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Hodge classes on self-products of a variety with an automorphism

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0. Introduction

This article is concerned with the question of which elements in the cohomology ring of a connected, complex, projective, n-dimensional manifold Y are fundamental classes of algebraic cycles. The cycle class map

$$\psi_p\colon CH^p(Y) \otimes \mathbb{Q} \to Hd^p(Y) := H^{p,p}(Y) \cap H^{2p}(Y, \mathbb{Q})$$

is well known to be surjective when $p = 1$ or $n - 1$. Although the "Hodge conjecture" [H] asserts that ψ_p is surjective for all p, this remains very much in doubt. Recent attempts to verify this assertion in concrete, but non-trivial, specific cases have met with mixed results. The goal has been to find Y such that $Hd^p(Y)$ is large and then try to produce enough condimension p algebraic cycles to show that ψ_p is surjective. When Y is a Fermat hypersurface of dimension $p \equiv 0 \bmod 2$, $Hd^p(Y)$ is generally large. Ran [R] and Shioda [S] have been able to show that ψ_p is surjective in many instances. In contrast to these positive results, is the situation with abelian varieties. Let R be an order in a CM field, K, and let (Y, θ) be a $p[K\colon \mathbb{Q}]$-dimensional abelian variety together with an embedding $\theta\colon R \to \mathrm{End}(Y)$ which induces on $H^0(Y, \Omega_Y)$ the structure of a free $K \otimes \mathbb{C}$-module. If $p > 1$, Weil [W] showed that Hd^p is not generally generated by products of divisors. For general varieties of this type, the image of ψ_p has remained mysterious. In fact, this particular class of varieties has often been suggested as a good place to begin to search for a possible counterexample to the Hodge conjecture. A consequence of the results to be presented here is that the Hodge conjecture is true at a general point in moduli for at least one of these families. For other work on the Hodge conjecture for abelian varieties the reader is referred to [D–M], [M], [Ri], [S2], and [S3].

In the first section of this paper we consider a Hodge structure on certain self-products of a curve with an automorphism, which is isomorphic to the Hodge structure studied by Weil when K is cyclotomic. This Hodge structure may be described via the representation of the automorphism group of the product variety on the cohomology. In fact the method also yields interesting Hodge classes on self-products of certain higher dimensional varieties. The next step is to produce algebraic cycles whose cohomology classes generate this Hodge structure. This is accomplished in Theorem 2.0 for a substantial, but by no means exhaustive, fraction of the Weil-type Hodge structures on products of curves. The proof involves an explicit and rather natural geometric construction. Finally, Theorem 2.0 is applied to verify the Hodge conjecture in the case of the Weil Hodge structure on abelian 4-folds with complex multiplication from the cyclotomic field of cube roots of one (Theorem 3.2).

The reader who has many pressing obligations can most quickly get a feeling for the general flavor of the techniques in this paper by restricting attention to the case where the automorphisms have no fixed points. By initially adopting this somewhat narrow focus he or she may entirely dispense with the more technical considerations including all of §1 after the proof of (1.6a) and all of §2 following the paragraph which preceeds (2.3) with the exception of the first case in Lemma 2.6. This is sufficient to understand the proof of Theorem 2.0 in the special case $r = 0$. The final section concerning cycles on abelian varieties has been written so that it essentially depends only on this special case of Theorem 2.0.

1. Hodge classes

The purpose of this section is to introduce a particular Hodge substructure in the middle dimensional cohomology of certain self-products of complex projective manifolds. Although eventually attention shall be restricted to the case of Riemann surfaces, the initial considerations go through for certain higher dimensional manifolds without additional effort.

Let C be a complex, connected submanifold of projective space, m an integer greater than one, and $\sigma: \mathbb{Z}/m \to \mathrm{Aut}(C)$ an injective homomorphism. Fix an integer k and let V denote the largest $\mathbb{Z}/m$-submodule of $H^k(C, \mathbb{Q})$ with the property that $V \otimes \mathbb{C}$ is a sum of weight spaces for primitive $\mathbb{Z}/m$-characters. Then V is, in a natural way, a vector space over the cyclotomic field $\mathbb{Q}(\mu_m)$ whose dimension will be denoted by h. In this paper it is further assumed that k is odd, $k = \dim C$, $V \otimes \mathbb{C}$ has a two-step Hodge decomposition (i.e. $V \otimes \mathbb{C} \simeq V^{p,k-p} \otimes V^{k-p,p}$) and that primitive $\mathbb{Z}/m$-characters do not contribute to the $\mathbb{Z}/m$-representation $H^i(C, \mathbb{C})$ for $i \neq k$. These hypotheses are satisfied when C is a curve or a Fano threefold with rank $\mathrm{Pic}(C) = 1$ and in other situations as well. The image $\tilde{N}$ of the obvious homomorphism

$$\sigma \times \cdots \times \sigma: (\mathbb{Z}/m)^h \to \mathrm{Aut}(C^h)$$

is normalized by the symmetric group, $\mathcal{S}_h$, of all permutations of the factors in the product. The same holds for the image N of the subgroup $\{(v_1, \ldots, v_h) \in (\mathbb{Z}/m)^h: \Sigma\, v_i = 0\}$. Then $G := N\mathcal{S}_h$ is normal in $\tilde{G} := \tilde{N}\mathcal{S}_h$ and $\tilde{G}/G \simeq \mathbb{Z}/m$. Finally write $\delta: \mathbb{Z}/m \to \mathrm{Aut}(C^h)$ for the homomorphism $\delta(v) = (\sigma(v), \sigma(0), \ldots, \sigma(0))$. This paper is concerned with the largest $\tilde{G}$-submodule $U \subset H^{kh}(C^h, \mathbb{Q})^G$ for which $U \otimes \mathbb{C}$ is a sum of weight spaces for primitive $\tilde{G}/G$-characters. The Hodge structure U may be better understood with the help of the Künneth formula. Define $\xi: H^k(C, \mathbb{Q})^{\otimes h} \to H^{kh}(C^h, \mathbb{Q})$ by $\xi(\eta_1 \otimes \ldots \otimes \eta_h) = p_1^*\eta_1 \cup \ldots \cup p_h^*\eta_h$ where $p_i: C^h \to C$ is projection on the ith factor.

LEMMA 1.1: $U = (\xi(V^{\otimes h}))^G$.

Proof: Künneth pieces of the form $H^{k-j}(C) \otimes H^{(k-1)h+j}(C^{h-1})$ with $j \neq 0$ are orthogonal to the representation U since δ, or some permutation composed with δ, does not act by a primitive character on such pieces. Thus $U \subset \mathrm{im}(\xi)$. Let $W \subset H^k(C, \mathbb{Q})$ denote the $\mathbb{Z}/m$-submodule complementary to V. If $i \neq 0$ or h, $\xi(V^{\otimes i} \otimes W^{\otimes(h-i)})^N = 0$. Thus $U \subset (\xi(V^{\otimes h}))^G$. Since δ acts by primitive characters on $(\xi(V^{\otimes h}))^G$, the lemma follows. □

LEMMA 1.2: *The action of $\tilde{G}/G$ makes U a one dimensional $\mathbb{Q}(\mu_m)$-vector space.*

Proof: We may write $V \otimes \mathbb{C} \simeq \oplus_{1 \leqslant i \leqslant h} \oplus_\chi V_{i,\chi}$ where the inner sum is over primitive $\mathbb{Z}/m$-characters and each $V_{i,\chi}$ is a one dimensional representation on which $\mathbb{Z}/m$ acts by χ. Note that $\xi(V^{\otimes h})^N$ is isomorphic to the sum of those weight spaces $V_{i_1,\chi_1} \otimes \ldots \otimes V_{i_h,\chi_h}$ on which the element of $(\mathbb{Z}/m)^h$,

$(1, -1, 0, \ldots, 0)$, and permutations thereof act trivially. This implies $\chi_1 = \cdots = \chi_h$. Since k is odd, ξ induces an isomorphism from the anti-symmetric tensors $\Lambda^h V \subset V^{\otimes h}$ to $(\xi(V^{\otimes h}))^{\mathscr{S}_h}$. The projection

$$p(v_1 \otimes \ldots \otimes v_h) = (h!)^{-1} \sum_{\tau \in S_h} \operatorname{sgn}(\tau)\, v_{\tau(1)} \otimes \ldots \otimes v_{\tau(h)}$$

of $V^{\otimes h}$ onto the anti-symmetric tensors annihilates the subspace $V_{i_1,\chi} \otimes \ldots \otimes V_{i_h,\chi}$ unless $(i_1, \ldots, i_h)$ is a permutation of $(1, \ldots, h)$. Choose a basis $v_{i,\chi}$ for each $V_{i,\chi}$. Then

$$(\xi(V^{\otimes h}))^G = \xi(\operatorname{Span}\{p(v_{1,\chi} \otimes \ldots \otimes v_{h,\chi}) : \chi \text{ is primitive}\}). \tag{1.3}$$

The lemma follows.

In the decomposition $V \otimes \mathbb{C} \simeq \oplus_{i \leqslant i \leqslant h} \oplus_\chi V_{i,\chi}$ we may require that either $V_{i,\chi} \subset V^{p,k-p}$ or $V_{i,\chi} \subset V^{k-p,p}$ for every pair (i, χ). This makes it easy to read off the Hodge decomposition of $U \otimes \mathbb{C}$, once one knows the multiplicity v_χ with which each primitive $\mathbb{Z}/m$-character χ appears in the representation $V^{p,k-p}$. In fact $\mathbb{Z}/m$ acts on $U \otimes \mathbb{C}$ via δ and the one dimensional weight space corresponding to the character χ has Hodge type $(pv_\chi + (k-p)(h - v_\chi), (k-p)v_\chi + (h - v_\chi)p)$. It follows without difficulty that $U \otimes \mathbb{C}$ has pure Hodge type $(kh/2, kh/2)$ exactly when $v_\chi = h/2$ for all χ. In particular, h must be even.

REMARK 1.4: Suppose dim. $C = 1$. Now the symmetric product, $S^h C$, is smooth and U may be regarded as belonging to the subalgebra of $H^{\cdot}(S^h C, \mathbb{Q})$ generated by $H^1(S^h C, \mathbb{Q})$. In fact, if one defines $\omega_{i,\chi} = \Sigma_{1 \leqslant j \leqslant h} p_j^* v_{i,\chi} \in H^1(C^h, \mathbb{C})^{\mathscr{S}_h}$, then it is straightforward to check that $(h!)\xi \circ p(v_{1,\chi} \otimes \ldots \otimes v_{h,\chi}) = \omega_{1,\chi} \cup \ldots \cup \omega_{h,\chi}$. Notice that $\{\omega_{1,\chi}, \ldots, \omega_{h,\chi}\}$ is a basis for the subspace of $H^1(C^h, \mathbb{C})^{\mathscr{S}_h}$ on which $\mathbb{Z}/m$, through its diagonal action on C^h, acts by the character χ.

Let $\mathscr{L}$ denote the universal line bundle on $C \times \operatorname{Pic}^0(C)$ which is uniquely defined up to translation on $\operatorname{Pic}^0(C)$ and write $\operatorname{Alb}(C) := \operatorname{Pic}^0(\operatorname{Pic}^0(C))$. The choice of a base point c_0 on C gives rise to a morphism

$$\Xi\colon C^h \to \operatorname{Alb}(C)$$

$$\Xi(c_1, \ldots, c_h) = \bigotimes_{1 \leqslant i \leqslant h} (\mathscr{L}_{|c_i \times \operatorname{Pic}0(C)} \otimes \mathscr{L}^{-1}_{|c_0 \times \operatorname{Pic}0(C)}).$$

Writing the functorial action of $t \in \mathbb{Z}/m$ on Alb(C) by juxtaposition we have

$$\Xi(\sigma(t)c_1, \ldots, \sigma(t)c_h) = t\Xi(c_1, \ldots, c_h) + (\mathscr{L}_{|\sigma(t)c_0 \times \mathrm{Pic}0(C)} \otimes \mathscr{L}^{-1}_{|c_0 \times \mathrm{Pic}0(C)})^{\otimes h}.$$

Since Ξ is $\mathbb{Z}/m$-equivariant up to translation in Alb(C), pullback Ξ^*: $H^1(\mathrm{Alb}(C), \mathbb{Q}) \to H^1(C^h, \mathbb{Q})^{\mathscr{S}_h}$ is a $\mathbb{Z}/m$-equivariant isomorphism. For each primitive character χ, let $\{\omega'_{1,\chi}, \ldots, \omega'_{h,\chi}\}$ be a basis for the χ-eigenspace of $H^1(\mathrm{Alb}(C), \mathbb{C})$. Set $U'' = \mathrm{Span}\{\omega'_{1,\chi} \cup \ldots \cup \omega'_{h,\chi}\}$ as χ ranges over the primitive characters. Then $\Xi^* U'' = U$. Thus the Hodge structure U is pulled back from the Jacobian of C.

The following lemma connects the numerical invariant, h, of the Hodge structure, V, with the topology of C. Consider the canonical quotient map of analytic spaces π: $C \to \langle\sigma\rangle\backslash C =: X$ and the open subset $\mathring{X} = \{x \in X$: $\#(\pi^{-1}(x)) = m\}$.

LEMMA 1.5: $h = -e(\mathring{X})$, *where e denotes the topological Euler characteristic.*

Proof: Fix a primitive $\mathbb{Z}/m$-character χ and let h_i denote the multiplicity with which this character appears in the $\mathbb{Z}/m$-module $H^i(C, \mathbb{C})$. By standard character theory $h_i = m^{-1} \Sigma_{t\in\mathbb{Z}/m} \chi(-t) \operatorname{tr}(\sigma_i(t))$, where σ_i is the representation on $H^i(C, \mathbb{C})$ induced by σ. Since we are assuming $h_i = h\delta_{ik}$ and k is odd,

$$-h = \sum_i (-1)^i h_i = m^{-1}\left[e(C) + \sum_{t\in\mathbb{Z}/m-\{0\}} \chi(-t)\left(\sum_i (-1)^i \operatorname{tr}\sigma_i(t)\right)\right].$$

The fixed locus, $C^{\sigma(t)}$, is a compact submanifold whose normal bundle has a canonical complex structure. Thus the sign term, sgn(det($Id - D\sigma$)), in the Lefschetz fixed point formula [Gi, p. 93] is 1. Substitution into this formula yields

$$-h = m^{-1}\left[e(C) + \sum_{t\in\mathbb{Z}/m-\{0\}} \chi(-t)\, e(C^{\sigma(t)})\right].$$

For each non-zero subgroup $H < \mathbb{Z}/m$, let C^H denote the subset of points whose stabilizer is exactly H. Then

$$-h = m^{-1}\left[e(C) + (-1)\sum_{H\neq\{0\}} e(C^H)\right] = e(\mathring{X}). \qquad \square$$

Recall that the Hodge decomposition of U is determined by the multiplicities, ν_χ. An appropriate form of the holomorphic Lefschetz fixed point theorem may be used to obtain a formula for ν_χ. The next two lemmas are special cases of this general phenomenon.

LEMMA 1.6a: *Suppose that* (C, σ) *satisfy the hypotheses set forth at the beginning of this section. If in addition,* $\mathbb{Z}/m$ *operates without fixed points, then* $\nu_\chi = h/2$ *for each primitive character,* χ.

Proof: Let $(\nu_\chi)_i$ denite the multiplicity with which χ appears in the $\mathbb{Z}/m$-representation $H^i(C, \Omega_C^{k-p})$, which will be denoted by σ_i. Then

$$(-1)^p \nu_\chi = \sum_i (-1)^i (\nu_\chi)_i = \sum_i (-1)^i m^{-1} \sum_{t \in \mathbb{Z}/m} \chi(-t) \operatorname{tr} \sigma_i(t)$$

$$= m^{-1} \sum_i (-1)^i h^i(\Omega_C^{k-p}) + m^{-1} \sum_{t \in \mathbb{Z}/m - \{0\}} \chi(-t) \sum_i (-1)^i \operatorname{tr} \sigma_i(t).$$

Since there are no fixed points, the final term vanishes by the Lefschetz fixed point theorem [A–B, Thm. 2]. This shows that ν_χ is independent of the choice of primitive character, χ. Since all primitive characters occur with the same multiplicity, h, in $V \otimes \mathbb{C}$ the lemma follows.

It is also possible to deduce the lemma from the Riemann–Roch theorem by dealing directly with the weight space $H^p(C, \Omega_C^{k-p})_\chi$. For an appropriate invertible sheaf $\mathscr{L}$ on X with $\mathscr{L}^{\otimes m} \simeq \mathcal{O}_X$ we have

$$\nu_\chi = \dim.\ H^p(C, \Omega_C^{k-p})_\chi = (-1)^p \sum_i (-1)^i \dim.\ H^i(C, \Omega_C^{k-p})_\chi$$

$$= (-1)^p \sum_i (-1)^i \dim.\ H^i(X, \mathscr{L} \otimes \Omega_X^{k-p}).$$

By the Riemann–Roch theorem this expression depends only on the numerical equivalence class of the total Chern class of $\mathscr{L} \otimes \Omega_X^{k-p}$. In particular it is independent of the choice of torsion line bundle, $\mathscr{L}$, and hence independent of the choice of χ. □

EXAMPLE: Let x_j, $0 \leqslant j \leqslant 4$ be homogeneous coordinates on $\mathbb{P}_{\mathbb{C}}^4$. Define σ: $\mathbb{Z}/5 \to \operatorname{Aut}(\mathbb{P}_{\mathbb{C}}^4)$ by $x_j \circ \sigma(t) = \exp(2\pi\sqrt{-1}\, jt/5) x_j$. Take C to be any non-singular quintic hypersurface which is invariant under the $\mathbb{Z}/5$-action and disjoint from the five fixed points in $\mathbb{P}_{\mathbb{C}}^4$. Since $H^0(C, \Omega_C^3)$ is invariant, the hypotheses of this section are satisfied. One computes $h = -e(X) = -e(C)/5 = 40$. By the above considerations, the Hodge sub-structure

$U \subset H^{120}(C^{40}, \mathbb{Q})$ has pure Hodge type (60, 60). It is unknown to the author whether or not U is generated by fundamental classes of algebraic cycles.

We shall now give a formula for v_χ in the case dim. $C = 1$. Let s denote the number of points in the branch locus $B = \{b_1, \ldots, b_s\} \subset X$ of the map $\pi: C \to X$. There is a natural way to associate to π an orbit in $(\mathbb{Z}/m - \{0\})^s$ for the diagonal action of $(\mathbb{Z}/m)^*$. In fact the field extension $\mathbb{C}(X) \subset \mathbb{C}(C)$ is determined by a cyclic, order m, subgroup of $\mathbb{C}(X)^*/\mathbb{C}(X)^{*m}$. A generator $\bar{f}$ gives rise to a mod m divisor, $(\bar{f}) = \Sigma_{1 \leqslant j \leqslant s} \bar{\alpha}_j b_j \in \mathrm{Div}(X)/m$, and hence to an s-tuple $(\bar{\alpha}_1, \ldots, \bar{\alpha}_s) \in (\mathbb{Z}/m - \{0\})^s$. A change in the choice of generator $\bar{f}$ changes the s-tuple by the diagonal action of an element of $(\mathbb{Z}/m)^*$. For computational purposes fix $f \in \mathbb{C}(X)^*$ such that $\mathbb{C}(C) \simeq \mathbb{C}(X)[y]/y^m - f$ and $y \circ \sigma(t) = \varepsilon(t) y$ where ε is a primitive $\mathbb{Z}/m$-character. We may arrange, by replacing f with $e^m f$ and y with ey for appropriate $e \in C(X)^*$, that $(f) = \Sigma_{1 \leqslant j \leqslant s} \alpha_j b_j + mD$ with $0 < \alpha_j < m$ and $B \cap \mathrm{Supp}(D) = \phi$. Let g denote the genus of C. Given a coset $\ell \in \mathbb{Z}/m$, $\langle \ell \rangle_m \in \mathbb{Z}$ will denote the unique representative in the interval $[0, m - 1]$. Note that any primitive $\mathbb{Z}/m$-character is of the form $\chi(t) = \varepsilon(nt)$ for some $n \in (\mathbb{Z}/m)^*$. With this notation we prove

LEMMA 1.6b:[1] $v_\chi = h/2 - s/2 + m^{-1} \Sigma_{1 \leqslant i \leqslant s} \langle -n\bar{\alpha}_i \rangle_m$.

Proof: First observe that the right hand side is an integer, since $(-h + s)/2$ is the holomorphic Euler characteristic $h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X)$ and $\Sigma\, \alpha_i = 0 \bmod m$ because deg. $(f) = 0$.

Let $(v_\chi)_i$ denote the multiplicity with which χ appears in the representation $\sigma_i|_{H^i(\mathcal{O}_C)}$. Then

$$-v_\chi = \sum_{0 \leqslant i \leqslant 1} (-1)^i (v_\chi)_i = \sum_{0 \leqslant i \leqslant 1} (-1)^i m^{-1} \sum_{t \in \mathbb{Z}/m} \chi(-t) \operatorname{tr} \sigma_i(t)|_{H^i(\mathcal{O}_C)}$$

$$= m^{-1} \left(1 - g + \sum_{t \in \mathbb{Z}/m - \{0\}} \chi(-t)(1 - \operatorname{tr} \sigma_1(t)|_{H^1(\mathcal{O}_C)}) \right). \tag{1.7}$$

By the holomorphic Lefschetz fixed point theorem [GH, p. 422]

$$1 - \operatorname{tr} \sigma_1(t)|_{H^1(\mathcal{O}_C)} = \sum_{c \in C^{\sigma(t)}} (1 - \chi_c(t))^{-1}$$

[1] I have learned that this lemma has a long history, dating all the way back to Hurwitz. See I. Morrison and H. Pinkham, Galois Weierstrass points and Hurwitz characters, *Ann. of Math.* 124 (1986) 591–625, Thm. 3.5 for a different proof, references, and discussion of related results.

where $\chi_c(t) \cdot Id$ is the map induced on the contangent space at c by $\sigma(t)$. Suppose that $\pi(c) = b_i$, and $d_i = \gcd(\alpha_i, m)$. Then c will be fixed by $\sigma(t)$ exactly when $t \in d_i\mathbb{Z}/m$. There is a local parameter w at b_i such that $w^{\alpha_i} = f$. Thus C is locally isomorphic to the normalization of $y^{m/d_i} - w^{\alpha_i/d_i} = 0$. In particular there is a local parameter v at c with $v^{\alpha_i/d_i} = y$. One has $v \circ \sigma(t) = \varepsilon(td_i/\alpha_i)v$, which computes $\chi_c(t)$. Rewrite (1.7),

$$= m^{-1}\left(1 - g + \sum_{t \in \mathbb{Z}/m - \{0\}} \sum_i d_i\varepsilon(-nt)(1 - \varepsilon(td_i/\alpha_i))^{-1}\right)$$

where the inner sum is over those i such that $1 \leqslant i \leqslant s$ and $d_i | t$,

$$= m^{-1}\left(1 - g + \sum_{1 \leqslant i \leqslant s} d_i \sum_{t \in d_i\mathbb{Z}/m - \{0\}} \varepsilon(-nt)(1 - \varepsilon(td_i/\alpha_i))^{-1}\right). \tag{1.8}$$

The inner sum is $\lambda(\ell) = \Sigma_{t \in d_i\mathbb{Z}/m - \{0\}}\, \varepsilon(t\ell)(1 - \varepsilon(t))^{-1}$ where $\ell = -n\alpha_i/d_i \in (\mathbb{Z}/(m/d_i))^*$. This expression may be evaluated as follows [H–Z, p. 171]: First $\lambda(0) = (-1 + m/d_i)/2$. Then $\lambda(1) - \lambda(0) = (1 - m/d_i)$ and $\lambda(j) - \lambda(j-1) = 1$ if $1 < j < m/d_i$. Summing yields,

$$\lambda(\ell) = (1 - m/d_i)/2 + \langle \ell \rangle_{m/d_i} - 1$$

so (1.8) becomes

$$= m^{-1}\left(1 - g + \sum_{1 \leqslant i \leqslant s} (-(d_i + m)/2 + \langle -n\bar{\alpha}_i \rangle_m)\right)$$

$$= m^{-1}\left(1 - g - \sum_{1 \leqslant i \leqslant s} d_i/2\right) + m^{-1} \sum_{1 \leqslant i \leqslant s} \langle -n\bar{\alpha}_i \rangle_m - s/2.$$

Since $-h/2 = e(\mathring{X})/2 = m^{-1}(1 - g - \Sigma_{1 \leqslant i \leqslant s}\, d_i/2)$, the lemma follows. □

COROLLARY 1.9: *$v_\chi = h/2$ for every primitive $\mathbb{Z}/m$-character χ if and only if $s = 2r$ and $\Sigma_{1 \leqslant i \leqslant 2r} \langle n\bar{\alpha}_i \rangle_m = mr$ for all $n \in (\mathbb{Z}/m)^*$.*

REMARK 1.10: The set of $2r$-truples $(\bar{\alpha}_1, \ldots, \bar{\alpha}_{2r}) \in (\mathbb{Z}/m - \{0\})^{2r}$ satisfying the condition of the Corollary will be denoted $\mathscr{B}_m^{2r-2}$. Evidently, $\mathscr{B}_m^{2r-2}$ is stable under the diagonal action of $(\mathbb{Z}/m)^*$. This set has been closely studied in the context of the degree m, dimension $2r - 2$, Fermat hypersurface,

F_m^{2r-2}. In fact $\mathscr{B}_m^{2r-2}$ may be viewed as a subset of the characters on $(\mathbb{Z}/m)^{2r}/$ (diagonal). This group acts on $H^{2r-2}(F_m^{2r-2}, \mathbb{C})$. A weight space is contained in $(H^{2r-2}(F_m^{2r-2}, \mathbb{Q}) \cap H^{r-1,r-1}(F_m^{2r-2})) \otimes \mathbb{C}$ exactly when the corresponding character lies in $\mathscr{B}_m^{2r-2}$. (See [R], [S] for details.)

A $2r$-tuple in $(\mathbb{Z}/m - \{0\})^{2r}$ is called simple, if after a permutation of the indices, $\bar{\alpha}_{2i} = -\bar{\alpha}_{2i-1}$. The set of such is denoted, $\mathscr{D}_m^{2r-2}$. It is invariant under the diagonal action of $(\mathbb{Z}/m)^*$ and is contained in $\mathscr{B}_m^{2r-2}$. In practice $\mathscr{D}_m^{2r-2}$ seems to be a rather large subset of $\mathscr{B}_m^{2r-2}$. The conditions under which these sets coincide has been determined by Aoki.

AOKI'S THEOREM [A]: *In order that $\mathscr{D}_m^{2r-2} = \mathscr{B}_m^{2r-2}$, it is necessary and sufficient that m satisfies one of the following conditions:*
(i) m is prime or m = 4,
(ii) Every prime divisor of m is greater than 2r.

REMARK 1.11: Much of the foregoing makes sense for C a smooth irreducible variety over an algebraically closed field of characteristic prime to m. In particular, the definitions of h and $U \subset H^{kh}(C^h, \mathbb{Q}_\ell)^G \simeq H^{kh}(C^h, \mathbb{Q}_\ell(kh/2))^G$ go through in the context of étale cohomology. If C is a curve, there is, as before, an associated $(\mathbb{Z}/m)^*$-orbit in $(\mathbb{Z}/m - \{0\})^{2r}$.

2. Algebraic cycles

Let C be an irreducible, smooth, projective curve over an algebraically closed field k of characteristic prime to a fixed integer $m > 1$. Suppose that an embedding σ: $\mathbb{Z}/m \to \mathrm{Aut}(C)$ is given. Assume that the invariant h associated to (C, σ) in §1 is even. This implies that the number of branch points of the canonical quotient map π: $C \to X$ is an even number (1.5), which will be denoted $2r$. The purpose of this section is to prove

THEOREM 2.0: *If the $(\mathbb{Z}/m)^*$ orbit in $(\mathbb{Z}/m - \{0\})^{2r}$ associated to (C, σ) is simple, then the subspace $U \subset H^h(C^h, \mathbb{Q}_\ell(h/2))$ is generated by fundamental classes of algebraic cycles.*

Here ℓ is a prime different from char(k). We carry over the notations of §1 to the present situation. In particular $k(C) \simeq k(X)[y]/(y^m - f)$ where

$$(f) = \sum_{1 \leq j \leq 2r} \alpha_j b_j + mD$$

with $1 \leqslant \alpha_j \leqslant m-1$, $\alpha_{2i-1} \equiv -\alpha_{2i} \bmod m$, and $\mathrm{Supp}(D) \cap \{b_1, \ldots, b_{2r}\} = \phi$. Note that the simplicity hypothesis is automatically satisfied if $r = 0$ or 1.

In order to construct the desired codimension $h/2$ algebraic cycles on C^h we consider the commutative diagram

$$\begin{array}{ccc} C^h & \xrightarrow{\gamma_C} & S^h C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\ N\backslash C^h & \longrightarrow & W_0 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta_0} \\ X^h & \xrightarrow[\gamma_X]{} & S^h X \end{array} \tag{2.1}$$

where $W_0 = G\backslash C^h$ and all morphisms except α_0 are canonical quotient maps for the obvious finite group action. Since the subspace of cohomology U is G-invariant, it is appropriate to look for the cycles on W_0. In fact, we intend to produce an $h/2$ dimensional projective space $P_0 \subset S^h X$ for which $P_0 \times_{\beta_0} W_0$ consists of m irreducible components. These components will in essence be the sought after algebraic cycles. Consider the morphism

$$\eta\colon S^h X \to \mathrm{Pic}^0 X, \quad \eta(d) = d - \left(\sum_{1 \leqslant j \leqslant 2r} b_j\right) - K_X.$$

The fiber $\eta^{-1}(0)$, which will be denoted P_∞, is a projective space of dimension $h - q = q - 2 + 2r$ (unless $r = 0$, in which case the dimension is $q - 1$) which contains the $q - 1$ dimensional linear space (empty if $q = 0$) $P_{-\infty} = \{(\Sigma_{1 \leqslant j \leqslant 2r} b_j) + d'\colon d' \in |K_X|\}$.

It is necessary to understand the branch locus of β_0. This will be done by giving an explicit local description of β_0 in a neighborhood of an arbitrary point of W_0. To this end let $X' = X - \mathrm{Supp}(D)$, and denote by $p_i\colon C^h \to C$ (resp. $pr_i\colon X^h \to X$) projection onto the ith factor. Note that $\hat{y} = \Pi_{1 \leqslant i \leqslant h}\, y \circ p_i \in k(C^h)^G$ and $\hat{y} \circ \delta(t) = \varepsilon(t)\hat{y}$, so $\hat{y}$ generates $k(W_0)$ over $k(S^h X)$. Since $\hat{y}^m - \Pi_{1 \leqslant i \leqslant h} f \circ pr_i = 0$, $\beta_0^{-1}(S^h X')$ is the normalization of Spec $k[S^h X'][v]/(v^m - \Pi_{1 \leqslant i \leqslant h} f \circ pr_i)$. It is possible to find a cover $S^h X'_{(1)} \cup \ldots \cup S^h X'_{(h+1)}$ of $S^h X$ where $X'_{(\ell)} = X - \mathrm{Supp}(D_\ell)$, $\{\mathrm{Supp}\, D_1, \ldots, \mathrm{Supp}\, D_{h+1}, B\}$ are pairwise disjoint and $k(C) \simeq k(X)[y_\ell]/(y_\ell^m - f_\ell)$ where $(f_\ell) = \Sigma_{1 \leqslant j \leqslant 2r}\, \alpha_j b_j + mD_\ell$.

Let $S_j = \gamma_X(b_j \times X^{h-1})$. Then $S_j \simeq S^{h-1} X$ is a smooth divisor on $S^h X$. The following lemma shows that arbitrary intersections of S_j's are transverse and that $\bigcup_{1 \leqslant j \leqslant 2r} S_j$ is the branch locus of β_0.

LEMMA 2.2: *Given $\vec{x} \in S^h X'(k)$, let $J = \{1 \leqslant j \leqslant 2r: \vec{x} \in S_j\}$. Then there are functions $\{s_j\}_{j \in J}$ vanishing at $\vec{x}$ such that*

(*i*) *$\{s_j\}_{j \in J}$ gives rise to a linearly independent subset of the cotangent space, $m_{\vec{x}}/m_{\vec{x}}^2$,*

(*ii*) *there is an open affine neighborhood Spec A of $\vec{x}$ such that $s_j = 0$ defines $S_j \cap$ Spec A and $W_0|_{\mathrm{Spec}\,A}$ is the normalization of Spec $A[v]/(v^m - (unit)\Pi_{j \in J}\, s_j^{\alpha_j})$.*

Proof: For $j \in J$ let μ_j denote the multiplicity with which b_j appears in the unordered h-tuple $\vec{x}$ and define $\theta_j = \Sigma_{\ell \leqslant j}\, \mu_\ell$. If τ_j denotes a local parameter on X at b_j, then a set of elements of $m_{\vec{x}}$ whose images in $m_{\vec{x}}/m_{\vec{x}}^2$ are linearly independent is given by

$$\bigcup_{j \in J} \{\text{elementary symmetric functions in } \tau_j \circ pr_i \text{ where } \theta_{j-1} < i \leqslant \theta_j\}.$$

In particular, setting $s_j = \Pi_{\theta_{j-1} < i \leqslant \theta_j}\, \tau_j \circ pr_i$, we see that (i) holds. Also, $s_j = 0$ defines S_j locally. Since $f = (\mathrm{unit})\tau_j^{\alpha_j}$ locally on X near b_j, $\Pi_{1 \leqslant i \leqslant h}\, f \circ pr_i = (\mathrm{unit})\ \Pi_{j \in J}\, s_j^{\alpha_j}$ in a neighborhood of $\vec{x}$. □

If $r = 0$, it is now a simple matter to construct the desired algebraic cycles. Let $P_0 = P_\infty = P_{-\infty} \subset S^{2q-2}X$. Since β_0 is étale, P_0 splits into m connected components, one of which will be denoted Q. Associated to a primitive $\mathbb{Z}/m$-character χ is an algebraic cycle $z_\chi = \Sigma_{t \in \mathbb{Z}/m}\, \chi(-t)\delta(t)_* Q$ with coefficients in $\mathbb{Q}(\mu_m)$ which satisfies $\delta(t)_* z_\chi = \chi(t) z_\chi$. By (2.6) the highest Chern class of the normal bundle $N_{P_0/S^hX} \simeq N_{Q/W_0}$ does not vanish, so $Q \cdot z_\chi \neq 0$. Thus the cohomology class of z_χ is non-trivial and lies in $U \otimes \mathbb{Q}_\ell(\mu_m)$. It is also evident that $U \otimes \mathbb{Q}_\ell(\mu_m)$ is generated by the "eigenvectors" z_χ as χ ranges over the primitive $\mathbb{Z}/m$-characters.

When $r > 0$, the trick is to choose the projective space P_0 so that it splits into irreducible components in W_0 and then to compute some intersection numbers, in spite of the singularities of W_0. Of course, the choice of P_0 is rather delicate, since it must meet the branch locus in a very special way. We first define linear subspaces $V_i = (P_\infty) \cap S_1 \cap \ldots \cap \hat{S}_{2i-1} \cap \hat{S}_{2i} \cap \ldots \cap S_{2r-1} \cap S_{2r}$ of (P_∞) and then set $P_0 = \mathrm{Span}\{V_1, \ldots, V_r\}$. Basic facts about these subspaces are gathered together in

LEMMA 2.3:

(*i*) *$P_{-\infty} = P_\infty \cap S_1 \cap \ldots \cap S_{2r} = P_\infty \cap S_1 \cap \ldots \cap \hat{S}_j \cap \ldots \cap S_{2r}$ for any j.*

(*ii*) If $J \subset \{1, \ldots, 2r\}$ *is a proper subset, the intersection* $(\bigcup_{j \in J} S_j) \cap P_\infty$ *is transverse.*

(*iii*) *If* $K \subset \{1, 2, \ldots, r\}$ *and* $i \notin K$, then $V_i \not\subset \mathrm{Span}\{\bigcup_{\ell \in K} V_\ell\}$.
(*iv*) $P_0 \cap S_{2i} = P_0 \cap S_{2i-1} = \mathrm{Span}\{V_1, \ldots, \hat{V}_i, \ldots, V_r\}$.
(*v*) $\dim P_0 = \mathrm{condim.}\ P_0 = h/2$.
(*vi*) *If* K *and* K' *are disjoint with* $K \cup K' = \{1, 2, \ldots, r\}$, *then* $P_0 \cap (\bigcap_{i \in K} S_{2i}) = \mathrm{Span}\{\bigcup_{i \in K'} V_i\}$.

Proof:
(i) The first equality is obvious. As for the second, one inclusion is clear, and the dimensions are the same by Riemann–Roch.
(ii) Since $h - |J| > 2q - 2$, $\bigcap_{j \in J} S_j \simeq S^{h-|J|}X$ is smooth over $\mathrm{Pic}^0 X$ with fiber a projective space.
(iii) By (i) and (ii) the linear space V_i contains $P_{-\infty}$ as a codimension one subspace. The assumption $V_i \subset \mathrm{Span}\{\bigcup_{\ell \in K} V_\ell\}$ implies $V_i \subset S_{2i-1} \cap S_{2i}$ which implies $V_i \subset \bigcap_{1 \leqslant j \leqslant 2r} S_j \cap P_\infty = P_{-\infty}$. This is a contradiction.
(iv) The assumption $P_0 \subset S_{2i}$ implies $V_i \subset S_{2i}$ and hence that $V_i \subset \bigcap_{j \neq 2i-1} S_j \cap P_\infty = P_{-\infty}$. This is a contradiction. Thus $S_{2i} \cap P_0$ has codimension one in P_0. By the same argument $S_{2i-1} \cap P_0$ is a codimension one linear subspace. But $S_{2i} \cap P_0$ and $S_{2i-1} \cap P_0$ each contain $\mathrm{Span}\{V_1, \ldots, \hat{V}_i, \ldots, V_r\}$ which also has codimension one.
(v) Since $P_{-\infty}$ is a codimension one subspace of each V_i, one has by (iii) that $\dim.\ P_0 = \dim.\ P_{-\infty} + r = q - 1 + r = h/2$.
(vi) This follows from (iv) and (iii). □

We shall show that P_0 splits into m irreducible components in W_0 by successively blowing up certain smooth codimension two subvarieties of $S^h X$ until the strict transform of P_0 is disjoint from the branch locus. Although this is not the quickest approach, it has the advantage of making intersection computations possible.

Let $(S^h X)_0 = S^h X$, $S_{i,0} = S_i$ and $K_0 = S_{1,0} \cap S_{2,0}$. For $i \leqslant r$ define inductively ϱ_i: $(S^h X)_i \to (S^h X)_{i-1}$ to be the blow up along $K_{i-1} = S_{2i-1,i-1} \cap S_{2i,i-1}$ where $S_{j,i-1}$ is the strict transform of $S_{j,0}$ with respect to $\varrho_1 \circ \ldots \circ \varrho_{i-1}$. Since for each subset $J \subset \{1, \ldots, 2r\}$ the intersection $\bigcap_{j \in J} S_{j,0}$ is transverse, an induction argument shows that each $S_{j,i}$ is nonsingular and the intersection $\bigcap_{j \in J} S_{j,i}$ is transverse. Let P_i denote the strict transform of P_0 in $(S^h X)_i$.

LEMMA 2.4: *For each* $i \leqslant r$, P_i *is isomorphic to* P_0 *and its disjoint from* $\bigcup_{j \leqslant 2i} S_{j,i}$.

Proof: Given $\vec{x} \in P_0(k)$, let $J = \{1 \leqslant j \leqslant 2r: \vec{x} \in S_j\}$ and let $J_2 = \{j \in J: j \equiv 0 \bmod 2\}$. Choose functions $t_1, \ldots, t_{h/2}$ in the local ring of $S^h X$ at $\vec{x}$

such that $(t_1, \ldots, t_{h/2})$ is the ideal of P_0. Let s_j define S_j locally. Then $\{s_j\}_{j \in J_2} \cup \{t_1, \ldots, t_{h/2}\}$ gives rise to a linearly independent set in $m_{\vec{x}}/m_{\vec{x}}^2$ (2.3vi). Of course, $J_2 = \phi \Leftrightarrow J = \phi \Leftrightarrow$ *a* neighborhood of $\vec{x}$ in $S^h X$ is not affected by any of the blow ups, ϱ_i. Assume that J_2 is not empty and that $2i$ is the smallest element. There is an equality of ideals $(s_{2i-1}, t_1, \ldots, t_{h/2}) = (s_{2i}, t_1, \ldots, t_{h/2})$ (2.3ii, iv). After a linear change in the coordinates $t_1, \ldots, t_{h/2}$ we arrange that $s_{2i-1} = as_{2i} + t_1 \bmod m_x^2$ with $a \neq 0$ (2.2i, 2.3iv). Locally, ϱ_i is the blow up of the ideal (s_{2i-1}, s_{2i}). Near $\vec{x}$, the center of the blow up meets $P_{i-1}(=P_0$ locally) in codimension one. Thus $\varrho_i|_{P_i}$: $P_i \to P_{i-1}$ is an isomorphism in a neighborhood of $\vec{x}$. An immediate computation in local coordinates shows that $S_{2i-1,i}$ and $S_{2i,i}$ do not meet P_i at the point above $\vec{x}$. If we denote the inverse image of $\vec{x}$ in P_i by $\vec{x}$ again, then the situation is the same as before except J must be replaced by $J - \{2i - 1, 2i\}$ and J_2 by $J_2 - \{2i\}$. Repeating the above argument finitely many times until J becomes empty verifies that the assertions of the lemma hold above the point $\vec{x} \in P_0(k)$. But $\vec{x}$ was arbitrary. □

For $1 \leqslant i \leqslant r$ define inductively W_i to be the normalization of $W_{i-1} \times_{\varrho_i} (S^h X)_i$. Write φ_i: $W_i \to W_{i-1}$ for the composition of normalization and projection onto the first factor in the fiber product and β_i: $W_i \to (S^h X)_i$ for the composition of normalization and projection onto the second factor.

LEMMA 2.5: *Given $\vec{x} \in (S^h X)_i$, define $J = \{1 \leqslant j \leqslant 2r$: $\vec{x} \in S_{j,i}\}$. Choose functions s_j which define $S_{j,i}$ locally. Then there is an affine open neighborhood, Spec A, of $\vec{x}$ such that $W_i|_{\mathrm{Spec}\,A}$ is the normalization of Spec $A[v]/(v^m -$ (unit) $\Pi_{j \in J} s_j^{\alpha_j})$.*

Proof: The exceptional divisor for each blow up, ϱ_ℓ, has multiplicity $\alpha_{2\ell-1} + \alpha_{2\ell} = m$ in the total transform of the branch locus of $\beta_{\ell-1}$ because $(\alpha_1, \ldots, \alpha_{2r})$ is a simple $2r$-tuple. Thus the exceptional divisor does not contribute to the branch locus of β_ℓ. Now (2.5) follows from (2.2). □

By (2.5), the branch locus of the map β_i: $W_i \to (S^h X)_i$ is supported on $\bigcup_{1 \leqslant j \leqslant 2r} S_{j,i}$. By (2.4), β_r is étale over a neighborhood of P_r. Since P_r is simply connected, $\beta_r^{-1}(P_r)$ is the union of m distinct connected components, one of which is denoted Q. As W_r is constructed via a series of base changes and normalizations, the action of $\tilde{G}/G$ on W_0 lifts to W_r. Thus, we may associate to each primitive $\mathbb{Z}/m$-character χ the cycle with cyclotomic coefficients $z_\chi = \Sigma_{t \in \mathbb{Z}/m}\, \chi(-t)\delta(t)_* Q$. Observe that for all $n \in \mathbb{Z}/m$, $\delta(n)_* z_\chi = \chi(n) z_\chi$. The pullback of the cohomology class $[z_\chi]$ via the inclusion i: $Q \to W_r$ is the highest Chern class of the normal bundle $c_{h/2}(\mathcal{N}_{Q/W_r}) \in H^h(Q, \mathbb{Q})$. Since

$\mathcal{N}_{Q/W_r} \simeq \mathcal{N}_{P_r/(S^hX)_r}$, the next lemma shows that the cohomology class $[z_\chi] \in H^h(W_r, \mathbb{Q}_\ell(h/2))$ is not zero.

LEMMA 2.6: *Let* $\mathbf{H} \in H^2(P_r, \mathbb{Q}_\ell(1))$ *denote the class of a hyperplane. Then* $c_{h/2}(\mathcal{N}_{P_r/(S^hX)_r}) = \pm(\mathbf{H})^{h/2}$.

Proof: Consider first the case $r = 0$. Then $h = 2q - 2$, $P_0 = P_\infty$, and β_0: $W_0 \to S^hX$ is étale. The choice of a point on X gives an embedding $S^hX \to S^{h+1}X$ which realizes P_∞ as a fiber of the obvious smooth morphism $S^{h+1}X \to \mathrm{Pic}^0 X$. In the exact sequence of normal bundles

$$0 \to \mathcal{N}_{P_\infty/S^hX} \to \mathcal{N}_{P_\infty/S^{h+1}X} \to \mathcal{N}_{S^hX/S^{h+1}X|P_\infty} \to 0,$$

the right most term is isomorphic to the hyperplane bundle. The relation between total Chern classes, $c(\mathcal{N}_{P_\infty/S^hX}) = c(\mathcal{O}_{\mathbb{P}^{q-1}}(1))^{-1}$ follows, and the case $r = 0$ of the lemma is proved.

If $r = 1$, $h = 2q$ and we still have $P_0 = P_\infty$. The exceptional divisor E_1 for the blow up ϱ_1: $(S^hX)_1 \to S^hX$ is isomorphic to $\mathbb{P}(\mathcal{N}_{K_0/S^hX}) \simeq \mathbb{P}((\mathcal{N}_{S_1/S^hX} \oplus \mathcal{N}_{S_2/S^hX})|_{K_0})$. Let R_1 denote the restriction of this $\mathbb{P}^1$ bundle to $P_{-\infty} \subset K_0$. Since $\mathcal{N}_{S_1/S^hX|P_{-\infty}} \simeq \mathcal{N}_{S_2/S^hX|P_{-\infty}}$, we have $R_1 \simeq P_{-\infty} \times \mathbb{P}^1$. Apply the following sublemma 2.7 to the case $V = (S^hX)_1$, $T = (\eta \circ \varrho_1)^{-1}(0)$, $T_1 = P_1$, $T_2 = R_1$, $S = P_{-\infty}$ to produce an exact sequence

$$0 \to \mathcal{N}^\vee_{(\eta\circ\varrho_1)^{-1}(0)/(S^hX)_1|P_1} \to \mathcal{N}^\vee_{P_1/(S^hX)_1} \to \mathcal{N}^\vee_{P_{-\infty}/R_1} \to 0.$$

The first term is isomorphic to the structure sheaf of P_1 and the last term to the structure sheaf of $P_{-\infty}$. Hence the following equalities hold in the Chow ring of P_1.

$$c(\mathcal{N}^\vee_{P_1/(S^hX)_1}) = c(\mathcal{N}^\vee_{P_{-\infty}/R_1}) = c(\mathcal{O}_{P_{-\infty}}) = c(\mathcal{O}_{P_1}(-1))^{-1}.$$

To complete the proof of the lemma when $r = 1$ it remains only to establish

SUBLEMMA 2.7: *Let T_1 and T_2 denote two distinct non-singular codimension p subvarieties of a smooth variety V. Suppose that the subschemes T and S defined by the ideal sheaves $I_T = I_{T_1} \cap I_{T_2}$ and $I_S = I_{T_1} + I_{T_2}$ are local complete intersections. Then there is an exact sequence of conormal sheaves,*

$$0 \to \mathcal{N}^\vee_{T/V|T_1} \to \mathcal{N}^\vee_{T_1/V} \to \mathcal{N}^\vee_{S/T_2} \to 0.$$

Proof: Tensor the exact sequence

$$0 \to I_T \to I_{T_1} \to I_{T_1}/I_T \to 0$$

with $\mathcal{O}_V/I_{T_1}$. The resulting term on the right is

$$I_{T_1}/(I_{T_1} \cap I_{T_2}) \otimes \mathcal{O}_V/I_{T_1} \simeq (I_{T_1} + I_{T_2})/I_{T_2} \otimes \mathcal{O}_V/(I_{T_1} + I_{T_2})$$

$$\simeq I_S/I_{T_2} \otimes \mathcal{O}_V/I_S = \mathcal{N}^v_{S/T_2}.$$

The map $I_T \otimes \mathcal{O}_V/I_{T_1} \to I_{T_1} \otimes \mathcal{O}_V/I_{T_1}$ induces a map

$$\psi\colon I_T/I_T^2 \otimes \mathcal{O}_V/I_{T_1} \to I_{T_1} \otimes \mathcal{O}_V/I_{T_1}.$$

By hypothesis both sides are locally free $\mathcal{O}_V/I_{T_1}$ modules of the same rank and the cokernel is supported on S. It follows that ψ is injective. Q.E.D.

If $r > 1$, P_0 is a proper linear subspace of P_∞. Consider the exact sequence

$$0 \to \mathcal{N}_{P_i/(P_\infty)_i} \to \mathcal{N}_{P_i/(S^hX)_i} \to \mathcal{N}_{(P_\infty)_i/(S^hX)_i|P_i} \to 0$$

where the subscript i denotes, as usual, the strict transform with respect to $\varrho_1 \ldots \circ \varrho_i\colon (S^hX)_i \to S^hX$. Since $h > 2q$, $S_{2i-1} \cap S_{2i}$ meets every fiber of η in a codimension two linear space. Thus $(P_\infty)_i$ is a fiber of the smooth morphism $\eta \circ \varrho_1 \circ \ldots \circ \varrho_i\colon (S^hX)_i \to \operatorname{Pic}^0 X$, so the right most sheaf in the sequence is globally free for each i. As for the first sheaf in the sequence, $(P_\infty)_i$ (for $i \leqslant r - 1$) is obtained by blowing up $P_\infty \simeq \mathbb{P}^{2r+q-2}$ along a sequence of i codimension two linear spaces which are in general position subject to the constraint that they contain the codimension $2r - 2$ linear space $P_{-\infty}$ (2.3ii). Each time such a codimension two linear subspace is blown up one gets a morphism to $\mathbb{P}^1$ with some fiber containing the strict transform of P_0. Thus there is a morphism $(P_\infty)_{r-1} \to (\mathbb{P}^1)^{r-1}$ where a fiber contains P_{r-1}. Since the codimension of P_{r-1} is $r - 1$, the fiber is in fact equal to P_{r-1}. After the final blow up, the fiber of the morphism $(P_\infty)_r \to (\mathbb{P}^1)^{r-1}$ containing P_r is the union of two irreducible components $P_r \cup R_r$, where $R_r \simeq \mathbb{P}^{r+q-2} \times \mathbb{P}^1$ meets P_r along $\mathbb{P}^{r+q-2} \times \{0\}$. The considerations of sublemma 2.7 apply and, exactly as in the case $r = 1$, we find $c(\mathcal{N}^v_{P_r/(P_\infty)_r}) = (1 - \mathbf{H})^{-1}$ where $\mathbf{H}$ is the class of a hyperplane in P_r. □

Until further notice we restrict to the case $k = \mathbb{C}$.

Because W_0 is the quotient of a smooth variety by a finite group, Poincaré duality between singular homology and cohomology with $\mathbb{Q}$-coefficients holds [St. Prop. 1.4]. Furthermore, from (2.5) it is evident that each W_i is also locally analytically a quotient, so Poincaré duality with $\mathbb{Q}$-coefficients holds here as well. By resolution of singularities, a subvariety, Z, of W_i maybe regarded as the image of a non-singular variety, $\tilde{Z}$, under a birational morphism $g: \tilde{Z} \to Z$. The fundamental homology class of $\tilde{Z}$ is denoted $\langle \tilde{Z} \rangle$. The element of $H^{\cdot}(W_i, \mathbb{Q})$ which corresponds to $g_* \langle \tilde{Z} \rangle$ by Poincaré duality is independent of the choice of desingularization and is denoted $[Z]$. The natural map φ: $W_r \to W_0$, gives rise, via Poincaré duality, to a Gysin map $\varphi_*: H^h(W_r, \mathbb{Q}) \to H^h(W_0, \mathbb{Q})$ which is compatible with the cycle class map and the push forward of dimension $h/2$ algebraic cycles.

Since φ: $W_r \to W_0$ is $\tilde{G}/G$ equivariant, the cohomology class $[\varphi_*(z_\chi)]$ lies in the subspace $U \otimes \mathbb{C} \subset H^h(W_0, \mathbb{C}) \simeq H^h(C^h, \mathbb{C})^G$. In order to show $[\varphi_*(z_\chi)] \neq 0$, consider the class $[z'_\chi] := \varphi^* \circ \varphi_*[z_\chi] - [z_\chi] \in H^h(W_r, \mathbb{C})$. By the projection formula, this is an element of Ker $\varphi_* \simeq H^h(W_r, \mathbb{C})/\varphi^* H^h(W_0, \mathbb{C})$. It is also an eigenvector on which $\mathbb{Z}/m$ acts by the primitive character χ. The following lemma shows tht $[z'_\chi] = 0$. Since it has already been shown that $[z_\chi] \neq 0$, this allows one to conclude that $\varphi^*[\varphi_*(z_\chi)] \neq 0$.

Lemma 2.8: *Primitive characters of $\tilde{G}/G$ do not occur in the decomposition of the representation $H^{\cdot}(W_i, \mathbb{C})/\varphi_i^* H^{\cdot}(W_{i-1}, \mathbb{C})$.*

Proof: Let $K'_{i-1} \subset W_{i-1}$ denote the inverse image of $K_{i-1} \subset (S^h X)_{i-1}$, viewed as an analytic set. Write E_i for the $\mathbb{P}^1$-bundle over K_{i-1} which is the exceptional divisor for ϱ_i: $(S^h X)_i \to (S^h X)_{i-1}$. Let $F_i \subset W_i$ be the inverse image of E_i. The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\to H^n_c(W_i - F_i, \mathbb{C}) & \to & H^n(W_i, \mathbb{C}) & \to & H^n(F_i, \mathbb{C}) & \to & H^{n+1}_c(W_i - F_i, \mathbb{C}) & \to \\
\uparrow\wr & & \uparrow & & \uparrow & & \uparrow\wr & \\
\to H^n_c(W_{i-1} - K'_{i-1}, \mathbb{C}) & \to & H^n(W_{i-1}, \mathbb{C}) & \to & H^n(K'_{i-1}, \mathbb{C}) & \to & H^{n+1}_c(W_{i-1} - K'_{i-1}, \mathbb{C}) & \to
\end{array}$$

gives rise, in a standard way, to a long exact sequence

$$\to H^n(W_{i-1}, \mathbb{C}) \to H^n(W_i, \mathbb{C}) \oplus H^n(K'_{i-1}, \mathbb{C}) \to H^n(F_i, \mathbb{C}) \to .$$

To prove the lemma it suffices to show that no primitive $\tilde{G}/G$-character appears in the decomposition of the representation $H^n(F_i, \mathbb{C})$. For this, choose a prime $p|m$ such that the $\mathbb{Z}/p$ branched cover π_p: $\langle \sigma(p) \rangle \backslash C \to X$ is ramified over b_{2i-1}. Since $\alpha_{2i} \equiv -\alpha_{2i-1} \bmod m$, b_{2i} is also a branch point of π_p. Let $\bar{F}_i$ denote the quotient of F_i by the unique index p subgroup of $\tilde{G}/G$.

SUBLEMMA 2.9: *Let* $\kappa\colon \bar{F}_i \to E_i$ *be the obvious* $\mathbb{Z}/p$*-branched cover. Then* $\kappa^*\colon H^{\cdot}(E_i, \mathbb{C}) \to H^{\cdot}(\bar{F}_i, \mathbb{C})$ *is an isomorphism.*

Proof: The two disjoint sections of the $\mathbb{P}^1$-bundle $\varrho_i|_{E_i}\colon E_i \to K_{i-1}$ given by $E_i \cap S_{2i-1,i}$ and $E_i \cap S_{2i,i}$ contribute to the branch locus of κ. The remainder of the branch locus consists of the divisor $R = \Sigma_{j\in J} E_i \cap S_{j,i}$, where $J = \{j\colon 1 \leqslant j \leqslant 2r, j \neq 2i \text{ or } 2i - 1, \text{ and } \pi_p \text{ is branched over } b_j\}$. Observe that $\kappa\colon \kappa^{-1}(R) \to R$ is a homeomorphism. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\to H^n_c(\bar{F}_i - \kappa^{-1}(R), \mathbb{C}) & \to & H^n(\bar{F}_i, \mathbb{C}) & \to & H^n(\kappa^{-1}(R), \mathbb{C}) & \to & H^{n+1}_c(\bar{F}_i - \kappa^{-1}(R), \mathbb{C}) \to \\
\uparrow \kappa_c^* & & \uparrow & & \uparrow & & \kappa_c^* \\
\to H^n_c(E_i - R, \mathbb{C}) & \to & H^n(E_i, \mathbb{C}) & \to & H^n(R, \mathbb{C}) & \to & H^{n+1}_c(E_i - R, \mathbb{C}) \to
\end{array}.$$

Let $\mathscr{R} = \Sigma_{j\in J} K_{i-1} \cap S_{j,i-1}$. Note that $R = (\varrho_i|_{E_i})^{-1}(\mathscr{R})$. So both $E_i - R$ and $\bar{F}_i - \kappa^{-1}(R)$ are $\mathbb{P}^1$-bundles, with section, over $K_{i-1} - \mathscr{R}$. By the Leyray–Hirsch theorem [BT, Thm. 5.11], the degree n cohomology of each of these spaces is isomorphic to $\oplus_{a+b=n} H^a(K_{i-1} - \mathscr{R}, \mathbb{C}) \otimes H^b(\mathbb{P}^1, \mathbb{C})$. It is easy to deduce that $\kappa^*\colon H^{\cdot}(E_i - R, \mathbb{C}) \to H^{\cdot}(\bar{F}_i - \kappa^{-1}(R), \mathbb{C})$ is an isomorphism. Averaging a C^∞ differential form under the action of $\mathbb{Z}/p$ gives rise to a map $\kappa_*\colon H^{\cdot}(\bar{F}_i - \kappa^{-1}(R), \mathbb{C}) \to H^{\cdot}(E_i - R, \mathbb{C})$, which is an isomorphism since $\kappa_* \circ \kappa^* = p \text{ Id}$.

The Poincaré dual of κ_* is κ_c^* which must also be an isomorphism. The sublemma follows from the five lemma applied to the preceding commutative diagram.

Now the sublemma tells us that

$$H^{\cdot}(F_i, \mathbb{C})^{\mathbb{Z}/m} \simeq H^{\cdot}(E_i, \mathbb{C}) \simeq H^{\cdot}(\bar{F}_i, \mathbb{C}) \simeq H^{\cdot}(F_i, \mathbb{C})^{\mathbb{Z}/(m/p)}$$

from which (2.8) follows. □

It has now been shown that $[\varphi_*(z_\chi)] \in U \otimes \mathbb{C}$ is not zero for an arbitrary primitive character χ on $\tilde{G}/G$. Since $U \otimes \mathbb{C}$ is the sum of weight spaces for primitive characters, each weight space having dimension one (1.2), the proof of Theorem 2.0 is now complete in the case $k = \mathbb{C}$. The case when k is algebraically closed of characteristic zero follows immediately.

Now suppose that k is an algebraically closed field of positive characteristic prime to m. In order to push through the preceding argument in this case one would need to define a Gysin map $\varphi_*\colon H^h(W_r, \mathbb{Q}_\ell(h/2)) \to H^h(W_0, \mathbb{Q}_\ell(h/2))$, presumably by proving Poincaré duality for these mildly singular varieties.

Then one would need to construct an appropriately functorial cycle class map to the cohomology of these singular varieties without resorting to resolution of singularities. I find it simpler to deduce the theorem in positive characteristic from the result in characteristic zero via a specialization argument. The first step is to show that π: $C \to X$ lifts to chararacteristic zero.

Let A be a complete discrete valuation ring with residue field k and fraction field K of characteristic zero. By deformation theory there exists a scheme $\underline{X}$ smooth and projective over A with special fiber X. The branch points $b_1, \ldots, b_{2r} \in X(k)$ may be lifted to sections $\underline{b}_1, \ldots, \underline{b}_{2r} \in \underline{X}(A)$ by Hensel's Lemma. Write $\mathbf{X}$ (resp. $\mathbf{b}_i$) for the generic fibers of $\underline{X}$ (resp. $\underline{b}_i$).

By replacing K by a finite extension, we may find a divisor $\mathbf{D}$ on $\mathbf{X}$ such that $(\Sigma_{1 \leqslant i \leqslant 2r} \alpha_i \mathbf{b}_i) + m\mathbf{D}$ is linearly equivalent to zero. Let $\mathbf{f}$ be a function on $\mathbf{X}$ with this divisor. Adjoining an m'th root of $\mathbf{f}$ gives rise to a Galois extension $\boldsymbol{\pi}$: $\mathbf{C} \to \mathbf{X}$. Since the closed fiber of $\underline{X}$ is rationally equivalent to zero, $\mathbf{f}$ may be chosen so that its divisor on $\underline{X}$ is $(\Sigma\ \alpha_i \underline{b}_i) + m\underline{D}$, where $\underline{D}$ denotes the closure of $\mathbf{D}$ in $\underline{X}$. Since we are free to replace K by a finite extension, it is easy to find a divisor $\mathbf{D}'$ on $\mathbf{X}$ linearly equivalent to $\mathbf{D}$ and to arrange that the supports of the divisors $\underline{D}$, $\underline{D}'$ and $\Sigma\ \alpha_i \underline{b}_i$ are pairwise disjoint. Let $\mathbf{g}$ be a function with divisor $\underline{D}' - \underline{D}$. By adjoining to the coordinate ring of $\underline{X} - \underline{D}$ an m'th root of $\mathbf{f}$ and to $\underline{X} - \underline{D}'$ an m'th root of $\mathbf{g}^m\mathbf{f}$ and gluing and normalizing, we construct a model $\underline{C}$ of $\mathbf{C}$ finite over $\underline{X}$ and smooth over the integers in a finite extension of the original field, K.

It is not necessarily the case that the closed fiber of $\underline{C}$ is isomorphic to our original curve C. However, the $\mathbb{Z}/m$-covers of X with a fixed associated mod m divisor class $\Sigma\ \bar{\alpha}_i b_i \in (\mathrm{Div}(X)/m)/(\mathrm{diagonal}(\mathbb{Z}/m)^*$ action) are transitively permuted by the action of the m-torsion subgroup, $\mathrm{Pic}^0(X)[m]$. Given such a cover obtained by taking an m'th root of a function f' with divisor $(f') = \Sigma\ \alpha_i b_i + mD'$, we may find an m-torsion divisor class $(D - D')$ and a function f with $(f) = \Sigma\ \alpha_i b_i + mD$ such that the cover associated to $f^{1/m}$ is the original cover π: $C \to X$. Using the well known fact that the specialization map $\mathrm{Pic}^0(\mathbf{X}) \to \mathrm{Pic}^0(X)$ is an isomorphism on torsion prime to char. k, we may (and will) choose $\mathbf{D}$ so that the restriction of $\underline{\pi}$: $\underline{C} \to \underline{X}$ to the special fiber is the original covering π: $C \to X$.

Consider the commutative diagram of specialization maps [F, 4.4]

$$\begin{array}{ccc} CH^{h/2}(\mathbf{C}^h_{\bar{K}}) & \longrightarrow & CH^{h/2}(C^h) \\ {\scriptstyle c\ell}\downarrow & & \downarrow{\scriptstyle c\ell} \\ H^h(\mathbf{C}^h_{\bar{K}}, \mathbb{Q}_\ell(h/2)) & \xrightarrow{\sim} & H^h(C^h, \mathbb{Q}_\ell(h/2)). \end{array}$$

The interesting piece of the cohomology $U_{C_{\bar{k}}}$ is contained in the image of the left hand verticle map. Thus the image of $c\ell$ contains U_C, which proves (2.0) when char $k > 0$.

REMARK 2.10: (Concerning the case of non-simple $2r$-tuples) Suppose that the ground field is $\mathbb{C}$ and that the $2r$-tuple associated to the pair (C, σ) satisfies the conditions of Corollary (1.9) but is not simple. In this case I do not know how to explicitly construct cycles who cohomology classes generate the Hodge structure U. Nonetheless, if $h = 4$, we can verify the Hodge conjecture for U in a significant number of cases. Since $h = -e(X - B) = 2q - 2 + 2r$, there are only four possibilities for the pair (q, r), namely $(3, 0)$, $(2, 1)$, $(1, 2)$, $(0, 3)$. In the first two cases, the $2r$-tuple is necessarily simple. In the third case, we shall argue that W_0 is uniruled. Since the Hodge conjecture is known for smooth uniruled 4-folds [C–M] or [B–S], it follows from standard arguments that $U \subset H^4(C^4, \mathbb{Q})$ is contained in the image of the cycle class map.

To construct a rational curve through a given point $w \in W_0(\mathbb{C})$, consider the image $\vec{x} \in S^4X(\mathbb{C})$. The fiber $\eta^{-1}(\eta(\vec{x}))$ is isomorphic to $\mathbb{P}^3$. There is a line in the fiber through $\vec{x}$ which meets the linear spaces $S_1 \cap S_2 \cap \eta^{-1}(\eta(\vec{x}))$ and $S_3 \cap S_4 \cap \eta^{-1}(\eta(\vec{x}))$. The inverse image of this line in W_0 is a rational curve through w, since it is a cover of $\mathbb{P}^1$ ramified in at most two points.

If $(q, r) = (0, 3)$ and if $\alpha_1 + \alpha_2 = m$ then an argument analogous to the one above shows that W_0 is again uniruled.

REMARK 2.11: (On the image of the Abel–Jacobi homomorphism for 1-cycles on C^3). Suppose for a moment that the base field is $\mathbb{C}$ and that the invariant h associated to the pair (C, σ) is odd and greater than one. Then the number of branch points of the canonical quotient map is an odd number, $s > 3$. We shall assume that an associated s-tuple $(\alpha_1, \ldots, \alpha_s) \in (\mathbb{Z}/m - \{0\})^s$ is such that $(\alpha_1, \ldots, \alpha_{s-2}, \alpha_{s-1} + \alpha_s)$ is a simple $(s - 1)$-tuple. By Lemma (1.6), $v_\chi \in \{(h - 1)/2, (h + 1)/2\}$ for each primitive $\mathbb{Z}/m$ character χ. The Hodge structure $U \subset H^h(C^h)$ has Hodge type $((h - 1)/2, (h + 1)/2) + ((h + 1)/2, (h - 1)/2)$. A more general form of the Hodge conjecture than that discussed in the introduction implies that the image of the Abel–Jacobi homomorphism on null-homologous, dimension $(h - 1)/2$ cycles on C^h contains the subtorus, $JU := U_{\mathbb{C}}/(F^{(h+1)/2}U_{\mathbb{C}} + U_{\mathbb{C}} \cap H^h(C^h, \mathbb{Z}))$, of the intermediate Jacobian of C^h.

Although I do not presently see how to prove this in general, when $h = 3$ one need only verify that W_0 is uniruled. In fact the Abel–Jacobi map is well known to be surjective for smooth uniruled 3-folds, so the desired result will follow from standard facts about resolution of singularities and functoriality of the Abel–Jacobi homomorphism. Now the above assumptions actually imply that W_0 is uniruled for any odd h. To see this observe that any closed fiber of η: $S^h X \to \mathrm{Pic}^0(X)$ is isomorphic to $\mathbb{P}^{q-2+s}$. Given any closed point $\vec{x}$ in the fiber, there is a line ℓ through $\vec{x}$ which meets each of the $(s - 1)/2$ linear spaces $S_1 \cap S_2 \cap \eta^{-1}(\eta(\vec{x})), \ldots, S_{s-2} \cap S_{s-1} \cap \eta^{-1}(\eta(\vec{x}))$. The normalization of $\beta_0^{-1}(\ell)$ can be ramified only over the points $S_s \cap \ell$ and $S_{s-2} \cap S_{s-1} \cap \ell$. Any cover of $\mathbb{P}^1_{\mathbb{C}}$ ramified in at most two points is rational. Thus W_0 is covered by rational curves.

Remark 2.12: (Extra automorphisms of curves) Let C be a smooth projective curve and σ: $\mathbb{Z}/m \to \mathrm{Aut}(C)$ an embedding with associated invariant $h = 2$ and associated $2r$-tuple simple. In this case the proof of Theorem 2.0 shows that the automorphism group of C is larger than $\mathbb{Z}/m$. Since $h = 2q - 2 + 2r$, (q, r) is one of the following three pairs: (0, 2), (1, 1) or (2, 0). In each case there is an involution of the quotient curve X which lifts to C. If $q = 2$, take the hyperelliptic involution. If $q = 1$, take the involution which switches the two branch points and has quotient P^1. When $q = 0$ there are four branch points. There is at least one involution of P^1 which stabilizes the branch locus, leaves no branch point fixed and acts by multiplication by -1 on the associated element of $(\mathbb{Z}/m - \{0\})^4$. (Indeed, with notation for the branch locus $\{b_1, b_2, b_3, b_4\}$ as in the second paragraph of this section choose $\varphi, \kappa \in \mathrm{Aut}(\mathbb{P}^1)$ with $\kappa(b_1) = 0, \kappa(b_2) = \infty, \kappa(b_3) = 1$ and $\varphi(x) = \kappa(b_4)/x$ for all $x \in \mathbb{P}^1$. Then $\kappa^{-1} \circ \varphi \circ \kappa$ is an involution of the desired type.) In each case the image of the graph of this involution in $S^2 X$ is the projective space which we have referred to as P_0 in the proof of (2.0). The m irreducible components of the pullback of P_0 in C^2 are graphs of automorphisms. In fact, $\mathrm{Aut}(C)$ contains the dihedral group D_m (by convention $D_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$). The elements of the non-trivial coset of $\mathrm{im}(\sigma$: $\mathbb{Z}/m \to D_m)$ give rise to the graphs lying over P_0. Because the map $D_m \to \mathrm{Aut}(\mathrm{Jac}(C))$ is injective, we have for $m > 2$ a factorization

$$\mathbb{Q}[D_m] \to M_2(\mathbb{Q}(\mu_m)^+) \to \mathrm{End}(\mathrm{Jac}(C)) \otimes \mathbb{Q}$$

where $\mathbb{Q}(\mu_m)^+$ denotes the maximal totally real subfield of the cyclotomic field. Thus $\mathrm{Jac}(C)$ contains two distinct isogenous abelian subvarieties with real multiplication. The fact that $M_2(\mathbb{Q}(\mu_m)^+$ is not a division algebra is consistent with the observation that our family of curves contains degenerate members whose Jacobians have multiplicative reduction. The degenerate curves are easily constructed as cyclic covers of singular curves, X.

This peculiar geometric phenomenon is related to the following equally surprising fact concerning endomorphism rings of abelian varieties. Let F be a degree $2n$ CM-field which is contained in the endomorphism algebra of a $2n$-dimensional complex abelian variety, A. Suppose that the action of F on Lie(A) makes the latter into a free rank one $F \otimes \mathbb{C}$ algebra. Then the endomorphism algebra of A actually contains a quaternion algebra with center the maximal real subfield of F [Sh, Prop. 18].

REMARK 2.13: (Concerning fields of definition of cycles) Let C be a smooth projective geometrically irreducible curve defined over a field L. Suppose that L contains m distinct mth roots of unity, that there is an embedding $\sigma: \mathbb{Z}/m \to \mathrm{Aut}(C_L)$, that the invariant $h = -e(X_{\bar{L}} - B_{\bar{L}})$ is even, and that the associated $2r$-tuple over $\bar{L}$ is simple. In addition we shall assume that the individual branch points are L-rational. With these assumptions I claim that the cycles produced on C^h in the proof of (2.0) are defined over an extension L' of L with $\mathrm{Gal}(L'/L) \simeq \mathbb{Z}/m$. In any case, it is clear that the construction of the projective space P_0 goes through over L. One can produce an element f, of the function field $L(X)$ such that $L(C) = L(X)[y]/(y^m - f)$ and such that $\hat{f} := \Pi_{1 \leqslant i \leqslant h} f \circ pr_i$ gives an element of the local ring of S^hX at the codimension $h/2$ point P_0 (see the discussion just prior to Lemma 2.2). Write $\bar{f}$ for the image of $\hat{f}$ in the function field $L(P_0)$. By the construction of P_0, the divisor of $\bar{f}$ is the mth power of an L-rational divisor on P_0. Since the sequence

$$1 \to H^0(P_{0_{\bar{L}}}, \mathcal{O}^*) \to \bar{L}(P_0)^* \to \mathrm{Div}_0(P_{0_{\bar{L}}}) \to 0$$

remains exact after taking $\mathrm{Gal}(\bar{L}/L)$ invariants (Hilbert's Theorem 90), the mth power of an L-rational degree zero divisor on P_0 is the divisor of the mth power of a function in $L(P_0)^*$. Thus there is $\ell \in L^*$ such that $\bar{f}\ell \in L(P_0)^{*m}$. The fiber over the generic point of P_0 may be written Spec $L(P_0) \times_{S^hX} W_0 \simeq$ Spec $L(P_0)[\hat{y}]/\hat{y}^m - \bar{f}$ by the discussion prior to (2.2). So the cycles produced in the proof of (2.0) are rational over $L(\ell^{1/m})$.

If no restrictions are placed on the rationality of the individual branch points other than that the branch divisor B be L-rational, then the above claim will not in general be true. Even the cohomology classes of the cycles will generally not be defined over $L(\ell^{1/m})$.

3. An application to certain abelian varieties

In this section the results of §2 are used to construct explicit algebraic cycles which generate the Hodge substructure of Weil for certain abelian varieties. Our technique applies most directly when the associated complex multiplication field is cyclotomic. It seems to be of little or no use when the *CM* field is non-abelian or non-Galois over $\mathbb{Q}$. Even in the cyclotomic case the results are not as complete as one might hope unless both the degree of the *CM* field and the dimension of the abelian variety are small. Very loosely speaking, Theorem 2.0 allows us to construct algebraic cycles on certain "generalized Prym varieties". In most, but not all cases, such varieties form a subspace of positive codimension in the moduli of all Weil abelian varieties of fixed dimension with multiplication by a fixed cyclotomic field.

Let m be an integer greater than two and $\varphi(m) = \#(\mathbb{Z}/m)^*$. Write R for an order in the cyclotomic field, $K = \mathbb{Q}(\mu_m)$, and let (A, θ) denote a complex abelian variety together with a ring homomorphism θ: $R \to \mathrm{End}(A)$ satisfying $\theta(1) = Id$. Then θ induces the structure of K-vector space on $H^1(A, \mathbb{Q})$. Let $h = \dim_K H^1(A, \mathbb{Q})$. Following Weil [W] (see also [G, §1] and [D–M, §4]) we may associate to this data a Hodge substructure $U' \subset \Lambda^h H^1(A, \mathbb{Q}) \simeq H^h(A, \mathbb{Q})$. To describe U' let $\psi_1, \ldots, \psi_{\varphi(m)}$ denote the distinct embeddings of K into $\mathbb{C}$. Observe that each character appears with multiplicity h in the representation of the torus K^* in $H^1(A, \mathbb{Q}) \otimes \mathbb{C}$. Then U' is the unique K^*-sub-representation of $\Lambda^h H^1(A, \mathbb{Q})$ which after tensoring with $\mathbb{C}$ becomes isomorphic to the sum of weight spaces. $\oplus_{1 \leqslant i \leqslant \varphi(m)} \psi_i^h$. Let $\{\omega'_{1,\psi_i}, \ldots, \omega'_{h,\psi_i}\}$ be a basis for the ψ_i-eigenspace of $H^1(A, \mathbb{Q}) \otimes \mathbb{C}$. Then $\{\omega'_{1,\psi_i} \wedge \ldots \wedge \omega'_{h,\psi_i}\}_{1 \leqslant i \leqslant \varphi(m)}$ is a basis for $U' \otimes \mathbb{C}$. Suppose now that $H^0(A, \Omega_A)$ is a free $K \otimes \mathbb{C}$ module. In other words each ψ_i appears with multiplicity $h/2$ in the representation of K^* on $H^{1,0}(A)$. We may then choose ω'_{j,ψ_i} to lie in $H^{1,0}(A)$ when $j \leqslant h/2$ and to lie in $H^{0,1}(A)$ when $j > h/2$. In this case $U' \otimes \mathbb{C}$ has pure Hodge type $(h/2, h/2)$. This construction generalizes to the case that K is an arbitrary *CM* field, but we shall ignore this.

Now let (C, σ) be a smooth, irreducible, complex, projective curve together with an embedding σ: $\mathbb{Z}/m \to \mathrm{Aut}(C)$. We shall be interested in the abelian subvariety $B \subset \mathrm{Alb}(C)$ whose tangent space is the subspace of $T_e\mathrm{Alb}(C) \simeq H^0(C, \Omega_C)^*$ where $\mathbb{Z}/m$ acts by primitive characters. More precisely, B is the image of the composition, P, of all elements $\Sigma_{t \in H}(Id - t) \in \mathrm{End}(\mathrm{Alb}(C))$ as H ranges over the non-zero subgroups of $\mathbb{Z}/m$. Since B has the correct sort of multiplication from K, $U' \subset H^h(B, \mathbb{Q})$ is defined. In the notation of (1.4), $U'' = P^*U'$, $U = \Xi^*U''$. We can now state the following corrollary of (2.0).

COROLLARY 3.1: *If* (C, σ) *satisfies the hypotheses of* (2.0), *then* $U' \subset H^h(B, \mathbb{Q})$ *is generated by fundamental classes of codimension* $h/2$ *algebraic cycles.*

Proof: Let j: $B \to \mathrm{Alb}(C)$ denote the inclusion. Then $P \circ j \in \mathrm{End}(B)$ is multiplication by a non-zero scalar. Thus it suffices to show that $U'' = P^*U'$

is generated by algebraic cycles. Recall the notation $\Xi\colon C^h \to \mathrm{Alb}(C)$ from (1.4) and write W for the cohomology class of the canonical polarization on $\mathrm{Alb}(C)$. Given $\alpha \in H^{\cdot}(\mathrm{Alb}(C), \mathbb{Q})$ the projection formula and Poincaré's formula [G–H, p. 350] yield

$$\Xi_*\Xi^*(\alpha) \;=\; \alpha \cdot \Xi_*(1) \;=\; (h!/(g-h)!)\alpha \cdot (W)^{g-h}.$$

By the hard Lefschetz theorem $\cdot\, W^{g-h}\colon H^h(\mathrm{Alb}(C), \mathbb{Q}) \to H^{2g-h}(\mathrm{Alb}(C), \mathbb{Q})$ is an isomorphism. According to a theorem of Lieberman [K; 2A11.2, 2.3] the inverse to the hard Lefschetz isomorphism on an abelian variety is "algebraic". In particular there is an algebraic cycle on $\mathrm{Alb}(C) \times \mathrm{Alb}(C)$ whose cohomology class, T, satisfies $pr_{2*}(T \cdot pr_1^*(\alpha \cdot W^{g-h})) = \alpha$ for all α, where pr_i denotes projection on the ith factor in the product. Thus

$$U'' \;=\; pr_{2*}(T \cdot pr_1^*(U'' \cdot W^{g-h})) \;=\; pr_{2*}(T \cdot pr_1^*(\Xi_*(U))).$$

The Corrollary follows since U is generated by algebraic cycles (2.0). □

Suppose that A is a complex abelian variety with a $\mathbb{Z}/m$-action such that $H^1(A, \mathbb{Q})$ inherits the structure of an h-dimensional K-vector space. In this case the fixed locus of any non-trivial element of $\mathbb{Z}/m$ consists of finitely many points. If $\dim(A) > 1$, A will contain $\mathbb{Z}/m$-stable smooth curves C disjoint from the fixed locus. In general it is too much to expect that the cycle classes constructed in (2.0) in the middle dimensional cohomology of some power of C will give rise to interesting cycle classes on A. One problem is that if C has large intersection number with an ample divisor on A, the correct power of C to take in (2.0) will be much larger than $\dim(A)$.

Nonetheless it is true that (2.0) may be used to prove the Hodge conjecture at the general point in moduli for a few of Weil's families of abelian varieties provided both m and h are sufficiently small. We may get some idea as to how small these invariants must be by counting moduli. Suppose that A is a complex, polarized, abelian variety with a $\mathbb{Z}/m$-action such that $H^0(A, \Omega_A)$ becomes a free $\mathbb{Q}(\mu_m) \otimes \mathbb{C}$ module of rank $h/2$. The cotangent space to the space of those deformations of A to which the polarization and the group action lift is isomorphic to the $\mathbb{Z}/m$-invariant subspace of $\mathrm{Sym}^2 H^0(A, \Omega_A)$. This has dimension $(\varphi(m)/2)(h/2)^2$. On the other hand we may determine the number of moduli of pairs, (C, σ), where $\sigma\colon \mathbb{Z}/m \to \mathrm{Aut}(C)$ is injective and the associated invariant is an even number, h. Recall that h is related to the genus, q, of the quotient curve, X, and the number of branch points, $2r$, by the formula $h = 2q - 2 + 2r$. If $q \geqslant 2$, X moves in a family of dimension $3q - 3$. It is easy to see that the number of moduli for pairs (C, σ) is

$3q - 3 + 2r$. In fact this formula is valid when $q = 0$ or 1 as well. For fixed h the number of moduli is largest when $r = 0$, in which case it is $(3/2)h$. Since the dimension of the moduli space for polarized abelian varieties, A, with $\mathbb{Z}/m$ action is actually equal to the dimension of the cotangent space, it is apparent that the general abelian variety in this family will not be dominated by the Jacobian of one of our curves, C, unless $3/2 \geqslant h\varphi(m)/8$. Now the Hodge structure $H^h(C, \mathbb{Q})$ is only interesting from the point of view of the Hodge conjecture when $h \geqslant 4$. Thus the only pairs (h, m) for which the method of proof of (3.1) has a chance to establish the Hodge conjecture at a general point in the moduli of a Weil family are (4, 3), (4, 4), (6, 3), (6, 4). (The case $m = 6$ is ignored, since it is equivalent to the case $m = 3$).

Most of the remainder of this section is devoted to proving the Hodge conjecture in the case (4, 3).

THEOREM 3.2: *Let $K = \mathbb{Q}(\mu_3)$, R an order in K, A a complex 4-dimensional abelian variety, and θ: $R \to \mathrm{End}(A)$ a unitary ring homomorphism. Assume that each embedding ψ_i: $K \to \mathbb{C}$, $i \in \{1, 2\}$, appears with multiplicity two in the K^*-representation $H^{1,0}(A)$. Then Weil's Hodge substructure, U', is generated by fundamental classes of two-dimensional algebraic cycles on A.*

Proof: In approximate terms the idea is to dominate A by a fourfold self-product of a genus seven curve with a fixed point free, order three, automorphism. The cycles on this product are constructed by means of (2.0). They push forward to give the required cycles on A.

First construct a suitable family of genus seven curves. We begin with a connected component of the fine moduli space for genus three curves whose jacobians have a full level 3-structure. After an étale base change we may construct

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\pi} & \mathscr{X} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle r} & \\ & \mathscr{N} & \end{array}$$

where r is the pullback of the universal family of genus three curves, π is an étale $\mathbb{Z}/3$-cover corresponding to a 3-torsion section of $\mathrm{Pic}^0_{\mathscr{X}/\mathscr{N}}$, and the family of genus 7 curves γ has a section, s_1. The element $P \in \mathrm{End}(\mathrm{Alb}_{\mathscr{C}/\mathscr{N}})$ defined earlier in this section is simply $P = 3 \cdot Id - \Sigma_{t \in \mathbb{Z}/3}\, \sigma(t)$ in the present context. The image of P, denoted $\mathscr{B}$, is an abelian subscheme of relative dimension 4 over $\mathscr{N}$. Now $\mathscr{B}$ inherits a polarization from $\mathrm{Alb}_{\mathscr{C}/\mathscr{N}}$ and we may assume, after an additional étale base extension if necessary, that $\mathscr{B}$ has a full level n-structure for some prime $n > 3$.

Next we recall the existence of a moduli space parametrizing the abelian varieties of interest.

LEMMA 3.3: *There is a polarized abelian scheme* $\alpha\colon \mathscr{A} \to \mathscr{W}$ *having the following properties*:

(*i*) $\mathscr{W}$ *is a smooth irreducible quasi-projective variety of dimension* 4.

(*ii*) α *has relative dimension* 4.

(*iii*) *There is an injective homomorphism* $\sigma\colon \mathbb{Z}/3 \to \mathrm{Aut}(\mathscr{A}/\mathscr{W})$. *The automorphisms fix the identity section and the polarizing class. Furthermore, the rank* 4 *vector bundle* $\alpha_*\Omega_{\mathscr{A}/\mathscr{W}}$ *is the direct sum of rank two sub-bundles on which* $\mathbb{Z}/3$ *acts by nontrivial conjugate characters.*

(*iv*) *The kernel of multiplication by n*, $\mathscr{A}[n]$, *is isomorphic to* $(\mathbb{Z}/n)^8$ *as a group scheme over* $\mathscr{W}$.

(*v*) *If* (A, θ) *satisfies the hypothesis of Theorem* 3.2, *then there is a closed point* $w \in \mathscr{W}$ *and an isogeny* $j\colon A \to \mathscr{A}_w$ *which is compatible with the action of the order* $R \subset \mathbb{Z}[\mu_3]$ *on the two varieties.*

(*vi*) *There is a Cartesian diagram*

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\Phi} & \mathscr{A} \\ \downarrow & & \downarrow \\ \mathscr{N} & \xrightarrow{\varphi} & \mathscr{W} \end{array}$$

where Φ *is a morphism of polarized abelian schemes with level n-structure.*

Proof: The reader is referred to [W] and [G, §1], but will have to fill in some details himself.

The existence of the section s_1 of $\mathscr{C}/\mathscr{N}$ enables one to define the standard morphism

$$\Xi\colon \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \to \mathrm{Alb}_{\mathscr{C}/\mathscr{N}}.$$

Our intention is to show that the composition

$$\Phi \circ P \circ \Xi\colon \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \times_{\mathscr{N}} \mathscr{C} \to \mathscr{A} \tag{3.4}$$

is dominant. Since the image of Ξ is, up to a non-zero scalar multiple, an intersection of relatively ample divisors, $P \circ \Xi$ is certainly dominant. By (3.3, vi) it suffices to produce a closed point $\mathbf{o} \in \mathscr{N}$ where the tangent map $T_{\mathbf{o}}\varphi$ is surjective.

Eventually, $T_0\varphi$ will be described in terms of an easily computed map between sheaf cohomology vector spaces. In order to effect this reduction, we recall the yoga of the Kodaira–Spencer map. Let $S/\mathbb{C}$ be a finite type scheme, $s \in S(\mathbb{C})$, and $f\colon Z \to S$ a smooth proper morphism with connected fibers. The Kodaira–Spencer map

$$\kappa_Z\colon T_sS \simeq H^0(Z_s, f^*\theta_{S|Z_s}) \to H^1(Z_s, \theta_{Z_s})$$

is constructed from the coboundary in the exact cohomology sequence associated to

$$0 \to \theta_{Z_s} \to \theta_{Z|Z_s} \to f^*\theta_{S|Z_s} \to 0. \tag{3.5}$$

Given a group action $\sigma\colon G \to \mathrm{Aut}(Z/S)$, there is an induced action on (3.5) and the image of κ_Z lies in the invariant subspace $H^1(Z_s, \theta_{Z_s})^\sigma$.

Given a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\Psi} & Z' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{\psi} & S' \end{array} \tag{3.6}$$

where $\psi(s) = s'$ with f and f' as above, there is induced a commutative diagram

$$\begin{array}{ccccc} H^1(Z_s, \theta_{Z_s}) & \xrightarrow{T\Psi} & H^1(Z_s, \Psi^*\theta_{Z'_{s'}}) & \xleftarrow{\Psi^\#} & H^1(Z'_{s'}, \theta_{Z'_{s'}}) \\ {\scriptstyle \kappa_Z}\uparrow & & & & \uparrow{\scriptstyle \kappa_{Z'}} \\ T_sS & & \xrightarrow[T_s\psi]{\qquad\qquad} & & T_{s'}S' \end{array} \tag{3.7}$$

where $\Psi^\#$ denotes the compositum of the obvious maps

$$H^1(Z'_{s'}, \theta_{Z'_{s'}}) \to H^1(Z'_{s'}, \Psi_*\Psi^*\theta_{Z'_{s'}}) \to H^1(Z_s, \Psi^*\theta_{Z'_{s'}}).$$

If (3.6) is Cartesian, $\Psi^\#$ and $T\Psi$ are ismorphisms. Using the resulting identification we may abbreviate (3.7) with the commutative diagram,

$$\begin{array}{ccc} T_sS & \xrightarrow{\kappa_Z} & H^1(Z_s, \theta_{Z_s}) \\ {\scriptstyle T_s\psi}\downarrow & \nearrow{\scriptstyle \kappa_{Z'}} & \\ T_{s'}S' & & \end{array} \tag{3.8}$$

Write C (resp. B) for the fiber $\mathscr{C}_{\mathbf{o}}$ (resp. $\mathscr{B}_{\mathbf{o}}$) over a closed point $\mathbf{o} \in \mathscr{N}$. An application of (3.8) to the diagram (3.3vi) gives rise to a commutative diagram

$$\begin{array}{ccc} T_{\mathbf{o}}\mathscr{N} & \xrightarrow{\kappa_{\mathscr{B}}} & H^1(B, \theta_B) \\ {\scriptstyle T_{\mathbf{o}}\varphi}\downarrow & \nearrow{\scriptstyle \kappa_{\mathscr{A}}} & \\ T_{\varphi(\mathbf{o})}\mathscr{W} & & \end{array} \tag{3.9}$$

Now $\kappa_{\mathscr{A}}$ maps $T_{\varphi(\mathbf{o})}\mathscr{W}$ isomorphically to the invariants, H^σ, of the subspace $H \subset H^1(B, \theta_B)$ which is annihilated by cup product with the polarizing class under the pairing

$$H^1(B, \theta_B) \otimes H^1(B, \Omega_B) \to H^2(B, \mathscr{O}_B).$$

Thus to show that $T_{\mathbf{o}}\varphi$ is surjective it suffices to show that the image of $\kappa_{\mathscr{B}}$ which is contained in H^σ is in fact all of H^σ.

From (3.7) there is a commutative diagram

$$\begin{array}{ccc} T_{\mathbf{o}}\mathscr{N} & \xrightarrow[\sim]{\kappa_{\mathscr{C}}} & H^1(C, \theta_C)^\sigma \\ {\scriptstyle \kappa_{\mathscr{B}}}\downarrow & & \downarrow{\scriptstyle T\Psi} \\ H^\sigma & \xrightarrow[\Psi_\#]{} & H^1(C, \Psi^*\theta_B)^\sigma \simeq (H^1(C, \mathscr{O}_C) \otimes \mathbf{t})^\sigma, \end{array} \tag{3.10}$$

where $\mathbf{t}$ denotes the Lie algebra of B and the top horizontal map may be factored as

$$T_{\mathbf{o}}\mathscr{N} \overset{\kappa_{\mathscr{X}}}{\simeq} H^1(X, \theta_X) \simeq H^1(X, \theta_X \otimes \pi_* \mathscr{O}_C)^\sigma \simeq H^1(C, \pi^*\theta_X)^\sigma \simeq H^1(C, \theta_C)^\sigma.$$

We have the following identifications of dual spaces:

(i) $(H^1(C, \theta_C)^\sigma)^* \simeq H^0(C, \omega_C^{\otimes 2})^\sigma \simeq H^0(X, \omega_X^{\otimes 2})$ (Serre duality)

(ii) $((H^1(C, \mathscr{O}_C) \otimes \mathbf{t})^\sigma)^* \simeq (H^0(C, \omega_C) \otimes \mathbf{t}^*)^\sigma \simeq (\mathbf{t}^* \otimes \mathbf{t}^*)^\sigma$ since $\mathbf{t}$ is canonically a subpace of $H^0(C, \omega_C)^*$, the Lie algebra of $\mathrm{Alb}(C)$.

(iii) $H^* \simeq \mathrm{Sym}^2\mathbf{t}^*$ [B, 7.3.5]. If $\mathscr{L}$ is a 3-torsion invertible sheaf on X such that $\pi_*\mathscr{O}_C \simeq \oplus_{i \in \mathbb{Z}/3} \mathscr{L}^i$, then $(\mathrm{Sym}^2\mathbf{t}^*)^\sigma \simeq H^0(X, \mathscr{L} \otimes \omega_X) \oplus H^0(X, \mathscr{L}^{-1} \otimes \omega_X)$.

LEMMA 3.11: *With these identifications the dual of* (3.10) *may be written*

$$\begin{array}{ccc} T_{\mathbf{o}}\mathscr{N}^* & \xleftarrow[\sim]{\kappa_{\mathscr{C}}^*} & H^0(C, \omega_C^{\otimes 2})^\sigma \\ {\scriptstyle \kappa_{\mathscr{B}}^*}\uparrow & \nearrow{\scriptstyle \bar{m}} & \uparrow{\scriptstyle m} \\ (\mathrm{Sym}^2\mathbf{t}^*)^\sigma & \xleftarrow[\text{can}]{} & (\mathbf{t}^* \otimes \mathbf{t}^*)^\sigma. \end{array} \tag{3.12}$$

where m and $\bar{m}$ are the tautological multiplication maps.

Proof: This is essentially proved in [B, 7.5].

Thus to show that $\kappa_{\mathscr{B}}$ surjects onto H^{σ} we must show that the natural multiplication map $\bar{m}$ injects or equivalently that

$$m': H^0(X, \mathscr{L} \otimes \omega_X) \otimes H^0(X, \mathscr{L}^{-1} \otimes \omega_X) \to H^0(X, \omega_X^{\otimes 2})$$

is injective. Since each factor on the left has dimension two and since $\mathscr{L} \otimes \omega_X \otimes (\mathscr{L}^{-1} \otimes \omega_X)^{-1}$ has no global sections, the base point free pencil trick [A–C–G–H, p. 126] assures injectivity once we have checked that the linear system $H^0(X, \mathscr{L}^{-1} \otimes \omega_X)$ has no base points. If p were a base point, then

$$\begin{aligned} h^0(\mathscr{L}^{-1} \otimes \omega_X(-p)) &= 2 \Rightarrow h^1(\mathscr{L}^{-1} \otimes \omega_X(-p)) \\ &= 1 \Rightarrow h^0(\mathscr{L}(p)) = 1. \end{aligned}$$

So we could find a point $q \neq p$ with $\mathscr{L} \simeq \mathscr{O}_X(q - p)$. Since $\mathscr{L}^{\otimes 3} \simeq \mathscr{O}_X$, there is some $f \in \mathbb{C}(X)$ with $\operatorname{div}(f) = 3q - 3p$. The resulting degree 3 morphism $f\colon X \to \mathbb{P}^1$ is branched above at most six points other than 0 and ∞ (Hurwitz). Modulo the action of $\operatorname{Aut}(\mathbb{P}^1, \{0, \infty\})$ the branch locus is free to move in a 5-parameter family. But the moduli space for genus 3 curves has dimension 6. Hence a general genus 3 curve X can not be realized as a degree 3 cover of $\mathbb{P}^1$ which is totally ramified at two points. The base point freeness of the linear system $H^0(X, \mathscr{L}^{-1} \otimes \omega_X)$ for general X follows. Thus for a general closed point $\mathbf{o} \in \mathscr{N}$ the tangent map $T_{\mathbf{o}}\varphi$ is surjective. This completes the proof that $\varphi\colon \mathscr{N} \to \mathscr{W}$, and hence $\Phi\colon \mathscr{B} \to \mathscr{A}$, is dominant.

Now given an Abelian four-fold with complex multiplication (A, θ) satisfying the hypotheses of Theorem 3.2, choose a closed point $w \in \mathscr{W}$ and an isogeny $j\colon A \to \mathscr{A}_w$ satisfying (3.3v). If $w \in \operatorname{im}(\varphi)$ for some closed point $\mathbf{o} \in \mathscr{N}$, the Hodge structure U' maybe regarded as a subspace of $H^4(\mathscr{B}_{\mathbf{o}}, \mathbb{Q})$. In this case (3.2) follows from Corollary 3.1.

Now suppose $w \notin \operatorname{im}(\varphi)$. Choose an irreducible curve $T \subset \mathscr{N}$ for which $w \in \overline{\operatorname{im} \varphi(T)}$ and write ε for the generic point of T. Let $\varepsilon' \to \varepsilon$ correspond to a finite field extension with the property that cycles constructed in the proof of Theorem 2.0 on $(\mathscr{C}_{\bar{\varepsilon}})^4$ are rational over ε'. Let T' be a smooth curve with generic point ε' which maps surjectively to $\overline{\operatorname{im} \varphi\ (T)}$ and write $\mathscr{A}_{T'}$ for the pullback of $\mathscr{A}$ to T'. Let $w' \in T'$ map to w. There is a specialization map $CH^2(\mathscr{A}_{\varepsilon'}) \to CH^2(\mathscr{A}_{w'})$ [F] and a specialization isomorphism $H^4(\mathscr{A}_{\bar{\varepsilon}}, \mathbb{Q}_\ell(2)) \xrightarrow{\sim} H^4(\mathscr{A}_{w'}, \mathbb{Q}_\ell(2))$. These are compatible with cycle class maps [SGA 6, X.7.13–7.16]. It follows that $j^*U' \subset H^4(\mathscr{A}_w, \mathbb{Q}_{\mathscr{L}}(2))$ is generated by fundamental classes of algebraic cycles. This completes the proof of (3.2).

REMARK 3.13: Let A' denote a complex abelian 3-fold with an action of $\mathbb{Z}/3$ which respects the group law. Suppose that in the resulting representation on the tangent space at the identity one primitive $\mathbb{Z}/3$-character appears with multiplicity one and the other with multiplicity two. Let A'' denote the blow up of A' at the finite set of points which comprise the fixed locus of the $\mathbb{Z}/3$-action. The resulting $\mathbb{Z}/3$-action on A'' fixes certain smooth rational curves and isolated points in the exceptional locus. By blowing these up and taking the quotient, one arrives at a smooth threefold, Y, of Kodaira dimension zero, with Hodge numbers $h^{3,0} = h^{1,0} = 0$, $h^{2,1} = 1$, $h^{2,0} = 2$. Theorem (3.2) may be used to show that the Abel–Jacobi homomorphism for null-homologous, codimension two cycles on Y is surjective. In fact one need only apply (3.2) to the Abelian 4-fold $E \times A'$ where $E \simeq \mathbb{C}/\mathbb{Z}[\mu_3]$. If A' is the Jacobian of a smooth genus three curve C with function field $\mathbb{C}(C) \simeq \mathbb{C}(t)[y]/y^3 - (t - b_1)^2 \Pi_{2 \leqslant i \leqslant 5} (t - b_i)$, then the surjectivity also follows from Remark 2.11. Details are left to the reader.

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