ be the category of all sheaves of $\mathcal{O}$-modules on a ringed space $(X, \mathcal{O})$. Let $\mathfrak{P}$ be the class of all complexes $\mathscr{P}^{\cdot} \in \mathfrak{C}$ which are bounded above and such that each $\mathscr{P}^i$ is a direct sum of sheaves of the form $\mathcal{O}_{U\subset X}$ with U open in X. Then $\mathfrak{P}$ satisfies 3.1(1). More generally, for each $i \in \mathbb{Z}$, let $\mathfrak{U}^i$ be a basis of the topology of X, and let $\mathfrak{P}(\mathfrak{U}^{\cdot})$ be the class of all bounded above complexes $\mathscr{P}^{\cdot} \in \mathfrak{P}$ such that each $\mathscr{P}^i$ is a direct sum of sheaves of the form $\mathcal{O}_{U\subset X}$ with $U \in \mathfrak{U}^i$. Then $\mathfrak{P}(\mathfrak{U}^{\cdot})$ satisfies 3.1(1).

3.3. LEMMA. *Let* $A^{\cdot} \in \mathfrak{C}$. *Then under the assumption* 3.1(1) *there exists a* $\mathfrak{P}$-*special direct system* $(P_n^{\cdot})_{n\geqslant -1}$ *and a direct system of chain maps* $f_n\colon P_n^{\cdot} \to \tau_{\leqslant n}A^{\cdot}$ *such that* f_n *is a quasi-isomorphism for every* $n \geqslant 0$.

We construct $(P_n^{\cdot})_{n\geqslant -1}$ and $(f_n)_{n\geqslant -1}$ by induction. As -1 has no predecessor in our indexing set, we must take $P_{-1}^{\cdot} = 0$, $f_{-1} = 0$, and by 3.1(1) we can find a quasi-isomorphism $f_0\colon P_0^{\cdot} \to \tau_{\leqslant 0}A^{\cdot}$ with $P_0^{\cdot} \in \mathfrak{P}$. Let now $n \geqslant 1$, and suppose that $P_{-1}^{\cdot}, \ldots, P_{n-1}^{\cdot}$ and $f_{-1}, \ldots, f_{n-1}$ are already constructed. Let $P^{\cdot} = P_{n-1}^{\cdot}$, $B^{\cdot} = \tau_{\leqslant n}A^{\cdot}$ and $f\colon P^{\cdot} \to B^{\cdot}$ the chain map induced by f_{n-1}. By 3.1(1) we can find a quasi-isomorphism, $g\colon Q^{\cdot} \to C_f^{\cdot}[-1]$ with $Q^{\cdot}[1] \in \mathfrak{P}$.

As $C_f^{\cdot}[-1] = P^{\cdot} \oplus B^{\cdot}[-1]$, g gives two maps $g'\colon Q^{\cdot} \to P^{\cdot}$ and $g''\colon Q^{\cdot} \to B^{\cdot}[-1]$, and g' is a chain map. Let then $h\colon C_{-g'}^{\cdot} = Q^{\cdot}[1] \oplus P^{\cdot} \to B^{\cdot}$ be defined by $h(x, y) = g''[1](x) + f(y)$. It is easily checked that h is a chain map and that $C_h^{\cdot} = C_g^{\cdot}[1]$. Since g is a quasi-isomorphism, so is therefore h, and we may take $P_n^{\cdot} = C_{-g'}^{\cdot}$, $f_n = h$.

3.4. THEOREM. *Assume that direct limits exist in* $\mathfrak{A}$ *and that* $\varinjlim$ *is exact. Let the class* $\mathfrak{P} \subset \mathfrak{C}$ *satisfy* 3.1(1). *Then every complex in* $\mathfrak{C}$ *has a left* $\underrightarrow{\mathfrak{P}}$-*resolution.*

This follows immediately from 3.3.

3.5. COROLLARY. *Assume that direct limits exist in $\mathfrak{A}$ and that $\varinjlim$ is exact. Assume moreover that $\mathfrak{A}$ has enough projectives. Then every complex in $\mathfrak{C}$ has a left K-projective resolution.*

Indeed, taking $\mathfrak{P}$ as in 3.2(a), every complex in $\underrightarrow{\mathfrak{P}}$ is K-projective.

B. Right resolutions

3.6. Let $\mathfrak{J} \subset \mathfrak{C}$ be a class of complexes. Assume that $\mathfrak{J}$ has the following property

(1) *Every bounded below complex $A^{\cdot} \in \mathfrak{C}$ has a right resolution $A^{\cdot} \to I^{\cdot}$* with $I^{\cdot} \in \mathfrak{J}$.

Notice that this holds for example if $\mathfrak{J}$ is the class of all bounded below complexes of injective objects of $\mathfrak{A}$, in case $\mathfrak{A}$ has enough injectives.

Dualizing the arguments used in part A, we get immediately:

3.7. LEMMA. *Let $A^{\cdot} \in \mathfrak{C}$. Then under the assumption* 3.6(1) *there exist a $\mathfrak{J}$-special inverse system $(I_n^{\cdot})_{n \geqslant -1}$ and an inverse system of chain maps f_n: $\tau^{\geqslant -n} A^{\cdot} \to I_n^{\cdot}$ such that f_n is a quasi-isomorphism for every $n \geqslant 0$.*

3.8. PROPOSITION. *Assume that inverse limits exist in $\mathfrak{A}$ and that $\varprojlim$ is exact. Let the class $\mathfrak{J} \subset \mathfrak{C}$ satisfy* 3.6(1). *Then every complex in $\mathfrak{C}$ has a right $\underleftarrow{\mathfrak{J}}$-resolution.*

3.9. COROLLARY. *Assume that inverse limits exist in $\mathfrak{A}$, that $\varprojlim$ is exact and that $\mathfrak{A}$ has enough injectives. Then every complex in $\mathfrak{C}$ has a right K-injective resolution.*

3.10. This unfortunately does not even apply to modules over a ring R, since in this case $\varprojlim$ is not exact. The assumption on $\varprojlim$ in 3.9 is however used only to ensure that in the situation of 3.7 the chain map $f = \varprojlim f_n$: $A^{\cdot} \to \varprojlim I_n^{\cdot}$ is a quasi-isomorphism. As the inverse system $(I_n^{\cdot})_{n \geqslant -1}$ has a very special form, a much weaker assumption is already sufficient. We get in particular:

3.11. PROPOSITION. *Let R be a ring and let $\mathfrak{A}$ be the category of all left R-modules. Then every complex in $\mathfrak{C}$ has a right K-injective resolution.*

This follows from 3.7, 2.5 and 0.11.

REMARK. Together with 3.5, this proves theorem C.

3.12. Let X be a topological space, $\mathcal{O}$ a sheaf of rings on X (not necessarily commutative) and $\mathfrak{A} = \mathfrak{Mod}(X)$ the category of all sheaves of left $\mathcal{O}$-modules on X. Let $\mathfrak{B} \subset \mathfrak{Mod}(X)$ be a class of sheaves. We consider the following condition on $\mathfrak{B}$.

(1) For every $x \in X$, there exist a fundamental system $\mathfrak{U}_x$ of open neighborhoods of x and an integer d_x such that $H^i(U; \mathscr{B}) = 0$ for every $\mathscr{B} \in \mathfrak{B}$, $U \in \mathfrak{U}_x$, $i > d_x$.

EXAMPLES. (a) Let $(X, \mathcal{O})$ be a scheme and let $\mathfrak{B}$ be the class of all quasi-coherent $\mathcal{O}$-modules on X. Then (1) is satisfied.

(b) Assume that every point of X has a fundamental system of contractible open neighborhoods and let the class $\mathfrak{B}$ consist of constant sheaves on X. Then (1) is satisfied.

3.13. PROPOSITION. *Assume that* $\mathfrak{B}$ *satisfies* 3.12(1) *and let* $\mathscr{A}^{\cdot} \in \mathfrak{C}$ *be such that* $\mathscr{H}^i(\mathscr{A}^{\cdot}) \in \mathfrak{B}$ *for every* $i \in \mathbb{Z}$. *Let* $\mathfrak{J}$ *be the class of all bounded below complexes of injective* $\mathcal{O}$*-modules. Then the chain map* $f = \varprojlim f_n$ *given by 3.7 is a quasi-isomorphism. In particular* $\mathscr{A}^{\cdot}$ *has a right* $\mathfrak{J}$*-resolution, hence also a right K-injective resolution.*

We use the notation of 3.7. Let $\mathscr{I}^{\cdot} = \varprojlim \mathscr{I}_n^{\cdot}$. We have a commutative diagram

$$\begin{tikzcd} & \mathscr{I}^{\cdot} \arrow[dr] & \\ \mathscr{A}^{\cdot} \arrow[ur, "f"] \arrow[rr, "f_n"'] & & \mathscr{I}_n^{\cdot} \end{tikzcd} \qquad . \tag{1}$$

Let $m \in \mathbb{Z}$. For $n \geqslant -m$, $n \in \mathbb{N}$, f_n induces an isomorphism

$$\mathscr{H}^m(\mathscr{A}^{\cdot}) \cong \mathscr{H}^m(\mathscr{I}_n^{\cdot}). \tag{2}$$

In view of (1), the homomorphism

$$\mathscr{H}^m(\mathscr{A}^{\cdot}) \to \mathscr{H}^m(\mathscr{I}^{\cdot}) \tag{3}$$

induced by f is therefore injective.

For surjectivity, let $x \in X$, $V \subset X$ an open neighborhood of x and $\gamma \in \Gamma(V; \mathscr{I}^m)$ satisfy $d\gamma = 0$. By definition of $\mathscr{I}^{\cdot}$, $\gamma = (\gamma_n)_{n \geqslant -1}$ with $\gamma_n \in \Gamma(U; \mathscr{I}_n^m)$ and $d\gamma_n = 0$. Let $\mathfrak{U}_x$, $d_x \geqslant 0$ be as in 3.12(1), and choose $N \in \mathbb{N}$ such that $N > d_x - m$.

For $n \in \mathbb{N}$, let $\mathscr{C}_n^{\cdot}$ be the kernel of $\mathscr{I}_n^{\cdot} \to \mathscr{I}_{n-1}^{\cdot}$. For $n > N$ and $U \in \mathfrak{U}_x$, the sequence

$$\Gamma(U; \mathscr{C}_n^{m-1}) \to \Gamma(U; \mathscr{C}_n^{m}) \to \Gamma(U; \mathscr{C}_n^{m+1}) \to \Gamma(U; \mathscr{C}_n^{m+2}) \tag{4}$$

is then exact. It follows therefore from 0.11 that the homomorphism

$$H^m(\Gamma(U; \mathscr{I}^{\cdot})) \to H^m(\Gamma(U; \mathscr{I}_N^{\cdot})) \tag{5}$$

is an isomorphism.

In view of (2), there exists $U \in \mathfrak{U}_x$, $U \subset V$, such that the image of $\gamma_N|_U$ in $H^m(\Gamma(U; \mathscr{I}_N^{\cdot}))$ is contained in the image of $H^m(\Gamma(U; \mathscr{A}^{\cdot}))$. By the isomorphism (5), the image of $\gamma|_U$ in $H^m(\Gamma(U; \mathscr{I}^{\cdot}))$ is therefore also contained in the image of $H^m(\Gamma(U; \mathscr{A}^{\cdot}))$. This implies that (3) is surjective.

4. K-injective resolutions of complexes of sheaves

In this paragraph X is a topological space, $\mathscr{O}$ is a sheaf of rings (not necessarily commutative) on X, and $\mathfrak{A}$ is the category $\mathfrak{Mod}(X)$ of all sheaves of left $\mathscr{O}$-modules on X.

4.1. LEMMA. *Let $\mathfrak{T} = (\mathscr{T}_e^{\cdot})_{e \in E}$ be a family of acyclic complexes, and let $\mathscr{A}^{\cdot} \in \mathfrak{C}$. Then $\mathscr{A}^{\cdot}$ has a right resolution s: $\mathscr{A}^{\cdot} \to \mathscr{B}^{\cdot}$ such that for every $e \in E$ and every chain map ω: $\mathscr{T}_e^{\cdot} \to \mathscr{A}^{\cdot}$, the chain map $s \circ \omega$ is homotopic to* 0.

For each chain map ω: $\mathscr{T}_e^{\cdot} \to \mathscr{A}^{\cdot}$, there exists a quasi-isomorphism s_ω: $\mathscr{A}^{\cdot} \to \mathscr{A}_\omega^{\cdot}$ such that $s_\omega \circ \omega$ is homotopic to 0.

Let

$$\Omega = \coprod_{e \in E} \mathrm{Mor}_{\mathfrak{C}}(\mathscr{T}_e^{\cdot}, \mathscr{A}^{\cdot}).$$

We may assume that Ω is well ordered. Let Δ be the set of all initial intervals of Ω, and let θ: $\Omega \to \Delta$ be defined by $\theta(\omega) = \{\omega' \in \Omega \mid \omega' \leqslant \omega\}$.

We define now by induction a direct system of complexes $(\mathscr{B}_\delta^{\cdot})_{\delta \in \Delta}$ in $\mathfrak{C}$. We set $\mathscr{B}_\varnothing^{\cdot} = \mathscr{A}^{\cdot}$.

If $\delta \in \Delta$ has no predecessor and $\delta \neq \varnothing$, define

$$\mathscr{B}_\delta^{\cdot} = \varinjlim_{\gamma < \delta} \mathscr{B}_\gamma^{\cdot}.$$

If $\delta \in \Delta$ has a predecessor $\delta - 1$, then $\delta = \theta(\omega)$ for some chain map ω: $\mathscr{T}_e^\cdot \to \mathscr{A}^\cdot$. As $\mathscr{A}^\cdot = B_\varnothing^\cdot$ and $\varnothing \leqslant \delta - 1$, we have already a chain map from $\mathscr{A}^\cdot$ to $\mathscr{B}_{\delta-1}^\cdot$. We can then find a homotopy commutative diagram in $\mathfrak{C}$, with t_ω a quasi-isomorphism,

$$\begin{tikzcd} \mathscr{A}^\cdot \arrow[r] \arrow[d, "s_\omega"'] & \mathscr{B}_{\delta-1}^\cdot \arrow[d, "t_\omega"] \\ \mathscr{A}_\omega^\cdot \arrow[r] & \mathscr{B}_\delta^\cdot \end{tikzcd}$$

which we use to define $\mathscr{B}_\delta^\cdot$ and the chain map $\mathscr{B}_{\delta-1}^\cdot \to \mathscr{B}_\delta^\cdot$ (hence also the chain maps $\mathscr{B}_\gamma^\cdot \to \mathscr{B}_\delta^\cdot$ for $\gamma < \delta$).

Let then $\mathscr{B}^\cdot = \varinjlim \mathscr{B}_\delta^\cdot$, and let s: $\mathscr{A}^\cdot = \mathscr{B}_\varnothing^\cdot \to \mathscr{B}^\cdot$ be the natural chain map. Then s is a quasi-isomorphism, and it has obviously the required properties.

4.2. LEMMA. *Let $\mathscr{B}^\cdot \in \mathfrak{C}$. Then there exists an injective chain map m: $\mathscr{B}^\cdot \to \mathscr{C}^\cdot$ which is a quasi-isomorphism and such that for every $j \in \mathbb{Z}$, $m(\mathscr{B}^j)$ is contained in an injective submodule of $\mathscr{C}^j$.*

For each $j \in \mathbb{Z}$ choose an injective map f_j: $\mathscr{B}^j \to \mathscr{I}^j$, with $\mathscr{I}^j$ injective. Let $\mathscr{J}^\cdot$ be the complex defined by $\mathscr{J}^j = \mathscr{I}^j \oplus \mathscr{I}^{j+1}$ and differential $d(x, y) = (y, 0)$. We have then an injective chain map u: $\mathscr{B}^\cdot \to \mathscr{J}^\cdot$, $u(b) = (f_j(b), f_{j+1}(db))$ $(b \in \mathscr{B}^j)$. Let v: $\mathscr{J}^\cdot \to \operatorname{Coker} u$ be the natural chain map and let $\mathscr{C}^\cdot = \mathscr{C}_v^\cdot[-1]$. It is then easily checked that the natural chain map m: $\mathscr{A}^\cdot \to \mathscr{C}^\cdot$ induced by u has the required properties.

4.3. LEMMA. *Let $\mathfrak{T} = (\mathscr{T}_e^\cdot)_{e \in E}$ be a family of acyclic complexes, and let $\mathscr{A}^\cdot \in \mathfrak{C}$. Then $\mathscr{A}^\cdot$ has a right resolution t: $\mathscr{A}^\cdot \to \mathscr{I}^\cdot$ such that:*
(a) *$\operatorname{Hom}^\cdot(\mathscr{T}_e^\cdot, \mathscr{I}^\cdot)$ is acyclic for every $e \in E$.*
(b) *Each $\mathscr{I}^j$ is an injective $\mathscr{O}$-module ($j \in \mathbb{Z}$).*

Let α be an infinite cardinal such that $\alpha \geqslant \operatorname{Card}(\mathscr{T}_e^j)$ for every $e \in E, j \in \mathbb{Z}$, and $\alpha \geqslant \operatorname{Card}(\mathscr{O})$ (we consider here the sheaves as "espaces étalés" [3, II.1.2]). Let β be the smallest cardinal strictly greater than α, and let Γ be the smallest ordinal whose cardinal is β.

We construct by induction a direct system of complexes $(\mathscr{I}_\gamma^\cdot)_{\gamma \in \Gamma}$ in $\mathfrak{C}$.

Let 0 be the smallest element of Γ. We set $\mathscr{I}_0^\cdot = \mathscr{A}^\cdot$.

If $\gamma \in \Gamma$ has no predecessor, and $\gamma \neq 0$, let $\mathscr{I}_\gamma^\cdot = \varinjlim_{\gamma' < \gamma} \mathscr{I}_{\gamma'}^\cdot$.

If $\gamma \in \Gamma$ has a predecessor $\gamma - 1$, we can find by 4.1 a quasi-isomorphism s_γ: $\mathscr{I}_{\gamma-1}^\cdot \to \mathscr{B}_\gamma^\cdot$ such that for every chain map ω: $\mathscr{T}_e^\cdot \to \mathscr{I}_{\gamma-1}^\cdot$, $s_\gamma \circ \omega$ is homotopic to 0. Replacing if necessary $\mathscr{B}_\gamma^\cdot$ by $\mathscr{B}_\gamma^\cdot \oplus \mathscr{T}_\gamma^\cdot$, where $\mathscr{T}_\gamma^\cdot$ is the cone over Id: $\mathscr{I}_{\gamma-1}^\cdot \to \mathscr{I}_{\gamma-1}^\cdot$, we may assume that s_γ is injective. By 4.2. we can find a quasi-isomorphism $u_\gamma\colon \mathscr{B}_\gamma^\cdot \to \mathscr{C}_\gamma^\cdot$ which is injective and such that $u_\gamma(\mathscr{B}_\gamma^j)$ is contained in an injective submodule of $\mathscr{C}_\gamma^j$ ($j \in \mathbb{Z}$). We take then $\mathscr{I}_\gamma^\cdot = \mathscr{C}_\gamma^\cdot$ and define the maps $\mathscr{I}_{\gamma'}^\cdot \to \mathscr{I}_\gamma^\cdot$ for $\gamma' < \gamma$ by the requirement that $\mathscr{I}_{\gamma-1}^\cdot \to \mathscr{I}_\gamma^\cdot$ is $u_\gamma \circ s_\gamma$.

We let then $\mathscr{I}^\cdot = \varinjlim \mathscr{I}_\gamma^\cdot$ and take for t the natural chain map $\mathscr{A}^\cdot = \mathscr{I}_0^\cdot \to \mathscr{I}^\cdot$. Notice that the natural chain maps $\mathscr{I}_\gamma^\cdot \to \mathscr{I}^\cdot$ are all injective. For notational simplicity we shall consider the complexes $\mathscr{I}_\gamma^\cdot$ as subcomplexes of $\mathscr{I}^\cdot$.

It is clear that t is a quasi-isomorphism. Let now $\phi\colon \mathscr{T}_e^\cdot \to \mathscr{I}^\cdot$ be a chain map. By choice of Γ, $\phi(\mathscr{T}_e^\cdot) \subset \mathscr{I}_\gamma^\cdot$ for some $\gamma \in \Gamma$. By construction, $\phi\colon \mathscr{T}_e^\cdot \to \mathscr{I}_{\gamma+1}^\cdot$ is then homotopic to 0. This implies (a).

For similar reasons, if $\mathfrak{a}$ is an ideal of $\mathcal{O}$ and $f\colon \mathfrak{a} \to \mathscr{I}^j$ is a homomorphism, then $f(\mathfrak{a}) \subset \mathscr{I}_\gamma^j$ for some $\gamma \in \Gamma$, and by construction f extends to $\mathcal{O} \to \mathscr{I}_{\gamma+1}^j$. This implies (b).

4.4. LEMMA. *Let* $\alpha = \max(\mathrm{Card}(\mathcal{O}), \mathrm{Card}(\mathbb{N}))$.
(a) *For every surjective homomorphism* $f\colon \mathscr{A} \to \mathscr{B}$ *in* $\mathfrak{Mod}(X)$ *there exists a subsheaf* $\mathscr{A}_0$ *of* $\mathscr{A}$ *such that* $f\colon \mathscr{A}_0 \to \mathscr{B}$ *is surjective and* $\mathrm{Card}(\mathscr{A}_0) \leqslant \alpha\,\mathrm{Card}(\mathscr{B})$.
(b) *Let* $\mathscr{T}^\cdot \neq 0$ *be an acyclic complex in* $\mathfrak{C}$. *Then* $\mathscr{T}^\cdot$ *has an acyclic subcomplex* $\mathscr{S}^\cdot \neq 0$ *such that* $\mathrm{Card}(\mathscr{S}^j) \leqslant \alpha$ *for every* $j \in \mathbb{Z}$.

(a) We can find a commutative diagram

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "g"'] \arrow[dr, "h"] & \\ \mathscr{A} \arrow[rr, "f"'] & & \mathscr{B} \end{tikzcd}$$

with h surjective and $\mathscr{C}$ a direct sum of at most $\mathrm{Card}(\mathscr{B})$ sheaves of the form $\mathcal{O}_{U\subset X}$, U open in X. Let $\mathscr{A}_0 = \mathrm{Im}(g)$. Then $\mathrm{Card}(\mathscr{A}_0) \leqslant \mathrm{Card}(\mathscr{C}) \leqslant \mathrm{Card}(X)\mathrm{Card}(\mathcal{O})\mathrm{Card}(\mathscr{B}) \leqslant \alpha\mathrm{Card}(\mathscr{B})$.

(b) Since $\mathscr{T}^\cdot \neq 0$, we can find $i \in \mathbb{Z}$, an open subset U of X and a section $s \in \Gamma(U; \mathscr{T}^i)$, with $s \neq 0$. Let then $\mathscr{S}^i$ be the subsheaf of $\mathscr{T}^i$ generated by s. For $j > i + 1$, let $\mathscr{S}^j = 0$. For $j = i + 1$, let $\mathscr{S}^j = d(\mathscr{S}^i)$. For $j < i$, we construct $\mathscr{S}^j$ by descending induction. If $\mathscr{S}^{j+1}$ is already constructed, with $\mathrm{Card}(\mathscr{S}^{j+1}) \leqslant \alpha$, let $\mathscr{T}'^j = d^{-1}(\mathscr{S}^{j+1})$. Since $\mathscr{T}^\cdot$ is acyclic, (a) shows that we can find $\mathscr{S}^j \subset \mathscr{T}'^j$ in such a way that $d(\mathscr{S}^j) = d(\mathscr{T}'^j)$ and $\mathrm{Card}(\mathscr{S}^j) \leqslant \alpha$. It is clear that this procedure gives a subcomplex $\mathscr{S}^\cdot$ of $\mathscr{T}^\cdot$ with the required properties.

4.5. THEOREM. *Every complex in $\mathfrak{C}$ has a right K-injective resolution.*

There exists a family $(\mathcal{T}_e^{\cdot})_{e\in E}$ of acyclic complexes such that every acyclic complex $\mathcal{T}^{\cdot} \in \mathfrak{C}$ with $\mathrm{Card}(\mathcal{T}^j) \leqslant \max(\mathrm{Card}(\mathcal{O}), \mathrm{Card}(\mathbb{N}))$ is isomorphic to some $\mathcal{T}_e^{\cdot}$, $e \in E$. Let then $\mathcal{A}^{\cdot} \in \mathfrak{C}$ be an arbitrary complex, and let t: $\mathcal{A}^{\cdot} \to \mathcal{I}^{\cdot}$ be the resolution constructed in 4.3. We show that $\mathcal{I}^{\cdot}$ is K-injective.

Let $\mathcal{S}^{\cdot} \in \mathfrak{C}$ be acyclic. Using Zorn's lemma and 4.4, we find that there exist a well ordered set Ψ and an increasing family $(\mathcal{S}_\psi^{\cdot})_{\psi\in\Psi}$ of subcomplexes of $\mathcal{S}^{\cdot}$ with the following properties.

(1) If $\psi \in \Psi$ has no predecessor, then $\mathcal{S}_\psi^{\cdot} = \bigcup_{\phi<\psi}\mathcal{S}_\phi^{\cdot}$

(2) If $\psi \in \Psi$ has a predecessor $\psi - 1$, then $\mathcal{S}_\psi^{\cdot}/\mathcal{S}_{\psi-1}^{\cdot}$ is acyclic and $\mathrm{Card}(\mathcal{S}_\psi^j/\mathcal{S}_{\psi-1}^j) \leqslant \max(\mathrm{Card}(\mathcal{O}), \mathrm{Card}(\mathbb{N}))$ for every $j \in \mathbb{Z}$.

It follows from 4.3 that the inverse system of complexes of abelian groups $(\mathrm{Hom}^{\cdot}(\mathcal{S}_\psi^{\cdot}, \mathcal{I}^{\cdot}))_{\psi\in\Psi}$ satisfies the conditions of 0.11. Therefore $\mathrm{Hom}^{\cdot}(\mathcal{S}^{\cdot}, \mathcal{I}^{\cdot}) = \varprojlim \mathrm{Hom}^{\cdot}(\mathcal{S}_\psi^{\cdot}, \mathcal{I}^{\cdot})$ is acyclic.

4.6. The method used here to construct K-injective resolutions uses only the following properties.

(a) The existence and exactness of direct limits in $\mathfrak{A}$.

(b) The existence of a family $(\mathcal{T}_e^{\cdot})_{e\in E}$ of non-zero acyclic complexes such that every non-zero acyclic complex has a subcomplex isomorphic to one of the $\mathcal{T}_e^{\cdot}$'s.

(c) The existence of a well-ordered set I without largest element having the following properties.

(c1) For every direct system $(\mathcal{A}_i^{\cdot})_{i\in I}$ (in which all the maps are injective), every chain map $\mathcal{T}_e^{\cdot} \to \varinjlim \mathcal{A}_i^{\cdot}$ factors through some $\mathcal{A}_i^{\cdot}$.

(c2) For every direct system $(\mathcal{A}_i)_{i\in I}$ in $\mathfrak{A}$ and every ideal $\mathfrak{a}$ of the structure sheaf, every morphism $\mathfrak{a} \to \varinjlim \mathcal{A}_i$ factors through some $\mathcal{A}_i$.

As these conditions are also fulfilled when $\mathfrak{A}$ is the category of sheaves on an etale site, every complex of such sheaves has therefore also a K-injective resolution.

5. Some classes of complexes of sheaves

In this section all ringed spaces are assumed to tbe commutative.

Unless otherwise stated, we consider a fixed ringed space $(X, \mathcal{O})$, and $\mathfrak{A}$ is the category $\mathfrak{Mod}(X)$ of all sheaves of $\mathcal{O}$-modules on X.

5.0. We let $\mathfrak{P}(X)$ (resp. $\mathfrak{Q}(X)$) be the class of all complexes $\mathcal{A}^{\cdot} \in \mathfrak{C}$ satisfying the following conditions.

(a) $\mathscr{A}^\cdot$ is bounded above.
(b) For each $i \in \mathbb{Z}$, $\mathscr{A}^i$ is a direct sum of sheaves of the form $\mathcal{O}_{Z \subset X}$, with Z open (resp. locally closed) in X.

If $\mathfrak{U}^\cdot = (\mathfrak{U}^i)_{i \in \mathbb{Z}}$ is a family of families of open subsets of X, we let also $\mathfrak{P}(\mathfrak{U}^\cdot)$ be the class of all complexes $\mathscr{A}^\cdot \in \mathfrak{P}$ such that each $\mathscr{A}^i$ is a direct sum of sheaves of the form $\mathcal{O}_{U \subset X}$ with $U \in \mathfrak{U}^i (i \in \mathbb{Z})$.

A. K-flat complexes

5.1. Definition. A complex $\mathscr{A}^\cdot \in \mathfrak{C}$ is *K-flat* if for every acyclic complex $\mathscr{S}^\cdot \in \mathfrak{C}$, $\mathscr{A}^\cdot \otimes_{\mathcal{O}} \mathscr{S}^\cdot$ is acyclic.

5.2. Proposition. *Let $\mathscr{A}^\cdot \in \mathfrak{C}$. Suppose that $\mathscr{A}^i = 0$ for $i \neq 0$. Then $\mathscr{A}^\cdot$ is K-flat if and only if $\mathscr{A}^0$ is a flat $\mathcal{O}$-modules.*

This is clear.

5.3. Proposition. *Let $\mathscr{A}^\cdot \in \mathfrak{C}$. Then the following conditions are equivalent.*
(a) *$\mathscr{A}^\cdot$ is K-flat.*
(b) *For every $x \in X$, the complex $\mathscr{A}^\cdot_x$ of $\mathcal{O}_x$-modules is K-flat.*
(c) *$\mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathscr{I}^\cdot)$ is K-injective for every K-injective complex $\mathscr{I}^\cdot \in \mathfrak{C}$.*

The equivalence of (a) and (b) is clear, and that of (a) and (c) follows from 0.10 and the natural isomorphism

$$\mathrm{Hom}^\cdot(\mathscr{S}^\cdot \otimes_{\mathcal{O}} \mathscr{A}^\cdot, \mathscr{I}^\cdot) = \mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathscr{I}^\cdot)). \tag{1}$$

5.4. Proposition. (a) *If $\mathscr{A}^\cdot, \mathscr{B}^\cdot \in \mathfrak{C}$ are K-flat, then so is $\mathscr{A}^\cdot \otimes_{\mathcal{O}} \mathscr{B}^\cdot$.*
(b) *If $(Y, \mathcal{O}_Y)$ is a ringed space and $f: Y \to X$ is a morphism of ringed spaces, then f^* transforms K-flat complexes of $\mathfrak{C}(X)$ into K-flat complexes of $\mathfrak{C}(Y)$.*
(c) *The class of all K-flat complexes is closed under filtered direct limits in $\mathfrak{C}$.*
(d) *If in a distinguished triangle of $\mathfrak{K}$ two of the vertices are K-flat, then so is the third one.*

This is clear.

5.5. Corollary. *All the complexes in $\mathfrak{P}$ and $\mathfrak{Q}$ are K-flat.*

5.6. Proposition. *Every complex in $\mathfrak{C}$ has a left $\mathfrak{P}$-resolution, hence also a left $\mathfrak{Q}$-resolution and a left K-flat resolution.*

This follows from 3.4 and 3.2(b).

5.7. PROPOSITION. *If $\mathscr{A}^{\cdot} \in \mathfrak{C}$ is K-flat and acyclic, then $\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{B}^{\cdot}$ is acyclic for every $\mathscr{B}^{\cdot} \in \mathfrak{C}$*

Let $\mathscr{P}^{\cdot}$ be a K-flat resolution of $\mathscr{B}^{\cdot}$. Then $\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{B}^{\cdot}$ is quasi-isomorphic to $\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{P}^{\cdot}$ since $\mathscr{A}^{\cdot}$ is K-flat, and $\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{P}^{\cdot}$ is acyclic since $\mathscr{P}^{\cdot}$ is K-flat and $\mathscr{A}^{\cdot}$ acyclic.

5.8. PROPOSITION. *If $\mathscr{A}^{\cdot} \in \mathfrak{C}$ is K-projective, then it is K-flat.*

Let $\mathscr{S}^{\cdot} \in \mathfrak{C}$ be acyclic, and let $\mathscr{I}^{\cdot} \in \mathfrak{C}$ be K-injective. We use the natural isomorphism

$$\operatorname{Hom}^{\cdot}(\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot}, \mathscr{I}^{\cdot}) \;=\; \operatorname{Hom}^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{H}om^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{I}^{\cdot})) \tag{1}$$

If $\mathscr{A}^{\cdot}$ is K-projective, the right hand side is acyclic, and it follows from 0.10 that $\mathscr{A}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot}$ is acyclic.

This is of course relevant mainly in the case where X is discrete.

5.9. PROPOSITION. *Let $\mathscr{A}^{\cdot} \in \mathfrak{C}$. Then $\mathscr{A}^{\cdot}$ has a left resolution $\mathscr{Q}^{\cdot} \to \mathscr{A}^{\cdot}$ with the following property: for every $x \in X$, $\mathscr{Q}^{\cdot}_x \to \mathscr{A}^{\cdot}_x$ is a K-projective resolution of the complex $\mathscr{A}^{\cdot}_x$ of $\mathcal{O}_x$-modules. Moreover $\mathscr{Q}^{\cdot}$ is K-flat, and every $\underrightarrow{\mathfrak{Q}}$-resolution of $\mathscr{A}^{\cdot}$ has this property.*

That $\mathscr{Q}^{\cdot}$ is K-flat follows from 5.8 and 5.3. In view of 5.6, it remains only to check that if $\mathscr{Q}^{\cdot}$ is a $\underrightarrow{\mathfrak{Q}}$-resolution of $\mathscr{A}^{\cdot}$, then $\mathscr{Q}^{\cdot}_x$ is a K-projective resolution of $\mathscr{A}^{\cdot}_x$.

It follows immediately from the definitions that if $(Y, \mathcal{O}_Y)$ is a ringed space and f: $Y \to X$ is a morphism of ringed spaces, then f^* maps $\mathfrak{P}(X)$, $\underrightarrow{\mathfrak{P}}(X)$, $\mathfrak{Q}(X)$, $\underrightarrow{\mathfrak{Q}}(X)$ into $\mathfrak{P}(Y)$, $\underrightarrow{\mathfrak{P}}(Y)$, $\mathfrak{Q}(Y)$ and $\underrightarrow{\mathfrak{Q}}(Y)$ respectively. Consider now the case where $(Y, \mathcal{O}_Y) = (\{x\}, \mathcal{O}_x)$ and f is the obvious map. Then f^* is exact and transforms $\underrightarrow{\mathfrak{Q}}$-resolutions into $\underrightarrow{\mathfrak{Q}}(Y)$-resolutions. But over a point $\underrightarrow{\mathfrak{Q}}(Y)$-resolutions are K-projective, by 2.5. The result follows.

5.10. REMARK. Let $\mathfrak{U}^{\cdot}$ be as in 5.0 and suppose that each $\mathfrak{U}^i$ is a basis of the topology of X. Then the construction of $\underrightarrow{\mathfrak{P}}$-resolutions can easily be adapted to prove that every complex in $\mathfrak{C}$ has a left $\underrightarrow{\mathfrak{P}}(\mathfrak{U}^{\cdot})$-resolution.

B. Sections over open subsets

We investigate here conditions which ensure that a complex of sheaves behaves well under the functors $\Gamma(U; -)$ or $\Gamma_{\Phi}(U; -)$, where U is open in X and Φ is a family of supports.

5.11. DEFINITIONS. Let $\mathscr{A}^{\cdot} \in \mathfrak{C}$.

(a) $\mathscr{A}^{\cdot}$ is *K-limp* if $\operatorname{Hom}^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})$ is acyclic for every acyclic complex $\mathscr{S}^{\cdot} \in \mathfrak{P}$.

(b) $\mathscr{A}^{\cdot}$ is *K-flabby* if $\operatorname{Hom}^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})$ is acyclic for every acyclic complex $\mathscr{S}^{\cdot} \in \mathfrak{Q}$.

(c) $\mathscr{A}^{\cdot}$ is *weakly K-injective* if $\operatorname{Hom}^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})$ is acyclic for every acyclic K-flat complex $\mathscr{S}^{\cdot}$.

5.12. It is clear that

K-injective $\Rightarrow$ weakly K-injective $\Rightarrow$ K-flabby $\Rightarrow$ K-limp.

As every complex in $\mathfrak{C}$ has a K-injective right resolution, every complex has therefore also a weakly K-injective, a K-flabby and a K-limp right resolutions.

5.13. PROPOSITION. *Let $\mathscr{A}^{\cdot} \in \mathfrak{C}$ be such that $\mathscr{A}^i = 0$ for $i \neq 0$. Then the following hold.*

(a) *$\mathscr{A}^{\cdot}$ is K-flabby if and only if $\mathscr{A}^0$ is a flabby sheaf.*

(b) *$\mathscr{A}^{\cdot}$ is K-limp if and only if $H^j(U; \mathscr{A}^0) = 0$ for every open set $U \subset X$ and every $j \geqslant 1$.*

Suppose that $\mathscr{A}^{\cdot}$ is K-flabby, and let $U \subset X$ be open. Let $Z = X - U$. The natural exact sequence

$$0 \to \mathcal{O}_{U \subset X} \to \mathcal{O} \to \mathcal{O}_{Z \subset X} \to 0 \tag{1}$$

may be considered as an acyclic complex $\mathscr{S}^{\cdot} \in \mathfrak{Q}$, and the acyclicity of $\operatorname{Hom}^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})$ is equivalent to the exactness of

$$0 \to \Gamma_Z(X; \mathscr{A}^0) \to \Gamma(X; \mathscr{A}^0) \to \Gamma(U; \mathscr{A}^0) \to 0. \tag{2}$$

Thus $\mathscr{A}^0$ is flabby.

Conversely, suppose that $\mathscr{A}^0$ is flabby. Let $f: \mathscr{A}^{\cdot} \to \mathscr{I}^{\cdot}$ be a resolution of $\mathscr{A}^{\cdot}$ by a bounded below complex of injective $\mathcal{O}$-modules. Then $\mathscr{C}_f^{\cdot}$ is an acyclic complex of flabby sheaves, and $\mathscr{C}_f^{\cdot}$ is bounded below. Therefore $\operatorname{Hom}^{\cdot}(\mathcal{O}_{Z \subset X}, \mathscr{C}_f^{\cdot})$ is acyclic for every locally closed subset Z of X. It follows then from 2.2 and 2.8 that $\operatorname{Hom}^{\cdot}(\mathscr{Q}^{\cdot}, \mathscr{C}_f^{\cdot})$ is acyclic for every $\mathscr{Q}^{\cdot} \in \mathfrak{Q}$. As a consequence $\operatorname{Hom}^{\cdot}(\mathscr{Q}^{\cdot}, \mathscr{A}^{\cdot})$ is quasi-isomorphic to $\operatorname{Hom}^{\cdot}(\mathscr{Q}^{\cdot}, \mathscr{I}^{\cdot})$ for every $\mathscr{Q}^{\cdot} \in \mathfrak{Q}$, and $\mathscr{A}^{\cdot}$ is K-flabby since $\mathscr{I}^{\cdot}$ is so. This proves (a).

A similar argument shows that $\mathscr{A}^{\cdot}$ is K-limp if $H^j(U; \mathscr{A}^0) = 0$ for every $j \geqslant 1$ and every open subset U of X.

We use Čech complexes to prove the converse. Let $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of some open subset U of X. Choose a total order on the set I. If $\alpha, \ldots, \gamma \in E$, let $U_{\alpha\ldots\gamma} = U_\alpha \cap \ldots \cap U_\gamma$. We have then a natural acyclic complex $\mathscr{C}^\cdot(\mathfrak{U})$

$$\cdots \to \bigoplus_{\alpha<\beta<\gamma} \mathcal{O}_{U_{\alpha\beta\gamma}\subset X} \to \bigoplus_{\alpha<\beta} \mathcal{O}_{U_{\alpha\beta}\subset X} \to \bigoplus_{\alpha} \mathcal{O}_{U_\alpha\subset X} \to \mathcal{O}_{U\subset X} \to 0$$

where the differentials are defined in the usual way, and for every sheaf $\mathscr{B}$ the complex $\mathrm{Hom}(\mathscr{C}^\cdot(\mathfrak{U}), \mathscr{B})$ is the augmented Čech complex

$$0 \to \Gamma(U; \mathscr{B}) \to \prod_\alpha \Gamma(U_\alpha; \mathscr{B}) \to \prod_{\alpha<\beta} \Gamma(U_{\alpha\beta}; \mathscr{B}) \to \prod_{\alpha<\beta<\gamma} \Gamma(U_{\alpha\beta\gamma}; \mathscr{B}) \to \cdots$$

The sheaves for which all these augmented Čech complexes are acyclic are precisely those which are acyclic for all functors $\Gamma(U; -)$ (U open in X). As $\mathscr{C}^\cdot(\mathfrak{U}) \in \mathfrak{P}$, it follows that if $\mathscr{A}^\cdot$ is K-limp, then $H^i(U; \mathscr{A}^0) = 0$ for every $i > 0$ and every open subset U of X.

5.14. PROPOSITION. *Let $\mathscr{I}^\cdot \in \mathfrak{C}$ be K-injective. Then for every $\mathscr{A}^\cdot \in \mathfrak{C}$, $\mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathscr{I}^\cdot)$ is weakly K-injective.*

Let $\mathscr{S}^\cdot$ be an acyclic K-flat complex. We have a natural isomorphism

$$\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathscr{I}^\cdot)) = \mathrm{Hom}^\cdot(\mathscr{S}^\cdot \otimes_{\mathcal{O}} \mathscr{A}^\cdot, \mathscr{I}^\cdot). \qquad (1)$$

By 5.7, $\mathscr{S}^\cdot \otimes_{\mathcal{O}} \mathscr{A}^\cdot$ is acyclic. Hence the right hand side of (1) is acyclic, and $\mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathscr{I}^\cdot)$ is therefore weakly K-injective.

5.15. PROPOSITION. (a) *If two of the vertices of a distinguished triangle of $\mathfrak{K}$ are K-limp (resp. K-flabby, resp. weakly K-injective), then so is the third one.*
(b) *Let $(Y, \mathcal{O}_Y)$ be a ringed space and $f\colon X \to Y$ a morphism of ringed spaces. If $\mathscr{A}^\cdot \in \mathfrak{C}(X)$ is K-limp (resp. K-flabby, resp. weakly K-injective), then so is $f_*\mathscr{A}^\cdot$ in $\mathfrak{C}(Y)$.*
(c) *The class of all K-limp (resp. K-flabby, resp. weakly K-injective) complexes in $\mathfrak{C}$ is closed under special inverse limits.*

(a) is clear, and (c) follows from 2.5. For (b) we note that if $\mathscr{S}^\cdot \in \mathfrak{C}(Y)$ is an acyclic K-flat complex, then $f^*\mathscr{S}^\cdot$ is acyclic by 5.7, and K-flat by 5.4(b). Moreover $f^*(\mathfrak{P}(Y)) \subset \mathfrak{P}(X), f^*(\mathfrak{Q}(Y)) \subset \mathfrak{Q}(X)$. It remains then only to use the adjunction formula

$$\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, f_*\mathscr{A}^\cdot) = \mathrm{Hom}^\cdot(f^*\mathscr{S}^\cdot, \mathscr{A}^\cdot) \quad (\mathscr{A}^\cdot \in \mathfrak{C}(X), \mathscr{S}^\cdot \in \mathfrak{C}(Y)).$$

5.16. PROPOSITION. *Let $\mathscr{A}^\cdot \in \mathfrak{C}$ be a K-limp acyclic complex and let $U \subset X$ be open. Then $\Gamma(U; \mathscr{A}^\cdot)$ is acyclic.*

Let $\gamma \in \Gamma(U; \mathscr{A}^n)$ satisfy $d\gamma = 0$. We must find a section $\sigma \in \Gamma(U; \mathscr{A}^{n-1})$ such that $d\sigma = \gamma$.

Define a complex $\mathscr{C}^\cdot$ as follows. For $i > n$ let $\mathscr{C}^i = 0$. Let $\mathscr{C}^n$ be the subsheaf of $\mathscr{A}^n$ generated by γ. For $i < n$ define inductively $\mathscr{C}^i = d^{-1}(\mathscr{C}^{i+1})$. This defines an acyclic subcomplex $\mathscr{C}^\cdot$ of $\mathscr{A}^\cdot$. There exists a $\mathfrak{P}$-resolution $f: \mathscr{P}^\cdot \to \mathscr{C}^\cdot$ such that $\mathscr{P}^i = 0$ for $i > n$, $\mathscr{P}^n = \mathcal{O}_{U \subset X}$ and $f_n(1_U) = \gamma$. Now $\operatorname{Hom}^\cdot(\mathscr{P}^\cdot, \mathscr{A}^\cdot)$ is acyclic since $\mathscr{A}^\cdot$ is K-limp and $\mathscr{P}^\cdot \in \mathfrak{P}$ is acyclic. Considering f as a chain map from $\mathscr{P}^\cdot$ to $\mathscr{A}^\cdot$, there exists therefore $g = (g_i)_{i\in\mathbb{Z}} \in \operatorname{Hom}^{-1}(\mathscr{P}^\cdot, \mathscr{A}^\cdot)$ such that $dg = f$, where $g_i \in \operatorname{Hom}(\mathscr{P}^i, \mathscr{A}^{i-1})$. On $\mathscr{P}^n$, dg is the map $d_{\mathscr{A}^\cdot} \circ g_n + g_{n+1} \circ d_{\mathscr{P}^\cdot} = d_{\mathscr{A}^\cdot} \circ g_n$ since $\mathscr{P}^{n+1} = 0$. Thus $\gamma = f_n(1_U) = d_{\mathscr{A}^\cdot}(g_n(1_U)) = d\sigma$, where $\sigma = g_n(1_U) \in \Gamma(U; \mathscr{A}^{n-1})$.

5.17. COROLLARY. *Let $f: \mathscr{A}^\cdot \to \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{A}^\cdot \in \mathfrak{C}$. Then the following are equivalent.*
(i) *$\mathscr{A}^\cdot$ is K-limp.*
(ii) *For every open subset U of X, the map $\Gamma(U; \mathscr{A}^\cdot) \to \Gamma(U; \mathscr{I}^\cdot)$ induced by f is a quasi-isomorphism.*

Applying 5.16 to the cone $\mathscr{C}^\cdot$ of f, we find that (ii) follows from (i). Conversely, if (ii) holds, then $\operatorname{Hom}^\cdot(\mathcal{O}_{U \subset X}, \mathscr{C}^\cdot)$ is acyclic for every open subset U of X. Hence $\operatorname{Hom}^\cdot(\mathscr{P}^\cdot, \mathscr{C}^\cdot)$ is acyclic for every $\mathscr{P}^\cdot \in \mathfrak{P}$ by 2.7. Therefore $\operatorname{Hom}^\cdot(\mathscr{P}^\cdot, \mathscr{A}^\cdot)$ is quasi-isomorphic to $\operatorname{Hom}^\cdot(\mathscr{P}^\cdot, \mathscr{I}^\cdot)$ for every $\mathscr{P}^\cdot \in \mathfrak{P}$, and (i) follows.

5.18. PROPOSITION. *Let $\mathscr{A}^\cdot \in \mathfrak{C}$ be a K-flabby acyclic complex, U an open subset of X and Φ a family of supports in U. Then $\Gamma_\Phi(U; \mathscr{A}^\cdot)$ is acyclic.*

Let $\gamma \in \Gamma_\Phi(U; \mathscr{A}^n)$ be such that $d\gamma = 0$. Let $Z \subset U$ be the support of γ, $V = U - Z$. We have an exact sequence

$$0 \to \mathcal{O}_{V \subset X} \to \mathcal{O}_{U \subset X} \to \mathcal{O}_{Z \subset X} \to 0 \tag{1}$$

which we may view as an acyclic complex $\mathscr{Q}^\cdot \in \mathfrak{Q}$, with $\mathcal{O}_{Z \subset X}$ in degree 0. By assumption $\operatorname{Hom}^\cdot(\mathscr{Q}^\cdot, \mathscr{A}^\cdot)$ is therefore acyclic.

Since $d\gamma = 0$, 5.16 shows that there exists $\beta \in \Gamma(U; \mathscr{A}^{n-1})$ such that $d\beta = (-1)^n\gamma$. Then $d\beta|_V = (-1)^n\gamma|_V = 0$, and by 5.16 again there exists $\alpha \in \Gamma(V; \mathscr{A}^{n-2})$ such that $d\alpha = (-1)^n\beta|_V$. The triple (α, β, γ) gives an element $f \in \operatorname{Hom}^n(\mathscr{Q}^\cdot, \mathscr{A}^\cdot) = \operatorname{Hom}(\mathscr{Q}^{-2}, \mathscr{A}^{n-2}) \times \operatorname{Hom}(\mathscr{Q}^{-1}, \mathscr{A}^{n-1}) \times \operatorname{Hom}(\mathscr{Q}^0, \mathscr{A}^n)$, and $df = 0$. Thus there exists $g \in \operatorname{Hom}^{n-1}(\mathscr{Q}^\cdot, \mathscr{A}^\cdot)$ such that $dg = f$. Consider the component $g_0 \colon \mathcal{O}_{Z\subset X} \to \mathscr{A}^{n-1}$ of g. Setting $s = g_0(1_Z)$, we have $ds = \gamma$, and $s \in \Gamma_Z(U; \mathscr{A}^{n-1}) \subset \Gamma_\Phi(U; \mathscr{A}^{n-1})$. Thus $\Gamma_\Phi(U; \mathscr{A}^\cdot)$ is acyclic.

5.19. Corollary. *Let $f \colon \mathscr{A}^\cdot \to \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{A}^\cdot \in \mathfrak{C}$. Then the following are equivalent.*
(i) *$\mathscr{A}^\cdot$ is K-flabby.*
(ii) *For every open subset U of X and every family of supports Φ in U, the map $\Gamma_\Phi(U; \mathscr{A}^\cdot) \to \Gamma_\Phi(U; \mathscr{I}^\cdot)$ induced by f is a quasi-isomorphism.*

The argument is the same as for 5.17.

5.20. Proposition. *Let $\mathscr{A}^\cdot, \mathscr{S}^\cdot \in \mathfrak{C}$. Assume that $\mathscr{A}^\cdot$ or $\mathscr{S}^\cdot$ is acyclic and that one of the following conditions holds.*
(a) *$\mathscr{A}^\cdot$ is weakly K-injective and $\mathscr{S}^\cdot$ is K-flat.*
(b) *$\mathscr{A}^\cdot$ is K-flabby and $\mathscr{S}^\cdot \in \vec{\mathfrak{Q}}$.*
(c) *$\mathscr{A}^\cdot$ is K-limp and $\mathscr{S}^\cdot \in \vec{\mathfrak{P}}$.*
Then $\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{A}^\cdot)$ *is acyclic.*

Suppose first that $\mathscr{A}^\cdot$ is acyclic and that (b) (resp. (c)) holds. Let $\mathfrak{E} \subset \mathfrak{C}$ be the class of all complexes $\mathscr{E}^\cdot$ such that $\mathrm{Hom}^\cdot(\mathscr{E}^\cdot, \mathscr{A}^\cdot)$ is acyclic. By 5.18 (resp. 5.16), $\mathfrak{E}$ contains the single degree complexes whose only non-zero term is of the form $\mathcal{O}_{Z\subset X}$ with Z locally closed (resp. open) in X. By 2.8, $\mathfrak{E}$ is closed under special direct limits. It follows that $\mathfrak{E} \supset \vec{\mathfrak{Q}}$ (resp. $\mathfrak{E} \supset \vec{\mathfrak{P}}$). Thus $\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{A}^\cdot)$ is acyclic in this case.

Let now $\mathscr{I}^\cdot$ be a K-injective resolution of $\mathscr{A}^\cdot$. The results just obtained imply in particular that under the assumption (b) (resp. (c)) the complexes $\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{A}^\cdot)$ and $\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{I}^\cdot)$ are quasi-isomorphic, and the latter is acyclic if $\mathscr{S}^\cdot$ is acyclic. This settles the cases (b) and (c).

Suppose now that (a) holds. Let $\mathscr{P}^\cdot$ be a $\vec{\mathfrak{P}}$-resolution of $\mathscr{S}^\cdot$. Since $\mathscr{A}^\cdot$ is weakly K-injective, the complexes $\mathrm{Hom}^\cdot(\mathscr{S}^\cdot, \mathscr{A}^\cdot)$ and $\mathrm{Hom}^\cdot(\mathscr{P}^\cdot, \mathscr{A}^\cdot)$ are quasi-isomorphic. By (c), the latter is acyclic if $\mathscr{A}^\cdot$ is acyclic. The case where $\mathscr{S}^\cdot$ is acyclic follows immediately from the definition of weak K-injectivity.

5.21. Proposition. *Let $\mathscr{A}^\cdot \in \mathfrak{C}$. Then the following conditions are equivalent.*
(a) *$\mathscr{A}^\cdot$ is K-flabby (resp. K-limp).*
(b) *$\mathscr{A}^\cdot$ is K-flabby (resp. K-limp) as a complex of sheaves of abelian groups.*
(c) *For every K-injective resolution $\mathscr{A}^\cdot \to \mathscr{I}^\cdot$ and every locally closed (resp. open) subset Z of X, the morphism* $\mathrm{Hom}^\cdot(\mathcal{O}_{Z\subset X}, \mathscr{A}^\cdot) \to \mathrm{Hom}^\cdot(\mathcal{O}_{Z\subset X}, \mathscr{I}^\cdot)$ *is a quasi-isomorphism.*

(a) $\Rightarrow$ (b) is a special case of 5.15(b). To prove (b) $\Rightarrow$ (c), let $\mathscr{I}^{\cdot}$ be a K-injective resolution of $\mathscr{A}^{\cdot}$. Then $\mathscr{I}^{\cdot}$ is weakly K-injective as a complex of sheaves of abelian groups. It follows then from 5.19 (resp. 5.17) used for complexes of sheaves of abelian groups, that the morphism

$$\mathrm{Hom}^{\cdot}_{\mathbb{Z}}(\mathbb{Z}_{Z\subset X}, \mathscr{A}^{\cdot}) \to \mathrm{Hom}^{\cdot}_{\mathbb{Z}}(\mathbb{Z}_{Z\subset X}, \mathscr{I}^{\cdot})$$

is a quasi-isomorphism. Since $\mathrm{Hom}^{\cdot}_{\mathbb{Z}}(\mathbb{Z}_{Z\subset X}, \mathscr{S}^{\cdot}) = \mathrm{Hom}^{\cdot}(\mathcal{O}_{Z\subset X}, \mathscr{S}^{\cdot})$ for every $\mathscr{S}^{\cdot} \in \mathfrak{C}$, the result follows.

Finally (c) $\Rightarrow$ (a) by 5.19 (resp. 5.17).

5.22. PROPOSITION. *Let $\mathscr{A}^{\cdot}$, $\mathscr{S}^{\cdot} \in \mathfrak{C}$ and let $\mathscr{B}^{\cdot} = \mathscr{H}om^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})$.*
(a) *If $\mathscr{A}^{\cdot}$ is weakly K-injective and $\mathscr{S}^{\cdot}$ is K-flat, then $\mathscr{B}^{\cdot}$ is weakly K-injective.*
(b) *If $\mathscr{A}^{\cdot}$ is K-flabby and $\mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{Q}}$, then $\mathscr{B}^{\cdot}$ is K-flabby.*
(c) *If $\mathscr{A}^{\cdot}$ is K-limp and $\mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{P}}$, then $\mathscr{B}^{\cdot}$ is K-limp.*

Notice first that if $\mathscr{S}^{\cdot}$, $\mathscr{T}^{\cdot} \in \mathfrak{C}$ are K-flat and $\mathscr{T}^{\cdot}$ is acyclic, then $\mathscr{T}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot}$ is a K-flat acyclic complex. In view of the isomorphism

$$\mathrm{Hom}^{\cdot}(\mathscr{T}^{\cdot}, \mathscr{H}om^{\cdot}(\mathscr{S}^{\cdot}, \mathscr{A}^{\cdot})) = \mathrm{Hom}^{\cdot}(\mathscr{T}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot}, \mathscr{A}^{\cdot}),$$

we get then (a). For (b) (resp. (c)), we use moreover 5.20 and the fact that if $\mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{Q}}$ and $\mathscr{T}^{\cdot} \in \mathfrak{Q}$ (resp. $\mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{P}}$ and $\mathscr{T}^{\cdot} \in \mathfrak{P}$), then $\mathscr{T}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{Q}}$ (resp. $\mathscr{T}^{\cdot} \otimes_{\mathcal{O}} \mathscr{S}^{\cdot} \in \underrightarrow{\mathfrak{P}}$).

REMARK. One can show more generally that if $\mathscr{S}^{\cdot}$, $\mathscr{T}^{\cdot} \in \underrightarrow{\mathfrak{Q}}$ (resp. $\mathscr{S}^{\cdot}$, $\mathscr{T}^{\cdot} \in \underrightarrow{\mathfrak{P}}$), then $\mathscr{S}^{\cdot} \otimes_{\mathcal{O}} \mathscr{T}^{\cdot} \in \underrightarrow{\mathfrak{Q}}$ (resp. $\mathscr{S}^{\cdot} \otimes_{\mathcal{O}} \mathscr{T}^{\cdot} \in \underrightarrow{\mathfrak{P}}$).

6. Some formulae for complexes of sheaves

We discuss here some standard formulae for complexes of sheaves (see e.g. [1, V.10]), using the results of the previous paragraphs to get rid of some finiteness conditions.

As in paragraph 5, all ringed spaces are assumed to be commutative.

If $(X, \mathcal{O}_X)$ is a ringed space, we let $\mathfrak{P}(X)$ and $\mathfrak{Q}(X)$ denote the classes of complexes of sheaves of $\mathcal{O}_X$-modules defined in 5.0.

A. The functors $\mathrm{RHom}^{\cdot}$ *and* $\mathrm{R}\mathscr{H}om^{\cdot}$.

6.1. PROPOSITION. *Let* $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot} \in \mathfrak{D}(X)$. *Then* $\mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot})$ *and* $\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot})$ *are defined and can be computed by anyone of the following methods.*

(i) *Using a K-injective resolution of* $\mathscr{B}^{\cdot}$.
(ii) *Using a K-flat resolution of* $\mathscr{A}^{\cdot}$ *and a weakly K-injective resolution of* $\mathscr{B}^{\cdot}$.
(iii) *Using a* $\underrightarrow{\mathfrak{Q}}(X)$*-resolution of* $\mathscr{A}^{\cdot}$ *and a K-flabby resolution of* $\mathscr{B}^{\cdot}$.
(iv) *Using a* $\underrightarrow{\mathfrak{P}}(X)$*-resolution of* $\mathscr{A}^{\cdot}$ *and a K-limp resolution of* $\mathscr{B}^{\cdot}$.

This follows from 4.5 and 5.20.

6.2. REMARK. It is well known that in some cases $\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot})$ can be computed by means of a locally finitely generated free resolution of $\mathscr{A}^{\cdot}$ [4, II.7.4]. I am however unable to get any improvement in this direction.

B. Hypercohomology

6.3. As every complex in $\mathfrak{C}(X)$ has a K-injective resolution, the functor $\mathrm{R}\Gamma(U; -)$ (U open in X) is defined and can be computed by means of K-injective resolutions. The same applies to $\mathrm{R}\Gamma_{\Phi}(U; -)$ if Φ is a family of supports in U. It follows that for $\mathscr{A}^{\cdot} \in \mathfrak{D}(X)$ the hypercohomology groups

$$\mathbb{H}^{\cdot}(U; \mathscr{A}^{\cdot}) = H^{\cdot}(\mathrm{R}\Gamma(U; \mathscr{A}^{\cdot}))$$

and more generally

$$\mathbb{H}^{\cdot}_{\Phi}(U; \mathscr{A}^{\cdot}) = H^{\cdot}(\mathrm{R}\Gamma_{\Phi}(U; \mathscr{A}^{\cdot}))$$

are well-defined.

6.4. PROPOSITION. *Let* $U \subset X$ *be open and let* $\mathscr{A}^{\cdot} \in \mathfrak{D}(X)$. *Then* $\mathrm{R}\Gamma(U; \mathscr{A}^{\cdot})$ *and* $\mathbb{H}^{\cdot}(U; \mathscr{A}^{\cdot})$ *may be computed by means of right K-limp resolutions of* $\mathscr{A}^{\cdot}$. *Similarly, if* Φ *is a family of supports in U, then* $\mathrm{R}\Gamma_{\Phi}(U; \mathscr{A}^{\cdot})$ *and* $\mathbb{H}^{\cdot}_{\Phi}(U; \mathscr{A}^{\cdot})$ *may be computed by means of right K-flabby resolutions of* $\mathscr{A}^{\cdot}$.

This follows from 5.17 and 5.19.

C. Tensor products.

6.5. PROPOSITION. (a) *For every* $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot} \in \mathfrak{D}(X)$, $\mathscr{A}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{B}^{\cdot}$ *is defined and may be computed by means of a K-flat resolution of either of the factors.*
(b) *For every* $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot} \in \mathfrak{D}(X)$ *there is a natural isomorphism*

$$\mathscr{A}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} (\mathscr{B}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{C}^{\cdot}) = (\mathscr{A}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{B}^{\cdot}) \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{C}^{\cdot} \tag{1}$$

(a) follows from 5.6 and 5.7, and (b) follows from (a) and the associativity of the tensor product.

6.6. PROPOSITION. *Let* $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot} \in \mathfrak{D}(X)$. *Then*

$$\mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot}) = \mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}\mathscr{H}om(\mathscr{B}^{\cdot}, \mathscr{C}^{\cdot})) \tag{1}$$

$$\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot} \otimes^{\mathsf{L}}_{\mathscr{O}_X} \mathscr{B}^{\cdot}, \mathscr{C}^{\cdot}) = \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{B}^{\cdot}, \mathscr{C}^{\cdot})). \tag{2}$$

We may assume that $\mathscr{B}^{\cdot}$ is K-flat and $\mathscr{C}^{\cdot}$ K-injective. By 5.3, $\mathscr{H}om^{\cdot}(\mathscr{B}^{\cdot}, \mathscr{C}^{\cdot})$ is then K-injective, and we are reduced to the corresponding formulae in $\mathfrak{C}(\mathfrak{Ab})$ and $\mathfrak{C}(X)$.

D. Inverse and direct images

6.7. PROPOSITION. *Let* $f: X \to Y$ *and* $g: Y \to Z$ *be morphisms of ringed spaces. Then the following hold.*
(a) *The derived functors* $\mathrm{L}f^*$ *and* $\mathrm{R}f_*$ *are defined. Moreover* $\mathrm{L}f^*$ (*resp.* $\mathrm{R}f_*$) *may be computed by means of left K-flat resolutions* (*resp. right K-limp resolutions*).
(b) $\mathrm{L}(g \circ f)^* = \mathrm{L}f^* \circ \mathrm{L}g^*$,
$\mathrm{R}(g \circ f)_* = \mathrm{R}g_* \circ \mathrm{R}f_*$
(c) *Let* $\mathscr{A}^{\cdot} \in \mathfrak{D}(Y)$, $\mathscr{B}^{\cdot} \in \mathfrak{D}(X)$. *Then*

$$\mathrm{RHom}^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}f_*\mathscr{B}^{\cdot}) = \mathrm{RHom}^{\cdot}(\mathrm{L}f^*\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}) \tag{1}$$

$$\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathrm{R}f_*\mathscr{B}^{\cdot}) = \mathrm{R}f_*\mathrm{R}\mathscr{H}om^{\cdot}(\mathrm{L}f^*\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}) \tag{2}$$

(a) follows from 5.6, 4.5 and 5.16. The first assertion of (b) follows from 5.4(b), and the second from (a) and 5.15(b). For (c) we may take $\mathscr{A}^{\cdot}$ K-flat and $\mathscr{B}$ K-injective. Using then 5.15(b) and 6.1(ii) for the left hand side, and 5.4(b) and 5.3 for the right hand side, we are reduced to the corresponding formulae in $\mathfrak{C}(\mathfrak{Ab})$ and $\mathfrak{C}(Y)$.

6.8. PROPOSITION. *Let f: $X \to Y$ be a morphism of ringed spaces, and let $\mathscr{A}^{\cdot}$, $\mathscr{B}^{\cdot} \in \mathfrak{D}(X)$. Then*

$$\mathrm{L}f^*(\mathscr{A}^{\cdot} \otimes^{\mathrm{L}}_{\mathscr{O}_Y} \mathscr{B}^{\cdot}) = \mathrm{L}f^*\mathscr{A}^{\cdot} \otimes^{\mathrm{L}}_{\mathscr{O}_X} \mathrm{L}f^*\mathscr{B}^{\cdot} \tag{1}$$

This follows from 5.4 and the corresponding formula in $\mathfrak{C}(X)$.

E. Hypercohomology with compact supports and the functors $f_!$ and $f^!$

In this section we consider functors for which the methods devised in this paper are much less adequate.

All the topological spaces are assumed to be Hausdorff and locally compact, and f: $X \to Y$ is a morphism of ringed spaces. Recall that a sheaf $\mathscr{A}$ on X is c-soft if $H^i_c(U; \mathscr{A}) = 0$ for every $i > 0$, U open in X.

6.9. For $V \subset Y$ open, let $\Phi(V)$ be the set of all (necessarily closed) subspaces C of $f^{-1}(V)$ such that the restriction of f to $C \to V$ is proper. For $\mathscr{A} \in \mathfrak{Mod}(X)$, the assignment

$$V \mapsto \Gamma_{\Phi(V)}(f^{-1}(V); \mathscr{A}) \ (V \text{ open in } Y) \tag{1}$$

defines a sheaf $f_!\mathscr{A}$ on Y which is in an obvious way an $\mathscr{O}_Y$-module. For $y \in Y$ there is a natural isomorphism

$$(f_!\mathscr{A})_y \cong \Gamma_c(f^{-1}(y); \mathscr{A}|_{f^{-1}(y)}). \tag{2}$$

6.10. Let $\mathscr{K} \in \mathfrak{Mod}(X)$ be c-soft. For $\mathscr{B} \in \mathfrak{Mod}(Y)$, the assignment

$$U \mapsto \mathrm{Hom}_{\mathscr{O}_Y}(f_!(\mathscr{K}_{U\subset X}), \mathscr{B}) \ (U \text{ open in } X) \tag{1}$$

defines a presheaf $f^!_{\mathscr{K}}\mathscr{B}$ on X which is in an obvious way an $\mathscr{O}_X$-module. This presheaf is actually a sheaf, as is easily deduced from the following observation.

Let $U \subset X$ be open and $(U_\alpha)_{\alpha\in I}$ an open covering of U, with I totally ordered. We have a long exact sequence

$$\cdots \to \bigoplus_{\alpha<\beta<\gamma} \mathscr{K}_{U_\alpha\cap U_\beta\cap U_\gamma\subset X} \xrightarrow{d_2} \bigoplus_{\alpha<\beta} \mathscr{K}_{U_\alpha\cap U_\beta\subset X} \xrightarrow{d_1} \bigoplus_{\alpha} \mathscr{K}_{U_\alpha\subset X} \xrightarrow{d_0} \mathscr{K}_{U\subset X} \to 0 \tag{2}$$

where the differential is defined in the usual way. *Then* Ker d_i *is c-soft for every* $i \in \mathbb{N}$. (We need actually to known this only for $i = 0, 1$.)

To check this, notice first that (2) is an acyclic finite complex of c-soft sheaves if I is finite. That Ker d_i is c-soft in this case is then obvious. In general, if J is a finite subset of I, we have an analogue of (2) with U replaced by $U_J = \bigcup_{\alpha \in J} U_\alpha$, and differentials d_i^J. We know already that Ker d_i^J is c-soft. As Ker $d_i = \varinjlim$ Ker d_i^J, where J runs over the finite subsets of I, Ker d_i is also c-soft.

If $\mathscr{K}^\cdot$ is a complex of c-soft sheaves, we define in a similar way a functor $f^!_{\mathscr{K}^\cdot}: \mathfrak{C}(Y) \to \mathfrak{C}(X)$ by $f^!_{\mathscr{K}^\cdot}(\mathscr{B}^\cdot)(U) = \mathrm{Hom}^\cdot(f_!(\mathscr{K}^\cdot_{U \subset X}), \mathscr{B}^\cdot)(\mathscr{B}^\cdot \in \mathfrak{C}(Y)$, U open in X).

REMARK. It follows from 6.9(2) that we could as well start with $\mathscr{K}$ such that $\mathscr{K}|_{f^{-1}(y)}$ is c-soft for every $y \in Y$.

6.11. LEMMA. (a) $f_!$ *commutes with direct limits.*
(b) $f^!_{\mathscr{K}}$ *commutes with inverse limits.*

(a) follows from 6.9(2) and the analogous property of Γ_c, and (b) is obvious from the definitions.

6.12. Let $\mathscr{K} \in \mathfrak{Mod}(X)$ be c-soft. For every $\mathscr{A} \in \mathfrak{Mod}(X)$, $\mathscr{B} \in \mathfrak{Mod}(Y)$, there is a natural homomorphism

$$\mathscr{H}om\,(f_!(\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{K}), \mathscr{B}) \to f_*\, \mathscr{H}om\,(\mathscr{A}, f^!_{\mathscr{K}}\mathscr{B}), \tag{1}$$

which is easily seen to be an isomorphism, when $\mathscr{A}$ is a direct sum of sheaves of the form $\mathscr{O}_{Z \subset X}$ (Z locally closed in X), c.f. [1, V.7.5].

If $\mathscr{K}^\cdot$ is a complex of c-soft sheaves, we have in a similar way a natural chain map

$$\mathscr{H}om^\cdot\,(f_!(\mathscr{A}^\cdot \otimes_{\mathscr{O}_X} \mathscr{K}^\cdot), \mathscr{B}^\cdot) \to f_*\, \mathscr{H}om^\cdot\,(\mathscr{A}^\cdot, f^!_{\mathscr{K}^\cdot}\mathscr{B}^\cdot) \tag{2}$$

for $\mathscr{A}^\cdot \in \mathfrak{C}(X)$, $\mathscr{B}^\cdot \in \mathfrak{C}(Y)$, and (2) is an isomorphism if each $\mathscr{A}^i$ is a direct sum of sheaves of the form $\mathscr{O}_{Z \subset X}$ (Z locally closed in X), in particular if $\mathscr{A}^\cdot \in \underrightarrow{\mathfrak{Q}}(X)$.

6.13. Let now $\mathscr{K}^\cdot \in \mathfrak{C}(X)$ be a right resolution of $\mathscr{O}_X$ by a bounded below complex of c-soft sheaves. We set then

$$f^! = \mathrm{R}f^!_{\mathscr{K}^\cdot}. \tag{1}$$

This actually does not depend on the choice of $\mathscr{K}^{\cdot}$, up to canonical isomorphism. To see this, notice that if $\mathscr{L}^{\cdot}$ is an acyclic bounded below complex of c-soft sheaves, then for every open subset U of X the complex $f_!(\mathscr{L}^{\cdot}_{U\subset X})$ is still acyclic, and therefore so is $\mathrm{Hom}^{\cdot}(f_!(\mathscr{L}^{\cdot}_{U\subset X}), \mathscr{I}^{\cdot})$ for every K-injective complex $\mathscr{I}^{\cdot} \in \mathfrak{C}(Y)$.

6.14. Our aim is to prove that the functors $\mathrm{R}f_!$ and $f^!$ are adjoint and that they satisfy various formulae. Typically, if Z is a closed subset of X and $U = X - Z$, then for every complex $\mathscr{A}^{\cdot}$ there is a distinguished triangle in $\mathfrak{D}$ with vertices $\mathscr{A}^{\cdot}$, $\mathscr{A}^{\cdot}_{U\subset X}$ and $\mathscr{A}^{\cdot}_{Z\subset X}$, and we can expect this triangle to induce a long exact hypercohomology sequence

$$\cdots \to \mathbb{H}^i_c(U; \mathscr{A}^{\cdot}) \to \mathbb{H}^i_c(X; \mathscr{A}^{\cdot}) \to \mathbb{H}^i_c(Z; \mathscr{A}^{\cdot}|_Z) \to \cdots \qquad (1)$$

What seems to be needed to carry out this program is a good analogue of the notion of c-softness. The various results discussed below would follow without any assumption on the underlying topological spaces (apart from local compactness) from the existence of a class $\mathfrak{S}$ of complexes having the following properties.

(a) If $\mathscr{I}^{\cdot}$ is an injective resolution of $\mathscr{A}^{\cdot} \in \mathfrak{S}$, then for every open subset U of X the induced map $\mathrm{R}\Gamma_c(U; \mathscr{A}^{\cdot}) \to \mathrm{R}\Gamma_c(U; \mathscr{I}^{\cdot})$ is a quasi-isomorphism.
(b) If Y is a locally closed subset of X and $\mathscr{A}^{\cdot} \in \mathfrak{S}$, then $\mathscr{A}^{\cdot}_{Y\subset X} \in \mathfrak{S}$.
(c) If $(\mathscr{A}^{\cdot}_i)_{i\in I}$ is a direct system in $\mathfrak{C}$ and all the $\mathscr{A}^{\cdot}_i$'s belong to $\mathfrak{S}$, then so does their direct limit.
(d) The K-injective complexes are contained in $\mathfrak{S}$.
(e) If two of the vertices of a distinguished triangle of $\mathfrak{K}$ belong to $\mathfrak{S}$, then so does the third.
(f) The class $\mathfrak{S}$ is stable under $f_!$.

The author's efforts to construct such a class having failed, the discussion will be carried out under the assumption that the functors under consideration can be evaluated on complexes of c-soft sheaves.

We are therefore led to consider the following condition on a ringed space $(T, \mathscr{O}_T)$:

(2) *For every acyclic complex of c-soft sheaves* $\mathscr{L}^{\cdot} \in \mathfrak{C}(T)$, *with differential* d_i: $\mathscr{L}^i \to \mathscr{L}^{i+1}$ $(i \in \mathbb{Z})$, $\mathrm{Ker}\, d_i$ *is c-soft for every* $i \in \mathbb{Z}$.

This condition is fulfilled in particular if T is locally finite dimensional in the following sense: every point $t \in T$ has an open neighborhood U such that for some $n \in \mathbb{N}$, $H^{n+1}_c(U; \mathscr{A}) = 0$ for every $\mathscr{A} \in \mathfrak{Mod}(T)$.

Notice that the existence of right resolutions by complexes of c-soft sheaves is ensured by 4.3. In particular, if $(X, \mathscr{O}_X)$ satisfies (2), then the sequence (1) is exact and hypercohomology with compact supports on X commutes with direct limits. If f: $X \to Y$ is a morphism of ringed spaces and for every $y \in Y$ the ringed space $(f^{-1}(y), \mathcal{O}_X|_{f^{-1}(y)})$ satisfies 6.14(2), then the complexes of c-soft sheaves on X can be used to evaluate $\mathrm{R}f_!$.

6.15. PROPOSITION. *Assume that for every $y \in Y$, the ringed space $(f^{-1}(y), \mathcal{O}_X|_{f^{-1}(y)})$ satisfies 6.14(2). Then for every $\mathscr{A}^\cdot \in \mathfrak{D}(X)$, $\mathscr{B}^\cdot \in \mathfrak{D}(Y)$, we have natural isomorphisms*

$$\mathrm{R}\mathscr{H}om^\cdot(\mathrm{R}f_!\mathscr{A}^\cdot, \mathscr{B}^\cdot) = \mathrm{R}f_*\mathrm{R}\mathscr{H}om^\cdot(\mathscr{A}^\cdot, f^!\mathscr{B}^\cdot). \tag{1}$$

$$\mathrm{RHom}^\cdot(\mathrm{R}f_!\mathscr{A}^\cdot, \mathscr{B}^\cdot) = \mathrm{RHom}^\cdot(\mathscr{A}^\cdot, f^!\mathscr{B}^\cdot) \tag{2}$$

$$\mathrm{Mor}_{\mathfrak{D}(Y)}(\mathrm{R}f_!\mathscr{A}^\cdot, \mathscr{B}^\cdot) = \mathrm{Mor}_{\mathfrak{D}(X)}(\mathscr{A}^\cdot, f^!\mathscr{B}^\cdot) \tag{3}$$

We need only to prove (1). Let $\mathscr{K}^\cdot$ be a right resolution of $\mathcal{O}_X$ by a bounded below complex of c-soft sheaves.

Let $\mathscr{I}^\cdot \in \mathfrak{C}(Y)$ be K-injective and let $\mathscr{Q}^\cdot \in \underrightarrow{\mathfrak{Q}}(X)$. By 6.12, we have a natural isomorphism in $\mathfrak{C}(Y)$

$$\mathscr{H}om^\cdot(f_!(\mathscr{Q}^\cdot \otimes_{\mathcal{O}_X} \mathscr{K}^\cdot), \mathscr{I}^\cdot) = f_*\mathscr{H}om^\cdot(\mathscr{Q}^\cdot, f^!_{\mathscr{K}^\cdot}\mathscr{I}^\cdot), \tag{4}$$

hence also an isomorphism

$$\mathrm{Hom}^\cdot(f_!(\mathscr{Q}^\cdot \otimes_{\mathcal{O}_X} \mathscr{K}^\cdot), \mathscr{I}^\cdot) = \mathrm{Hom}^\cdot(\mathscr{Q}^\cdot, f^!_{\mathscr{K}^\cdot}\mathscr{I}^\cdot) \tag{5}$$

If $\mathscr{Q}^\cdot$ is acyclic, then $\mathscr{Q}^\cdot \otimes_{\mathcal{O}_X} \mathscr{K}^\cdot$ is an acyclic complex of c-soft sheaves, and by 6.9(2) and the assumption on the fibres the complex $f_!(\mathscr{A}^\cdot \otimes_{\mathcal{O}_X} \mathscr{Q}^\cdot)$ is therefore also acyclic. Since $\mathscr{I}^\cdot$ is K-injective, the left hand side of (5) is then acyclic, and $f^!_{\mathscr{K}^\cdot}\mathscr{I}^\cdot$ is therefore K-flabby.

Let now $\mathscr{Q}^\cdot$ be a $\underrightarrow{\mathfrak{Q}}(X)$-resolution of $\mathscr{A}^\cdot$ and $\mathscr{I}^\cdot$ a K-injective resolution of $\mathscr{B}^\cdot$. By 6.1, $5.22(\overrightarrow{\mathrm{b}})$ and 6.7(a), the right hand side of (4) represents then $\mathrm{R}f_*\mathrm{R}\mathscr{H}om^\cdot(\mathscr{A}^\cdot, f^!\mathscr{B}^\cdot)$. Our assumptions on the fibres and 6.9(2) imply also that $f_!(\mathscr{Q}^\cdot \otimes_{\mathcal{O}_X} \mathscr{K}^\cdot)$ represents $\mathrm{R}f_!\mathscr{A}^\cdot$. Thus the left hand side of (4) represents $\mathrm{R}\mathscr{H}om^\cdot(\mathrm{R}f_!\mathscr{A}^\cdot, \mathscr{B}^\cdot)$. This proves (1), and (2), (3) follow.

6.16. REMARKS. (a) Let $\mathscr{K}^\cdot$ be a complex of c-soft sheaves, and let $f_!^{\mathscr{K}^\cdot}$: $\mathfrak{C}(X) \to \mathfrak{C}(Y)$ be the functor $\mathscr{A}^\cdot \to f_!(\mathscr{A}^\cdot \otimes_{\mathcal{O}_X} \mathscr{K}^\cdot)$. Under the hypothesis of the proposition, the formula which comes naturally out of 6.12 is

$$\mathrm{R}\mathscr{H}om^\cdot(\mathrm{L}f_!^{\mathscr{K}^\cdot}\mathscr{A}^\cdot, \mathscr{B}^\cdot) = \mathrm{R}f_*\mathrm{R}\mathscr{H}om^\cdot(\mathscr{A}^\cdot, \mathrm{R}f^!_{\mathscr{K}^\cdot}\mathscr{B}^\cdot)$$

which is more similar to 6.7(2).

(b) In the proof we have checked also that if $\mathscr{I}^\cdot \in \mathfrak{C}(Y)$ is K-injective and $\mathscr{K}^\cdot \in \mathfrak{C}(X)$ is a complex of c-soft sheaves, then $f^!_{\mathscr{K}^\cdot}\mathscr{I}^\cdot$ is K-flabby.

6.17. PROPOSITION. *Let $g\colon Y \to Z$ be a morphism of ringed spaces. Then the following hold.*
(a) *If for every $z \in Z$ the ringed space $(g^{-1}(z), \mathscr{O}_Y|_{g^{-1}(z)})$ satisfies* 6.14(2), *then*

$$\mathrm{R}(g \circ f)_! \;=\; \mathrm{R}g_! \circ \mathrm{R}f_! \tag{1}$$

(b) *If moreover for every $z \in Z$ the ringed space $((g\circ f)^{-1}(z), \mathscr{O}_X|_{(g\circ f)^{-1}(z)})$ satisfies* 6.14(2), *then*

$$(g\circ f)^! \;=\; f^! \circ g^! \tag{2}$$

Let $\mathscr{A}^\cdot \in \mathfrak{D}(X)$ and let $\mathscr{I}^\cdot$ be the K-injective resolution of $\mathscr{A}^\cdot$ constructed in 4.5. We certainly have

$$(g\circ f)_!(\mathscr{I}^\cdot) \;=\; g_!(f_!(\mathscr{I}^\cdot)). \tag{3}$$

The left hand side represents $\mathrm{R}(g\circ f)_!(\mathscr{A}^\cdot)$, and $\mathrm{f}_!(\mathscr{I}^\cdot)$ represents $\mathrm{R}f_!(\mathscr{A}^\cdot)$. It remains to check that $g_!(f_!(\mathscr{I}^\cdot))$ represents $\mathrm{R}g_!(f_!(\mathscr{I}^\cdot))$. By construction in 4.5, $\mathscr{I}^\cdot$ is a complex of injective sheaves, hence a complex of c-soft sheaves. Then $f_!\mathscr{I}^\cdot$ is also a complex of c-soft sheaves. It can therefore be used to compute $\mathrm{R}g_!$.

Under the hypothesis of (b), 6.16 shows that $f^!$, $g^!$, $(g\circ f)^!$ are right adjoint to $\mathrm{R}f_!$, $\mathrm{R}g_!$ and $\mathrm{R}(g\circ f)_!$ respectively. Thus (2) follows from (1).

6.18. PROPOSITION. *Assume that for every $y \in Y$ the ringed space $(f^{-1}(y), \mathscr{O}_X|_{f^{-1}(y)})$ satisfies* 6.14(2). *Then for every $\mathscr{A}^\cdot \in \mathfrak{D}(X)$, $\mathscr{B}^\cdot \in \mathfrak{D}(Y)$, there is a natural isomorphism*

$$\mathrm{R}f_!(\mathscr{A}^\cdot \otimes^{\mathrm{L}}_{\mathscr{O}_X} \mathrm{L}f^*\mathscr{B}^\cdot) \;=\; \mathrm{R}f_!\mathscr{A}^\cdot \otimes^{\mathrm{L}}_{\mathscr{O}_Y} \mathscr{B}^\cdot. \tag{1}$$

For every $\mathscr{A}^\cdot \in \mathfrak{D}(X)$, $\mathscr{B}^\cdot \in \mathfrak{D}(Y)$, there is a natural chain map

$$f_!\mathscr{A}^\cdot \otimes_{\mathscr{O}_Y} \mathscr{B}^\cdot \to f_!(\mathscr{A}^\cdot \otimes_{\mathscr{O}_X} f^*\mathscr{B}^\cdot). \tag{2}$$

We need only to check that (2) is an isomorphism when $\mathscr{A}^\cdot$ is a complex of c-soft sheaves and $\mathscr{B}^\cdot \in \underrightarrow{\mathfrak{P}}(Y)$. As $f_!$, f^* and tensor product commute with direct limits, it is even sufficient to check this when $\mathscr{B}^{\cdot}$ is a single sheaf $\mathscr{O}_{V\subset Y}$ with V open in Y. In this case it remains to check

$$(f_!\mathscr{A}^{\cdot})_{V\subset Y} = f_!(\mathscr{A}^{\cdot}_{f^{-1}(V)\subset X}), \tag{3}$$

which follows immediately from 6.9(2).

6.19. PROPOSITION. *Suppose that for every $y \in Y$ the ringed space $(f^{-1}(y), \mathscr{O}_X|_{f^{-1}(y)})$ satisfies 6.14(2). Then for every pair of complexes $\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot} \in \mathfrak{D}(Y)$ there is a natural isomorphism*

$$f^!\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}) = \mathrm{R}\mathscr{H}om^{\cdot}(\mathrm{L}f^*\mathscr{A}^{\cdot}, f^!\mathscr{B}^{\cdot}) \tag{1}$$

Let $\mathscr{S}^{\cdot} \in \mathfrak{D}(X)$. Using 6.6, 6.16 and 6.18, we get

$$\mathrm{RHom}^{\cdot}(\mathscr{S}^{\cdot}, f^!\mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot})) = \mathrm{RHom}^{\cdot}(\mathrm{R}f_!\mathscr{S}^{\cdot}, \mathrm{R}\mathscr{H}om^{\cdot}(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}))$$

$$= \mathrm{RHom}^{\cdot}(\mathrm{R}f_!\mathscr{S}^{\cdot} \otimes^{\mathrm{L}}_{\mathscr{O}_Y} \mathscr{A}^{\cdot}, \mathscr{B}^{\cdot}) = \mathrm{RHom}^{\cdot}(\mathrm{R}f_!(\mathscr{S}^{\cdot} \otimes^{\mathrm{L}}_{\mathscr{O}_X} \mathrm{L}f^*\mathscr{A}^{\cdot}), \mathscr{B}^{\cdot})$$

$$= \mathrm{RHom}^{\cdot}(\mathscr{S}^{\cdot} \otimes^{\mathrm{L}}_{\mathscr{O}_X} \mathrm{L}f^*\mathscr{A}^{\cdot}, f^!\mathscr{B}^{\cdot})$$

$$= \mathrm{RHom}^{\cdot}(\mathscr{S}^{\cdot}, \mathrm{R}\mathscr{H}om^{\cdot}(\mathrm{L}f^*\mathscr{A}^{\cdot}, f^!\mathscr{B}^{\cdot})),$$

and the result follows.

6.20. PROPOSITION. *For every cartesian diagram of commutative ringed spaces*

$$\begin{tikzcd} X' \arrow[r, "q'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

where q is flat and every fiber of f, equipped with the restriction of the structure shead of X, satisfies 6.14(2), we have natural isomorphisms of functors

$$\mathrm{L}q^* \circ \mathrm{R}f_! \cong \mathrm{R}f'_! \circ \mathrm{L}q'^*, \tag{1}$$

$$q^! \circ \mathrm{R}f_* \cong \mathrm{R}f'_* \circ q'^!. \tag{2}$$

These two formulae can be deduced from each other by adjointness. We need therefore only to prouve the first one. Notice also that since q is flat, then so is q', and we may therefore use q^* and q'^* instead of $\mathrm{L}q^*$ and $\mathrm{L}q'^*$ respectively.

There is a natural transformation of the functors in (1) from the left hand side to the right hand side defined as follows. Let $\mathscr{A}$ be an $\mathcal{O}_X$-module. If V is an open subset of Y and $V' \subset V$ is an open subset of Y', then every section γ contained in $\Gamma(V; f_!(\mathscr{A})) = \Gamma_{\Phi(V)}(f^{-1}(V); \mathscr{A})$ induces a section of $q^*f_!\mathscr{A}$ on V', and $q^*f_!\mathscr{A}$ is the sheaf generated by these sections. But we also have $q'(f'^{-1}(V')) \subset f^{-1}(V)$, and therefore γ induces a section of $q'^*\mathscr{A}$ over $f'^{-1}(V')$. This section has a support which is proper over V', and therefore can be considered as a section of $f'_!\, q'^*\mathscr{A}$.

It is therefore sufficient to prove that we have an isomorphism at the level of stalks. We may thus assume that $Y' = \{y\}$ is reduced to a single point of Y. If moreover the structure sheaf of Y' is given by $\mathcal{O}_{Y,y}$, then the assertion reduces to 6.9(2). This allows to reduce the problem further to the case where Y also consists of the single point y. In this case the problem is to show that if R is a ring, $\mathcal{O}_X$ is a sheaf of R-algebras, S is a flat R-algebra and X satisfies 6.14(2), then tensor product with S commutes with hypercohomology with compact support on X. This is obviously true when S is free on R, hence also when S is flat on R since both tensor product and hypercohomology with compact supports commute with direct limits.

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